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13. ABSTRACT: Let W=UV where U and V are independent random variables. It is shown that if V is distributed according to the non-central Chi-square distribution, then W is distributed according to the Chi-square distribution if and only if U=1 with probability 1. If V is normally distributed, it is shown that W is normally distributed if and only if the distribution of U is a two-point distribution.

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ON THE DISTRIBUTION OF A PRODUCT OF TWO RANDOM VARIABLES

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On The Distribution Of A Product

Of Two Random Variaoles

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Summary. Let U and V be two independent random variables, and let W = UV. It is shown that if V is distributed as $X^2_{2m,\delta}$, a non-central Chi-square with 2m degrees of freedom and non-centrality parameter δ, then W is distributed as a Chi-square random variable if, and only if U = 1 with probability 1. If V is normally distributed with mean μ, say, it is shown that W is normally distributed if, and only if, the distribution of U is degenerate at a point distinct from the origin, when $\mu \neq 0$, or the distribution has probability mass at two points equidistant from the origin. This result is generalized for the case in which V is a random vector, distributed according to the multivariate normal distribution and U is an independent random matrix.

1. Main results. Let U and V be independent random variables, and let . be distributed as $\chi^2_{2m,\delta}$, that is, non-central Chi-square with 2m degrees of freedom and non-centrality parameter δ. Let $W = UV$. Theorem 1.1 below, gives a necessary and sufficient condition that W is distributed as a Chi-square random variable.

Theorem 1.1. W is distributed as a Chi-square random variable if, and only if, $U = 1$ with probability 1.

Proof: The sufficiency part of the theorem is trivial. We consider the necessary condition of the theorem. Let W be distributed as $\chi^2_{2m',\delta'}$, where $m' > 0$ and $\delta' \geq 0$. Clearly, $U > 0$ with probability 1. Suppose that $P\{U \geq 1+\zeta\} = \alpha$, where α and ζ are positive numbers. Let $C = \frac{1}{2}(1+\zeta)^{-1}$. Then

$$
\begin{aligned}
(1.1) \qquad E\, e^{CW} &= E(e^{CUV}) \\
&\geq \alpha E(e^{\frac{1}{2}V}).
\end{aligned}
$$

where E denotes expectation. From the distribution of Chi-square it is seen that the left hand side of (1.1) is finite, whereas the right hand side is infinite. Therefore, $P\{U > 1\} = 0$.

The density function of the distribution of $\chi^2_{2m,\delta}$ is given by

(1.2) $$g(x) = \sum_{r=0}^{\infty} e^{-\delta} \frac{\delta^r}{r!} f_{2m+2r}(x); \quad x > 0$$

where

(1.3) $$f_{2m}(x) = x^{m-1}e^{-x/2}/(2^m\Gamma(m))$$

denotes the density function of the distribution of χ^2_{2m}, a central Chi-square with 2m degrees of freedom. The Laplace transform of the distribution of $\chi^2_{2m,\delta}$ is given by

(1.4) $$\phi(\lambda) = \int_0^{\infty} e^{-\lambda x} g(x)dx$$

$$= \sum_{r=0}^{\infty} e^{-\delta} \frac{\delta^r}{r!} (1+2\lambda)^{-m-r}$$

$$= (1+2\lambda)^{-m}\exp(-2\lambda\delta/(1+2\lambda))$$

where $\lambda > -\frac{1}{2}$. Therefore, the Laplace transform of the distribution of $W = UV$ is given by

(1.5) $$\psi(\lambda) = E((1+2\lambda U)^{-m}\exp(-2\lambda\delta U/(1+2\lambda U))).$$

As W is distributed as $\chi^2_{2m',\delta'}$, we have from (1.4) that

$$(1.6) \qquad \psi(\lambda) = (1+2\lambda)^{-m'}\exp(-2\lambda\delta'/(1+2\lambda)).$$

We shall show that $m = m'$ and $\delta = \delta'$. Let $H(u)$ denote the distribution function of U, and let

(1.7)

$$h(u,\lambda)=(1+2\lambda u)^{-m}\exp(-2\lambda\delta u/(1+2\lambda u))/\int_0^1(1+2\lambda u)^{-m}\exp(-2\lambda\delta u/(1+2\lambda u))dH(u)$$

$$0 < u \leq 1.$$

Let $\lambda > 0$. It is seen that $h(u,\lambda)$ represents a density function with monotone likelihood ratio property in λ, that is,

$$\int_0^1 h(u,\lambda)dH(u) = 1$$

and

$$(1.8) \qquad h(u,\lambda)h(u',\lambda')-h(u',\lambda)h(u,\lambda') \geq 0$$

for $u' \geq u$ and $\lambda' \geq \lambda$. From the monotone likelihood ratio property, given by (1.8), it follows that

$$H'(x;\lambda) = \int_0^x h(u,\lambda)dH(u), \quad 0 < x \leq 1$$

is non-increasing in λ for each x. Therefore, for any non-decreasing (non-increasing) integrable function $\zeta(u)$, we have that

$$\eta(\lambda) = \int_0^1 \zeta(u)dH'(u;\lambda)$$

is non-decreasing (non-increasing) in λ. Thus

$$\eta(\lambda) \leq \int_0^1 \zeta(u)dH(u) \tag{1.9}$$

if $\zeta(u)$ is non-increasing in u.

From (1.5) we have

$$-\delta\log\psi(\lambda)/\delta\lambda = \int_0^1 \left(\frac{2mu}{1+2\lambda u} + \frac{2\delta u}{(1+2\lambda u)^2}\right)dH'(u;\lambda).$$

or

$$-\lambda\partial\log\psi(\lambda)/\partial\lambda = m + \int_0^1 \left(\frac{\delta-m}{1+2\lambda u} - \frac{\delta}{(1+2\lambda u)^2}\right)dH'(u;\lambda). \tag{1.10}$$

On the other hand, from (1.6) we have

$$-\lambda\partial\log\psi(\lambda)/\partial\lambda = m' + \frac{\delta'-m'}{1+2\lambda} - \frac{\delta'}{(1+2\lambda)^2} \tag{1.11}$$

From (1.9) we have that

$$\int_0^1 (1+2\lambda u)^{-1} dH'(u;\lambda) \leq \int_0^1 (1+2\lambda u)^{-1} dH(u)$$

for $\lambda > 0$. Therefore, the integral on the right hand side of (1.10) tends to zero as $\lambda \to \infty$. It follows from (1.10) and (1.11), letting $\lambda \to \infty$, that $m = m'$, and also $\delta - m = \delta' - m'$. Thus, $\delta = \delta'$.

As V and W have the same Chi-square distribution, it follows that $U = 1$ with probability 1. The Theorem is proved.

Let U and V be independent random variables as in the preceding. From Theorem 1.1 we have the following corollary.

Corollary 1.1. If V is normally distributed with mean μ then W is normally distributed if, and only if, the distribution of U is degenerate at a point distinct from the origin, when $\mu \neq 0$, or the distribution has probability mass at two points, equidistant from the origin.

Proof: The sufficiency part of the corollary is trivial. We consider the necessary condition of the corollary. It is given that V and W are normally distributed. Let ν denote the mean of W. Without loss of generality we may assume that $\mathrm{Var}(V) = \mathrm{Var}(W) = 1$. Then V^2 and W^2 are distributed as Chi-square with 1 degree of freedom. As $W^2 = U^2V^2$, from Theorem 1.1 we have $\mu^2 = \nu^2$, and

(1.12) $$P\{U^2 = 1\} = 1.$$

As $\nu = E(W) = \mu E(U)$, we have that $E(U) = 1$ or -1 if $\mu \neq 0$. Therefore, $P\{U = 1\} = 1$ or $P\{U = -1\} = 1$, if $\mu \neq 0$. Thus the distribution of U is degenerate if $\mu \neq 0$. This completes the proof of the corollary.

Next we generalize the results of Corollary 1.1. Let A be a $p \times p$ random matrix which is non-singular with probability 1, and let Y be a p-component random vector, distributed according to the multivariate normal distribution with mean ζ and covariance $\sum$, where $\sum$ is a diagonal matrix. Let ζ_i and α_i denote the ith component and the ith vector, respectively, of ζ and A. We shall say that the distribution of α_i has property π if $P\{\alpha_i = \pm\alpha\} = 1$ for some non-null vector α. Suppose that $\alpha_1, \ldots, \alpha_p$ and Y are mutually independent. Let $Z = AY$. We have the following result.

Corollary 1.2. The distribution of Z is multivariate normal if, and only if, for each $i = 1, \ldots, k$, the distribution of α_i is either degenerate at a point distinct from the origin if $\zeta_i \neq 0$, or if the distribution of α_i has property π.

Proof: It is known that a p-component vector X is multivariate normal if and only if the distribution of $\lambda' X$ is univariate normal for all non-null p-component row vector λ'. Therefore Z is normally distributed if and only if

$\lambda'Z$, given by

$$(1.13) \qquad \lambda'Z = \sum_{i=1}^{p} (\lambda'\alpha_i)Y_i$$

is normally distributed for all non-null vectors λ', where Y_i denotes the ith component of Y. The sufficiency part of the corollary is shown easily. We shall prove the necessary condition of the corollary.

Suppose that $\lambda'Z$ is normally distributed for all non-null vectors λ'. As the components of Y and the columns of A are independent, the terms of the summation on the right hand side of (1.13), are independently distributed. By the reproductive property of the normal distribution it follows that $(\lambda'\alpha_i)Y_i$ is normally distributed for each $i = 1, \ldots, p$. From Corollary 1.1 it follows that the distribution of $\lambda'\alpha_i$ is discrete which is either degenerate at a point distinct from the origin, or it has saltus at two points equidistant from the origin, and that the distribution is degenerate if $\zeta_i \neq 0$. This result, which holds for all non-null vector λ', implies the conclusion of the corollary.

The following result, due to Kingman and Graybill [1], is related to Corollary 1.2. Let Y be a p-component random vector, and let the components of Y be independently and identically distributed. Let $A = (a_{ij})$ be a $p \times p$ random matrix which is orthogonal with probability 1, and $E(\sum_{j=1}^{p} a_{ij}) \neq 0$ for some i. Let $Z = AY$. From Theorem 3.1 of [1] it follows that $Z \overset{d}{\sim} N(0, I_p)$ if, and only if, $Y \overset{d}{\sim} N(0, I_p)$ where I_p denotes the $p \times p$ identity matrix.

A plausible result supplementing Theorem 1.1 is given below as a conjecture. Let X and Y be independent random variables, and let $Z = XY$.

Conjecture I: If $E(Z) \neq 0$ then Z is normally distributed if and only if either X or Y is normally distributed.

Suppose that Conjecture I is true. From Corollary 1.1 we have the following proposition characterizing the normal distribution.

Proposition: If $E(Z) \neq 0$ then Z is normally distributed if, and only if, between the two random variables X and Y one of them is normally distributed and the other is degenerate at a point distinct from the origin.

The following example shows that the assumption $E(Z) \neq 0$ is necessary for Conjecture I to be true. Let X' be a random variable distributed as $(\chi^2_{2m})^{1/4}$, that is, 4th root of Chi-square with 2m degrees of freedom, and let U be an independent random variable taking values +1 and -1 with equal probability $\frac{1}{2}$. Set X = UX'. Let Y be distributed as $(4\chi^2_{2m+1})^{1/4}$ independent of U and X'. Clearly, the distribution of Z = XY is symmetrical about the origin.

Let $\Phi(t)$ denote the characteristic function of $\text{Log}Z^2$. As the characteristic function of $\log\chi^2_{2m}$ is equal to $2^{it}\Gamma(m+it)/\Gamma(m)$, and $Z^2 = (X')^2Y^2$, we have

$$(1.14) \quad \Phi(t) = 2^{2it}\Gamma(m+\tfrac{it}{2})\Gamma(m+\tfrac{it}{2} + \tfrac{1}{2})/\Gamma(m)\Gamma(m+1/2)$$

$$= 2^{it}\Gamma(2m+it)/\Gamma(2m).$$

The second line on the right hand side of (1.14) is obtained from Legendre's duplication formula for the gamma function. Let $m = \frac{1}{4}$ then (1.14) shows that Z^2 is distributed as χ^2_1. As the distribution of Z is symmetrical about the origin, it follows that Z is normally distributed with mean zero and variance 1, whereas neither X nor Y is normally distributed.

References

[1] Kingman, A. and Graybill, F. A. (1970). A non-linear characterization of the normal distribution. Ann. Math. Statist. (41) 1889-1895.