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POWER-SERIES SOLUTIONS FOR FLOWS OF AN IDEAL DISSOCIATING GAS

Final Report

Shock Wave Propagation and Diffraction

1 January 1969 - 15 September 1971

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13. ABSTRACT

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Forward

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POWER-SERIES SOLUTIONS FOR FLOWS OF AN IDEAL DISSOCIATING GAS

1. Introduction

This report deals with the solution of certain problems in fluid mechanics by power-series expansion of the solution in the independent variable(s). The method is directly related to the well-known Frobenius method for determining analytic solutions of linear ordinary differential equations (see, e.g., Agnew 1960). However, here we apply it to nonlinear systems of differential equations in as many as three independent variables. In consequence, the recursion formulas for the series coefficients are relatively complicated, and an electronic computer is required to effect and to store their solution.

There are some texts on numerical methods which discuss power-series methods for the solution of nonlinear ordinary differential equations (e.g., Henrici 1964), but usually only as a prelude to the description of more "practical methods. Most applications of series expansions to nonlinear problems have been for the purpose of clarifying some local behavior of the solutions and, thus requiring only the first few terms of the series, have been carried out analytically rather than numerically. Nevertheless, the use of series methods in fluid mechanics has a considerable history.

Most of this history is related to the problem of the steady supersonic flow of a perfect gas past a blunt body.

On prescribing the bow shock shape (analytically), one can substitute approximate series expansions for the flow variables into the governing equations and determine their coefficients recursively. It is hoped that the region in which the series converges encompasses the body contour (which must be determined as part of the solution) and the sonic line, beyond which the solution can be continued by the method of characteristics. First to suggest this approach were Lin and Rubinov (1948). Early attempts to implement it were made by Dugundji (1948), Lin and Shen (1951), and Cabannes (1953, 1955). The first to use a computer [illegible] to be able to generate a substantial number of series coefficients was Richtmyer (1957).

However, Van Dyke (1958) soon gave evidence that the series do not converge [illegible] the body. He ascribed this failure to the appearance in the analytic continuation of the series upstream of the shock a singularity which is closer to the center of the expansion than is the body. Subsequent studies were therefore directed at extending the region of convergence of the series so that they would have [illegible] physical [illegible]. Van Dyke (1966), [illegible] (1965), Leavitt (1968), and Moran (1970) all devised successful solutions of this problem.

The [illegible] noted above are all based on expansion of the flow properties in series in each of the independent variables. In 1908, Blasius suggested that the growth of a two-dimensional laminar boundary layer in a pressure gradient could be determined by expanding the solution in a power series in the variable running along the body. The coefficients

would depend on the coordinate normal to the body and would be determined by numerical solution of appropriate ordinary differential equations (see Schlicting 1958 for subsequent developments of this "Blasius series" approach). Van Dyke (1965) applied a similar partial-expansion method to the blunt-body problem with excellent results. He also used this procedure, which he calls the series truncation method, on the Navier-Stokes equations for the flow past a circular cylinder (Van Dyke 1964, Underwood 1969). Davis (1967) studied flow past a flat plate with the same method. In the meantime, Conti and Van Dyke (1966, 1969) have treated inviscid flows of dissociating gasses past blunt bodies with partial-expansion methods, while Moran and Van Moorhem (1969, 1970) used a full-expansion method for the three-dimensional (two space, one time) flows of a perfect inviscid gas which are generated when a plane shock wave strikes either a stationary blunt body or one that is in steady supersonic flow.

The above listed flow situations have in common a smooth variation of the flow properties with the variables in which they are expanded. To be sure, boundary layers, in which the properties vary rapidly, do develop in the flow of an inviscid dissociating gas past a blunt body and in the high-Reynolds number flow of a viscous fluid past a solid body. However, the expansions used by Conti and Van Dyke (1966, 1969), Davis (1967), and Underwood (1969) in treating these problems were only in the variables running parallel to these layers. On the other hand, while Conti, Van Dyke and Davis were able

to describe their boundary layers quite successfully, the series of Underwood converge only in the low-Reynolds-number regime, where the boundary layers are relatively thick and the property variations relatively smooth.

The present investigations were directed at elucidating the ability of the power-series methods to treat problems with embedded boundary layers (as distinguished from cases in which coordinate stretching has narrowed the focus to the boundary layer itself, as in Blasius's (1908) problem). The problems selected involved the non-equilibrium flow of a dissociating gas (specifically, the "ideal dissociating gas" of Lighthill (1957) and Freeman (1958)). Behind a strong shock, the finite-rate relaxation of such a gas towards equilibrium creates a region in which the flow properties may vary rapidly relative to their variations elsewhere in the flow. That is, the characteristic relaxation length may be small compared to the characteristic geometric length. Boundary layers also appear near stagnation points when the ratio of these characteristic lengths is at the other extreme.

The equations which govern all the flows studied are set down in Section 2. To illustrate the basic techniques we employ, the relatively simple problem of determining the flow properties behind a strong plane shock is examined in detail in the next three Sections. First the recursion formulae for the series coefficients are derived, then the manipulations required to make maximum use of the coefficients determined are described, and finally the results obtained are examined.

Sections 6 and 7 are devoted to two-dimensional steady flows past pointed and blunt bodies. Finally, in Section 8, we treat the three-dimensional (two space, one time) flow which results when a plane shock reflects from a blunt body.

It would appear from our results that series-expansion methods are of somewhat limited utility in describing flows with imbedded boundary layers. While a full-expansion approach is generally easier to program, it may be necessary to use a partial expansion, with no expansion in the variable running normal to the layer. This is certainly so if singularities appear in the flow field, and even then what amounts to a local analysis of the singularity (cf. Conti and Van Dyke 1966, 1969) is necessary to make the method useful. On the other hand, where the series can be made to converge, their accuracy is superb, this justifying much of the effort required.

2. Basic Equations

The forms of the conservation equations convenient for the present investigations are as follows (Vincenti and Kruger 1965, p. 246):

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \underline{v} = 0 \tag{1}$$

$$\rho\frac{D\underline{v}}{Dt} + \operatorname{grad} p = 0 \tag{2}$$

$$\rho\frac{Dh}{Dt} - \frac{Dp}{Dt} = 0 \tag{3}$$

Here p, ρ, h, $\underline{v}$ are the fluid pressure, density, enthalpy, and velocity, and D/Dt is the convective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}\cdot \operatorname{grad} \tag{4}$$

For Lighthill's ideal dissociating gas, the species conservation equation is

$$\frac{D\alpha}{Dt} = \frac{\alpha^*-\alpha}{\tau} \tag{5}$$

where α is the degree of dissociation, α^* its equilibrium value, and τ a relaxation time. The law of mass action becomes, in Lighthill's (1957) approximation,

$$\frac{\alpha^{*2}}{1-\alpha^*} = \frac{\rho_d}{\rho} \; e^{-\theta_d/T} \tag{6}$$

where T is the temperature and ρ_d, θ_d are constants of the gas. Freeman's (1958) equation for the relaxation time may be written

$$\tau = \frac{\rho_d}{C\rho^2 T^\beta} \; \frac{1-\alpha^*}{1-(1-\alpha^*)(1-\alpha)} \tag{7}$$

where C is a constant. Finally, the equations of state are, with R the gas constant,

$$p = \rho(1+\alpha)\; RT \tag{8}$$

$$h = (4+\alpha)\; RT + \alpha\theta_d \tag{9}$$

the latter of which also being an approximation due to Lighthill (1957).

Across shock waves, the conservation equations become the jump conditions

$$[\rho(v_n - \dot{x}_s)] = 0 \tag{10}$$

$$[P + \rho(v_n - \dot{x}_s)^2] = 0 \tag{11}$$

$$[v_t] = 0 \tag{12}$$

$$[h + \frac{1}{2}(v_n - \dot{x}_s)^2] = 0 \tag{13}$$

$$[\alpha] = 0 \tag{14}$$

Here $[(\ )]$ denotes the jump in $(\ )$ across the shock, v_n and v_t are the components of $\underline{v}$ normal and tangential to the shock, respectively, and $\dot{x}_s$ is the speed of the shock normal to itself. The use of Lighthill's caloric equation of state (9) implies that vibration is instantaneously half equilibrated immediately behind the shock.

3. Relaxation Behind a Steady Normal Shock: Development of Series Solution

The simplest case we have considered is the one-dimensional flow thru a standing normal shock wave. Let x be the distance downstream of the shock. With $D/Dt = v d/dx$, the conservation equations (1)-(3), together with the jump conditions (10), (11), and (13), may be integrated to yield

$$\rho v = \rho_\infty v_\infty \tag{15}$$

$$p + \rho_\infty v_\infty v + p_\infty + \rho_\infty v_\infty^2 \tag{16}$$

$$h + v^2/2 = h_\infty + v_\infty^2/2 \tag{17}$$

where the subscript ∞ denotes values upstream of the shock. The species conservation equation (5), however, remains in differential form:

$$\begin{aligned} v\frac{d\alpha}{dx} &= \frac{\alpha^* - \alpha}{\tau} \\ &= CT^{\beta}\left[(\rho - \rho\alpha)e^{-\theta_d/T} - \frac{\rho^2\alpha^2}{\rho_d}\right] \end{aligned} \tag{18}$$

in which we have incorporated equations (6) and (7). From the jump condition (14), we get the initial condition

$$\alpha(0) = \alpha_\infty \tag{19}$$

The series solution of these equations begins with the assumption that each of the variables may be expanded in a Taylor series of the form

$$A(x) = \sum_{j=0} A_j x^j \tag{20}$$

We then obtain recursion formulas for the coefficients A_j by substituting the series into the governing equations and collecting terms of like order in x^k. From the linear equation (16), this yields

$$p_m + \rho_\infty v_\infty v_m = (p_\infty + \rho_\infty v_\infty^2)\ \delta_{m0} \tag{21}$$

with no trouble whatsoever. Equations with terms quadratic in the unknowns are only a little more difficult. From (9), (15), and (17) we obtain

$$h_m = 4RT_m + 4\sum_{i=0}^{m} \alpha_i T_{m-i} + \theta_d \alpha_m \tag{22}$$

$$\sum_{i=0}^{m} \rho_i v_{m-i} = \rho_\infty v_\infty \delta_{m0} \tag{23}$$

$$h_m + \sum_{i=0}^{m} \frac{1}{2} v_i v_{m-i} = (h_\infty + \frac{1}{2} v_\infty^2)\delta_{m0} \tag{24}$$

However, equations like (8), with a triple product of unknowns, and the highly nonlinear (18), are rather nasty to work with. We generally find it worthwhile to introduce enough auxiliary variables so that no equation contains terms more nonlinear than quadratic before expanding the variables therein in Taylor series. Thus, with

$$b \equiv \rho\alpha \tag{25}$$

equation (8) becomes

$$p = (\rho + b)RT \tag{26}$$

and their Taylor-series expansions yield the recursion formulas

$$b_m = \sum_{i=0}^{m} \rho_i \alpha_{m-i} \tag{27}$$

$$p_m = R \sum_{i=0}^{m} (\rho_i + b_i) T_{m-i} \tag{28}$$

To simplify (18), we further define

$$f \equiv e^{-\theta_d/T} \tag{29}$$

$$g \equiv \theta_d/T \tag{30}$$

$$w \equiv T^{\beta} \tag{31}$$

$$d \equiv (\rho-b)f-b^2/\rho_d \tag{32}$$

Then (18) becomes

$$v\frac{d\alpha}{dx} = Cwd \tag{33}$$

Equations (30), (32), and (33) are easily expanded in series to give

$$\sum_{i=0}^{m} g_i T_{m-i} = \theta_d \delta_{m0} \tag{34}$$

$$d_m = \sum_{i=0}^{m} (\rho_i - b_i) f_{m-i} - b_i b_{m-i}/\rho_d \tag{35}$$

$$\sum_{i=0}^{m} i\alpha_i v_{m-i} = C \sum_{i=1}^{m} w_{i-1} d_{m-i} \tag{36}$$

To help expand (29) and (31), we follow Leavitt (1966) and first differentiate with respect to x:

$$\frac{df}{dx} = f\frac{dg}{dx} \tag{37}$$

$$\frac{dw}{dx} = \beta\frac{w}{T}\ \frac{dT}{dx} \tag{38}$$

These differential equations, on series expansion, yield the recursion formulas

$$mf_m = -\sum_{i=0}^{m} ig_i\ f_{m-i} \tag{39}$$

$$\sum_{i=1}^{m} iw_i\ T_{m-i} = \beta\sum_{i=1}^{m} iT_i w_{m-i} \tag{40}$$

Note that equations (36), (39), and (40) are valid only for $m > 0$. For $m = 0$, they are replaced by the initial conditions

$$\alpha_0 = \alpha_\infty \tag{36a}$$

$$f_0 = e^{-\theta_d/T_0} \tag{39a}$$

$$w_0 = T_0^{\beta} \tag{40a}$$

which are derived from equation (19) and (after setting x=0) from the definitions (29) and (31).

A problem often more difficult than the formation of the recursion formulas amonst the series coefficients is the determination of the order in which they may be solved. A

useful first step towards the solution is to list the coefficients of highest order in each of the recursion formulas. Since one hopes (as in the Frobenius method) to determine the coefficients in sequence according to their order, this procedure reveals what equations must be solved first. Thus, from Table 1, we see that equation (36) (or (36a)) determines c_m, after which ρ_m, p_m, v_m, T_m, h_m and b_m may be found by simultaneous solution of equations (21), (22), (23), (24), (27), and (28). Then g_m can be found from (34), f_m from (39) or (39a), d_m from (35), and w_m from (40) or (40a). This routine is followed for m = 0, 1, 2, ..., and thus yields coefficients of the series expansions of arbitrarily high order.

The equations to be solved are all linear, except when m = 0, and even then an analytic solution is not hard to obtain. However, the coefficients of the linear equations rapidly increase in complexity with increasing m. Thus, if one wishes to obtain more than the first few series coefficients, he must use a digital computer. Programming the recursion relations is straightforward. Hearist (1969) and others simplify the programming by writing subroutines for extracting the term of order x^m from series which are the double or triple product or exponential, etc., of other series. However, we do not find the direct programming of the recursion formulas so onerous as to be worth the pr[illegible] in storage and computation time one must pay for the use of simplifying subroutines.

4. Utilization of Series Solution

Once the series coefficients are known, the next step is to use the series to calculate the flow properties at points of interest. Where the series converge, they may be used directly, and the accuracy of the results may be estimated by examining the effect of varying the number of terms after which the series is truncated.

Unfortunately, the series we encounter typically have disappointingly small regions of convergence. The singularities which limit convergence are usually outside the domain in which the series is of physical interest.* The one which is troublesome in the relaxation problem appears (from examination of the series) to be located at some negative value of x, which would put it upstream of the shock, where the flow is uniform and not described by the series at all. This artificiality does not make the singularity any less dangerous; the region in which a series converges depends only on the distance from the center of the expansion to the nearest singularity, and is otherwise independent of the location of that singularity.

Thus, for the series solution to be useful, we must be able to continue it analytically beyond its basic region of convergence, given (numerically) only the first several coefficients of the series. Lewis (1965) studied a number of approaches to this problem. Probably the most powerful is simply to introduce a new independent variable so that, when the series is reexpanded

*When they are within that domain, our method generally fails; see Sections 6 and 7 below.

in the new variable, there is a more favorable relationship between the locations of the singularities of the function and the domain of interest. Such transformations were developed further by Leavitt (1968), who made use of Domb's (1965) procedure for locating and classifying the singularity closest the origin of a function known only by the coefficients of its power-series expansion.

Lewis and Leavitt both obtain splendid results for the supersonic flow of a perfect gas past a blunt body. The procedures we have employed, however, are based on the theory of continued fractions. Given the coefficients of a power series

$$a_o + a_1 x + a_2 x^2 + \ldots = f(x) \tag{41}$$

we can, under certain restrictions on the a_n's, find numbers $\bar{a}_n$ such that the power-series expansion of the continued fraction

$$\bar{a}_0/(1 + \bar{a}_1 x/(1 + \bar{a}_2 x/(1 + \bar{a}_3 x/(1 + \ldots \tag{42}$$

agrees with the left side of (41). An efficient procedure for so doing is the "quotient-difference algorithm" (Henrici 1963).

The theory of continued fractions is full of half-promises concerning the efficacy of representing functions by expansions like (42). It is known, for example, what conditions on the coefficients a_n of the power-series expansion of $f(x)$ are necessary and sufficient for it to possess a continued-fraction expansion like (42) (Wall 1948, Chap. XI). However, these conditions take the form that certain matrices whose elements are the power-series coefficients must be nonsingular. Such

conditions cannot be verified when, as in our case, the coefficients are known only numerically, because of roundoff errors. It is also known (Wall 1948, p. 399) that every function analytic at the origin has a continued-fraction expansion of a form somewhat more general than (42); specifically, the quantities $\bar{a}_k x$ must be replaced by $\bar{a}_k x^{\beta_k}$, where β_k is a positive integer not necessarily equal to unity. However, the numerical difficulties required to determine the β_k also appear to be insurmountable. Thus, all we can do is to assume that the power series with which we deal have continued-fraction expansions of the form (42).

The convergence properties of such expansions are rather spectacular in certain cases, which of course is what motivates us to use the expansions. If a function f(x) can be expanded in the form (42), and if the coefficients $\bar{a}_k$ approach a finite limit, $\bar{a}_\infty$ (say) it can be shown (Wall 1948, Chap. XI) that the expansion converges to f(x) everywhere in the complex x-plane except at the poles of the functions (if any) and along a branch cut which begins at the point $x = -1/4\bar{a}_\infty$ and proceeds to infinity along a ray from the origin. However, it is also possible to construct continued-fraction expansions which diverge within the circle of convergence of the corresponding power-series expansion. Fortunately, such examples appear to be pathological. We have yet to encounter a power series which converged better than the corresponding continued fraction.

5. Relaxation Behind a Steady Normal Shock: Results

Calculations have been carried out as described above for the three cases identified in Table 2, which were previously studied by Freeman (1958) with a more conventional numerical method. Before carrying out the computations, the equations were of course made dimensionless. In particular, the independent variable in which the series are expanded is not the physical distance downstream of the shock x, but

$$x^* \equiv C\left(\frac{V_\infty^2}{R}\right)^\beta \frac{\rho_\infty}{V_\infty} x \tag{43}$$

The first twenty-five coefficients of the power-series expansion of the temperature behind the shock for the first case of Table 2 are listed in Table 3, along with the coefficients of the corresponding continued-fraction expansion. Note that the latter tend to remain of the same order of magnitude, which is something of a convenience. However (unless the approach is completely obscured by roundoff errors), they evidently do not approach a limit, so that the theorems quoted in the previous section shed no light on the convergence properties of the continued-fraction expansion.

Of course, empirical evidence on the utility of both the power-series and continued-fraction expansions may be obtained by varying the number of terms used. Data relevant to the expansions whose coefficients are tabulated in Table 3 are contained in Tables 4 and 5. It is clear that the power series is useless even for $x^* = 8.0$, although examination of the

coefficients by Domb's (1965) procedures indicates that the series should converge up to about $x^* = 11.5$. The continued-fraction expansion, on the other hand, is quite satisfactory for x^* well above 100. Of course, its accuracy eventually diminishes with increasing x^*, due to roundoff errors if nothing else; the results listed certainly appear to be trying to converge. On the other hand, the fact that the continued-fraction results appear to oscillate with diminishing amplitude as the number of terms is increased is somewhat misleading. As can be seen from Table 5, the exact result (obtained by what is essentially Freeman's (1958) method of direct integration) does not necessarily fall within the range of the oscillations. This, again, is probably due to roundoff errors in the continued-fraction coefficients and/or in the evaluation of the expansion.

Results obtained from the continued-fraction expansion for T are compared with the numerically exact results for cases 1 and 2 in figure 1 (the results for cases 2 and 3 are much the same). To give some indication of the apparent convergence of the expansion, results obtained both by using 34 terms and by using 35 terms are plotted, the difference between them being indicated by a bar. It is seen that the results behave as described above in both of cases 1 and 2, except that the range of x^* for which the continued-fraction expansion is useful is more restricted in case 2 (and in case 3) than in case 1. However, the power-series expansions in the last two cases do not converge beyond $x^* = 0.5$, which is far below the largest x^* for which the corresponsing continued fraction is useful. Results for the other properties behind the shock are quite similar.

Of course, the series-expansion method is not really practical for the problem of finding the flow behind a shock in a relaxing gas; it cannot compete with ordinary numerical integration either in range of utility or in time (the present method took about 4 seconds on a CDC 6600 to obtain results equivalent to those obtained by a Runge-Kutta method in 2 seconds.) However, the series method is much more attractive in two- and three-dimensional problems, for which alternative procedures are at least as costly and much less accurate. For the present, the main conclusion we can draw is that the series method can be useful through the most interesting part of the shock layer. The failure of the method to describe the solution accurately as it approaches equilibrium may simply be due to the large arguments involved in the expansion.

6. Steady Supersonic Flow Past Wedges and Cones

The next most difficult problem we attempted was the steady flow behind a shock attached to a wedge or cone. In spherical coordinates, equations (1)-(5) reduce to

$$\frac{\partial(\rho u)}{\partial r} + (1+\varepsilon)\ \frac{\rho u}{r} + \frac{\rho v}{r\tan\theta} + \frac{1}{r}\frac{\partial \rho v}{\partial\theta} = 0 \tag{44}$$

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\ \frac{\partial u}{\partial\theta} - \frac{v^2}{r} + \frac{1}{\rho}\ \frac{\partial p}{\partial r} = 0 \tag{45}$$

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\ \frac{\partial v}{\partial\theta} + \frac{uv}{r} + \frac{1}{\rho r}\ \frac{\partial p}{\partial\theta} = 0 \tag{46}$$

$$h + \frac{1}{2}\ (u^2 + v^2) = \text{constant} \tag{47}$$

$$u\frac{\partial\alpha}{\partial r} + \frac{v}{r}\frac{\partial\alpha}{\partial\theta} = \frac{\alpha^* - \alpha}{\tau} \tag{48}$$

Here u, v are the velocity components in the r, θ directions, respectively, while

$$\begin{aligned} \varepsilon &= 0 && \text{for plane flow (wedge case)} \\ &= 1 && \text{for axisymmetric flow (cone case)} \end{aligned} \tag{49}$$

If the shock is located at $\theta = \theta_s(r)$, the shock jump conditions (10) - (12) become

$$\rho_\infty V_\infty \frac{d}{dr}\ (r \sin\theta_s) = \rho_s(r\ \theta_s'\ u_s - v_s) \tag{50}$$

$$V_\infty \frac{d}{dr}\ (r \cos\theta_s) = u_s + r\theta_s'\ v_s \tag{51}$$

$$\frac{p_s - p_\infty}{\rho_\infty V_\infty^2} = \frac{\left[\frac{d}{dr}(r \sin\theta_s)\right]^2}{1 + r^2(\theta_s')^2} \left(1 - \frac{1}{\rho_s}\right) \tag{52}$$

Again the subscript ∞ denotes properties upstream of the shock. Note that the shock shape $\theta_s(r)$ is to be determined as part of the solution.

Finally, we have the condition of no flow thru the body:

$$v = 0 \qquad \text{at} \qquad \theta = \theta_b \tag{53}$$

where $\theta = \theta_b$, a constant, locates the body.

Were dissociation absent, these equations would have the familiar r-independent similarity solution. However, the equation (48) introduces a characteristic length into the problem, and the solution does depend on both r and θ. On the other hand, the problem is hyperbolic, and has been treated successfully by the method of characteristics (Capiaux and Washington 1963; see also Sedney and Gerber, 1963, 1967).

Two versions of the series-expansion method were attempted for these problems. The first is what Van Dyke (1966) calls the "method of series truncation". The idea here is to expand the variables in r, e.g.,

$$p = \sum_{i=0} p_i(\theta)\, r^i \tag{54}$$

determining the θ-dependent coefficients like $p_i(\theta)$ by solving (numerically) the ordinary differential equations derived by substituting equations like (54) into the governing algebraic and partial differential equations and then collecting terms of

like order in r. Alternatively, one may develop the variables in double series in r and θ ; e.g.,

$$p = \sum_{i,j=0} p_{ij} r^i \eta^j \tag{55}$$

where, by using

$$\eta \equiv \frac{\theta - \theta_s(r)}{\theta_b - \theta_s(r)} \tag{56}$$

as an independent variable, we ease the satisfaction of the boundary conditions both at the shocks ($\eta = 0$) and at the body ($\eta = 1$).

In either approach, the hyperbolic nature of the problem is reflected in the determinability of the coefficients of r^i independent of those of r^j for $j>i$. However, as in equations like (22)-(24), the equations governing the coefficients of r^i do involve the coefficients of r^j for $j<i$, so that the coefficients must be determined in order; first those of r^0, then those of r^1, etc. In the full-expansion method, the coefficients like p_{ij} must also be determined in order of their second subscript for any fixed i, and $p_{i-1,j}$ must be known in order to find p_{ij}.

In both methods, the coefficients of r^i are found by solving a boundary-value problem, in which the shock shape must be determined so as to produce a flow field which meets the body boundary condition (53). Specifically, we expand the shock-shape function

$$\theta_s(r) = \sum_{i=0} \theta_i r^i \tag{57}$$

Then θ_i must be determined by iteration (we used the secant method) so that the coefficient of r^i in the expansion of equation (53) vanishes.

In the partial-expansion approach, this boundary condition becomes simply

$$v_i(\eta) = 0 \qquad \text{at } \eta = 1 \tag{58}$$

For an assumed θ_i, the shock-jump conditions fix the values of the coefficients like p_i and v_i at $\eta = 0$. The value of $v_i(1)$ for that θ_i was then determined by integrating, from $\eta = 0$ to $\eta = 1$, the system of ordinary differential equations formed by substituting expansions like (54) into the governing equations. While the lower-order coefficients on which those of r^i depend are already known in principle, to save storage they were regenerated by solution of their ordinary differential equations along with the coefficients of r^i. Of course, since θ_j is then known for $j<i$, the problem of finding the lower-order coefficients is initial-value in nature. Still, the partial-expansion approach was found to be much more time-consuming than the full-expansion procedure, and no more accurate. Perhaps the time problem could have been alleviated by storage of the known lower-order coefficients (at least to an extent which would have permitted their generation by interpolation). However, we decided to spend most of our effort on the full-expansion method.

The derivation of the recursion formulas for the series coefficients in the full-expansion method proceeded much as in

the one-dimensional problem discussed earlier. To illustrate, we shall consider in detail the treatment of the differential continuity equation (44), which, after introducing (56), becomes

$$\rho\tilde{u} + u\tilde{\rho} + (1+\varepsilon)(\theta_b-\theta_s)\,\tilde{u} - \varepsilon\rho v z + v\frac{\partial\rho}{\partial\eta} + \rho\frac{\partial v}{\partial\eta} = 0 \tag{60}$$

Here we have introduced the abbreviations

$$\tilde{A} \equiv r\left[(\theta_b-\theta_s)\frac{\partial A}{\partial r} + (\eta-1)\frac{d\theta_s}{dr}\frac{\partial A}{\partial\eta}\right] \tag{61}$$

A being any field variable, and

$$\begin{aligned} z &\equiv (\theta_s-\theta_b)\cot\theta \\ &= (\theta_s-\theta_b)\cot\left[(\theta_b-\theta_s)\eta + \theta_s\right] \end{aligned} \tag{62}$$

If A and $\tilde{A}$ are expanded as in equation (55), on substituting such expansions into equation (61), and collecting terms of order $r^i\eta^j$, we find

$$\begin{aligned} \tilde{A}_{ij} = \sum_{k=0}^{i} \{ & k\theta_k[i\,A_{i-k,j} - (j+1)\,A_{i-k,j+1}] \\ & + k\,A_{kj}(\theta_b\delta_{ki} - \delta_{i-k}) \end{aligned} \tag{63}$$

To help expand (62), we need some more auxiliary variables.

Let

$$c \equiv \cos\,(\theta_s-\theta_b)(1-\eta)$$
$$s \equiv \sin\,(\theta_s-\theta_b)(1-\eta) \tag{64}$$

and expand both functions in series like (55). Then

$$\sum_{i=0} c_{i0} r^i = \cos(\theta_s-\theta_b)$$
$$\sum_{i=0} s_{i0} r^i = \sin\,(\theta_s-\theta_b) \tag{65}$$

Following the strategy discussed in connection with equations (37) and (38), we differentiate equations (65) with respect to r and collect terms of order r^{i-1} to obtain

$$c_{i0} = \cos(\theta_0-\theta_b) \qquad \text{if } i = 0$$
$$= -\frac{1}{i} \sum_{k=0}^{i} k(\theta_k-\theta_b\delta_{k0})\, s_{i-k} \qquad \text{if } i > 0$$
$$s_{i0} = \sin\,(\theta_0-\theta_b) \qquad \text{if } i = 0$$
$$= \frac{1}{i} \sum_{k=0}^{i} k(\theta_k-\theta_b\delta_{k0})\, c_{i-k} \qquad \text{if } i > 0 \tag{66}$$

Similarly, after differentiating (64) with respect to η, we find

$$c_{ij} = \frac{1}{j} \sum_{k=0}^{i} (\theta_k - \theta_b \delta_{k0}) \; s_{i-k}$$

$$s_{ij} = -\frac{1}{j} \sum_{k=0}^{i} (\theta_k - \theta_b \delta_{k0}) \; c_{i-k} \tag{67}$$

Equation (62) may now be expanded in r and η to yield

$$\sum_{k=0}^{i} \sum_{\ell=0}^{j} z_{i-k,\, j-\ell} \; [\sin\theta_b \, c_{k\ell} + s_{k\ell} \cos\theta_b]$$

$$= \sum_{k=0}^{i} (\theta_{i-k} - \theta_b \delta_{ik})(c_{kj} \cos\theta_b - s_{kj} \sin\theta_b) \tag{68}$$

which may be solved for z_{ij} once lower-order coefficients are known.

The expansion of equation (60) itself is now relatively straightforward. On introducing the series expressions for the various quantities involved and collecting terms or order $r^i \eta^j$, we obtain

$$\sum_{k=0}^{i} \sum_{\ell=0}^{j} \{\rho_k \, u_{i-k,\, j-\ell} + u_{k\ell} \, \rho_{i-k,j-\ell}$$

$$+ (1+\varepsilon) \sum_{m=0}^{k} (\theta_b \delta_{m0} - \theta_m) \; \rho_{k-m,\ell} u_{i-k,j-\ell}$$

$$- \varepsilon \sum_{m=0}^{k} \sum_{n=0}^{\ell} \rho_{mn} v_{k-m,\ell-n} \; z_{i-k,j-\ell}$$

$$+ \ell \, \rho_{k,\ell+1} \, v_{i-k,j-\ell} + \ell \, v_{k,\ell+1} \rho_{i-k,j-\ell}\} = 0 \tag{69}$$

The other conservation equations are expanded in much the same way (though with much less difficulty!). Once the coefficients like p_{ij} are known for an assumed θ_i, we must calculate the coefficient of r^i in the expression for v (and, later, for other variables of interest) on the body surface, $\eta = 1$. From the equation (55), we see that this coefficient is the infinite series

$$\sum_{j=0} v_{ij}\eta^j \Big|_{\eta = 1} \tag{70}$$

which we recast as a continued fraction before evaluation at $\eta = 1$. In the case of flow past a wedge, this recasting proved to be unnecessary; the series (69) converged quite rapidly in its raw form. The cone case was much more difficult, as will be detailed below.

Thus, in essence, the full-expansion method simply uses power-series methods to integrate the ordinary differential equations which govern the θ-dependent coefficients of the expansions (54) used in the partial-expansion method. Once these coefficients are available at a fixed value of θ (or rather η), a recasting of the resulting series in r into a continued fraction in r was necessary to obtain final results.

Computations have been carried out for the four cases treated by Capiaux and Washington (1963) with the method of characteristics, see Table 6. Results from the two approaches are compared in Figures 2-4. Where they disagree, the present results are presumed to be correct, at least for relatively

small values of x*, the dimensionless distance along the wedge axis (see equation (43)), our justification being the convergence of our series expansions evident in Table 7*. The general limitation of the series method to small arguments, reflected in the figure by plotting the difference between 11- and 12- term results, has already been discussed. The characteristics method is not inherently so limited, but, being an initial-value method, it may become too costly to pursue the calculations to completion. This is especially true, if, as in the Capiaux-Washington calculations, one locates the points at which the solution is to be determined at the intersections of characteristics proceeding downstream from points at which the solution is known. The number of points at which calculations are made then increases rapidly with distance downstream of the wedge apex. It would have been more efficient for them to prescribe the solution points a priori, using Hartree's (1958) interpolating scheme to integrate the compatibility equations.

As it is, the series and characteristics methods are roughly comparable in their range of utility. The former method is much more limited than the latter in case 4, but does describe the entire relaxation zone rather well even then; it is only in the constant-property equilibrium region that it loses accuracy. In case 3, the present method is clearly superior. Note that, in this case, the shock wave is concave, and the shock-wave angle overshoots its

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*We assume that the results to which they converge are correct, so we can check our program by substituting the series results into every equation used.

equilibrium value. Capiaux and Washington obtained a similar result, but could not get convergence through refinement of their mesh size, and so discounted the over-shoot as a numerical inaccuracy (to be sure, the overshoot they obtained was much more severe than ours; see Figure 2(c), in which Δx^* is the distance between the first two points on the wedge at which calculations were made, and so is indicative of the general mesh spacing).

Equations (44)-(52) are equally applicable to flows past wedges and cones; the only difference is in the continuity equation (44), see equation (49). The methods used to solve those equations were exactly the same for both cases, and programs were written in which the input value of ε was the only problem identifier.

Unfortunately, the flow properties are not analytic at the cone surface, so that the series expansions cannot be made to converge there. To see this, consider equation (48), which, under the transformation (56), becomes

$$u\tilde{u} + v\frac{\partial\alpha}{\partial\eta} = Cr(\theta_b-\theta_s)T^{\beta}\ [(\rho-\rho\alpha)e^{-\theta_d/T} - \rho^2\alpha^2/\rho_d] \qquad (70)$$

The first two terms in the partial expansion (in r) of this equation are

$$v_0 \frac{d\alpha_0}{d\eta} = 0 \tag{72}$$

$$u_0 [(\theta_b - \theta_o)\alpha_1 + (\eta - 1)\theta_1 \frac{d\alpha_0}{d\eta}] + v_0 \frac{d\alpha_1}{d\eta} + v_1 \frac{d\alpha_0}{d\eta}$$

$$= C(\theta_b - \theta_0) T_0^{\beta} [(\rho_0 - \rho_0 \alpha_0) e^{-\theta_d / T_0} - \rho_0^2 \alpha_0^2 / \rho_d] \tag{73}$$

Equation (72) shows that, as would be expected, the flow is "frozen" (α = constant) in the immediate vicinity of the tip. As a result, in the zeroth-order approximation the flow behaves like a perfect gas, for which the flow properties are certainly analytic in θ and η .

Thus, using (72), we may write equation (73) as

$$v_0 \frac{d\alpha_1}{d\eta} + u_0 (\theta_b - \theta_0) \alpha_1 = f(\eta) \tag{74}$$

where f is an analytic function of T_o, ρ_0, and α_o, and so is an analytic function of as well. For the complementary solution of (74), we try a Frobenius-type expansion of the form

$$\alpha_1 = \sum_{i=0} \alpha_{0i} \xi^{i+\delta} \tag{75}$$

where $\xi = 1 - \eta$. Since v_0 and u_0 are analytic in η , they may be expanded in series like

$$u_0 = \sum_{i=0} u_{0i}\xi^i$$

$$v_0 = \sum_{i=1} v_{0i}\xi^i \tag{76}$$

where the form of the expansion for v_0 reflects the body boundary condition, $v_0 = 0$ at $\xi = 0$. Substituting these expansions into the homogeneous version of (73), we obtain

$$0 = \sum_{j=0} \xi^{i+j} \sum_{i=0}^{j} \alpha_{1i}[(i+\delta)v_{0,j+1-i} + (\theta_b-\theta_0)u_{0,j-i}] \tag{77}$$

To fix δ, we insist that $\alpha_{10} \neq 0$, and so obtain the indicial equation

$$\delta\, v_{01} + (\theta_b-\theta_0)\, u_{00} = 0 \tag{78}$$

But, collecting terms of order $r^0\xi^0$ in the continuity equation (60), we get

$$(1+\varepsilon)(\theta_b-\theta_0)u_{00} + v_{01} = 0 \tag{79}$$

so

$$\delta = \frac{1}{1+\varepsilon} \tag{80}$$

Thus, the degree of dissociation is analytic in the case of flow past a wedge ($\varepsilon=0$), but has a square-root singularity at the surface of a cone ($\varepsilon=1$).

It is rather more difficult to analyze the behavior of the other flow variables at the body surface. If we assume expansions like that of α_1 in (74) for all the variables, we find p_1 to be analytic at the cone surface, but ρ_1, T_1, and u_1 to have square-root singularities there. No result

for v_1 could be proved. However, our numerical calculations seem to indicate that the r-derivatives of p and v are indeed analytic in η, throughout the flow, but that the other variables do have singularities at $\eta = 1$. Our evidence for these conclusions is exhibited (partially) in Table 8, which lists the coefficients α_{1i} and v_{1i} of our original double-series expansions (55). Since the v_{1i}'s alternate in sign and decrease in magnitude with increasing i, convergence of the series

$$\sum_{i=0} v_{1i}\eta^i$$

is assured at least for $\eta < 1$. However, applying Domb's (1965) procedures to the coefficients α_{1i}, we obtain the sequences listed in Table 9 for the location and type of the singularity closest the origin of $\alpha_1(\eta)$. It seems clear from these data that this singularity is at $\eta = 1$, and it is not too hard to believe that it is indeed of square-root type.

The coefficients of r^2 and r^3 in the series (54) were also determined, and behave in the same way as do those of r^1. However, the higher the exponent of r, the less confidence we have in our determination of θ_i. The equations for v_{ij} apparently become less well conditioned as i increases, making our results well infected with roundoff errors, so that the series

$$\sum_{j=1} v_{ij}\eta^j$$

while not diverging, does not approach a fixed number to any acceptable degree of accuracy.

Sedney and Gerber (1967) determined the first two terms of an expansion of the flow past a cone of a gas out of vibrational equilibrium, and also found a singular behavior of the solution at the cone surface. However, their expansions were of the form

$$p = p_0(\zeta) + y\, p_1(\zeta) + \ldots \tag{81}$$

where

$$\zeta \equiv 2\psi/y^2 \tag{82}$$

ψ is a stream function,

$$\frac{1}{y}\frac{\partial\psi}{\partial x} = -\rho v_y, \quad \frac{1}{y}\frac{\partial\psi}{\partial y} = \rho v_x \tag{83}$$

x, y are Cartesian coordinates,

$$x = r\cos\theta\ , \quad y = r\sin\theta \tag{84}$$

and v_x, v_y are the velocity components in the x, y directions. It was not clear to us that the $\zeta^{1/2}$ singularity of their solution implied a $(1-\eta)^{1/2}$ singularity in ours, nor is singular behavior evident in the available characteristics calculations of reacting flow past cones (Sedney & Gerber 1963, Spurck, Gerber and Sedney 1967). It would be of interest to attempt the removal of these singularities, perhaps by matching asymptotic expansions.

7. Steady Supersonic Flow Past a Blunt Body

While the singularity which foiled our solution of the flow past a cone was something of a surprise, the existence of a singularity in the flow field of the corresponding blunt-body problem is well known (Conti and Van Dyke 1969, Vinokur 1970). Specifically, the stagnation point of such flows is singular. This phenomenon is connected with the fact that, since the residence time of the fluid is infinite at the stagnation point, the flow must reach equilibrium there, no matter how slow is the reaction rate.

Nevertheless, as noted in the Introduction, one of the most impressive successes of the series-expansion method in fluid mechanics has been its solution of the supersonic flow of a perfect gas past a blunt body. Therefore we wanted to see how it would perform in the present case of an ideal dissociating gas, whatever a priori misgivings we may have had. Conti and Van Dyke (1969) already employed partial expansions of the form

$$p = p_1(r) \cos^2 \theta + p_2(r) \sin^2\theta + O(\sin^4\theta)$$

where (r, θ) are polar coordinates centered at the center of curvature of the shock wave, which was prescribed as either circular (for a two-dimensional calculation) or spherical (the axisymmetric case). They did, in fact, obtain excellent results for the flow in the immediate vicinity of the stagnation point. To enable approach to the singularity, they had to introduce as an independent variable (in place of r) the

difference on the axis between the degree of dissociation and its equilibrium value (see Conti 1966).

The present calculations are based on full Taylor-series expansions of the form

$$p = \sum_{i,j=0} p_{ij}(x-x_s(z))^i z^j \tag{85}$$

where $x=x_s(z)$ locates the bow shock wave in cylindrical coordinates and z is the square of the distance from the axis of symmetry. Such an expansion eases the satisfaction of the shock jump conditions (10)-(14), and takes advantage of the fact that p is even in the cylindrical radius (to insure that all independent variables behaved similarly, we worked with $v/\sqrt{z}$ instead of the radial velocity component v, which itself is odd in z.) With

$$\begin{aligned} y &\equiv x-x_s(z) \\ w &\equiv v/\sqrt{z} \\ A &\equiv z\left(\frac{\partial A}{\partial t} - \frac{dxs}{dz}\frac{\partial A}{\partial y}\right) \end{aligned} \tag{86}$$

the conservation equations (1)-(5) become

$$u\frac{\partial \rho}{\partial y} + 2w\tilde{\rho} + \rho\frac{\partial u}{\partial y} + 2\rho w + 2\rho\tilde{w} = 0 \tag{87}$$

$$u\frac{\partial u}{\partial y} + 2w\tilde{u} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \tag{88}$$

$$u\frac{\partial w}{\partial y} + 2w\tilde{w} + w^2 + \frac{1}{\rho z}\frac{\partial p}{\partial z} = 0 \tag{89}$$

$$u\frac{\partial h}{\partial y} + 2w\tilde{h} = \frac{1}{\rho}\left(u\frac{\partial p}{\partial y} + 2w\tilde{p}\right) \tag{90}$$

$$u\frac{\partial \alpha}{\partial y} + 2w\tilde{\alpha} = \frac{\alpha^*-\alpha}{\tau} \tag{91}$$

The shock jump conditions (10)-(13) may be solved for the flow properties behind the bow shock as follows:

$$u_s = 1 + \frac{2\gamma}{\gamma+1}\frac{p_\infty}{\rho_\infty u_\infty^2} - \frac{2}{\gamma+1}\sin^2\theta_s \tag{92}$$

$$v_s = \frac{2}{\gamma+1}\left(\sin\theta_s\cos\theta_s - \frac{\cos\theta_s}{\sin\theta_s}\frac{\gamma p_\infty}{\rho_\infty u_\infty^2}\right) \tag{93}$$

$$\rho_s = \frac{(\gamma+1)\sin^2\theta_s}{(\gamma-1)\sin^2\theta_s + \frac{2\gamma p_\infty}{\rho_\infty u_\infty^2}} \tag{94}$$

$$p_s = \frac{2}{\gamma+1}\sin^2\theta_s - \frac{p_\infty}{\rho_\infty u_\infty^2} \tag{95}$$

Here

$$\gamma = \frac{4+\alpha_\infty}{3} \tag{96}$$

$$\sin\theta_s = \left[1 + 4z\left(\frac{dx_s}{dz}\right)^2\right]^{-1/2} \tag{97}$$

The boundary condition of no flow through the body is taken into account by working in terms of a stream function ψ defined by

$$\frac{\partial\psi}{\partial y} = -\rho_0 wz, \quad \frac{\partial\psi}{\partial z} = \rho_0 u + \frac{dx_s}{dz}\frac{\partial\psi}{\partial y}, \quad \psi = 0 \text{ at } y=z=0 \tag{98}$$

Then

$$\frac{\psi}{z} = 0 \tag{99}$$

on the body.

Following the usual inverse approach, the shock shape $x_s(z)$ is specified (in our calculations, only a spherical shock wave was studied) and the corresponding body shape determined as part of the solution by searching for the zeroes of ψ/z. The recursion relations formed by substituting

the series like (77) into the governing equations turn out to be solvable in sequence of the total order i+j of coefficients like p_{ij}. Amongst coefficients of the same total order, they are determined in order of the first subscript i. Specifically, the shock jump conditions (91)-(94) determine the coefficients with subscript i=0, and the differential equations (86)-(90) those with i>0.

For problems in more than one independent variable, there is no procedure directly analogous to the recasting of a one-variable series into a continued fraction. Thus, to improve the convergence of two-dimensional series like (84), we must somehow put them into one-dimensional forms. This can be done in any number of ways. Van Tuyl (1960, 1967, 1971) starts with the series for p (e.g.) in the form

$$p \cong \sum_{j=0}^{M} z^j \left[\sum_{i=0}^{M-j} p_{ij}\, y^i \right] \tag{100}$$

recasts the polynomials in square brackets into the continued-fractions $P_j(y)$ (say), and then calculates p from

$$p \cong \sum_{j=0}^{M} z^j\, P_j(y) \tag{101}$$

This is essentially the procedure we used in our analysis of flows past wedges and cones, as described in the preceding section, except that the series corresponding to equation (100) was itself recast into a continued fraction in what corresponded to the present variable z.

However, in the case at hand, the unavailability of the coefficients p_{ij} of total order i+j beyond some finite number (M in equation (99)) means that the coefficients $p_j(y)$ are known with decreasing accuracy as j increases. Thus, after experimentation with a variety of procedures similar to Van Tuyl's (Moran 1965), we found it preferable to rearrange series like (77) into the form

$$p = \sum_{i=0}^{\infty} y^i \left[\sum_{j=0}^{i} p_{i-j,j} \left(\frac{z}{y}\right)^j \right] \tag{102}$$

evaluate the polynomials in the square brackets for specified z/y, and then recast the resultant series in y into continued-fraction form. The advantage of so doing is that the coefficients of y^i in (101) are known exactly (within roundoff errors) up to the point at which the series in y is truncated, since the coefficient of y^i consists of a linear combination of the coefficients p_{kl} of total order i. Therefore, our application of continued fractions may be interpreted as looking for the analytic continuation of a power series in y on a curve of constant z/y.

Our main objective was to find the shape of the body which generated the prescribed (spherical) shock and the flow properties on that body. Once its series coefficients were available, the continued-fraction expansion of ψ/z was formed as described above and its zeroes determined on curves of specified z/y by the secant method.

While a similar procedure worked beautifully in the corresponding perfect-gas problem (Moran 1970), results for

the present situation were very poor. Three cases, described in Table 10, were examined in detail for comparison with Conti and Van Dyke's (1969) partial-expansion results. As shown in Figures 5, in the two cases with relatively high rea tion rate, we simply were unable to locate the body with any accuracy. In case 2, the present results give no evidence at all of the boundary layer at the stagnation point in which the fluid makes its final approach to equilibrium. Only in case 3, the one with lowest reaction rate, did we get any indication of the presence of this layer. Also in case 3, our results for the degree of dissociation seem to suggest the presence of singularities at (and beyond) the stagnat on point. While this is correct (Conti and Van Dyke 1969), our results for the standoff distance (the value of y at which $\psi/z = 0$ and $z = 0$) appear to converge to an answer different from Conti and Van Dyke's. We believe their results to be correct, but cannot find any error in our calculations either. Our program was checked by evaluating all the flow properties and their derivatives at some point within the shock layer from the series coefficients, and determining that the results do indeed satisfy the governing equations.

In any event, the apparent failure of the full series-expansion method in this problem was not entirely unexpected. As noted above, Conti and Van Dyke do show that the approach of the fluid properties to their stagnation values generally is singular, their normal derivatives being proportional to some negative fractional power of the distance from the body.

our partial success in describing this phenomenon in case 3 is just as surprising as the apparent convergence in that case to the wrong result for the standoff distance.

8. Diffraction of a Plane Shock by a Blunt Body

Our most ambitious project in this series was the determination of the time-dependent flow produced when a plane shock wave impinges on and reflects from a solid body - specifically, a sphere - a problem of interest in connection with the blast-wave loading of stationary structures and with the starting of shock tubes and tunnels. Here we break new ground; to the author's knowledge, this problem has not been solved by other methods. To be sure, it is related to the well-studied problem of finding the flow about a body impulsively accelerated from rest, which is mainly of interest for its final steady state.

The problem is illustrated in Figure 6. Properties are known in the undisturbed region 1, while in region 2, behind the incident shock, their dependence on the distance from that shock can be found by the procedure described in detail in Section 2 of this report. However , for all cases considered herein, the equilibrium degree of dissociation in region 2 was less than 10^{-6}. Since the other flow properties in that region then also differ negligibly from their equilibrium values, we ignored such differences in our analysis and took region 2 to be uniform.

The determination of the flow properties in region 3, between the body and the reflected shock, proceeds much as in the other problems described in this report, with the following major differences:

1. The properties depend on time as well as on two space coordinates, and so are expanded in three-dimensional series of the form

$$p = \sum_{ijk=0} p_{ijk} y^i z^j t^k \tag{103}$$

where (cf. equations (84) and (85))

$$z \equiv r^2, \; y \equiv x - x_r(z,t) \tag{104}$$

The three-dimensionality of the series coefficients strains the capacity of the computer, so that the total order $(i + j + k)$ of the coefficients computable is a mere 10, less than we are used to and less than we would like. However, the programming effort is not greatly influenced by the three-dimensionality per se. It is probably less than 3/2 the effort required in a comparable two-dimensional problem.

2. Of far greater impact is the boundary-value nature of the problem; we must satisfy shock jump conditions on part of the boundary $(x = x_r(z,t))$ of region 3, and impenetrability conditions on the rest (the body surface $x = x_b(z)$). The previous problems were of purely initial-value type, one consequence of which was that coefficients of a given order were calculable recursively in terms of coefficients of other orders. In the present case, coefficients of different orders are coupled to one another and must be

determined simultaneously. Specifically, in determining the coefficients of total order $i+j+k=n$, all those coefficients of the same order j with respect to the radial coordinate must be calculated at the same time, starting with $j=0$, then $j=1$, etc. The equations involved are linear and of order $6(n-j)$, the unknowns being coefficients of the series expansions of p, u, ρ, $1/\rho$, h, and T. Coefficients of the other series may be determined separately.

3. The shape of the boundary is not completely known *a priori*. The shock shape $x_r(z,t)$ must thus be determined as part of the solution, partly from the shock jump conditions and partly from the geometric condition that the incident and reflected shocks meet the body at the same point, where $r^2 = z_i(t)$ (say):

$$v_s t = x_b(z_i) = x_r(z_i, t) \tag{105}$$

Here v_s is the speed of the incident shock.

4. While the series expansion of most of the equations is straight forward, the body boundary condition

$$v = w\frac{dx_b}{dz} \qquad \text{at} \qquad x = x_b(z) \tag{106}$$

where v is the axial velocity and w the radial velocity (divided by the radius to produce an even function of the radius), is relatively troublesome. The problem is to evaluate v and w on the body surface in terms of their

series coefficients. Since, for example,

$$v\Big|_{x = x_b(z)} = \sum_{ijk=0} v_{ijk}(x_b(z) - x_r(z,t))^i z^j t^k \tag{107}$$

we must introduce the auxiliary functions

$$Y_i(z,t) \equiv (x_b(z) - x_r(z,t))^i = \sum_{ijk=0} y_{ijk} z^j t^k \tag{108}$$

Since

$$y_i = (O(x) + O(t))^i \tag{109}$$

we have

$$y_{ijk} = 0 \qquad \text{if} \qquad j+k < i \tag{110}$$

Thus

$$v\Big|_{x=x_b(z)} = \sum_{jk=0} z^j t^k \sum_{\ell=0}^{j} \sum_{m=0}^{k} \sum_{i=0}^{\ell+m} v_{i,j-\ell,k-m} y_{i\ell m} \tag{111}$$

The cases for which computations were carried out are identified in Table II. The gas properties associated with these cases are about the same as in Capiaux and Washington's (1963) study. The values chosen for the dissociation rate parameter C in equation (7) correspond roughly to body radii of 1 cm (cases 1 and 3) and 10 cm (case 2).

Figures 7 and 8 show the distribution of pressure and degree of dissociation along the body surface at two times early in the process, while their distributions along the axis of symmetry are shown in Figures 9 and 10. The convergence is seen to be slow even when $v_s t$ is but a tenth

of the sphere's radius. However, the strong effect of the dissociation rate parameter C is quite clear. In the corresponding perfect-gas problem, the method performed much better, yielding useful results in the vicinity of the stagnation point for $v_s t/r_b$ up to 2.0 or more (Moran and Van Moorhem 1969). Of course, it was to be expected that the present problem, with its relaxation process, would be much more difficult than the other.

REFERENCES

Agnew, R. P. (1960) Differential Equations, McGraw-Hill: New York.

Blasius, H. (1908) Grenzchichten in Flussigkeiten mit Kleiner Reibung. Z. Math. Phys, 56, 1-37.

Capiaux, R. and Washington, M. (1963) Nonequilibrium flow past a wedge. AIAA J. 1, 650-660.

Cabannes, H. (1951) Determination theorique de l'ecoulement d'un fluid derriere une onde de choc detachee, ONERA note tech. no. 5.

Cabannes, H. (1956) Tables pour la determination des onde de choc detachees. La Recherche Aero. No. 49, 11-15.

Conti, R. (1966) A Theoretical Study of Non-equilibrium Blunt-body Flows. J. Fluid Mech. 24, 65-88.

Conti, R. and Van Dyke, M. (1969) Inviscid reacting flow near a stagnation point. J. Fluid Mech. 35, 799-813.

Davis, R. T. (1967), Laminar Incompressible Flow Past a Semi-infinite Flat Plate. J. Fluid Mech. 27, 691-704.

Domb, C. (1965), Statistical Mechanics of Critical Behavior in Magnetic Materials. Magnetism II A, 1-42. New York: Academic Press.

Dugundji, J. (1948) An investigation of the detached shock in front of a body of revolution. J. Aero. Sci. 15, 699-705.

Freeman, N. C. (1958) Non-equilibrium flow of an ideal dissociating gas. J. Fluid Mech. 4, 407.

Hartree, D. R. (1958) Numerical Analysis, 2nd Ed. Oxford U. Press.

Henrici, P. (1963) Some applications of the quotient-difference algorithm. Proc. Symp. Appl. Math. 15, 159-183.

Henrici, P. (1964) Elements of Numerical Analysis. Wiley: New York.

Leavitt, J. A. (1966) Methods of applications of power series. Math. Comp. 20, 46-52.

Leavitt, J. A. (1968) Computational aspects of the detached shock problem. AIAA J. 6, 1084-1088.

Lewis, G. E. (1965) Analytic continuation using numerical methods. Methods in Comp Phys. 4, 45-81.

Lighthill, M. J. (1957) Dynamics of a dissociating gas. Part I. J. Fluid Mech. 2, 1-32.

Lin, C. C. and Rubinov, S. I. (1948) On the flow behind curved shocks. J. Math. and Phys. 27, 105-129.

Lin, C. C. and Shen, S. F. (1951) An analytic determination of the flow behind a symmetrical curved shock in a uniform stream. NACA Tech. Note 2506.

Moran, J. P. (1965) Two problems in gasdynamics. Part II. The inverse blunt-body problem. Cornell U. Ph.D. Thesis.

Moran, J. P. and Van Moorhem, W. K. (1969) Diffraction of a plane shock by an analytic blunt body. J. Fluid Mech. 38, 127-136.

Moran, J. P. (1970) Initial stages of axisymmetric shock-on-shock interaction for blunt bodies. Phys. Fluids 13, 237-248.

Richtmyer, R. D. (1957) Detached-shock calculations by power series. NYU Courant Inst., AEC Comp. and Appl. Math Center, NYO-7973.

Schlicting, H. (1958) Boundary Layer Theory. 4th Ed. McGraw Hill: New York.

Sedney, R. and Gerber, N. (1963) Nonequilibrium flow past a cone. AIAA J. 1, 2482-2486.

Sedney, R. and Gerber, N. (1967). Shock curvature and gradients at the tip of pointed axisymmetric bodies in non-equilibrium flow. J. Fluid Mech. 29, 765-779.

Spurk, J. H., Gerber, N. and Sedney, R. (1966) Characteristic calculation of flowfields with chemical reactions. AIAA J. 4, 30-37.

Underwood, R. L. (1969) Calculation of incompressible flow past a circular cylinder at moderate Reynolds numbers. J. Fluid Mech. 37, 95-114.

Van Dyke, M. D. (1958) A model of supersonic flow past blunt axisymmetric bodies, with application to Chester's solution. J. Fluid Mech. 3, 515-522.

Van Dyke, M. D. (1964) The circle at low Reynolds number as a test of the method of series truncation. Proc. 11th Int. Cong. Appl. Mech., Munich, 1165-1169.

Van Dyke, M. D. (1965) Hypersonic flow behind a paraboloidal bow wave. J. de Mecanique 4, 477-493.

Van Dyke, M. D. (1966) The blunt-body problem revisited. Fundamental Problems in Hypersonic Flow, 52-65. Ithaca: Cornell U. Press.

Van Tuyl, A. (1960) The use of rational approximations in the calculation of flows with detached shocks. J. Aero. Sci. 27, 559-560.

Van Tuyl, A. (1967) Use of rational approximations in the calculation of flows past blunt bodies. AIAA J. 5, 218-225.

Van Tuyl, A. H. (1971) Use of Pade fractions in the calculation of blunt body flows. AIAA J. 9, 1431-1433.

Vincenti, W. G. and Kruger, C. H. Jr. (1965) Introduction to Physical Gas Dynamics. Wiley: New York.

Vinoleur, M. (1970) On stagnation-point conditions in non-equilibrium inviscid blunt-body flows. J. Fluid Mech. 43, 49-75

Wall, H. S. (1948) Continued Fractions. New York: D. Van Nostrand.

Table 1

Equation	Coefficients of Highest order
(21)	p_m, v_m
(22)	h_m, T_m, α_m
(23)	ρ_m, v_m
(24)	h_m, v_m
(27)	b_m, ρ_m, α_m
(28)	p_m, ρ_m, b_m, T_m
(34)	g_m, T_m
(35)	d_m, ρ_m, b_m, f_m
(36)	α_m
(39)	f_m, g_m
(40)	w_m, T_m

Table 2

Identification of Cases Studied:

Relaxation Behind a Normal Shock

Case	$\rho d/\rho\alpha$	β	$\frac{R\theta_d}{v_\infty^2}$	α_∞	$\frac{v_\infty^2}{RT_\infty}$
1	10^6	0.0	1.0	0	∞
2	10^6	0.0	2.0	0	∞
3	10^7	0.0	2.0	0	∞

Coefficients of Expansions of Temperature Behind Normal Sho

(Case 1 of Table 2)

$$T = T_0 + T_1 x^* + T_2 x^{*2} + \ldots = \cfrac{\overline{T}_0}{1 + \cfrac{\overline{T}_1 x^*}{1 + \cfrac{\overline{T}_2 x^*}{1 + \ldots}}}$$

n	T_n	$\overline{T}_n$
1	1.22449×10^{-1}	.1224490
2	-3.79970×10^{-3}	.0310308
3	4.43516×10^{-4}	.0856934
4	-6.03850×10^{-5}	.0264638
5	8.77218×10^{-6}	.0568869
6	-1.32095×10^{-6}	.0335236
7	2.03542×10^{-7}	.0528393
8	-3.18718×10^{-8}	.0366453
9	5.05080×10^{-9}	.0517006
10	-8.07909×10^{-10}	.0375344
11	1.30204×10^{-10}	.0511719
12	-2.11143×10^{-11}	.0371503
13	3.44196×10^{-12}	.0514257
14	-5.63621×10^{-13}	.0362754
15	9.26553×10^{-14}	.0527321
16	-1.52846×10^{-14}	.0349608
17	2.52912×10^{-15}	.0551014
18	-4.19647×10^{-16}	.0326020
19	6.98043×10^{-17}	.0601612
20	-1.16376×10^{-17}	.0244839

Effect of Number of Terms Used on

Power-Series Result for Temperature Behind Shock

Number Of Terms	$x^* = 1.0$	$x^* = 4.0$	$x^* = 8.0$
1	.1224490	.1224490	.1224490
2	.1186593	.1072502	.0920514
3	.1190928	.1143465	.1204365
4	.1190324	.1104818	.0895194
5	.1190412	.1127275	.1254502
6	.1190399	.1113748	.0821652
7	.1190401	.1122085	.1355224
8	.1190400	.1116864	.0686824
9	.1190400	.1120174	.1534208
10	.1190400	.1118056	.0449851
11	.1190400	.1119421	.1847901
12	.1190400	.1118536	.0034195
13	.1190400	.1119133	.2399490
14	.1190400	.1118735	-.0699049
15	.1190400	.1118983	.3375974
16	.1190400	.1118819	-.2001800
17	.1190400	.1118928	.5117046
18	.1190400	.1118856	-.4332575
19	.1190400	.1118904	.8242251
20	.1190400	.1118872	-.8529350
21	.1190400	.1118893	1.388612
Exact Result	.1190400	.1118885	.1059518

Table 5

Effect of Number of Terms used on Continued-Fraction Result for Temperature behind Shock

Case 1 of Table 2

Number of Terms	$x^* = 4.0$	$x^* = 40.0$	$x^* = 400.0$
1	.1224490	.1224490	.1224490
2	.1089284	.0546346	.0091296
3	.1120878	.0956385	.0905788
4	.1118508	.0835421	.0296109
5	.1118920	.0909935	.0815088
6	.1118879	.0884608	.0469398
7	.1118885	.0899361	.0772184
8	.1118885	.0893760	.0576798
9	.1118885	.0896908	.0748744
10	.1118885	.0895659	.0638180
11	.1118885	.0896349	.0735434
12	.1118885	.0896073	.0673053
13	.1118885	.0896225	.0727986
14	.1118885	.0896166	.0693158
15	.1118885	.0896199	.0723992
16	.1118885	.0896186	.0705000
17	.1118885	.0896194	.0721986
18	.1118885	.0896191	.0712143
19	.1118885	.0896193	.0721100
20	.1118885	.0896192	.0716900
21	.1118885	.0896193	.0720915
Exact Result	.1118885	.0896192	.0722560

Table 6

Identification of Cases Studied

Steady Flow Past a Wedge

Case	ρ_d/ρ_0	β	$\frac{R\theta_d}{v_\infty^2}$	α_∞	M^*_∞	θ_b**
1	0.811×10^8	-2.5	0.15375	10^{-10}	32.0	25.1751
2	0.811×10^3	-2.5	0.15375	0.9	32.0	21.9985
3	0.811×10^8	-2.5	0.04	10^{-10}	32.0	25.175
4	0.811×10^8	-2.5	0.04	0.9	32.0	21.9985

* $M_\infty = \dfrac{V_\infty}{\sqrt{\dfrac{(4+\alpha_\infty)\ (1+\alpha_\infty)RT_\infty}{3}}}$

** Determined so that $\theta_s(0) = 30$

Table 7

Effect of Number of Terms Used on Continued -

Fraction Results for Steady Flow Past Wedge in Lighthill Gas

Case 1 of Table 6

			Number of Terms			
r^*	x^*		9	10	11	12
.0001		θ_s (rad)	.5229404	.5229404	.5229404	.5229404
.001		θ_s (rad)	.5174350	.5174350	.5174350	.5174350
.01		θ_s (rad)	.4990483	.4982539	.4986145	.4985156
	.0001	$P_{body}/\rho_\infty v_\infty^2$	.2130972	.2130972	.2130972	.2130972
	.001	$P_{body}/\rho_\infty v_\infty^2$	.2061449	.2061468	.2061464	.2061460
	.01	$P_{body}/\rho_\infty v_\infty^2$	.1992687	.1998255	.1993362	.1980560
	.0001	α_{body}	.0342119	.0342119	.0342119	.0342119
	.001	α_{body}	.1998346	.1999623	.1999538	.1999566
	.01	α_{body}	.3848967	.4235381	.3882949	.4076512

Table 8

Coefficients of r in Power Series (55)

Conditions Same as Case 1 of Table 6

i	α_{1i}	γ_{1i}
0	0	.0126422
1	-.9970547	-.0129672
2	-.3148299	-.0007958
3	-.1225967	+.0015306
4	-.0831509	-.0007508
5	-.0559873	+.0005975
6	-.0432364	-.0004271
7	-.0334583	+.0003252
8	-.0275312	-.0002418
9	-.0227804	+.0001872
10	-.0194839	-.0001406
11	-.0167675	+.0001103
12	-.0149197	-.0000833
13	-.0129975	+.0000660
14	-.0116260	-.0000499
15	-.0104534	+.0000399
16	-.0094833	-.0000302
17	-.0086422	+.0000244
18	-.0079270	[illegible]
19	+.0072[illegible]	[illegible]
20	-.0067[illegible]5	[illegible]

Table 9

Domb's Sequences for Analysis of the Series $\Sigma\, \alpha_{ij}\, \eta^j$

Coefficients α_{ij} from Table 8

Domb's analysis assumes

$$f(\eta) = \sum f_i \eta^i = (\eta - \eta_0)^q\, g(\eta) + h(\eta)$$

where g, h are regular for $\eta \leq \eta_0$. Then

$$b_j = \frac{1}{(j+1)\dfrac{f_{j+2}}{f_{j+1}} - j\dfrac{f_{j+1}}{f_j}} \longrightarrow \eta_0$$

$$c_j = j-1-j\, b_j\, (f_{j+1})/f_j \longrightarrow q$$

j	b_j	c_j
10	0.9363	1.1365
11	1.0587	-0.1529
12	0.9581	1.0022
13	1.0370	0.0139
14	0.9726	0.8824
15	1.235	0.1430
16	0.9822	0.7838
17	1.0151	0.2407
18	0.9885	0.7055
19	1.0097	0.3134
20	0.9925	0.6480
21	1.0062	0.3566
22	0.9952	0.6046

Table 10

Identification of Cases Studied:

Steady Flow Past Behind Spherical Shock of Radius 1

Case	$\frac{\rho_d}{\rho_\infty}$	β	$\frac{R\theta_d}{v_\infty^2}$	α_∞	M_∞	$\frac{C\rho_\infty}{v_\infty}$
1	8.4×10^6	0.0	0.8175	0.0	72.415	7.7333×10^5
2	8.4×10^6	0.0	0.8175	0.0	72.415	1.3899×10^5
3	8.4×10^6	0.0	0.8175	0.0	72.415	2.6545×10^3

Table II

Identification of Cases Studied:

Diffraction of Plane Shock by

Sphere of Radius 1

Case	$\frac{\rho_d}{\rho_1}$	β	$\frac{R\theta_d \rho_1}{p_1}$	$\frac{w_s}{p_1/\rho_1}$	$\frac{C\rho_1^{3/2}}{p_1^{1/2}}$	$\left(\frac{p_1}{\rho_1 \Gamma}\right)^{\beta}$
1	1.5×10^8	-2.5	218.0	10.0	10^7	
2	1.5×10^8	-2.5	218.0	10.0	10^8	
3	1.5×10^8	-2.5	218.0	15.0	10^7	

Figure Captions

Figure 1. Temperature behind normal shock in Lighthill gas (normalized by v_∞^2/R). Cases identified in Table 2. Solid lines are Freeman's (1958) results; circles, continued-fraction results (34 and 35 terms).

Figure 2. Angle between tangent to shock attached to wedge in Lighthill gas and freestream direction (radius). Solid lines are Capiaux and Washington's (1963) results; circles, continued-fraction results (11 and 12 terms). (a) Case 1 of Table 6.

Figure 2(b). Case 2 of Table 6.

Figure 2(c). Case 3 of Table 6.

Figure 3. Presume distribution on wedge in Lighthill gas (normalized by $\rho_\infty v_\infty^2$). For legend see Figure 2. (a). Case 1 of Table 6.

Figure 3(b) Case 2 of Table 6.

Figure 3(c) Case 4 of Table 6.

Figure 4. Temperature distribution on wedge in Lighthill gas (normalized by v_∞^2/R). For legend see Figure 2. (a). Case 1 of Table 6.

Figure 4(b). Case 2 of Table 6.

Figure 4(c). Case 4 of Table 6.

Figure 5. Streamfunction divided by square of radius (ψ/z) and degree of dissociation (α) on axis of symmetry in shock layer of blunt body which supports spherical shock wave in Lighthill gas. Y is distance downstream of nose of shock (normalized by shock radius).

Solid lines are Conti and Van Dyke's (1969) results; circles, continued-fraction results (34 and 35 terms).
(a). Case 1 of Table 10.

Figure 5(b). Case 2 of Table 10.

Figure 5(c). Case 3 of Table 10.

Figure 6. Diffraction of plane shock by solid body.

Figure 7. Pressure distribution on surface of sphere of radius 1 after impingement of shock (normalized by $\rho_2 v_2^2$). Times normalized by $1/v_s$. Continued-fraction results (10 and 11 terms). Circles, case 1 of Table II; triangles, case 2; squares, case 3.

Figure 8. Degree of dissociation on surface of sphere of radius 1 after impingement of shock. For legend see Figure 7.

Figure 9. Pressure distribution of axis between reflected shock and sphere of radius (normalized by $\rho_2 v_2^2$). For legend see Figure 7.

Figure 10. Degree of dissociation on axis between reflected shock and sphere of radius 1. For legend see Figure 7.

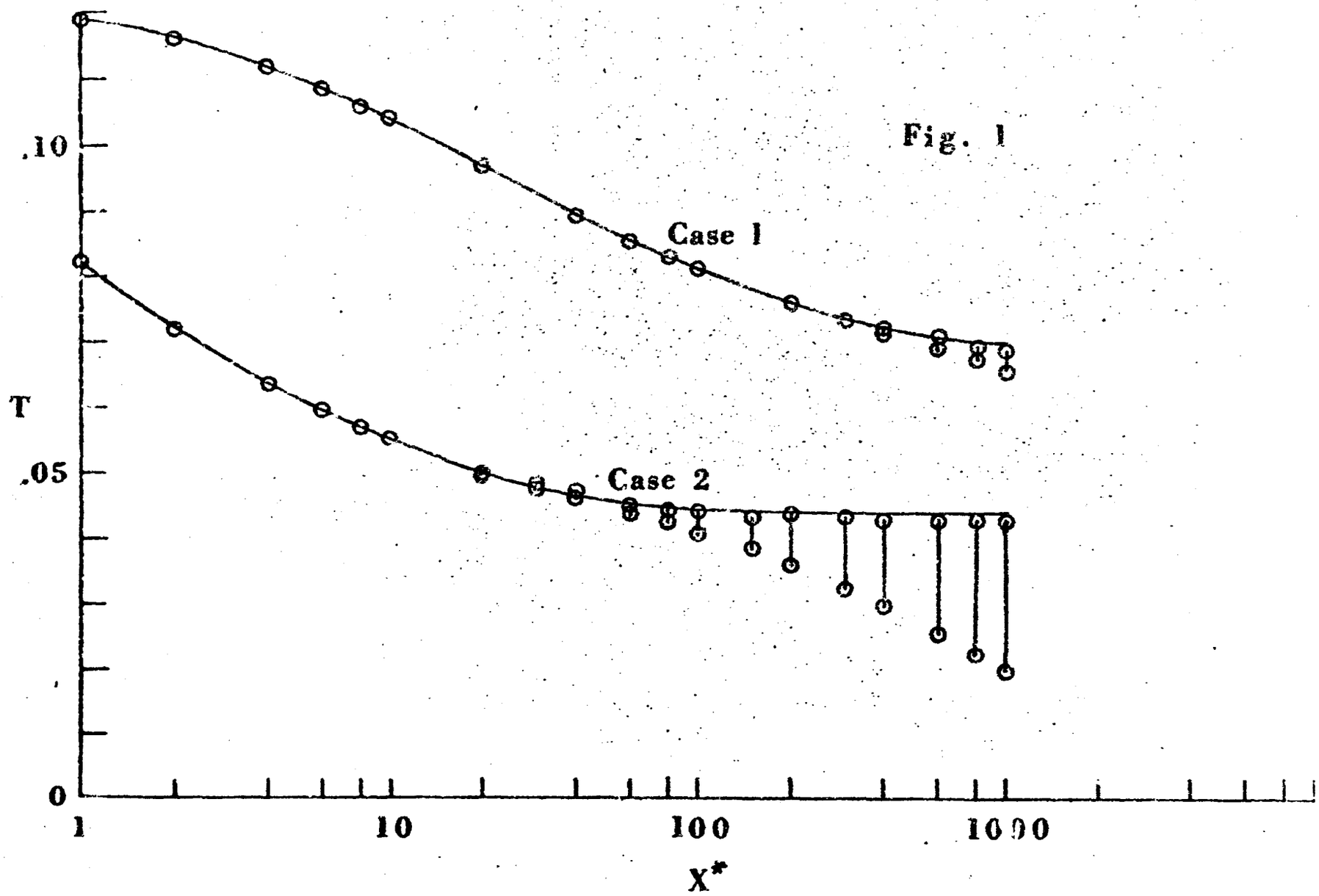


Fig. 1

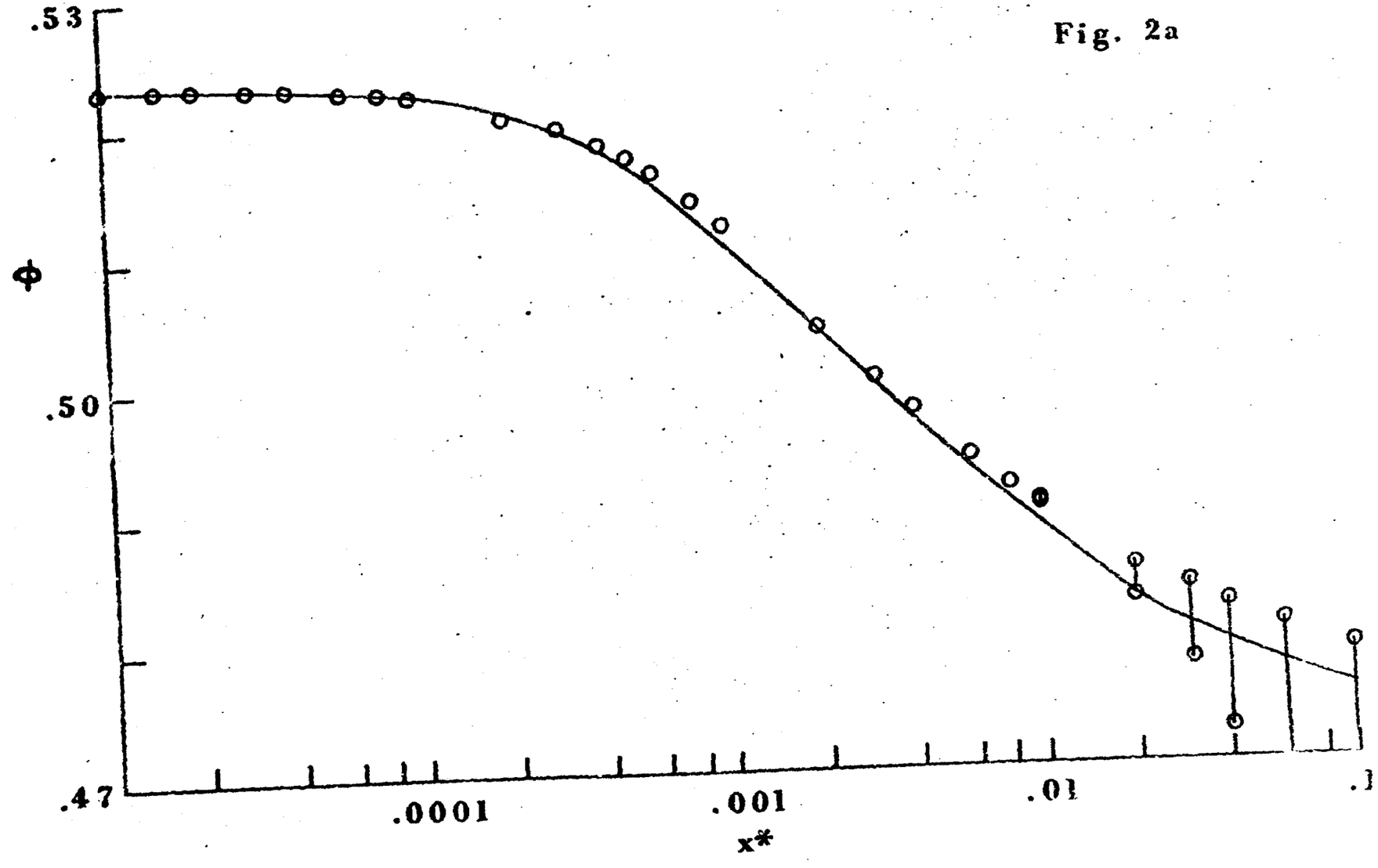


Fig. 2a

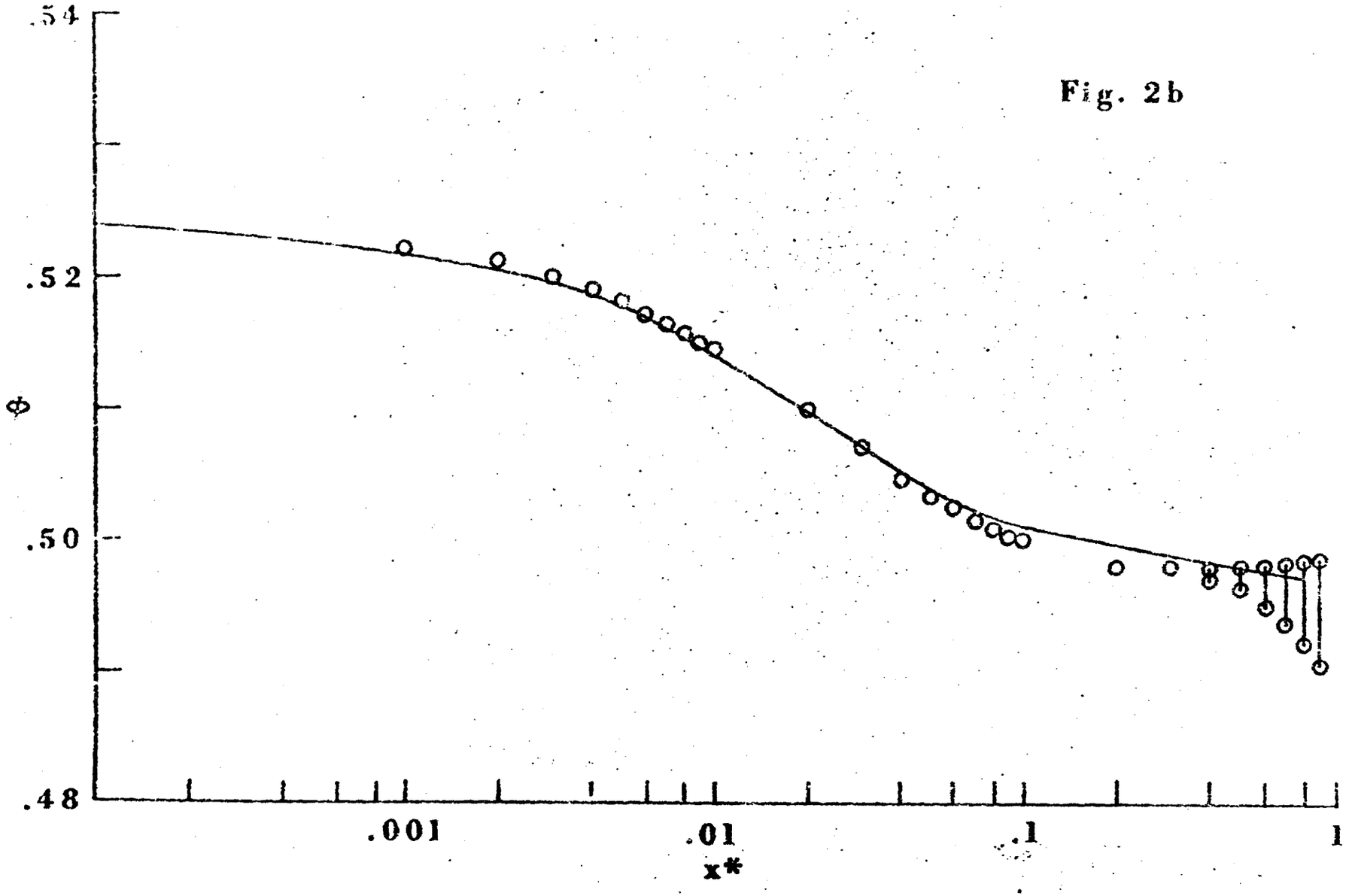


Fig. 2b

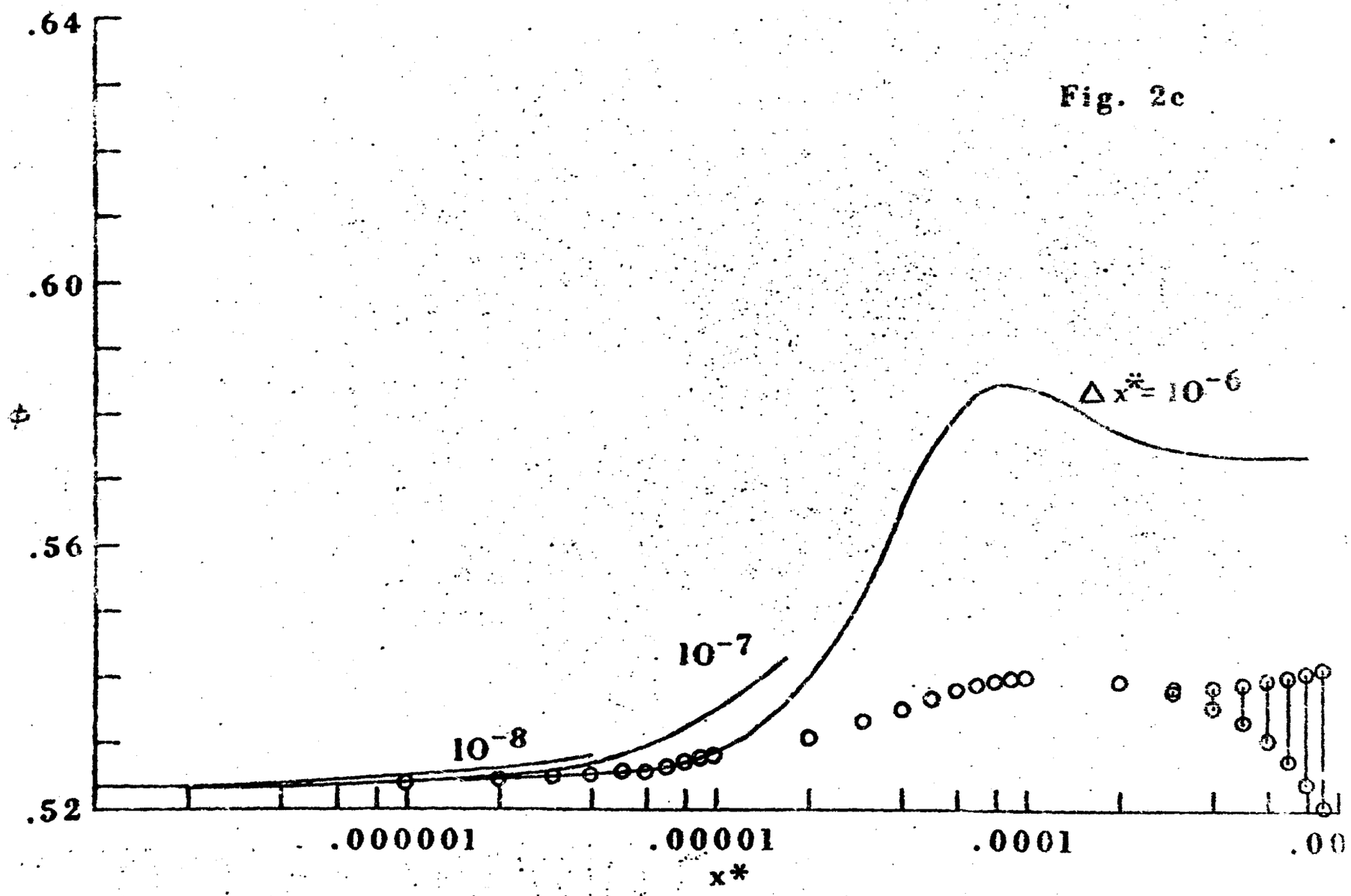


Fig. 2c

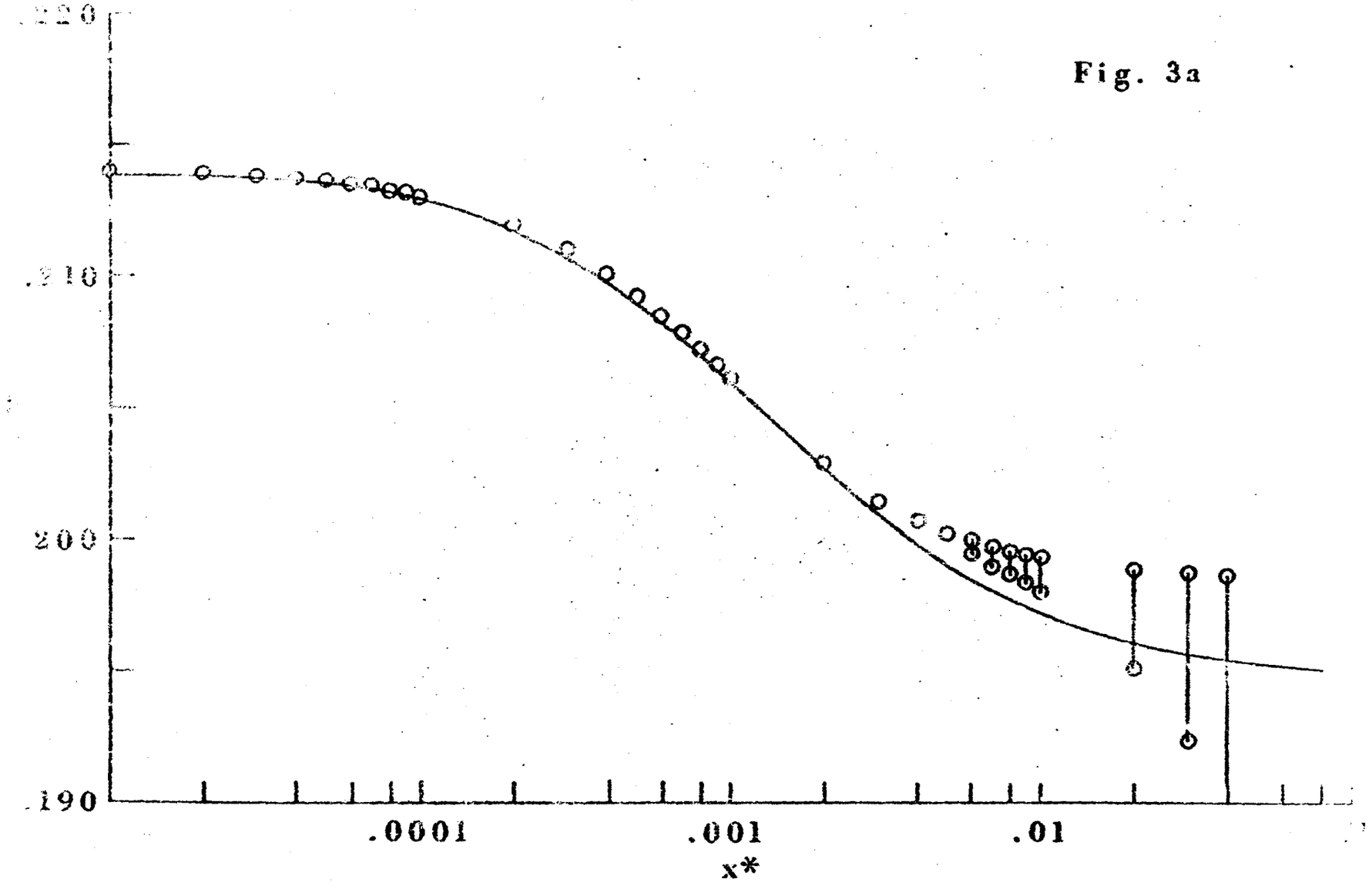


Fig. 3a

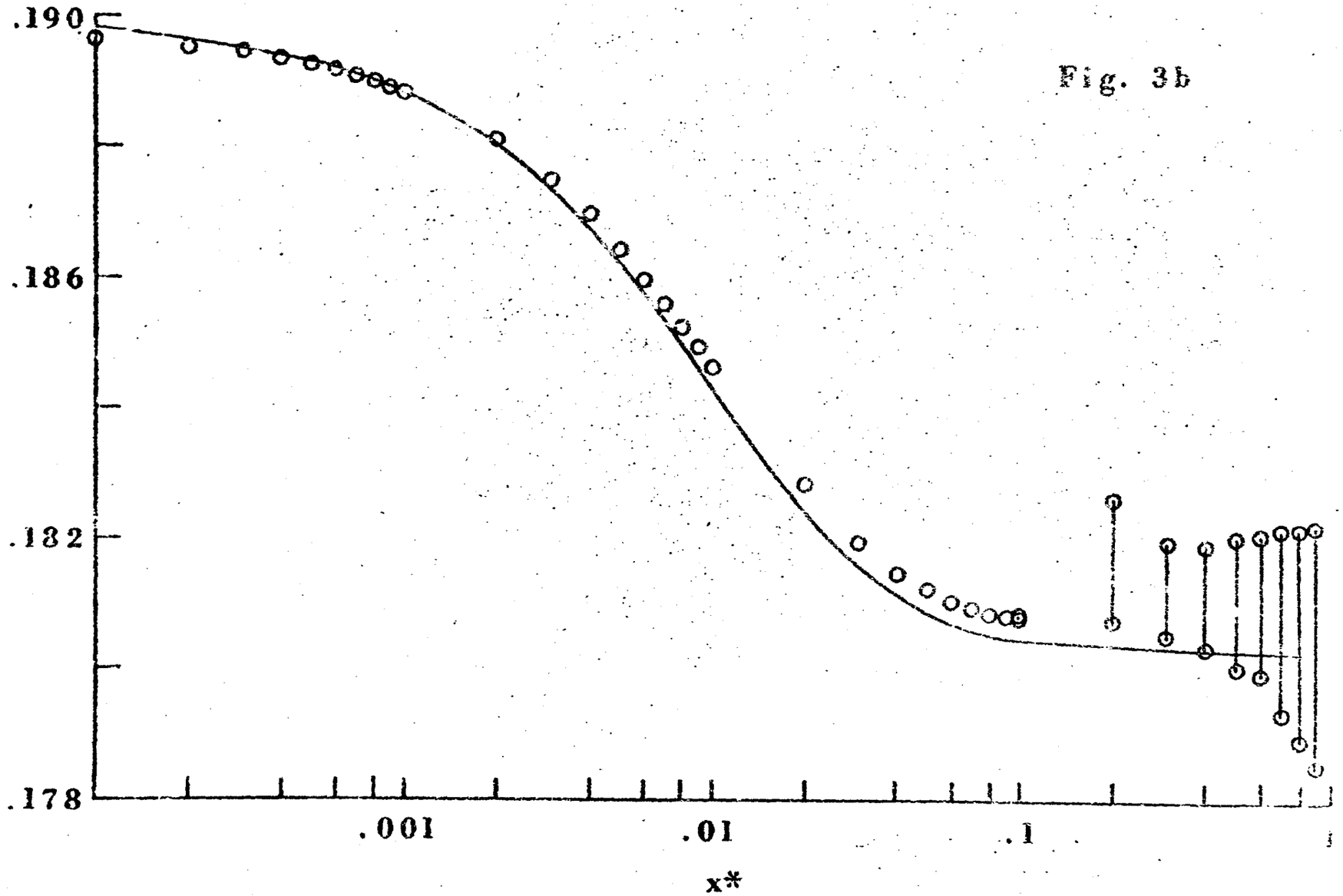


Fig. 3b

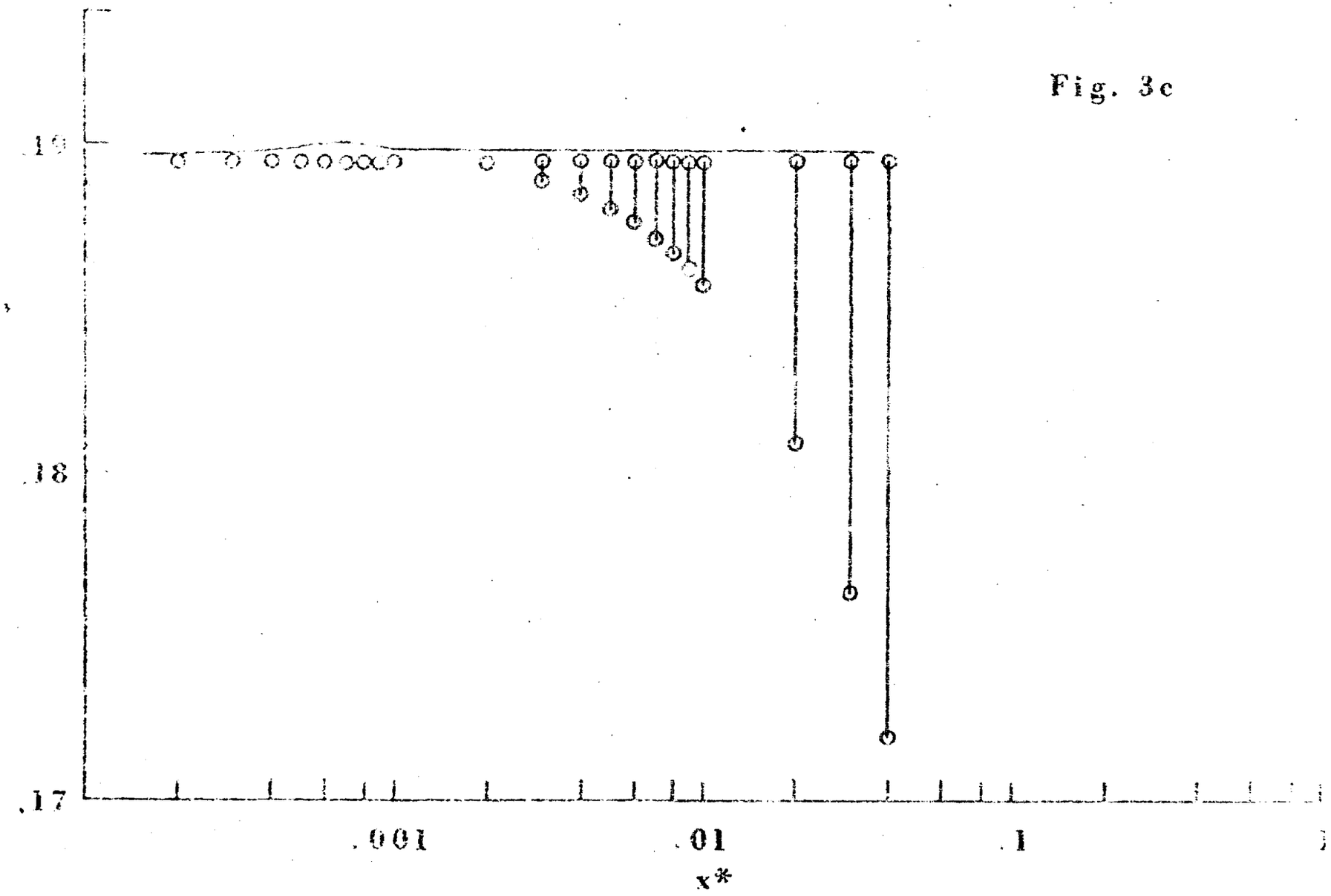


Fig. 3c

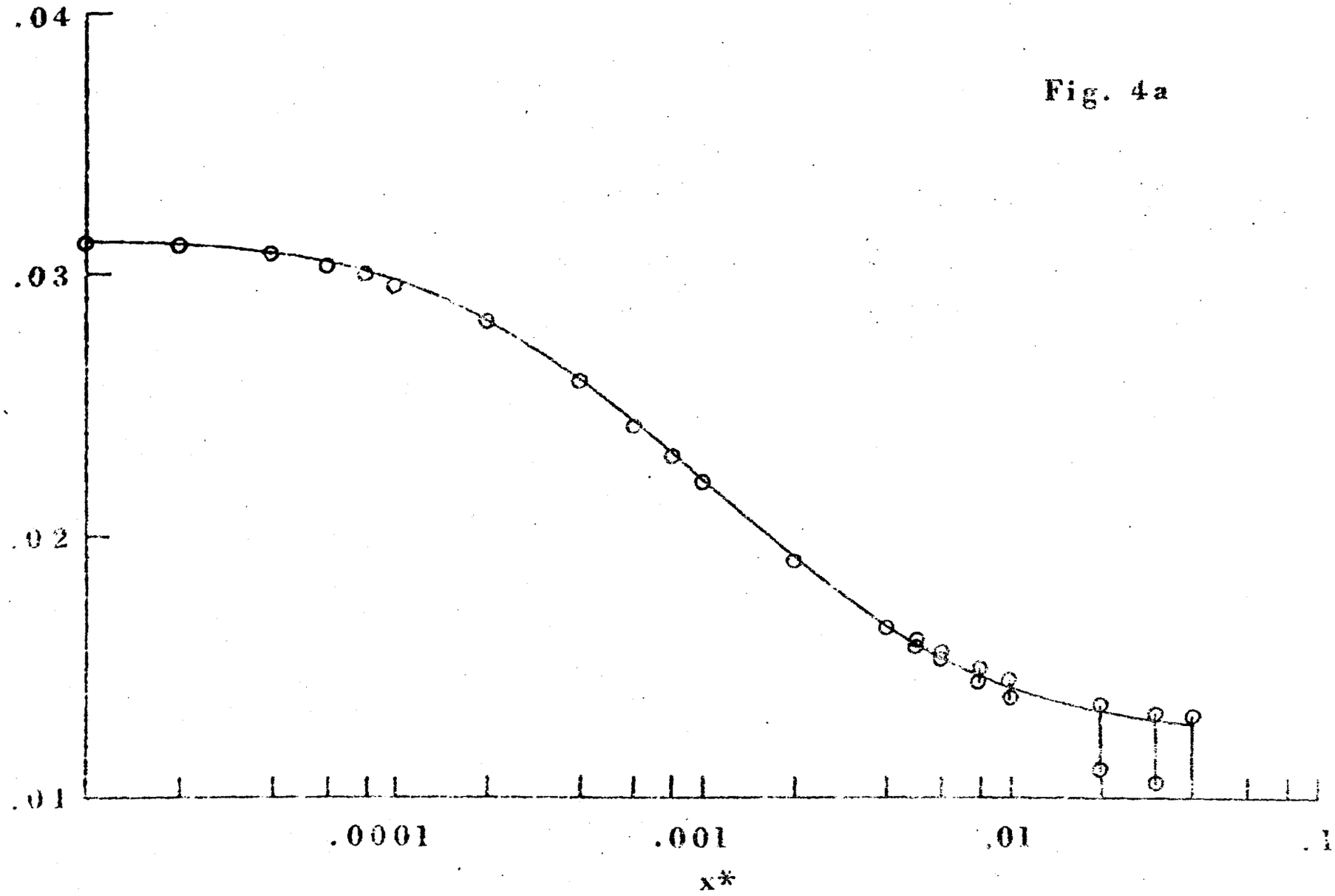
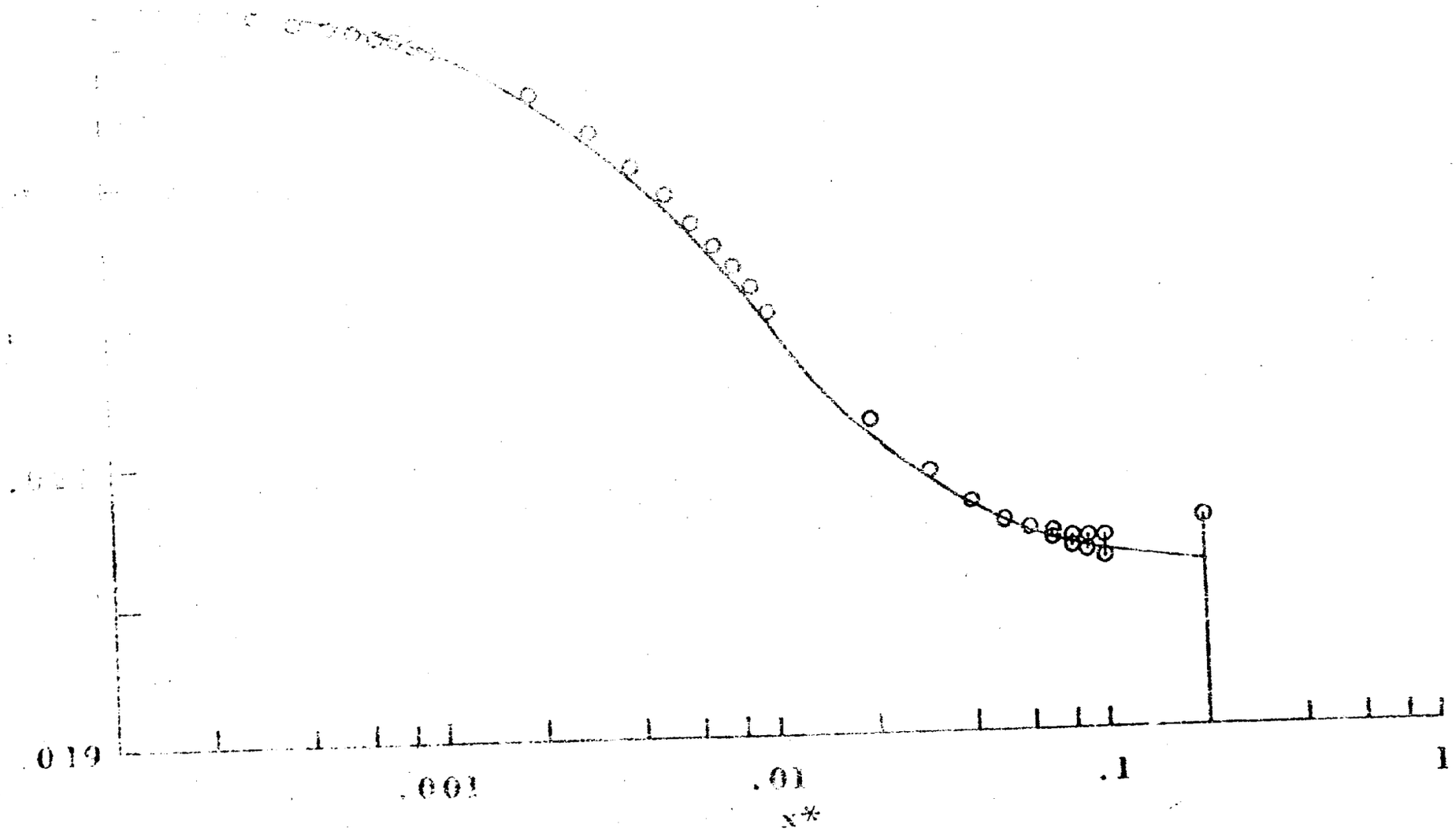


Fig. 4a

Fig. 4b



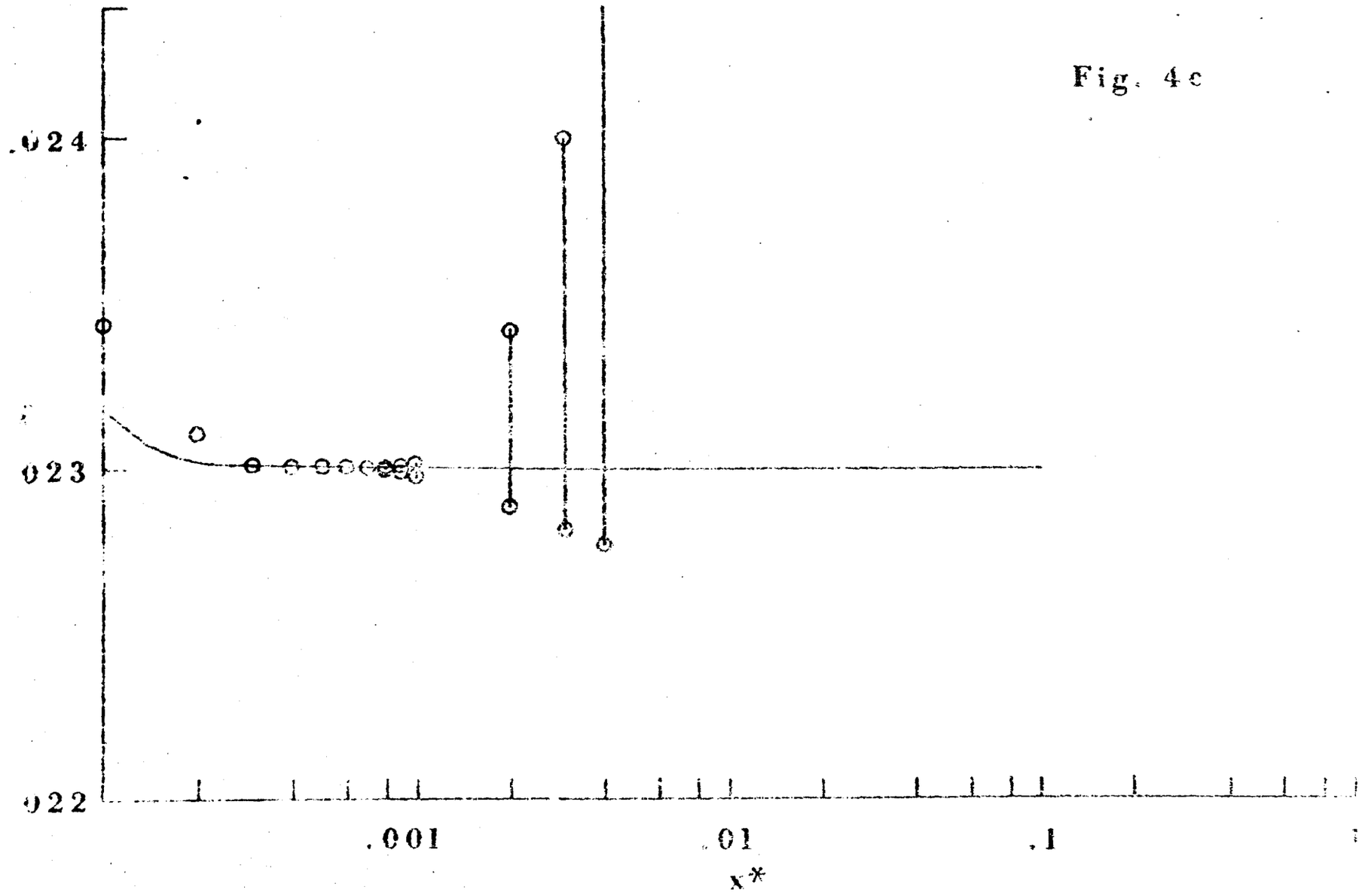


Fig. 4c

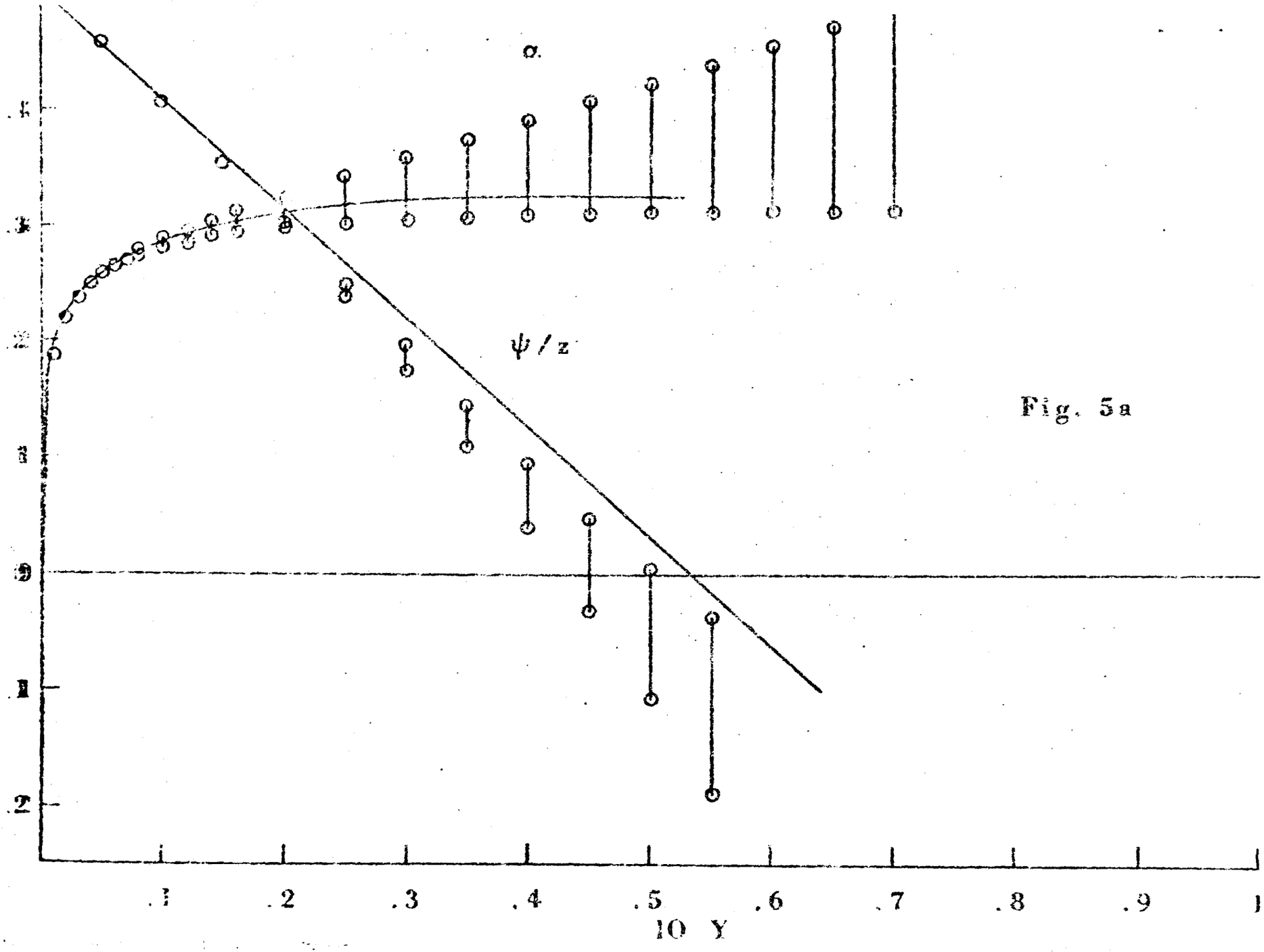


Fig. 5a

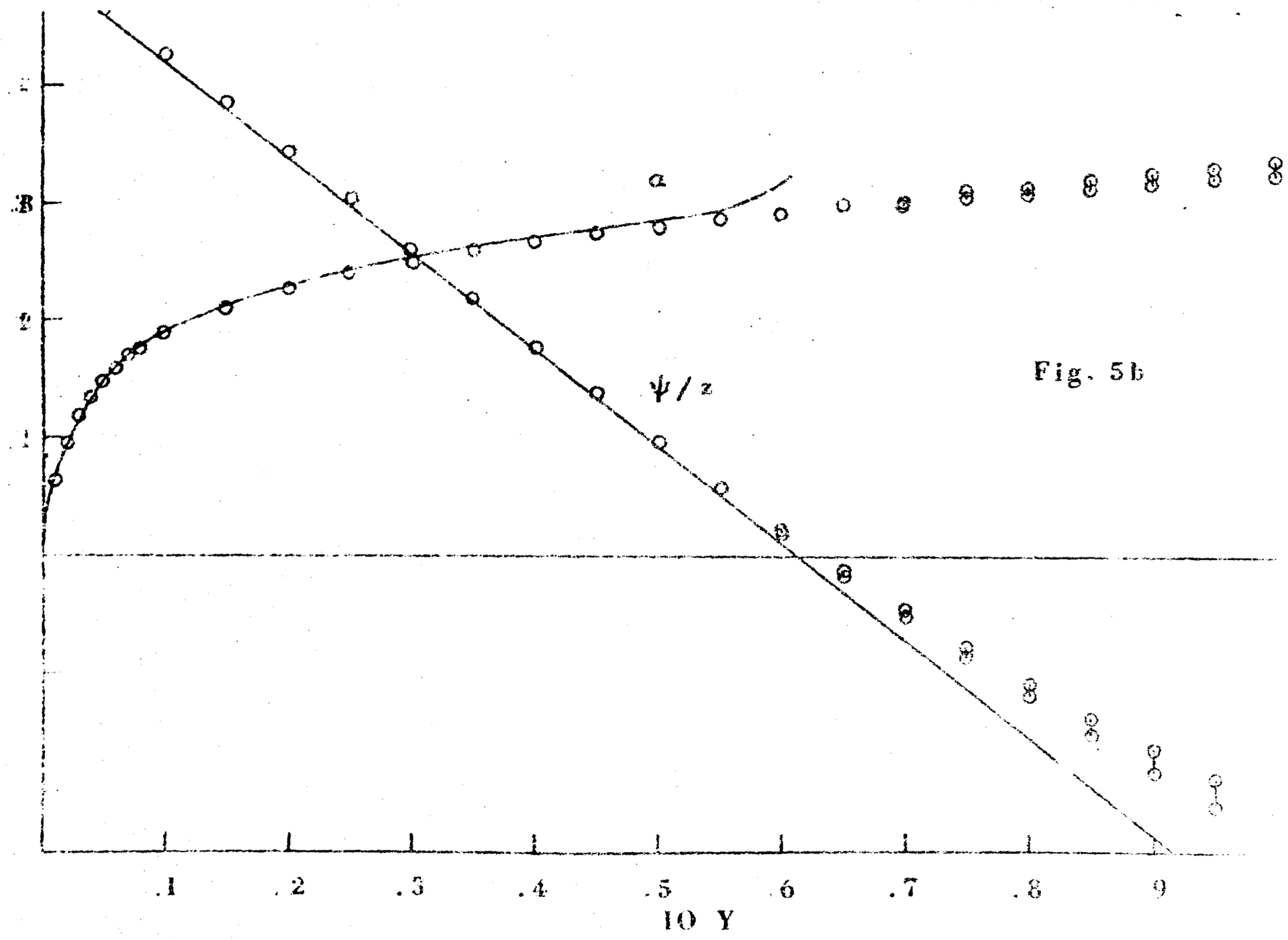


Fig. 5b

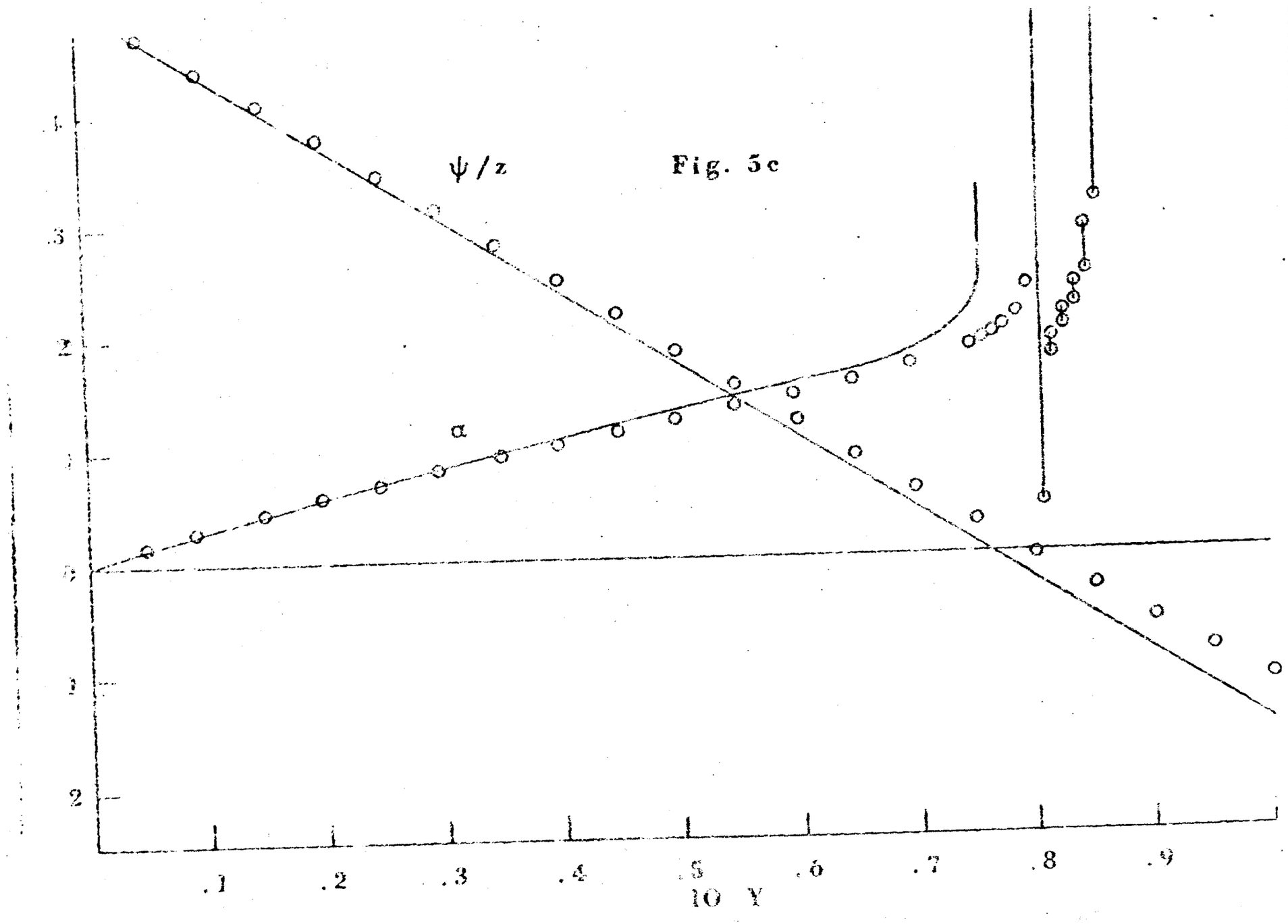


Fig. 5c

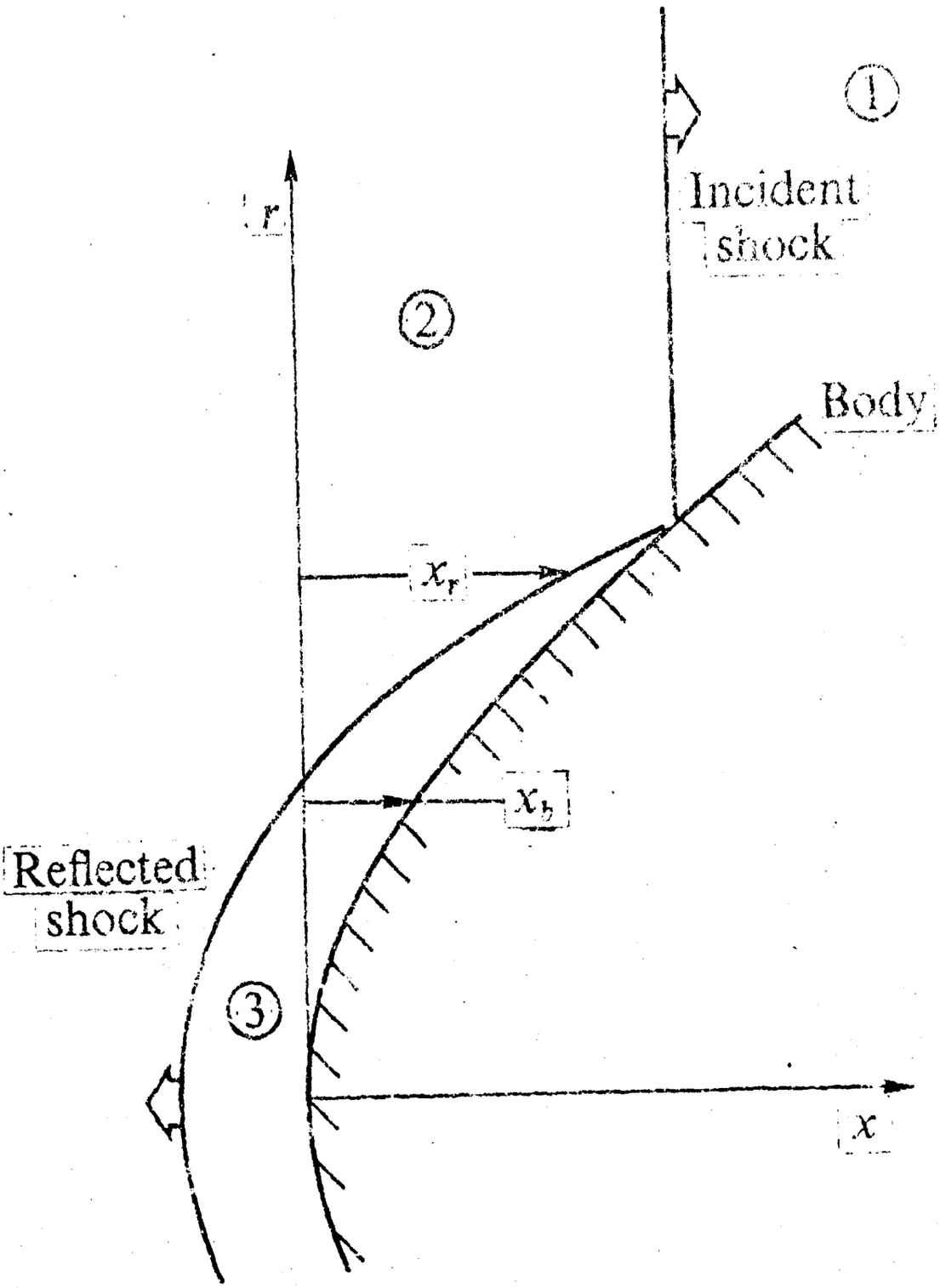


Fig. 6

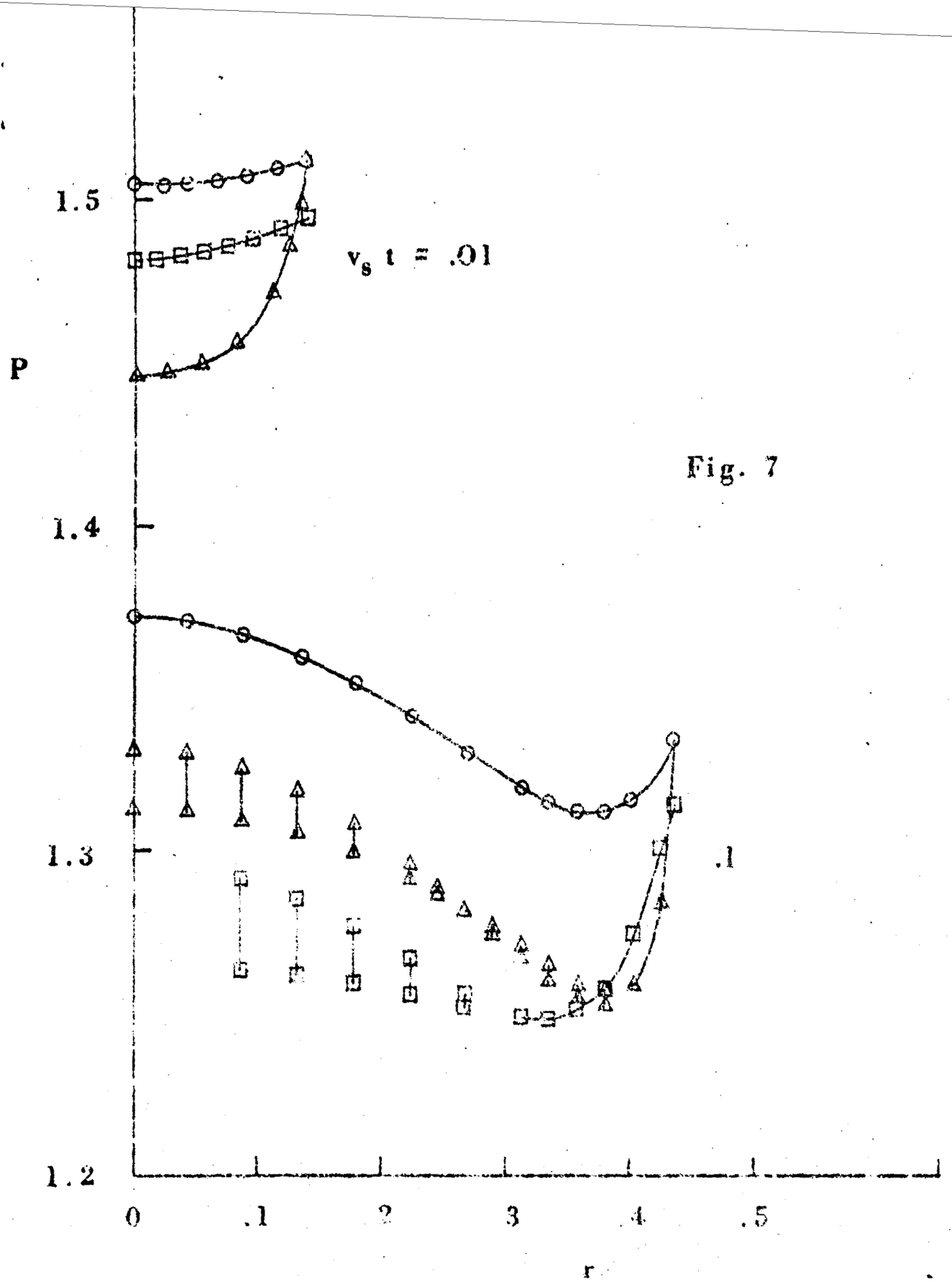


Fig. 7

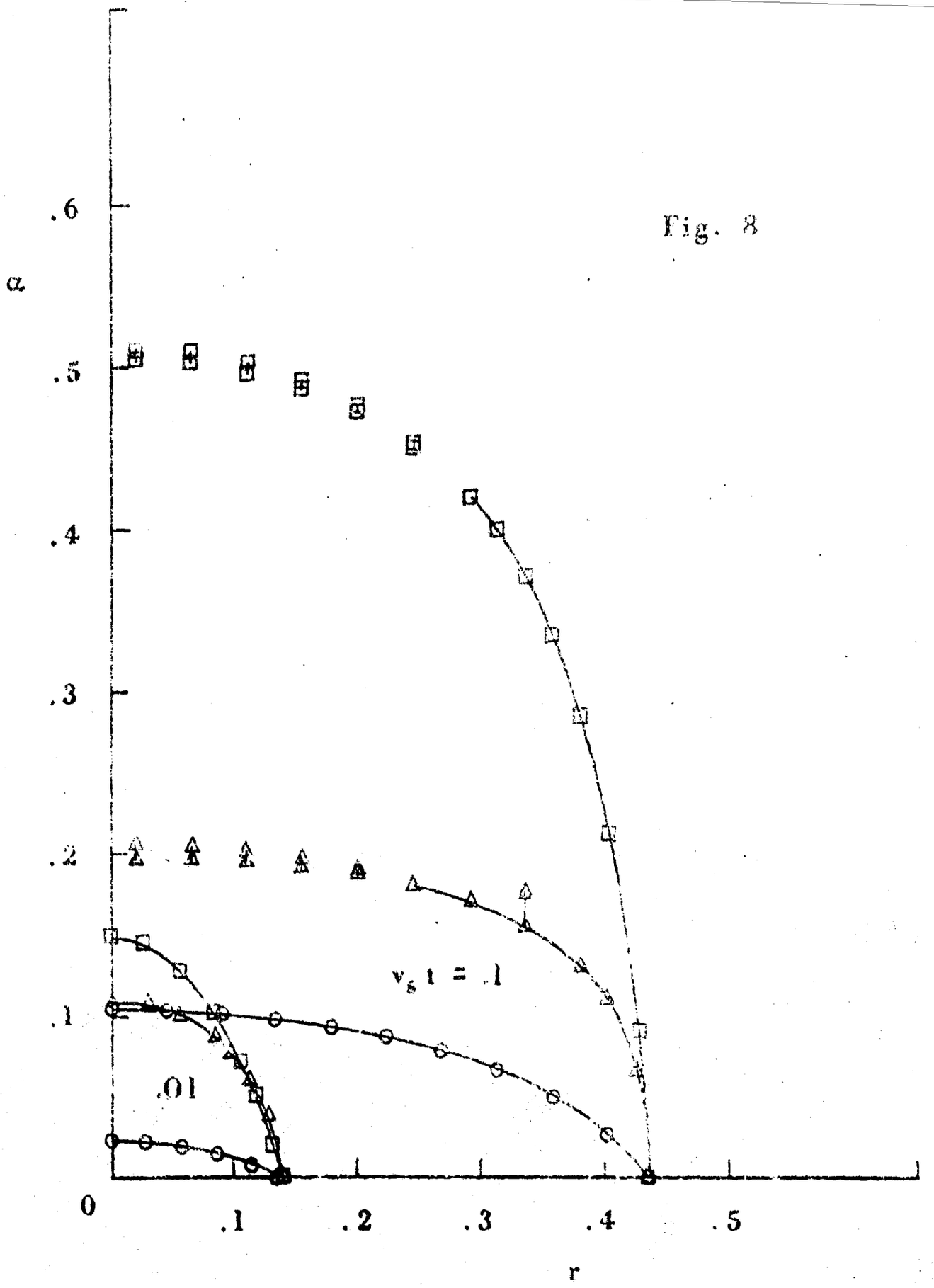


Fig. 8

Fig. 9

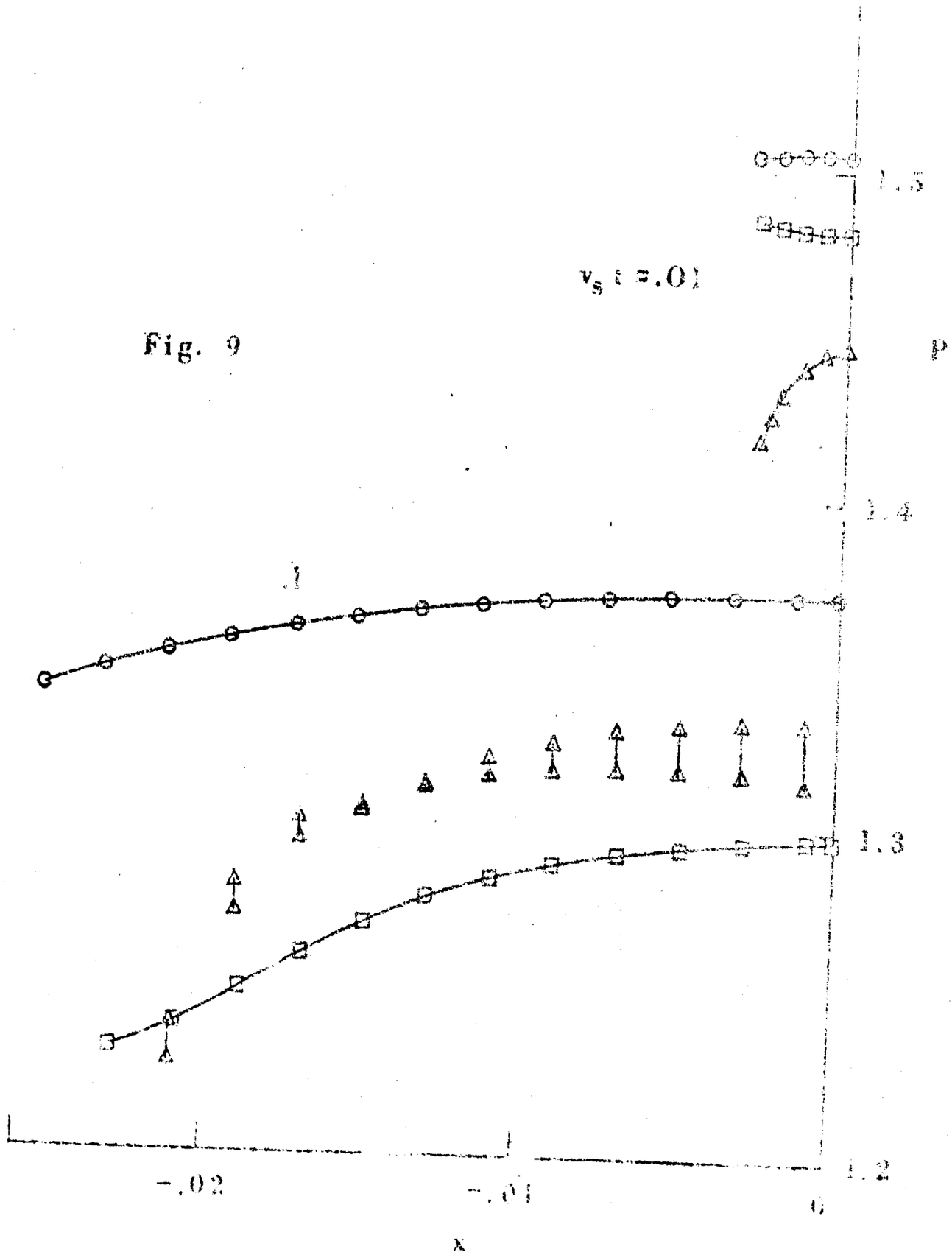
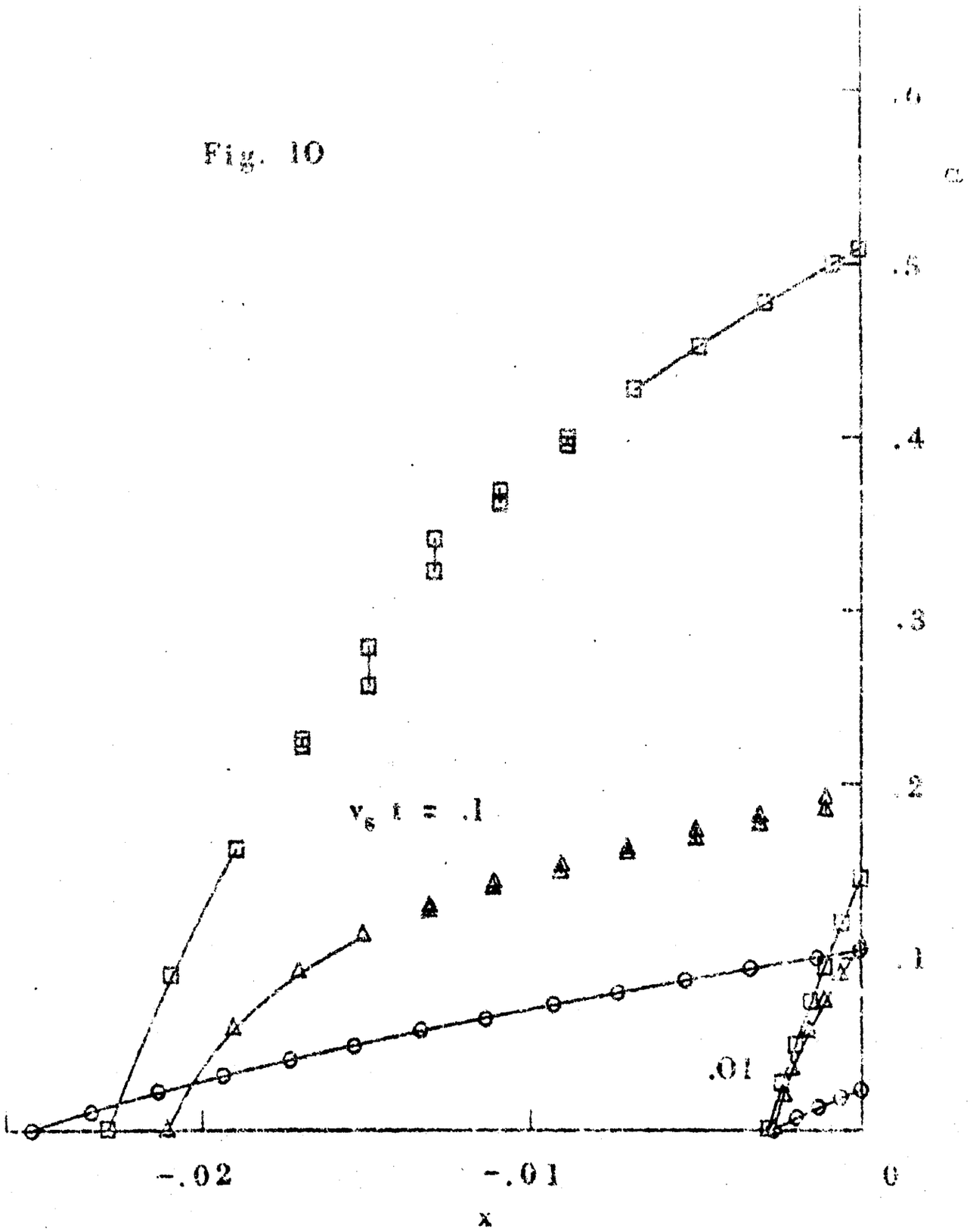



Fig. 10