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# ON GENERATING BESSEL FUNCTIONS BY USE OF THE BACKWARD RECURRENCE FORMULA

by

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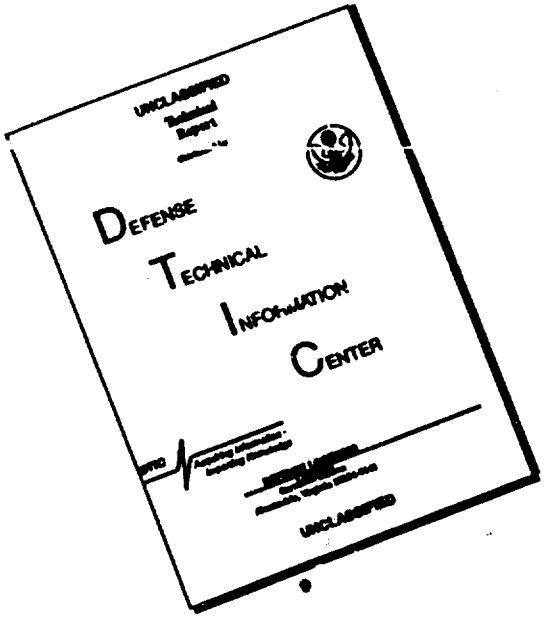


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13. ABSTRACT

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## INTRODUCTION

In my treatise on the special functions [1], a class of rational approximations for the generalized hypergeometric function ${}_pF_q$ was developed. These approximations depend on a number of free parameters. Since $I_\nu(z)$ can be expressed in terms of a ${}_0F_1$ or a ${}_1F_1$, there is a particular rational approximation corresponding to each of these hypergeometric forms and a choice of the aforementioned free parameters.

The idea of using the recursion formula for $I_\nu(z)$ in the backward direction to generate values of $I_\nu(z)$ is due to J. C. P. Miller [2]. It is a very powerful tool and the notion has created considerable interest; see [1, Vol. 2, pp. 159-166], [3,4] and the references quoted in these sources. The Miller scheme together with two certain normalization relations also gives rise to rational approximations.

In a conversation Jerry L. Fields conjectured that the specific rational approximations noted in the first paragraph are identical to the certain rational approximations which emerge by use of the backward recurrence scheme noted in the second paragraph. In the present paper, we verify this conjecture. In addition, we develop a new interpretation of the Miller method. We also study a third normalization technique which is sometimes used with the backward recursion scheme. A closed form analytical expression of the error for each case is derived. These equations are valuable as they lead to simple asymptotic estimates of the error which are very realistic and easy to apply in practice. It is demonstrated that the round-off error is insignificant. The paper closes with some numerical examples.

In the main body of the paper, we find it convenient to deal with the modified Bessel function $I_\nu(z)$. The results are valid for all $z$ in the cut complex z-plane $-\pi < \arg z \le \pi$ and in the cut complex $\nu$-plane, $|\arg \nu| < \pi$. In this connection, we should note that $I_{-\nu}(z) = I_\nu(z)$ if $\nu$ is an integer or zero. Thus we suppose throughout that $\nu$ is not a negative integer. Actually, it is sufficient to have $0 \le \arg z \le \pi/2$ in view of the definition of $I_\nu(z)$. Also it is sufficient to have $R(\nu) > -1$, for if $I_{1-\nu}(z)$ and $I_{-\nu}(z)$ are known, computations of $I_{-m-\nu}(z)$, $m$ = a positive integer, can be done by use of the recursion formula for $I_{-m-\nu}(z)$. All of this not withstanding, it is convenient to restate some of the key equations to facilitate application of our results to the Bessel function $J_\nu(z)$. This is done near the end of the paper.

## RATIONAL APPROXIMATIONS FOR $I_\nu(z)$

Case I. We begin with the representation

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\, {}_0F_1(\nu+1;z^2/4) \quad . \tag{1}$$

Theorem 1.

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \frac{\varphi_n(z)}{h_n(z)} + R_n(z),\ R_n(z) = \frac{S_n(z)}{h_n(z)} \quad , \tag{2}$$

$$\psi_n(z) = \sum_{k=0}^{n} \frac{(-n)_k(n+\lambda)_k}{(\nu+1)_k k!}\, {}_3F_0(-n+k,n+\lambda+k,1;X) \quad , \tag{3}$$

$$X = 4/z^2 \ ,\ \lambda = \nu+2-\delta \ ,\ \delta = 0 \text{ or } \delta = 1 \ , \tag{4}$$

$$h_n(z) = {}_3F_0(-n,n+\lambda,1;X) \quad , \tag{5}$$

or

$$h_n(z) = (-)^n n!(n+\nu+2-\delta)_n X^n\ {}_0F_1^n(-2n-1+\delta-\nu;X^{-1}) \quad . \tag{6}$$

Here for convenience we introduce the notation

$$ {}_pF_q^n\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) = \sum_{k=0}^{n} \frac{(\alpha_p)_k z^k}{(\rho_q)_k k!} \tag{7}$$

and our usual shorthand notation for generalized hypergeometric series applies, see [1, Vol. 1, pp. 41, 42]. Further,

$$S_n(z) = \frac{(z/2)^\nu O(n^\mu)}{\Gamma(\nu+1)} \quad ,$$

$$\mu = 1-\delta-R(\nu) \text{ if } -1 < R(\nu) \le 1-\delta \ ,\ \mu = 0 \text{ if } R(\nu) \ge 1-\delta \ , \tag{8}$$

$$h_n(z) = (-)^n n!(n+\nu+2-\delta)_n X^n \left[1+O(n^{-1})\right] , \tag{9}$$

whence

$$R_n(z) = \frac{(-)^n (z/2)^{\nu+2n}\Gamma(n+\lambda)}{\Gamma(\nu+1)n!\Gamma(2n+\lambda)} O(n^{\mu}) \left[1+O(n^{-1})\right] , \; R(\nu) > -1 , \tag{10}$$

and so, for $z$ and $\nu$ fixed, $R(\nu) > -1$, the approximation process is convergent.

Proof: Equations (2)-(5) follow from [1, Vol. 2, p. 96] with

$$a = 0 , \; f = g = 0 , \; p = 0 , \; q = 1 , \; \rho_1 = \nu+1 ,$$

$$\alpha = 1 - \delta , \; \beta = \nu , \; \lambda = \nu+2-\delta , \; \delta = 0 \text{ or } \delta = 1 , \; \gamma = z$$

and $z$ replaced by $z^2/4$ . Notice that the ${}_3F_0$ series for $h_n(z)$ in (5) turned around is the alternative form for $h_n(z)$ in (6).

Equation (8) follows from [1, Vol. 2, p. 103] while (9) follows from (6), see also [1, Vol. 1, pp. 259-261], and (10) is now obvious.

<u>Remark</u>: In the proof developed in the cited source, it was necessary to suppose that $R(\nu) > -1$ . Later, we present a new formulation of the error which shows that $\nu$ is unrestricted save that $\nu$ is not a negative integer. So throughout this work $\nu$ is arbitrary except as just indicated. Computation wise, the exception is no burden since $I_{-n}(z) = I_n(z)$ .

<u>Theorem 2</u>. Both $\psi_n(z)$ and $h_n(z)$ satisfy the same recurrence formula

$$h_n(z) + (C_1+XD_1)h_{n-1}(z) + (C_2+XD_2)h_{n-2}(z) + C_3 h_{n-3}(z) = 0 ,$$

$$C_1 = \frac{(2n+\lambda-2)(n-\lambda+1)}{(2n+\lambda-4)(n+\lambda-1)} , \; D_1 = \frac{n(2n+\lambda-2)(2n+\lambda-1)}{n+\lambda-1} ,$$

$$C_2 = \frac{-(n-1)(n+2\lambda-4)(2n+\lambda-1)}{(n+\lambda-1)(n+\lambda-2)(2n+\lambda-5)} , \; D_2 = \frac{-(n-1)(2n+\lambda-2)(2n+\lambda-1)(n+\lambda-3)}{(n+\lambda-1)(n+\lambda-2)} ,$$

$$C_3 = \frac{-(n-1)(n-2)(2n+\lambda-2)(2n+\lambda-1)}{(2n+\lambda-5)(2n+\lambda-4)(n+\lambda-1)(n+\lambda-2)} , \tag{11}$$

where $n \geq 3$ .

Proof: See [1, Vol. 2, Ch. 12].

Case II. Next we consider

$$I_\nu(z) = \frac{(z/2)^\nu e^z}{\Gamma(\nu+1)} \, {}_1F_1(\nu+\tfrac{1}{2};2\nu+1;-2z) \quad . \tag{12}$$

<u>Theorem 3.</u>

$$I_\nu(z) = \frac{(z/2)^\nu e^z}{\Gamma(\nu+1)} \frac{\xi_n(z)}{g_n(z)} + V_n(z) \ , \ V_n(z) = \frac{W_n(z)}{g_n(z)} \quad , \tag{13}$$

$$\xi_n(z) = \sum_{k=0}^{n} \frac{(-n)_k(n+2\nu+2)_k(\nu+\frac{1}{2})_k}{(2\nu+1)_k(\nu+3/2)_k k!} \, {}_3F_1\left(\begin{matrix}-n+k,n+2\nu+2+k,1\\ \nu+3/2+k\end{matrix}\middle| -\frac{1}{2z}\right), \tag{14}$$

$$g_n(z) = {}_3F_1\left(\begin{matrix}-n,n+2\nu+2,1\\ \nu+3/2\end{matrix}\middle| -\frac{1}{2z}\right), \tag{15}$$

or

$$g_n(z) = \frac{n!(n+2\nu+2)_n}{(2z)^n(\nu+3/2)_n} \, {}_1F_1^n\left(\begin{matrix}-n-\nu-\frac{1}{2}\\ -2n-2\nu-1\end{matrix}\middle| 2z\right) \quad . \tag{16}$$

Further,

$$W_n(z) = \frac{(z/2)^\nu e^z O(n^\omega)}{\Gamma(\nu+1)} ,$$

$$\omega = 1-2R(\nu) \text{ if } -1 < R(\nu) \leq \tfrac{1}{2} \ , \ \omega = 0 \text{ if } R(\nu) \geq \tfrac{1}{2} \ . \tag{17}$$

Also,

$$g_n(z) = \frac{n!(n+2\nu+2)_n e^z}{(2z)^n(\nu+3/2)_n} \left[1 - \frac{(z^2/4)}{n+\nu+1} + \frac{(z^2/32)(z^2-8)}{(n+\nu+1)^2} + O(n^{-3})\right] , \tag{18}$$

whence

$$V_n(z) = \frac{(z/2)^{\nu}(2z)^n(\nu+3/2)_n O(n^{\omega})}{\Gamma(\nu+1)(n+2\nu+2)_n n!} , \quad R(\nu) > -1 , \tag{19}$$

and for $z$ and $\nu$ fixed, $R(\nu) > -1$ , the approximation process is convergent.

Proof: Equations (13)-(15) follow from [1, Vol. 2, p. 96] with

$$a = 0,\ f = g = 0 ,\ p = q = 1 ,\ \alpha_1 = \nu+\tfrac{1}{2} ,\ \rho_1 = 2\nu+1 ,$$

$$\alpha = 1 ,\ \beta = 2\nu ,\ \lambda = 2\nu+2 ,\ \gamma = z$$

and $z$ replaced by $-2z$ . Equation (16) is equation (15) turned around. Equation (17) comes from [1, Vol. 2, p. 103] while (18) comes from (16), see also [1, Vol. 1, pp. 133, 259-261]. Thus (19) is at hand.

<u>Remark</u>: See the remark after Theorem 1.

<u>Theorem 4</u>. Both $\xi_n(z)$ and $g_n(z)$ satisfy the same recurrence formula

$$g_n(z) + (E_1+2F_1/z)g_{n-1}(z) + (E_2+2F_2/z)g_{n-2}(z) + E_3 g_{n-3}(z) = 0 ,$$

$$E_1 = \frac{-(n+\nu)(n+2\nu-1)}{(n+\nu-1)(n+2\nu+1)} ,\ F_1 = \frac{4n(n+\nu)}{n+2\nu+1} ,$$

$$E_2 = \frac{-n(n-1)}{(n+2\nu)(n+2\nu+1)} ,\ F_2 = \frac{-4(n-1)(n+\nu)(n+2\nu-1)}{(n+2\nu)(n+2\nu+1)} ,$$

$$E_3 = \frac{(n-1)(n-2)(n+\nu)}{(n+2\nu)(n+2\nu+1)(n+\nu-1)} , \tag{20}$$

where $n \geq 3$ .

Proof: See [1, Vol. 2, Ch. 12].

## BACKWARD RECURRENCE SCHEMATA FOR GENERATING $I_\nu(z)$

The technique for generating $I_\nu(z)$ by use of the recurrence formula for $I_\nu(z)$ employed in the backward direction is as follows. The recurrence formula

$$\varphi_{m,\nu}(z) = \frac{2(m+\nu+1)}{z}\varphi_{m+1,\nu}(z) + \varphi_{m+2,\nu}(z) \tag{21}$$

is satisfied by

$$I_{m+\nu}(z) \text{ and } e^{i(m+\nu)\pi}K_{m+\nu}(z) \ , \tag{22}$$

$$K_{m+\nu}(z) = \frac{(\pi/2)}{\sin(m+\nu)\pi}\left[I_{-m-\nu}(z) - I_{m+\nu}(z)\right] \ . \tag{23}$$

In this work, we always take $m$ a positive integer or zero. For later convenience, we also record the formula

$$\varphi_{m,\nu}(z) = \left\{\frac{4(m+\nu+1)(m+\nu+2)}{z^2} + \frac{2(m+\nu+2)}{m+\nu+3}\right\}\varphi_{m+2,\nu}(z) - \frac{(m+\nu+1)}{(m+\nu+3)}\varphi_{m+4,\nu}(z) \ . \tag{24}$$

Let $N$ be a positive integer and consider that solution of (21), call it $\varphi^{(N)}_{m,\nu}(z)$ with $m \le N+2$ such that

$$\varphi^{(N)}_{N+2,\nu} = 0 \ , \ \varphi^{(N)}_{N+1,\nu} = 1 \ . \tag{25}$$

Clearly $\varphi^{(N)}_{m,\nu}(z)$ is a linear combination of the solutions (22) subject to the conditions (25) and we readily find that

$$\varphi^{(N)}_{m,\nu}(z) = z\left[I_{m+\nu}(z)K_{N+2+\nu}(z) + e^{-i\pi(N+1-m)}K_{m+\nu}(z)I_{N+2+\nu}(z)\right] \tag{26}$$

in view of the Wronskian relation

$$I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = 1/z \ . \tag{27}$$

Suppose that we are given the normalization relation

$$\theta(z) = \sum_{k=0}^{\infty} \omega_k I_{k+\nu}(z) \quad . \tag{28}$$

Put

$$\theta^{(N)}(z) = \sum_{k=0}^{N+1} \omega_k \varphi_{k,\nu}^{(N)}(z) \tag{29}$$

and consider

$$i_{m+\nu}^{(N)}(z) = \frac{\theta(z)\varphi_{m,\nu}^{(N)}(z)}{\theta^{(N)}(z)} \ , \ m \le N+1 \quad . \tag{30}$$

We can now prove

<u>Theorem 5</u>. $$\lim_{N\to\infty} i_{m+\nu}^{(N)}(z) = I_{m+\nu}(z) \ , \ 0 \le m < N+1 \quad . \tag{31}$$

Proof: Using (26) and (29), we can write

$$i_{m+\nu}^{(N)}(z) = \frac{\theta(z)\left[I_{m+\nu}(z) - (-)^{N+m}\dfrac{I_{N+\nu+2}(z)K_{m+\nu}(z)}{K_{N+\nu+2}(z)}\right]}{\displaystyle\sum_{k=0}^{N+1} \omega_k I_{k+\nu}(z) - (-)^N \frac{I_{N+\nu+2}(z)}{K_{N+\nu+2}(z)} \sum_{k=0}^{N+1} (-)^k \omega_k K_{k+\nu}(z)} \ , \tag{32}$$

and the result follows from the known behavior of the Bessel functions for large order. That is,

$$I_{m+\nu}(z) = \frac{(z/2)^{m+\nu}}{\Gamma(m+\nu+1)}\left[1+O(m^{-1})\right] \ , \tag{33}$$

$$K_{m+\nu}(z) = \tfrac{1}{2}(z/2)^{-m-\nu}\Gamma(m+\nu)\left[1+O(m^{-1})\right] \ . \tag{34}$$

We next show that $\varphi_{m,\nu}^{(N)}(z)$ , $m = N+1, N, N-1, \ldots$ can be represented in terms of a generalized hypergeometric polynomial. We then prove that for two specific choices of $\theta(z)$ , the series (29) can also be expressed in terms of a generalized hypergeometric polynomial; and further, for the two choices of $\theta(z)$, respectively, $i_\nu^{(N)}(z)$ and the rational approximations $\psi_n(z)/h_n(z)$ and $\theta_n(z)/g_n(z)$ , respectively, are equal. Actually, we first state and prove theorems for the Case I situation in some detail. The corresponding theorems for Case II are stated and proofs are omitted as the details are much akin to their Case I analogs.

Another choice for $\theta(z)$ previously discussed in the literature is $I_\nu(z)$ . We call this Case III even though the corresponding $i_{m+\nu}^{(N)}(z) = \varphi_{m,\nu}^{(N)}(z) I_\nu(z) / \varphi_{o,\nu}^{(N)}(z)$ is not a member of the family of approximations from which Cases I and II were derived. We defer further analysis of Case III to a later discussion when we determine closed form error expressions for all the cases.

## HYPERGEOMETRIC REPRESENTATION FOR $\varphi_{m+\nu}^{(N)}(z)$

Theorem 6.

$$\varphi_{N+1,\nu}^{(N)}(z) = 1 \ , \ \varphi_{N,\nu}^{(N)}(z) = \frac{2(N+\nu+1)}{z} \ ,$$

$$\varphi_{N-1,\nu}^{(N)}(z) = 1 + \frac{4(N+\nu)_2}{z^2} \ , \ \varphi_{N-2,\nu}^{(N)}(z) = \frac{4(N+\nu)}{z} + \frac{8(N+\nu-1)_3}{z^3} \ ,$$

$$\varphi_{N-3,\nu}^{(N)}(z) = 1 + \frac{12(N+\nu-1)_2}{z^2} + \frac{16(N+\nu-2)_4}{z^4} \ ,$$

$$\varphi_{N-4,\nu}^{(N)}(z) = \frac{6(N+\nu-1)}{z} + \frac{32(N+\nu-2)_3}{z^3} + \frac{32(N+\nu-3)_5}{z^5} \ ,$$

$$\varphi_{N-5,\nu}^{(N)}(z) = 1 + \frac{24(N+\nu-2)_2}{z^2} + \frac{80(N+\nu-3)_4}{z^4} + \frac{64(N+\nu-4)_6}{z^6} \ ,$$

$$\varphi_{N-6,\nu}^{(N)}(z) = \frac{8(N+\nu-2)}{z} + \frac{80(N+\nu-3)_3}{z^3} + \frac{192(N+\nu-4)_5}{z^5} + \frac{128(N+\nu-5)_7}{z^7} \ ,$$

etc. (35)

$$\varphi^{(N)}_{2n-2k+\epsilon,\nu}(z) = (4/z)^{\eta} \sum_{m=0}^{k-\epsilon\delta} \frac{(2n-k-m+1+\nu+(1-\delta)\epsilon)_{2m+\eta}(k+1-m-\epsilon\delta)_{2m+\eta}(m+1)^{\eta}X^m}{(2m+2\eta)!}$$

$$= \left[\frac{2(k+1-\delta)(2n-k+1+\nu)}{z}\right]^{\eta} {}_4F_1\left(\begin{matrix}\epsilon\delta-k, k+1+\eta-\epsilon\delta, k-2n-\nu-(1-\delta)\epsilon, 2n-k+2+\nu+\epsilon\delta-\delta \\ \frac{1}{2}+\eta\end{matrix}\middle| X/4\right)$$

$$X = 4/z^2 \ , \ N = 2n-\delta \ , \ \delta = 0 \text{ or } \delta = 1 \ , \ \epsilon = 0 \text{ or } \epsilon = 1 \ ,$$

$$\eta = \epsilon\delta + (1-\epsilon)(1-\delta) \ , \tag{36}$$

where $\delta = 0$ or $1$ according as $N$ is even or odd, respectively. Also

$$\varphi^{(N)}_{m,\nu}(z) = \frac{(2/z)^{2n-m+1-\delta}\Gamma(2n+2-\delta+\nu)}{\Gamma(m+\nu+1)}$$

$$\times \ {}_2F_3^{\left[n-\frac{m-1+\delta}{2}\right]}\left(\begin{matrix}-n+\frac{m-1+\delta}{2}, -n+\frac{m+\delta}{2} \\ -2n-1+\delta-\nu, m+\nu+1, -2n-1+\delta+m\end{matrix}\middle| z^2\right) \ , \tag{37}$$

where $[p]$ is the largest integer $\leq p$ .

Proof: By induction: The Table (35) is readily developed by use of (21) and the starting conditions (25) and it is easily verified that (36) gives the polynomials listed in (35). Put (36) with $\delta = 1$ and $\epsilon = 0$ in (24) (in (24) replace $m$ by $2n-2k$ ) . Then after some algebra, it is seen that the coefficients of like powers of $z$ vanish, which proves (36) for $\delta = 1$ and $\epsilon = 0$ . To get (36) when $\delta = \epsilon = 1$ , use (21). The case $\delta = 0$ is similar and we omit the details. Finally, (37) is just a special case of (36). To connect these two equations, we set $2n-2k+\epsilon = m$ and choose $\epsilon = 0$ or $1$ according as $m$ is even or odd, respectively.

<u>Remark 1</u>: We have given more polynomials in (35) than are necessary for the proof. The additional entries are given for convenience.

Remark 2: If in (36), $\epsilon = \delta = 1$ , and if $k$ and $\nu$ are replaced by $k+1$ and $\nu+1$ , respectively, then we get (36) with $\epsilon = \delta = 0$ . Again, if in (36), $\epsilon = 0$ and $\delta = 1$ , and if $\nu$ is replaced by $\nu+1$ , then we get (36) with $\epsilon = \delta = 1$ .

## HYPERGEOMETRIC REPRESENTATIONS FOR $\theta^{(N)}(z)$ AND THE FORMS FOR $i_{m+\nu}^{(N)}(z)$

Case I. Consider the normalization relation

$$\theta(z) = \frac{(z/2)^{\nu}}{\Gamma(\nu+1)} = \sum_{k=0}^{\infty} \frac{(-)^k(2k+\nu)\Gamma(k+\nu)}{\Gamma(\nu+1)k!} I_{2k+\nu}(z) \ , \ \nu \neq 0 \ ,$$

$$= 1 = I_0(z) + 2\sum_{k=1}^{\infty} (-)^k I_{2k}(z) \ , \ \nu = 0 \ , \tag{38}$$

which is given in [1, Vol. 2, p. 45, Eq. (2)] .

Theorem 7.

$$\theta^{(N)}(z) = \sum_{k=0}^{n} \frac{(-)^k(2k+\nu)\Gamma(k+\nu)}{\Gamma(\nu+1)k!} \varphi_{2k,\nu}(z)$$

$$= (2/z)^{1-\delta} \frac{(-)^n(\nu+1)_{n+1-\delta}}{n!} {}_3F_0(-n,n+\lambda,1;X)$$

$$= (2/z)^{1-\delta} \frac{(-)^n(\nu+1)_{n+1-\delta}}{n!} h_n(z) \ ,$$

$$X = 4/z^2 \ , \ N = 2n-\delta \ , \ \lambda = \nu+2-\delta \ , \ \delta = 0 \text{ or } \delta = 1 \ . \tag{39}$$

Proof: We consider the case $\delta = 1$ only as the details for $\delta = 0$ are similar. We demonstrate that like powers of $X$ in the sums on the first two lines of (39) are equal. Thus we must show that

$$h_k(\nu) = B_k(\nu) \ , \tag{40}$$

$$h_k(\nu) = \sum_{r=0}^{k} b_r \ , \ b_r = \frac{(-)^r(2r+\nu)\Gamma(r+\nu)(-k)_r(s+1+\nu)_r}{r!\Gamma(\nu+1)(k+1+\nu)_r(-s)_r} \ ,$$

$$B_k(\nu) = \frac{(-)^k (s-k)!(\nu+1)_k}{s!} ,$$

$$n-m = k \ , \ n+m = s \ . \tag{41}$$

The case $k = 0$ is trivial. Assume $k > 0$. Clearly $B_k(\nu)$ is a polynomial in $\nu$ of degree $k$ which vanishes if $\nu = -u$, $u = 1,2,\ldots,k$. A straightforward calculation shows that $b_r + b_{u-r} = 0$ whence $h_k(\nu)$ also vanishes if $u = 1,2,\ldots,k$. Next multiply both sides of (40) by $(k+\nu+1)_{k-1}$. Then (40) and (41) take the form

$$h_k^*(\nu) = B_k^*(\nu) \ , \tag{42}$$

$$h_k^*(\nu) = \sum_{r=0}^{k} b_r^* \ , \ b_r^* = \frac{(-)^r (2r+\nu)\Gamma(r+\nu)(-k)_r (s+1+\nu)_r \Gamma(2k+\nu)}{r!\Gamma(\nu+1)(-s)_r \Gamma(k+1+\nu+r)}$$

$$B_k^*(\nu) = \frac{(-)^k (s-k)!\Gamma(2\lambda+\nu)}{k!\Gamma(\nu+1)} \tag{43}$$

Now each side of (42) is a polynomial in $\nu$ of degree $2k-1$ and $B_k^*(\nu)$ vanishes for $u = 1,2,\ldots,2k-1$. Also $h_k^*(\nu)$ vanishes for $u = 1,2,\ldots,k$. If $\nu = -k-v$, $v = 1,2,\ldots,k-1$, then $b_r^* = 0$ for $r = 0,1,\ldots,v-1$ and $b_{v+j}^* + b_{k-j}^* = 0$ for $j = 0,1,\ldots,(k-v)/2$. So $h_k^*(\nu)$ and $B_k^*(\nu)$ have the same zeros. Further, it is easy to see that the coefficients of $\nu^{2k-1}$ on both sides of (42) are equal. Hence $h_k^*(\nu) = B_k^*(\nu)$ and so also $h_k(\nu) = B_k(\nu)$ for all values of $\nu$ which proves the theorem.

<u>Theorem 8.</u>

$$\varphi_{o,\nu}^{(N)}(z) = (2/z)^{1-\delta} \frac{(-)^n (\nu+1)_{n+1-\delta}}{n!} \psi_n(z) \ ,$$

$$N = 2n-\delta \ , \ \delta = 0 \text{ or } \delta = 1 \ . \tag{44}$$

Proof: By induction: Using (4) and (37), we can readily verify the statement for $n = 0$, 1 and 2. A straightforward analysis shows that both sides of (44) satisfy the same recurrence formula which is easily deduced from (11).

From Theorems 7 and 8, and (32) and (37), we have

Theorem 9.

$$i_{\nu}^{(N)}(z) = \frac{\theta(z)\varphi_{o,\nu}^{(N)}(z)}{\theta^{(N)}(z)} = \frac{\psi_n(z)}{h_n(z)} \quad ,$$

$$i_{m+\nu}^{(N)}(z) = \frac{\theta(z)\varphi_{m,\nu}^{(N)}(z)}{\theta^{(N)}(z)} = \frac{(z/2)^{m+\nu}}{\Gamma(m+\nu+1)} \; \frac{{}_2F_3^{\left[n-\frac{m-1+\delta}{2}\right]}\left(\begin{matrix}-n+\frac{m-1+\delta}{2}, -n+\frac{m+\delta}{2} \\ -2n-1+\delta-\nu, m+\nu+1, -2n-1+\delta+m\end{matrix}\middle| z^2\right)}{{}_0F_1^n(-2n-1+\delta-\nu; z^2/4)} \quad . \tag{45}$$

Thus the result produced by use of the recurrence formula for $I_\nu(z)$ employed in the backward direction together with the normalization relation (38) and the rational approximation given in (2)-(5) are identical

Case II. From [1, Vol. 2, p. 45, Eq. (5)], we have the normalization relation

$$\Omega(z) = \frac{(z/2)^{\nu} e^{z}}{\Gamma(\nu+1)} = \sum_{k=0}^{\infty} \frac{(2k+2\nu)\Gamma(k+2\nu)}{\Gamma(2\nu+1)k!} I_{k+\nu}(z), \; \nu \neq 0 \; ,$$

$$= e^{z} = I_0(z) + 2\sum_{k=1}^{\infty} I_k(z) \; , \; \nu = 0 \; . \tag{46}$$

Here to avoid confusion, we replace $\theta$ by $\Omega$ in the notation of equations (28) and (29).

Proofs for Theorems 10, 11 and 12 given below are akin to those for Theorems 7, 8, and 9, respectively, and we skip the details.

Theorem 10.

$$\Omega^{(N)}(z) = \sum_{k=0}^{N+1} \frac{(2k+2\nu)\Gamma(k+2\nu)}{\Gamma(2\nu+1)k!} \varphi_{k,\nu}^{(N)}(z)$$

$$= \frac{(2\nu+2)_{N+1}}{(N+1)!}\, {}_3F_1\left(\begin{matrix}-N-1, N+2\nu+3, 1\\ \nu+3/2\end{matrix}\,\middle|\, -1/2z\right)$$

$$= \frac{(2\nu+2)_{N+1}}{(N+1)!}\, g_{N+1}(z) \quad . \tag{47}$$

<u>Theorem 11.</u>

$$\varphi_{0,\nu}^{(N)}(z) = \frac{(2\nu+2)_{N+1}}{(N+1)!}\, \theta_{N+1}(z) \quad , \tag{48}$$

and as a consequence of (44) and (48), we have

$$\frac{(2\nu+2)_{2n+1-\delta}\,\theta_{2n+1-\delta}(z)}{(2n+1-\delta)!} = \frac{(2/z)^{1-\delta}(-)^n(\nu+1)_{n+1-\delta}}{n!}\,\psi_n(z) \ , \ \delta = 0 \text{ or } \delta = 1 \ . \tag{49}$$

<u>Theorem 12.</u>

$$i_\nu^{(N)}(z) = \frac{\varphi_{0,\nu}^{(N)}(z)}{\Omega^{(N)}(z)} = \frac{\theta_{N+1}(z)}{g_{N+1}(z)} \quad ,$$

$$i_{m+\nu}^{(N)}(z) = \frac{\varphi_{m,\nu}^{(N)}(z)}{\Omega^{(N)}(z)} = \frac{(z/2)^{m+\nu}e^z}{\Gamma(m+\nu+1)}\ \frac{{}_2F_3^{\left[n-\frac{m-1+\delta}{2}\right]}\left(\begin{matrix}-n+\frac{m-1+\delta}{2}, -n+\frac{m+\delta}{2}\\ -2n-1+\delta+m, m+\nu+1, -2n-1+\delta-m\end{matrix}\,\middle|\, z^2\right)}{{}_1F_1^{2n+1-\delta}\left(\begin{matrix}-2n-3/2-\nu+\delta\\ -4n-3-2\nu+2\delta\end{matrix}\,\middle|\, 2z\right)} \ . \tag{50}$$

## A FURTHER INTERPRETATION OF THE BACKWARD RECURRENCE PROCESS

From Watson's treatise on Bessel functions [5, p. 295], we can write

$$I_{2n+2-\delta+\nu}(z) = T_{2n-\delta,\nu+2}(z)I_\nu(z) - T_{2n+1-\delta,\nu+1}(z)I_{\nu+1}(z) \ , \tag{51}$$

$$T_{2n+1-\delta,\nu+1}(z) = \varphi_{0,\nu}^{(N)}(z) \ , \ N = 2n-\delta, \ \ \delta = 0 \text{ or } \delta = 1 \ , \tag{52}$$

where $\varphi_{0,\nu}^{(N)}(z)$ is defined by (37). Consider the case $\delta = 0$ only. We put (51) in the form

$$\frac{I_\nu(z)}{I_{\nu+1}(z)} - Q_n(z) = \frac{I_{2n+\nu+2}(z)}{I_{\nu+1}(z)T_{2n,\nu+2}(z)} \ , \ Q_n(z) = \frac{T_{2n+1,\nu+1}(z)}{T_{2n,\nu+2}(z)} \ . \tag{53}$$

Further, we can put

$$Q_n(z) = \frac{E_n(z)}{F_n(z)} \ ,$$

$$E_n(z) = \frac{(z/2)^{2n+\nu+1}T_{2n+1,\nu+1}}{\Gamma(2n+\nu+2)} \ , \ F_n(z) = \frac{(z/2)^{2n+\nu+1}T_{2n,\nu+2}}{\Gamma(2n+\nu+2)} \ . \tag{54}$$

Now it is known [5, p. 302] that for $z$ and $\nu$ fixed, $z \neq 0$ ,

$$\lim_{n\to\infty} E_n(z) = I_\nu(z) \ , \ \lim_{n\to\infty} F_n(z) = I_{\nu+1}(z) \ . \tag{55}$$

From (44) and (52), we have

$$E_n(z) = L_n(z)\frac{(z/2)^\nu \psi_n(z)}{\Gamma(\nu+1)h_n(z)} \ , \ L_n(z) = \frac{(-)^n(z/2)^{2n}h_n(z)}{n!(n+\nu+2)_n} \ , \tag{56}$$

where $\psi_n(z)$ and $h_n(z)$ are given by (3) and (5), respectively, with $\delta = 0$ . But for $z$ and $\nu$ fixed, $z \neq 0$ ,

$$\lim_{n\to\infty} L_n(z) = 1 \ , \tag{57}$$

see (9). That is,

$$\lim_{n\to\infty} E_n(z) = \lim_{n\to\infty} \frac{(z/e)^\nu \psi_n(z)}{\Gamma(\nu+1)h_n(z)} = \lim_{n\to\infty} \frac{(z/2)^\nu \varphi_{0,\nu}^{(N)}(z)}{\Gamma(\nu+1)\theta^{(N)}(z)} = I_\nu(z) \ . \tag{58}$$

A similar analysis can be made for $F_n(z)$ . Also, a like study can be done for the Case II scheme. We omit the details.

Further, the above shows that the backward recurrence scheme for the computation of $zI_\nu(z)/I_{\nu+1}(z)$ is the same as the well-known truncated continued fraction representation which in turn is the same as the main diagonal Padé approximation for this function.

## ERROR ANALYSES

In the first part of this section we develop closed form representations of the error in $i_{m+\nu}^{(N)}(z)$ for Cases I-III under the assumption of exact arithmetic. This type of error arises because $N$ is finite and is called the truncation error. From each analytical representation of the error, we deduce an asymptotic estimate of the error which is very realistic and easy to apply in practice. The results for Cases I and II when $m = 0$ are much better than those given by (10) and (19), respectively. Further, for Cases I and II, if $z$ and $\nu$ are fixed and $n$ is sufficiently large with respect to $m$ , the relative error in the approximation for $I_{m+\nu}(z)$ is essentially independent of $m$ .

An analytical formulation of the round-off error is developed in the latter part of this section where it is shown that this source of error is insignificant.

We now turn to a study of the truncation errors.

Case I. Let

$$E_{m,\nu}^{(N)}(z) = I_{m+\nu}(z) - i_{m+\nu}^{(N)}(z) \tag{59}$$

where $i_{m+\nu}^{(N)}(z)$ is given in (45).

Theorem 13. If $\nu$ is not a positive integer or zero,

$$ {}_0F_1^n(-2n-1+\delta-\nu;z^2/4)E_{m,\nu}^{(N)}(z) - \frac{(-)^n(z/2)^{2n+2}\Gamma(n+1-\delta+\nu)}{(n+1)!\Gamma(2n+2-\delta+\nu)} I_{m+\nu}(z)\,{}_1F_2\left(\begin{matrix}1\\ n+2,-n+\delta-\nu\end{matrix}\middle| z^2/4\right) $$

$$ = \frac{(-)^\delta\pi(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)\sin\nu\pi} I_{-m-\nu}(z)I_{2n+2-\delta+\nu}(z) $$

$$ = \frac{2(-)^{m+\delta}(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)} I_{2n+2-\delta+\nu}(z)K_{m+\nu}(z) $$

$$ + \frac{(-)^\delta\pi(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)\sin\nu\pi} I_{2n+2-\delta+\nu}(z)I_{m+\nu}(z) \quad . \tag{60} $$

Equation (60) can be rearranged so that with the aid of L'Hospital's theorem, we can get a representation of the error when $\nu$ becomes a positive integer or zero. We do not give this result. However, for arbitrary $\nu$, we always have

$$ {}_0F_1^n(-2n-1+\delta-\nu;z^2/4)E_{m,\nu}^{(N)}(z) = \frac{(-)^n(z/2)^{2n+2}\Gamma(n+1-\delta+\nu)}{(n+1)!\Gamma(2n+2-\delta+\nu)} I_{m+\nu}(z)\,{}_1F_2^s\left(\begin{matrix}1\\ n+2,-n+\delta-\nu\end{matrix}\middle| z^2/4\right) $$

$$ + \frac{2(-)^{m+\delta}(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)} I_{2n+2-\delta+\nu}(z)K_{m+\nu}(z) + \frac{O\left((z/2)^{2n}\right)}{\Gamma(2n+2-\delta+\nu)\Gamma(2n+3-\delta+\nu)}, \tag{61} $$

where $s = n-\delta+\nu (s=\infty)$ if $\nu$ is (is not) a positive integer or zero. Clearly the backward recurrence scheme is convergent. Further, for $n$ sufficiently large, $n \gg m$, the relative error is essentially independent of $m$. For convenience in the applications we record the formula

$$ E_{m,\nu}^{(N)}(z) = \frac{(-)^n(z/2)^{2n+2}\Gamma(n+1-\delta+\nu)}{(n+1)!\Gamma(2n+2-\delta+\nu)} I_{m+\nu}(z)\left[1+O(n^{-1})\right] $$

$$ + \frac{2(-)^{m+\delta}(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)} I_{2n+2-\delta+\nu}(z)K_{m+\nu}(z) \quad . \tag{62} $$

Proof: We have need for the formula [1, Vol. 1, p. 216]

$$I_a(z)I_b(z) = \frac{(z/2)^{a+b}}{\Gamma(a+1)\Gamma(b+1)}\, {}_0F_1(a+1;z^2/4)\,{}_0F_1(b+1;z^2/4)$$

$$= \frac{(z/2)^{a+b}}{\Gamma(a+1)\Gamma(b+1)}\, {}_2F_3\left(\left.\begin{matrix}(a+b+1)/2,(a+b+2)/2\\ a+1,b+1,a+b+1\end{matrix}\right| z^2\right), \tag{63}$$

where it must be understood that none of the numbers $a+1$ , $b+1$ , $a+b+1$ is a negative integer or zero. Now let

$$A_m(z) = {}_0F_1(m+\nu+1;z^2/4)\,{}_0F_1(-2n-1+\delta-\nu;z^2/4)$$

$$= \frac{\Gamma(m+\nu+1)\Gamma(-2n-1+\delta-\nu)}{(z/2)^{-2n-2+\delta+m}}\, I_{m+\nu}(z)I_{-2n-1+\delta-\nu}(z)$$

$$= \frac{(-)^{\delta}\pi\Gamma(m+\nu+1)(z/2)^{2n+2-\delta-m}}{\Gamma(2n+2-\delta+\nu)\,\sin\nu\pi}\, I_{m+\nu}(z)I_{-2n-1+\delta-\nu}(z)$$

$$= \sum_{k=0}^{\infty} a_k z^{2k} \quad . \tag{64}$$

Then

$$a_k = \left[2^{2k}k!(-2n-1+\delta-\nu)_k\right]^{-1}\, {}_2F_1\left(\left.\begin{matrix}-k,2n+2-\delta+\nu-k\\ m+\nu+1\end{matrix}\right| 1\right)$$

$$= \frac{\left(-n+\frac{m-1+\delta}{2}\right)_k\left(-n+\frac{m+\delta}{2}\right)_k}{(-2n-1+\delta-\nu)_k(m+\nu+1)_k(-2n-1+\delta+m)_k} \quad , \tag{65}$$

whence

$$a_k = 0 \ , \ k = 1+\left[n-\frac{m-1+\delta}{2}\right] , \ . \ . \ . \ . \ , \ 2n+1-\delta-m \ ,$$

$$a_{k+2n+2-\delta-m} = \frac{(-)^{m+\delta}\Gamma(m+\nu)\Gamma(m+\nu+1)}{2^{4n+4-2\delta-2m}\Gamma(2n+2-\delta+\nu)\Gamma(2n+3-\delta+\nu)}$$

$$\times \frac{\left(n-\frac{m}{2}+\frac{3-\delta}{2}\right)_k\left(n-\frac{m}{2}+\frac{4-\delta}{2}\right)_k}{(2n+3-\delta+\nu)_k(1-m-\nu)_k(2n+3-\delta-m)_k k!}, \quad k \geq 0 . \tag{66}$$

Thus

$$A_m(z) = {}_2F_3^{\left[n-\frac{m-1+\delta}{2}\right]}\left(\begin{matrix}-n+\frac{m-1+\delta}{2}, -n+\frac{m+\delta}{2} \\ -2n-1+\delta-\nu, m+\nu+1, -2n-1+\delta+m\end{matrix}\middle| z^2\right)$$

$$+ \frac{(-)^{\delta}\pi\Gamma(m+\nu+1)(z/2)^{2n+2-\delta-m}}{\Gamma(2n+2-\delta+\nu)\sin\nu\pi} I_{-m-\nu}(z)I_{2n+2-\delta+\nu}(z) , \tag{67}$$

and in view of (23), we have the alternate representations

$$A_m(z) = \frac{(-)^{\delta}\pi\Gamma(m+\nu+1)(z/2)^{2n+2-\delta-m}}{\Gamma(2n+2-\delta+\nu)\sin\nu\pi} I_{m+\nu}(z)\left[-\frac{2}{\pi}(-)^{\delta}K_{2n+1-\delta+\nu}(z) + I_{2n+1-\delta+\nu}(z)\right]$$

$$= {}_2F_3^{\left[n-\frac{m-1+\delta}{2}\right]}\left(\begin{matrix}-n+\frac{m-1+\delta}{2}, -n+\frac{m+\delta}{2} \\ -2n-1+\delta-\nu, m+\nu+1, -2n-1+\delta+m\end{matrix}\middle| z^2\right)$$

$$+ \frac{(-)^{\delta}\pi\Gamma(m+\nu+1)(z/2)^{2n+2-\delta-m}}{\Gamma(2n+2-\delta+\nu)\sin\nu\pi} I_{2n+2-\delta+\nu}(z)\left[\frac{2}{\pi}(-)^{m}K_{m+\nu}(z) + I_{m+\nu}(z)\right] . \tag{68}$$

As a remark aside, the combination (26) and (68) yields (37) and so we have an alternative proof of (37). Using (45), (59) and (67), we can write

$$
{}_0F_1^n(-2n-1+\delta-\nu;z^2/4)E_{m,\nu}^{(N)}(z) = I_{m+\nu}(z)\,{}_0F_1^{(n)}(-2n-1+\delta-\nu;z^2/4)
$$

$$
- \frac{(z/2)^{m+\nu}}{\Gamma(m+\nu+1)}\Bigg[{}_0F_1(m+\nu+1;z^2/4)\,{}_0F_1(-2n-1+\delta-\nu;z^2/4)
$$

$$
- \frac{(-)^{\delta}\pi\Gamma(m+\nu+1)(z/2)^{2n+2-\delta-m}}{\Gamma(2n+2-\delta+\nu)\sin\nu\pi}\, I_{-m-\nu}(z)I_{2n+2-\delta+\nu}(z)\Bigg],
$$

and since

$$
{}_0F_1^n(-2n-1+\delta-\nu;z^2/4) - {}_0F_1(-2n-1+\delta-\nu;z^2/4) = \frac{(-)^n\Gamma(n+1-\delta+\nu)}{(n+1)!\,\Gamma(2n+2-\delta+\nu)}\, {}_1F_2\left(\begin{matrix}1\\ n+2,-n+\delta-\nu\end{matrix}\middle| z^2/4\right), \tag{69}
$$

we readily find the first part of (60). The second part of (60) follows from (23) or it could have been found by repeating the above analysis with the second equation of (68) in place of (67).

Next we briefly examine the situation when $\nu$ is a positive integer or zero. With $\nu = r+\varepsilon$, the ${}_1F_2$ on the right-hand side of (60) can be expressed as

$$
{}_1F_2\left(\begin{matrix}1\\ n+2,-n+\delta-\nu\end{matrix}\middle| z^2/4\right) = {}_1F_2^{n-\delta+r}\left(\begin{matrix}1\\ n+2,-n+\delta-\nu\end{matrix}\middle| z^2/4\right) + \sum_{k=n-\delta+r+1}^{\infty} \frac{(z/2)^{2k}}{(n+2)_k(-n+\delta-\nu)_k}
$$

$$
= {}_1F_2^{n-\delta+r}\left(\begin{matrix}1\\ n+2,-n+\delta-\nu\end{matrix}\middle| z^2/4\right) + \frac{(-)^{n+1+\delta}\pi(n+1)!\,(z/2)^{2n+2-2\delta+2r}}{\sin\pi\nu\,\Gamma(2n+3-\delta+r)\Gamma(n+1-\delta+\nu)\Gamma(1-\varepsilon)}
$$

$$
\times\ {}_1F_2\left(\begin{matrix}1\\ 2n+3-\delta+r,1-\varepsilon\end{matrix}\middle| z^2/4\right). \tag{70}
$$

The first term on the right-hand side of (70) is defined when $\epsilon = 0$. When the second term on the right-hand side of (70) is multiplied by the coefficient of the ${}_1F_2$ in (60) and the result is combined with the term involving $I_{m+\nu}(z)I_{2n+2-\delta+\nu}(z)$ in (60), it will be seen that one can pass to the limit as $\epsilon \to 0$. The final expression is not of great interest and we omit further details. Equation (61), which is important for practical considerations (see later numerical example) readily follows from (60) and the above remarks, and (62) is but a simplified version of (61).

<u>Remark</u>: Let $\nu$, $n$ and $z$ be fixed so that $E^{(N)}_{m,\nu}(z)$ is a function of $m$ only. Then $E^{(N)}_{m,\nu}(z)$ satisfies the recurrence formula for $\varphi_{m,\nu}(z)$, see (21). This is evident from (30) and confirmed by (60).

Case II. Let

$$F^{(N)}_{m,\nu}(z) = I_{m+\nu}(z) - i^{(N)}_{m+\nu}(z) \tag{71}$$

where $i^{(N)}_{m+\nu}(z)$ is given in (50).

Theorem 14. If neither $\nu$ nor $\nu+\frac{1}{2}$ is a positive integer or zero, then

$$e^{-z}\,{}_1F_1^{2n+1-\delta}\left(\begin{matrix}-2n-3/2+\delta-\nu\\-4n-3+2\delta-2\nu\end{matrix}\middle|\,2z\right) F^{(N)}_{m,\nu}(z)$$

$$+\frac{(2z)^{2n+2-\delta}(\nu+\frac{1}{2})_{2n+2-\delta}\,e^{-z}I_{m+\nu}(z)}{(2n+2-\delta)!\,(2n+2-\delta+2\nu)_{2n+2-\delta}}\;{}_2F_2\left(\begin{matrix}\frac{1}{2}-\nu,1\\2n+3-\delta,-2n-1+\delta-2\nu\end{matrix}\middle|\,2z\right)$$

$$=\frac{(-)^{\delta}\pi(z/2)^{2n+2-\delta+\nu}I_{-m-\nu}(z)I_{2n+2-\delta+\nu}(z)}{\Gamma(2n+2-\delta+\nu)\sin\nu\pi} \tag{72}$$

which is the same as the right-hand side of (60).

If $\nu+\frac{1}{2}$ is a positive integer or zero, call it $r$, we have

$$e^{-z}\,{}_1F_1^{2n+1-\delta}\left(\begin{matrix}-2n-2+\delta-r\\-4n-4+2\delta-2r\end{matrix}\middle|\,2z\right)F^{(N)}_{m+r+\frac{1}{2}}(z)$$

$$= -\frac{(2z)^{2n+2-\delta}(r+1)_{2n+2-\delta}}{(2n+2-\delta)!\,(2n+3-\delta+2r)_{2n+2-\delta}}\,e^{-z}I_{m+r+\frac{1}{2}}(z)\,{}_2F_2^{r}\left(\begin{matrix}-r,1\\2n+3-\delta,-2n-2+\delta-2r\end{matrix}\middle|\,2z\right)$$

$$+\frac{(-)^{\delta+r+1}\pi(z/2)^{2n+5/2-\delta+r}I_{2n+r+5/2-\delta}(z)}{\Gamma(2n+5/2-\delta+r)}\left[\mu_r I_{m+r+\frac{1}{2}}(z)-I_{-m-r-\frac{1}{2}}(z)\right],$$

$$r=0,1,2,\ldots\,,\ \mu_r=1 \text{ if } r=-1\,,\ \mu_r=2 \text{ if } r\geq 0\,. \tag{73}$$

In particular,

$$F^{(N)}_{0,-\frac{1}{2}}(z)=\frac{(-)^{\delta}(\pi/2z)^{\frac{1}{2}}e^{-z}I_{2n+3/2-\delta}(z)}{K_{2n+3/2-\delta}(z)}$$

$$=\frac{(-)^{\delta}(2\pi z)^{\frac{1}{2}}e^{-z}(z/2)^{4n+3-2\delta}}{\left[(2n+1-\delta)!\right]^2}\left[1+O(n^{-1})\right]. \tag{74}$$

As in Theorem 13, (72) can be rearranged to get a representation for the error when $\nu$ is a positive integer or zero. This result is omitted. However, for arbitrary $\nu$, $\nu\neq-\frac{1}{2}$, we always have

$$e^{-z}\,{}_1F_1^{2n+1-\delta}\left(\begin{matrix}-2n-3/2+\delta-\nu\\-4n-3+2\delta-2\nu\end{matrix}\middle|\,2z\right)F^{(N)}_{m,\nu}(z)$$

$$= -\frac{(2z)^{2n+2-\delta}(\nu+\frac{1}{2})_{2n+2-\delta}}{(2n+2-\delta)!\,(2n+2-\delta+2\nu)_{2n+2-\delta}}\,e^{-z}I_{m+\nu}(z)\,{}_2F_2^{t}\left(\begin{matrix}\frac{1}{2}-\nu,1\\2n+3-\delta,-2n-1+\delta-2\nu\end{matrix}\middle|\,2z\right)$$

$$+\frac{2(-)^{m+\delta}(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)}I_{2n+2-\delta+\nu}(z)K_{m+\nu}(z)$$

$$+\frac{O\left((z/2)^{2n}\right)}{\Gamma(2n+2-\delta+\nu)\Gamma(2n+3-\delta+\nu)}\,, \tag{75}$$

where $t = \nu-\frac{1}{2}(t=2n+1-\delta+2\nu)$ if $\nu$ is half a positive integer (is a positive integer or zero) and where $t = \infty$ for all other $\nu$. Clearly, the backward recurrence scheme is convergent.

Further, for $n$ sufficiently large, $n \gg m$, the relative error is independent of $m$. For convenience in the applications, we record the formula

$$F^{(N)}_{m,\nu}(z) = \frac{-(2z)^{2n+2-\delta}(\nu+\frac{1}{2})_{2n+2-\delta}e^{-z}}{(2n+2-\delta)!(2n+2-\delta+2\nu)_{2n+2-\delta}} I_{m+\nu}(z)\left[1 + O(n^{-1})\right]$$

$$+ \frac{2(-)^{m+\delta}(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)} I_{2n+2-\delta+\nu}(z)K_{m+\nu}(z) \ , \ \nu \neq -\tfrac{1}{2} . \quad (76)$$

Proof: Let

$$B_m(z) = \left\{e^{z}\,{}_1F_1\left(\begin{matrix} m+\nu+\frac{1}{2} \\ 2m+2\nu+1 \end{matrix}\middle| -2z\right)\right\}\left\{e^{-z}\,{}_1F_1\left(\begin{matrix} -2n-3/2+\delta-\nu \\ -4n-3+2\delta-2\nu \end{matrix}\middle| 2z\right)\right\} = \sum_{k=0}^{\infty} b_k z^k . \quad (77)$$

In view of (1) and (12), $B_m(z) = A_m(z)$ where $A_m(z)$ is given by (64). Hence $b_k = 0$ for $k$ odd and $b_{2k} = a_k$ where $a_k$ is given by (65)-(66). The analysis proceeds as for Case I and we find (72). Notice that the right-hand sides of (72) and (60) are identical.

For the proof of (73), let $V_{m,\nu}(z)$ be the entire first term on the right-hand side of (72), that is, the term involving the ${}_2F_2$. Let $\nu = r+\frac{1}{2}+\epsilon$, $r = 0,1,2,\ldots$. Then we can write

$$V_{m,\nu}(z) = -\zeta_\nu(z)(r+1+\epsilon)_{2n+2-\epsilon}\left[{}_2F_2\left(\begin{matrix} \frac{1}{2}-\nu, 1 \\ 2n+3-\delta, -2n-1+\delta-2\nu \end{matrix}\middle| 2z\right)\right.$$

$$\left. + \sum_{k=r+1}^{2n+2-\delta+2r} c_k z^{2k} + \sum_{k=2n+3-\delta+2r}^{\infty} c_k z^{2k}\right] ,$$

$$\zeta_\nu(z) = \frac{(2z)^{2n+2-\delta}e^{-z}I_{m+\nu}(z)}{(2n+2-\delta)!(2n+2-\delta+2\nu)_{2n+2-\delta}} , \quad c_k = \frac{(\frac{1}{2}-\nu)_k}{(2n+3-\delta)_k(-2n-1+\delta-2\nu)_k} .$$

It is easy to pass to the limit when $\epsilon \to 0$ and (73) readily follows.

Now for arbitrary $a$ and $b$,

$$aI_{-\frac{1}{2}}(z) + bI_{\frac{1}{2}}(z) = (2/\pi z)^{\frac{1}{2}}(a \cosh z + b \sinh z) .$$

Further, from (16), with $n$ replaced by $2n+1-\delta$, see also (17), and $\nu = -\frac{1}{2}$, $g_n(z)$ reduces to

$${}_2F_0\left(-2n-1+\delta, 2n+2-\delta; -\frac{1}{2z}\right) = (\pi/2z)^{-\frac{1}{2}}e^{z}K_{2n+3/2-\delta}(z) .$$

Use these data with $r = -1$ and $m = 0$ in (73) to get the first line of (74). Derivation of the second line of (74) is trivial and details are omitted. Since

$$I_{-\frac{1}{2}}(z) = (2\pi z)^{-\frac{1}{2}}e^{z}(1+e^{-2z}) ,$$

it is easy to show that $(2\pi z)^{\frac{1}{2}}e^{-z}i^{(N)}_{-\frac{1}{2}}(z)$ is the main diagonal Padé approximation to $1 + e^{-2z}$. When allowance is made for a change of notation, (74) is a previously obtained result [1, Vol. 2, p. 74, Eqs. (34), (35)].

When $\nu = r+\epsilon$, $r$ a positive integer or zero, we can rearrange (72) after the manner of the discussion surrounding (70) and use L'Hospital's theorem to get the limit as $\epsilon \to 0$. The result is not of immediate interest and we omit details. The statements (75) and (76) are readily derived and here too we skip details.

<u>Remark</u>: $F^{(N)}_{m,\nu}(z)$ with $n$, $\nu$ and $z$ fixed and $m$ variable satisfies the recurrence formula for $\varphi_{m,\nu}(z)$, see (21). This feature is clearly depicted by (72). In both (62) and (76), the term involving $K_{m+\nu}(z)$ is of lower order than the term involving $I_{m+\nu}(z)$. Neglecting the former term in each equation, we have

Theorem 15.

$$\frac{E^{(N)}_{m,\nu}(z)}{F^{(N)}_{m,\nu}(z)} = \frac{(-)^{n+1}\Gamma(n+1-\delta+\nu)(2n+2-\delta)!2^{2\nu-1}(\frac{1}{2})_\nu(z/2)^\delta e^z\left[1+O(n^{-1})\right]}{(n+1)!\Gamma(2n+2-\delta+2\nu)}$$

$$= \frac{(-)^{n+1}(\frac{1}{2})_\nu(z/2)^\delta e^z}{n^{\nu+\delta}}\left[1+O(n^{-1})\right] \quad , \nu \neq -\tfrac{1}{2} . \tag{78}$$

This shows that there is little difference in the accuracy of the two schemes for the evaluation of $I_{m+\nu}(z)$ . Computation-wise, if the backward recursion scheme is used, Case I requires less operations since the associated normalization relation, see (38) and (39), uses the sequence $\left\{\varphi^{(N)}_{k,\nu}(z)\right\}$ , $k = 0,2,4,\ldots$ , while the Case II normalization relation, see (46) and (47), employs $\left\{\varphi^{(N)}_{k,\nu}(z)\right\}$ , $k = 0,1,2,\ldots$ . Also to get $I_\nu(z)$ by the Case II scheme, $e^z$ must be evaluated. On the other hand, if $|z|$ is large, $R(z) > 0$ , one often wants not $I_\nu(z)$ , but $e^{-z}I_\nu(z)$ . The latter is automatically furnished by the Case II technique. It appears that *for the same* $n$ , *the Case* II procedure might be more accurate than the Case I scheme even for moderate values of $|z|$ , $R(z) > 0$ , in view of the presence of $e^z$ in the numerator of (78). Also, Case II is favored when $R(\nu+\delta) < 0$ . Improved information cannot be derived from (78) as the estimate is for fixed $m$ , $\nu$ and $z$ . For error analyses it is suggested that one use (62) or (76) as appropriate. Further discussion is deferred to a later part of the paper where numerical examples are presented.

If $z$ is pure imaginary and $\nu$ is real, then $z^{-\nu}I_\nu(z)$ is real and definitely the Case I procedure is better than the Case II scheme since the former requires real arithmetic while the latter demands complex arithmetic.

If only $I_\nu(z)$ or only $e^{-z}I_\nu(z)$ is required, use of the rational approximation scheme or the equivalent backward recursion scheme demands about the same number of operations. In the absence of a priori estimates of the error, the rational approximation scheme employed in the following fashion is preferred. It is sufficient to consider the Case I situation. Compute $\psi_n(z)$ from either (3) or the combination (37), (44), and $h_n(z)$ from (5), for $n = 0,1$ and $2$ . Compute subsequent values of $\psi_n(z)$ and $h_n(z)$ by use of the recursion formula (12). Comparison of $\psi_n(z)/h_n(z)$ with $\psi_{n+1}(z)/h_{n+1}(z)$ affords an estimate of the error. If one requires $I_{k+\nu}(z)$ or $e^{-z}I_{k+\nu}(z)$ for $k = 0,1,2,\ldots,r$ , then obviously the backward recursion scheme is highly advantageous.

Case III. Consider

$$G^{(N)}_{m,\nu}(z) = I_{m+\nu}(z) - i^{(N)}_{m+\nu}(z) \ , \tag{79}$$

$$i^{(N)}_{m+\nu}(z) = \frac{I_\nu(z)\varphi^{(N)}_{m,\nu}(z)}{\varphi^{(N)}_{0,\nu}(z)} \ , \tag{80}$$

where $\varphi^{(N)}_{m,\nu}(z)$ is given by (37).

<u>Theorem 16</u>.

$$G^{(N)}_{m,\nu}(z) = \frac{(-)^{\delta+m} I_{2n+2-\delta+\nu}(z)\varphi^{(m-2)}_{0,\nu}(z)}{\varphi^{(N)}_{0,\nu}(z)}$$

$$= \frac{(-)^{\delta+m}\Gamma(m+\nu)(z/2)^{2n+2-m-\delta}}{\Gamma(2n+2-\delta+\nu)} I_{2n+2-\delta+\nu}(z) L_{m,n}(z) \ ,$$

$$L_{m,n}(z) = \frac{{}_2F_3\left[\frac{m-1}{2}\right]\left(\begin{matrix}-\frac{m-1}{2}, -\frac{m-2}{2} \\ 1-\nu-m, \nu+1, 1-m\end{matrix}\middle| z^2\right)}{{}_2F_3\left[n-\frac{\delta-1}{2}\right]\left(\begin{matrix}-n+\frac{\delta-1}{2}, -n+\frac{\delta}{2} \\ -2n-1+\delta-\nu, \ \nu+1, -2n-1+\delta\end{matrix}\middle| z^2\right)} \ , \ 0 < m \le 2n+1-\delta \ , \tag{81}$$

and it is clear that the computational scheme is convergent. Further, if we treat $G^{(N)}_{m,\nu}(z)$ with $n$ , $\nu$ and $z$ fixed and only $m$ as a variable, then $(-)^m G^{(N)}_{m,\nu}(z)$ satisfies the recurrence formula for $\varphi_{m,\nu}(z)$ , see (21). Thus

$$G^{(N)}_{m+2,\nu}(z) = G^{(N)}_{m,\nu}(z) - \frac{2(m+\nu+1)}{z} G^{(N)}_{m+1,\nu}(z) \ ,$$

$$G^{(N)}_{0,\nu}(z) = 0 \ , \ G^{(N)}_{1,\nu}(z) = \frac{(-)^{\delta+1} I_{2n+2-\delta+\nu}(z)}{\varphi^{(N)}_{0,\nu}(z)} \ . \tag{82}$$

Finally, for convenience in the applications, we record the formulas

$$G^{(N)}_{m,\nu}(z) = \frac{(-)^{m+\delta}(z/2)^{2n+4-2\delta+2\nu-m}\Gamma(m+\nu)}{\Gamma(2n+2-\delta+\nu)\Gamma(2n+3-\delta+\nu)\Gamma(\nu+1)} \frac{{}_2F_3^{\left[\frac{m-1}{2}\right]}\left(\begin{matrix}-\frac{m-1}{2},-\frac{m-2}{2}\\ 1-\nu-m,\nu+1,1-m\end{matrix}\middle| z^2\right)}{I_\nu(z)}\left[1+O(n^{-1})\right] , \tag{83}$$

$$G^{(N)}_{m,\nu}(z) = \frac{(-)^{m+\delta}(z/2)^{2n+4-2\delta+\nu-m}\Gamma(m+\nu)}{\Gamma(2n+2-\delta+\nu)\Gamma(2n+3-\delta+\nu)}\left[1+O(m^{-1})\right]\left[1+O(n^{-1})\right] . \tag{84}$$

Proof: Using (37), (67) and (69), we find

$$\varphi^{(N)}_{0,\nu}(z)G^{(N)}_{m,\nu}(z) = \frac{(-)^{\delta}(\pi z/2)}{\sin \nu\pi} I_{2n+2-\delta+\nu}(z)\left[I_\nu(z)I_{-m-\nu}(z)-I_{-\nu}(z)I_{m+\nu}(z)\right] ,$$

and in view of (23)

$$\varphi^{(N)}_{0,\nu}(z)G^{(N)}_{m,\nu}(z) = (-)^{\delta} I_{2n+2-\delta+\nu}(z)\left\{z\left[(-)^m I_\nu(z)K_{m+\nu}(z) - I_{m+\nu}(z)K_\nu(z)\right]\right\} .$$

From (26), the portion in curly brackets in the latter equation is $(-)^m\varphi^{(m-2)}_{0,\nu}(z)$ . Hence the first line of (81) is at hand. The remainder of (81) follows from (37). The first line of (81) coupled with the discussion surrounding equations (21)-(25) produces (82). By the confluence principle, see [1, Vol. 1, p. 50]

$$\varphi^{(N)}_{0,\nu}(z) = \frac{(2/z)^{N+1}\Gamma(N+2+\nu)}{\Gamma(\nu+1)} {}_2F_3^{\left[\frac{N+1}{2}\right]}\left(\begin{matrix}-\frac{N+1}{2},-\frac{N}{2}\\ -N-1-\nu,\nu+1,-N-1\end{matrix}\middle| z^2\right)$$

$$= \frac{(2/z)^{N+1}\Gamma(N+\nu+2)}{\Gamma(\nu+1)} {}_0F_1(\nu+1;z^2/4)\left[1+O(N^{-1})\right]$$

$$= (2/z)^{N+\nu+1}\Gamma(N+\nu+2)I_\nu(z)\left[1+O(N^{-1})\right] , \tag{85}$$

whence (83) and (84) readily follow with the aid of (33).

Error-wise, it is difficult to compare Cases I or II with Case III without some simplifying assumptions. If $n \gg m$, using (33), (62) and (83), we get

$$\frac{G_{m,\nu}^{(N)}(z)}{E_{m,\nu}^{(N)}(z)} = \frac{(-)^{m+m+\delta}(z/2)^{2m+2-2\delta+2\nu-m}(n+1)!\Gamma(m+\nu)}{\Gamma(2n+3-\delta+\nu)\Gamma(n+1-\delta+\nu)\Gamma(\nu+1)} \frac{{}_2F_3\left(\left[\frac{m-1}{2}\right]\left(\begin{matrix}-\frac{m-1}{2},-\frac{m-2}{2}\\ 1-\nu-m,\nu+1,1-m\end{matrix}\middle| z^2\right)\right)}{I_\nu(z)I_{m+\nu}(z)}\left[1+O(n^{-1})\right]$$

$$= \frac{(z/2)^{2n}}{\Gamma(2n+3-\delta+\nu)} O(n^{\delta+1-\nu}) \quad , \tag{86}$$

and so Case III is superior to Case I. Now suppose $m$ is sufficiently large so that in (62), the second term dominates the first term. This is certainly the case if $m = 2n+1-\delta-d$, $d \ll n$, in view of (33) and (34). Then

$$\frac{G_{m,\nu}^{(N)}(z)}{E_{m,\nu}^{(N)}(z)} = \frac{(z/2)^{-\delta-m-\nu}\Gamma(m+\nu)}{2K_{m+\nu}(z)}\left[1+O(n^{-1})\right]\left[1+O(m^{-1})\right]$$

$$= (2/z)^{\delta}\left[1+O(n^{-1})\right]\left[1+O(m^{-1})\right] \quad , \tag{87}$$

and under these conditions there is little to choose between the two cases. Overall, it appears that Case III gives better accuracy than Case I. However, for Case III, one must know $I_\nu(z)$ while for Case I no such knowledge is required. For all $z > 0$ and all $\nu$, $0 \le \nu \le 1$, coefficients are available to facilitate the rapid evaluation of $J_\nu(z)$ and $I_\nu(z)$, see [1, Vol. 2; 6,7,8]. (Actually, much more information is given in these sources.) All of this can often make the Case III approach rather attractive. See the numerical examples.

Next we consider the round-off error.

Cases I,II. It is sufficient to trace the effect of a given round-off error in a single entry of the table generated by the backward recursion process. Thus let $\Delta\varphi$ be the symbol for the round-off error in $\varphi$. Suppose that

$$\Delta\varphi^{(N)}_{m,\nu}(z) = 0 \text{ for } m = N+2, N+1, \ldots, S+2 ,$$

$$\Delta\varphi^{(N)}_{m,\nu}(z) = \omega \text{ for } m = S+1 , \tag{88}$$

where $S = 2s-\gamma$ and $\gamma$ is 0 (is 1) if $S$ is even (is odd). Hence

$$\Delta\varphi^{(N)}_{m,\nu}(z) = \omega\varphi^{(S)}_{m,\nu} , \quad m \le 2s+1-\gamma , \tag{89}$$

$$\Delta\theta^{(N)}(z) = \omega\theta^{(S)}(z) . \tag{90}$$

For the Case I procedure, we have

$$\Delta i^{(N)}_{m+\nu}(z) = \frac{\varphi^{(N)}_{m,\nu}(z)}{\theta^{(N)}(z)} - \frac{\varphi^{(N)}_{m,\nu}(z) - \omega\varphi^{(S)}_{m,\nu}(z)}{\theta^{(N)}(z) - \omega\theta^{(S)}(z)}$$

$$= \frac{\omega\left\{\theta^{(N)}(z)\varphi^{(S)}_{m,\nu}(z) - \theta^{(S)}(z)\varphi^{(N)}_{m,\nu}(z)\right\}}{\left\{\theta^{(N)}(z)\right\}^2\left\{1 - \omega\dfrac{\theta^{(S)}(z)}{\theta^{(N)}(z)}\right\}}$$

$$= \frac{\omega\sigma E^{(S)}_{m,\nu}(z)\left[1 - \dfrac{E^{(N)}_{m,\nu}(z)}{E^{(S)}_{m,\nu}(z)}\right]}{1-\omega\sigma} , \quad \sigma = \frac{\theta^{(S)}(z)}{\theta^{(N)}(z)} , \tag{91}$$

and these equations also hold for Case II provided $E$ is replaced by $F$. If $S = N$, that is, $s = n$, the round-off error is nil. Indeed, this must be since the starting value $\varphi^{(N)}_{N+1,\nu}$ is immaterial. It is clear that if all parameters and $z$ are fixed, then the round-off error decays to zero as $n \to \infty$. For $n$ sufficiently large with respect to $z$, $|\sigma| < 1$ and we can take $\left|1-E^{(N)}_{m,\nu}(z)/E^{(S)}_{m,\nu}(z)\right| < 2$. Also it appears heuristically that

$$\left|E^{(S)}_{m,\nu}(z)\right| \le \left|E^{(S)}_{0,\nu}(z)\right| \le \left|I_\nu(z) - (z/2)^\nu/\Gamma(\nu+1)\right| . \tag{92}$$

Thus on this basis,

$$\left|\Delta i_{m+\nu}^{(N)}(z)\right| \le 2\omega\left|I_\nu(z) - (z/2)^\nu/\Gamma(\nu+1)\right| \tag{93}$$

is an approximate bound for the round-off in a single entry of the set of numbers generated by the backward recursion process. If $\omega$ is the maximum round-off error in each entry, then the total round-off error in $i_{m+\nu}^{(N)}(z)$ is approximately bounded by $N$ times the right-hand side of (93). Thus the round-off error is insignificant, and it is easy to estimate the number of extra decimals which must be carried so that the total round-off error in the process lies within the error when the arithmetic is exact. Equations analogous to (92) and (93) for Case II are easily derived and we omit details.

Case III. We have

$$\Delta i_{m+\nu}^{(N)}(z) = \frac{\omega I_\nu(z) A_{m,\nu}^{N,S}(z)}{\varphi_{0,\nu}^{(N)}(z)\left[\varphi_{0,\nu}^{(N)}(z) - \omega\varphi_{0,\nu}^{(S)}(z)\right]} ,$$

$$A_{m,\nu}^{N,S}(z) = \varphi_{0,\nu}^{(N)}(z)\varphi_{m,\nu}^{(S)}(z) - \varphi_{0,\nu}^{(S)}(z)\varphi_{m,\nu}^{(N)}(z) . \tag{94}$$

Using (26), a straightforward computation shows that

$$\begin{aligned} A_{m,\nu}^{N,S}(z) &= (-)^\gamma z^2\left[I_{2s+2-\gamma+\nu}(z)K_{2n+2-\delta+\nu}(z) + (-)^{\delta+\gamma+1} I_{2n+2-\delta+\nu}(z)K_{2s+2-\gamma+\nu}(z)\right] \\ &\quad \times \left[I_{m+\nu}(z)K_\nu(z) + (-)^{m+1} I_\nu(z)K_{m+\nu}(z)\right] \\ &= (-)^{m+\gamma+1}\varphi_{2s+2-\gamma,\nu}^{(N)}\varphi_{0,\nu}^{(m-2)}(z) , \quad m > 0 , \\ &= 0 , \quad m = 0 . \end{aligned} \tag{95}$$

Thus

$$\Delta i_{m+\nu}^{(N)}(z) = \frac{\omega(-)^{m+\gamma+1} i_{m+\nu}^{(N)}(z)\varphi_{2s+2-\gamma,\nu}^{(N)}(z)\varphi_{\nu}^{(m-2)}(z)}{\varphi_{0,\nu}^{(N)}(z)\varphi_{m,\nu}^{(N)}(z)\left[1 - \frac{\omega\varphi_{0,\nu}^{(S)}(z)}{\varphi_{0,\nu}^{(N)}(z)}\right]} . \tag{96}$$

Obviously, this is nil is $S = N$ or if $m = 0$. Now using (37), we find

$$\frac{\varphi_{2s+2-\gamma,\nu}^{(N)}(z)}{\varphi_{0,\nu}^{(N)}(z)\varphi_{m,\nu}^{(N)}(z)} = \frac{(z/2)^{2n+2s+3-\delta-\gamma-m}\Gamma(\nu+1)\Gamma(m+\nu+1)}{\Gamma(2n+2-\delta+\nu)\Gamma(2s+3-\delta+\nu)}$$

$$\times \frac{{}_2F_3\left[n-s-\frac{1+\delta-\gamma}{2}\right]\left(\begin{matrix}-n+s+\frac{1+\delta-\gamma}{2}, -n+s+1+\frac{\delta-\gamma}{2} \\ -2n-1+\delta-\nu, 2s+3-\gamma+\nu, -2n+2s+1+\delta-\gamma\end{matrix}\middle| z^2\right)}{{}_2F_3\left[n-\frac{\delta-1}{2}\right]\left(\begin{matrix}-n+\frac{\delta-1}{2}, -n+\frac{\delta}{2} \\ -2n-1+\delta-\nu, \nu+1, -2n-1+\delta\end{matrix}\middle| z^2\right){}_2F_3\left[n-\frac{m-1+\delta}{2}\right]\left(\begin{matrix}-n+\frac{m-1+\delta}{2}, -n+\frac{m+\delta}{2} \\ -2n-1+\delta-\nu, m+\nu+1, -2n-1+\delta+m\end{matrix}\middle| z^2\right)} , \tag{97}$$

and it is clear that rounding errors are insignificant. Indeed, if all parameters and $z$ are fixed, then the round-off error tends to zero as $n \to \infty$.

## FORMULAS FOR $J_\nu(z)$

As previously remarked, the analyses for $I_\nu(z)$ hold throughout the cut complex z-plane, $-\pi < \arg z \le \pi$, and throughout the cut complex $\nu$-plane $|\arg \nu| < \pi$, although it is sufficient to have $0 \le \arg z \le \pi/2$ and $R(\nu) > -1$. Nonetheless, we indicate how to get results for $J_\nu(z)$ directly and to facilitate use of our findings, it is convenient to restate some of the key equations. We omit discussion of Case II since it requires complex arithmetic to generate $J_\nu(z)$ which is real when $z$ and $\nu$ are real. In any event, the reader should have no difficulty in establishing the Case II equations for $J_\nu(z)$ once it is observed how this is done for Case I.

All developments for $J_\nu(z)$ are readily gotten by use of the equations

$$I_{m+\nu}(ze^{-i\pi/2}) = e^{-i(m+\nu)\pi/2} J_{m+\nu}(z) , \tag{98}$$

$$K_{m+\nu}(ze^{-i\pi/2}) = \tfrac{1}{2}\pi i e^{-i(m+\nu)\pi/2} H^{(1)}_{m+\nu}(z) , \tag{99}$$

$$H^{(1)}_{m+\nu}(z) = J_{m+\nu}(z) + iY_{m+\nu}(z) , \tag{100}$$

$$Y_{m+\nu}(z) = (-)^m \csc \nu\pi \left[(-)^m \cos \nu\pi \, J_{m+\nu}(z) - J_{-m-\nu}(z)\right] , \tag{101}$$

where now in the $J_\nu(z)$ analyses, $-\pi/2 < \arg z \le 3\pi/2$.

It is convenient to introduce the following notation. Unless indicated otherwise, if $A$ is used to signify some function or equation in the developments for $I_\nu(z)$, then $A^*$ is used to signify the corresponding function or equation in the developments for $J_\nu(z)$. In illustration

$$J_\nu(z) = (z/2)^\nu {}_0F_1(\nu+1;-z^2/4) , \tag{1}*$$

and both $J_{m+\nu}(z)$ and $Y_{m+\nu}(z)$ are solutions of the difference equation

$$\varphi^*_{m,\nu}(z) = \frac{2(m+\nu+1)}{z} \varphi^*_{m+1,\nu}(z) - \varphi^*_{m+2,\nu}(z) . \tag{21}*$$

Also

$$J_{m+\nu}(z) = \frac{(z/2)^{m+\nu}}{\Gamma(m+\nu+1)} 1+O(m^{-1}) \quad , \tag{33)*}$$

$$Y_{m+\nu}(z) = -\frac{(z/2)^{-m-\nu}}{\pi} \Gamma(m+\nu)\left[1+O(m^{-1})\right] \quad . \tag{34)*}$$

We now present the key results pertinent to $J_\nu(z)$ .

<u>Theorem 1*</u>.

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \frac{\psi_n^*(z)}{h_n^*(z)} + R_n^*(z) \ , \ R_n^*(z) = \frac{S_n^*(z)}{h_n^*(z)} \quad , \tag{2)*}$$

where $\psi_n^*(z)$ and $h_n^*(z)$ are given by $\psi_n(z)$ and $h_n(z)$ , see (3)-(5), provided there we replace $z^2$ by $-z^2$ , that is, replace $X$ by $-X$ . Further, $S_n^*(z)$ and $S_n(z)$ are both given by (8) and $R_n^*(z) = (-)^n R_n(z)$ , see (10).

<u>Theorem 2</u>.* Both $\psi_n^*(z)$ and $h_n^*(z)$ satisfy the same recurrence formula (11) if there we replace $X$ by $-X$ .

We state without proof the following equations.

$$\varphi_{m,\nu}^{(N)*}(z) = \frac{(z/2)^{2n-m+1-\delta}\Gamma(2n+2-\delta+\nu)}{\Gamma(m+\nu+1)} \times {}_2F_3^{\left[n-\frac{m-1+\delta}{2}\right]}\left(\begin{matrix}-n+\frac{m-1+\delta}{2}, -n+\frac{m+\delta}{2} \\ -2n-1-\nu+\delta, m+\nu+1, -2n-1+\delta+m\end{matrix}\middle| -z^2\right) . \tag{37)*}$$

$$\theta^*(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} = \sum_{k=0}^{\infty} \frac{(2k+\nu)\Gamma(k+\nu)}{\Gamma(\nu+1)k!} J_{2k+\nu}(z) \ . \tag{38)*}$$

$$\theta^{(N)*}(z) = (2/z)^{1-\delta} \frac{(\nu+1)_{n+1-\delta}}{n!} h_n^*(z) \ . \tag{39)*}$$

Theorem 9.*

$$j_\nu^{(N)}(z) = \frac{\varphi_{0,\nu}^{(N)*}(z)}{\theta^{(N)*}(z)} = \frac{\psi_n^*(z)}{h_n^*(z)} ,$$

$$j_{m+\nu}^{(N)}(z) = \frac{\varphi_{m,\nu}^{(N)*}(z)}{\theta^{(N)*}(z)} = (z/2)^{m+\nu} \frac{{}_2F_3^{\left[n-\frac{m-1+\delta}{2}\right]}\left(\begin{matrix}-n+\frac{m-1+\delta}{2}, -n+\frac{m+\delta}{2} \\ -2n-1-\nu+\delta, m+\nu+1, -2n-1+\delta+m\end{matrix}\middle| -z^2\right)}{{}_0F_1^{(n)}(-2n-1+\delta-\nu;-z^2/4)} . \quad (45)*$$

Let

$$E_{m,\nu}^{(N)*}(z) = J_{m+\nu}(z) - j_{m+\nu}^{(N)}(z) \qquad (59)*$$

where $j_{m+\nu}^{(N)}(z)$ is given in (45)*.

Theorem 13.* If $\nu$ is not a positive integer or zero,

$${}_0F_1^n(-2n-1+\delta-\nu;-z^2/4)E_{m,\nu}^{(N)*}(z) + \frac{(z/2)^{2n+2}\Gamma(n+1-\delta+\nu)}{(n+1)!\Gamma(2n+2-\delta+\nu)} J_{m+\nu}(z)\,{}_1F_2\left(\begin{matrix}1\\ n+2, -n+\delta-\nu\end{matrix}\middle| -z^2/4\right)$$

$$= \frac{(-)^m\pi(z/2)^{2n+2-\delta+\nu}}{\sin\nu\pi\Gamma(2n+2-\delta+\nu)} J_{-m-\nu}(z)J_{2n+2-\delta+\nu}(z)$$

$$= -\frac{\pi(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)} J_{2n+2-\delta+\nu}(z)Y_{m+\nu}(z)$$

$$+ \frac{(\pi/2)(z/2)^{2n+2-\delta+\nu}}{\tan\nu\pi\,\Gamma(2n+2-\delta+\nu)} J_{m+\nu}(z)J_{2n+2-\delta+\nu}(z) . \qquad (60)*$$

Equation (60)* can be rearranged so that with the aid of L'Hospital's theorem, we can get a representation of the error when $\nu$ becomes a positive integer or zero. This result is omitted. However, for arbitrary $\nu$, we always have

$$
{}_0F_1^n(-2n-1+\delta-\nu;-z^2/4)E_{m,\nu}^{(N)*}(z) = -\frac{(z/2)^{2n+2}\Gamma(n+1-\delta+\nu)}{(n+1)!\Gamma(2n+2-\delta+\nu)} J_{m+\nu}(z)\,{}_1F_2^s\left(\begin{matrix}1\\ n+2,-n+\delta-\nu\end{matrix}\middle|-z^2/4\right)
$$

$$
-\frac{\pi(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)} J_{2n+2-\delta+\nu}(z)Y_{m+\nu}(z)
$$

$$
+\frac{O\left((z/2)^{2n}\right)}{\Gamma(2n+2-\delta+\nu)\Gamma(2n+3-\delta+\nu)} , \qquad (61)*
$$

where $s = n-\delta+\nu$ $(s = \infty)$ if $\nu$ is (is not) a positive integer or zero. Clearly the backward recurrence scheme is convergent. Further, for $n$ sufficiently large, $n \gg m$, the relative error is essentially independent of $m$. For convenience in the applications we record the formula

$$
E_{m,\nu}^{(N)*}(z) = -\frac{(z/2)^{2n+2}\Gamma(n+1-\delta+\nu)}{(n+1)!\Gamma(2n+2-\delta+\nu)} J_{m+\nu}(z)\left[1+O(n^{-1})\right]
$$

$$
-\frac{\pi(z/2)^{2n+2-\delta+\nu}}{\Gamma(2n+2-\delta+\nu)} J_{2n+2-\delta+\nu}(z)Y_{m+\nu}(z) . \qquad (62)*
$$

<u>Remark</u>: Let $\nu$, $n$ and $z$ be fixed so that $E_{m,\nu}^{(N)*}(z)$ is a function of $m$ only. Then $E_{m,\nu}^{(N)*}(z)$ satisfies the recurrence formula for $\varphi_{m,\nu}^{*}(z)$, see (21)*.

Let

$$
G_{m,\nu}^{(N)*}(z) = J_{m+\nu}(z) - j_{m+\nu}^{(N)}(z) , \qquad (79)*
$$

$$
j_{m+\nu}^{(N)}(z) = \frac{J_\nu(z)\varphi_{m,\nu}^{(N)*}(z)}{\varphi_{0,\nu}^{(N)*}(z)} , \qquad (80)*
$$

where $\varphi_{m,\nu}^{(N)*}(z)$ is given by (37)*.

Theorem 16.*

$$G_{m,\nu}^{(N)*}(z) = \frac{J_{2n+2-\delta+\nu}(z)\varphi_{0,\nu}^{(m-2)*}(z)}{\varphi_{0,\nu}^{(N)*}(z)}$$

$$= \frac{\Gamma(m+\nu)(z/2)^{2n+2-m-\delta}}{\Gamma(2n+2-\delta+\nu)} J_{2n+2-\delta+\nu}(z) L_{m,n}^{*}(z) ,$$

$$L_{m,n}^{*}(z) = \frac{{}_2F_3\left[\frac{m-1}{2}\right]\left(\begin{matrix}-\frac{m-1}{2},-\frac{m-2}{2}\\ 1-\nu-m,\nu+1,1-m\end{matrix}\middle| -z^2\right)}{{}_2F_3\left[n-\frac{\delta-1}{2}\right]\left(\begin{matrix}-n+\frac{\delta-1}{2},-n+\frac{\delta}{2}\\ -2n-1+\delta-\nu,\nu+1,-2n-1+\delta\end{matrix}\middle| -z^2\right)} , \quad 0 < m \le 2n+1-\delta . \qquad (81)*$$

Further, $G_{m,\nu}^{(N)*}(z)$ satisfies the recurrence formula for $\overset{*}{\varphi}_{m+\nu}(z)$ , see (21)*. Thus

$$G_{m+2,\nu}^{(N)*}(z) = \frac{2(m+\nu+1)}{z} G_{m+1,\nu}^{(N)*}(z) - G_{m,\nu}^{(N)*}(z) ,$$

$$G_{0,\nu}^{(N)*}(z) = 0 , \quad G_{1,\nu}^{(N)*}(z) = \frac{J_{2n+2-\delta+\nu}(z)}{\varphi_{\nu}^{(N)*}(z)} \qquad (82)*$$

Finally, for convenience in the applications, we record the formulas

$$G_{m,\nu}^{(N)*}(z) = \frac{(z/2)^{4n+4-2\delta+2\nu-m}\Gamma(m+\nu)}{\Gamma(2n+2-\delta+\nu)\Gamma(2n+3-\delta+\nu)\Gamma(\nu+1)} \frac{{}_2F_3\left[\frac{m-1}{2}\right]\left(\begin{matrix}-\frac{m-1}{2},-\frac{m-2}{2}\\ 1-\nu-m,\nu+1,1-m\end{matrix}\middle| z^2\right)}{J_{\nu}(z)}\left[1+O(n^{-1})\right], \qquad (83)*$$

$$G_{m,\nu}^{(N)*}(z) = \frac{(z/2)^{4n+4-2\delta+\nu-m}\Gamma(m+\nu)}{\Gamma(2n+2-\delta+\nu)\Gamma(2n+3-\delta+\nu)} \left[1+O(m^{-1})\right]\left[1+O(n^{-1})\right] . \qquad (84)*$$

## NUMERICAL EXAMPLES

Let

$$N = 5 \ , \ n = 3 \ , \ \delta = 1 \ , \ z = 2/3 \ , \ \nu = 1/3 \ .$$

Values of $\varphi_{m+\nu}^{(N)}(z)$ , $\theta^{(N)}(z)$ and $\Omega^{(N)}(z)$ are given in the table below.

| $m$ | $\varphi_{m,\nu}^{(N)}(z)$ |
|---|---|
| 6 | 1 |
| 5 | 19 |
| 4 | 305 |
| 3 | 3984 |
| 2 | 40145 |
| 1 | 2 84999 |
| 0 | 11 80141 |

$\theta^{(N)}(z) = 880\ 75120/81$

$\Omega^{(N)}(z) = 1\ 38952\ 97360/6561$

Since

$$\frac{(z/2)^{\nu}}{\Gamma(\nu+1)} = 0.77645\ 82114, \ e^{-2/3} = 0.51341\ 71190 \ ,$$

the Case I and Case II approximations are

$$0.84272\ 08930 \text{ and } 0.84272\ 10326 \ ,$$

respectively. To 10 decimals

$$I_{1/3}(2/3) = 0.842772\ 08819 \ .$$

Thus the errors in the Case I and Case II approximations are

$$-0.111 \cdot 10^{-7} \text{ and } -0.151 \cdot 10^{-6} \ ,$$

respectively. Using (62) and (76) each with $O(n^{-1})$ and the term involving $K_{m+\nu}(z)$ neglected, the approximate Case I and Case II errors are

$$-0.110\cdot 10^{-7} \text{ and } -0.149\cdot 10^{-6} ,$$

respectively.

For a second example, let

$$N = 5 , n = 3 , \delta = 1 , z = 2 , \nu = 0 .$$

Again we illustrate the Case I and Case II schemes. We have the following data.

| $m$ | $\varphi_m^{(N)}(z)$ |
|---|---|
| 6 | 1 |
| 5 | 6 |
| 4 | 31 |
| 3 | 130 |
| 2 | 421 |
| 1 | 972 |
| 0 | 1393 |

$\theta^{(N)}(z) = 611$

$\Omega^{(N)}(z) = 4515$

$e^2 = 7.38905\ 6099$

| $m$ | $i_m^{(N)}(z)$ Case I | | $i_m^{(N)}(z)$ Case II | | $I_m(z)$ | |
|---|---|---|---|---|---|---|
| 0 | 2.27986 | 9067 | 2.27972 | 4285 | 2.27958 | 5302 |
| 1 | 1.59083 | 4697 | 1.59073 | 3672 | 1.59063 | 6855 |
| 2 | 0.68903 | 4370 | 0.68899 | 0613 | 0.68894 | 8448 |
| 3 | 0.21276 | 5957 | 0.21275 | 2446 | 0.21273 | 9959 |
| 4 | 0.50736 | $4975\cdot 10^{-1}$ | 0.50733 | $2755\cdot 10^{-1}$ | 0.50728 | $5700\cdot 10^{-1}$ |
| 5 | 0.98199 | $6727\cdot 10^{-2}$ | 0.98193 | $4365\cdot 10^{-2}$ | 0.98256 | $7932\cdot 10^{-2}$ |
| 6 | 0.16366 | $6121\cdot 10^{-2}$ | 0.16365 | $5728\cdot 10^{-2}$ | 0.16001 | $7336\cdot 10^{-2}$ |

| $m$ | Error Case I | Error Case II | Relative Error Case I | Relative Error Case II |
|---|---|---|---|---|
| 0 | $-0.284\cdot 10^{-3}$ | $-0.139\cdot 10^{-3}$ | $-0.124\cdot 10^{-3}$ | $-0.610\cdot 10^{-3}$ |
| 1 | $-0.198\cdot 10^{-3}$ | $-0.968\cdot 10^{-4}$ | $-0.124\cdot 10^{-3}$ | $-0.609\cdot 10^{-4}$ |
| 2 | $-0.859\cdot 10^{-4}$ | $-0.422\cdot 10^{-4}$ | $-0.125\cdot 10^{-3}$ | $-0.612\cdot 10^{-4}$ |
| 3 | $-0.260\cdot 10^{-4}$ | $-0.125\cdot 10^{-4}$ | $-0.122\cdot 10^{-3}$ | $-0.587\cdot 10^{-4}$ |
| 4 | $-0.793\cdot 10^{-5}$ | $-0.471\cdot 10^{-5}$ | $-0.156\cdot 10^{-3}$ | $-0.929\cdot 10^{-4}$ |
| 5 | $0.571\cdot 10^{-5}$ | $0.634\cdot 10^{-5}$ | $0.581\cdot 10^{-3}$ | $0.645\cdot 10^{-4}$ |
| 6 | $-0.365\cdot 10^{-4}$ | $-0.364\cdot 10^{-4}$ | $-0.228\cdot 10^{-1}$ | $-0.228\cdot 10^{-1}$ |

Here the entries in the $I_m(z)$ column are correct for the number of decimals given.

Using the first lines of (62) and (76), each with $O(n^{-1})$ and the term involving $K_{m+\nu}(z)$ neglected, the approximate relative error for Cases I and II, respectively, are $-0.116\cdot 10^{-3}$ and $-0.537\cdot 10^{-4}$, respectively.

In the table below, we record the approximate errors obtained by use of (62) with $O(n^{-1})$ omitted for $m = 6$ and 5 and by use of (21), see the remark following Theorem 13, for the lower values of $m$. This is called Case I, (62)-(21) in the table. We also present the analogous Case II, (76)-(21) data. In each instance known tabular values of $K_m(2)$ and $I_7(2)$ were used. In practice, we suggest using (34) or the lead term of the uniform asymptotic expansion of $K_{m+\nu}(z)$ developed by Olver [9]. For $I_{2n+2-\delta+\nu}(z)$, use (33) or the lead term in the uniform asymptotic expansion for this function which is also given in the source just cited. We also suggest that computation of the gamma functions be simplified as follows. With $|R(\alpha)| < 1$ and $r$ a positive integer, we have

$$\Gamma(r+\alpha+1) = r!\,\frac{\Gamma(r+\alpha+1)}{\Gamma(r+1)} = r!\,r^{\alpha}\left[1+O(r^{-1})\right]$$

and for $r$ sufficiently large, we neglect $O(r^{-1})$. The approximation is of course superfluous if $\alpha = 0$. If $\alpha = \pm\frac{1}{2}$, the approximation may still be used though known tables of the gamma function for half an odd integer may be preferred [10]. If more precise values of the gamma functions are required, see [11].

| | Approximate Error | |
|---|---|---|
| $m$ | Case I, (62)-(21) | Case II, (76)-(21) |
| 0 | $-0.264\cdot 10^{-3}$ | $-0.114\cdot 10^{-3}$ |
| 1 | $-0.184\cdot 10^{-3}$ | $-0.765\cdot 10^{-4}$ |
| 2 | $-0.797\cdot 10^{-4}$ | $-0.373\cdot 10^{-4}$ |
| 3 | $-0.242\cdot 10^{-4}$ | $-0.110\cdot 10^{-4}$ |
| 4 | $-0.723\cdot 10^{-5}$ | $-0.410\cdot 10^{-5}$ |
| 5 | $0.475\cdot 10^{-5}$ | $0.536\cdot 10^{-5}$ |
| 6 | $-0.310\cdot 10^{-4}$ | $-0.309\cdot 10^{-4}$ |

For a final example, we illustrate Case III using the data of our second example. We get the following numbers.

| $m$ | $i_m^{(N)}(z)$ | | $G_{m+\nu}^{(N)}(z)$ | Approximate Error (82)-(83) | Approximate Error (84) |
|---|---|---|---|---|---|
| 0 | 2.27958 | 5302 | 0 | 0 | 0 |
| 1 | 1.59063 | 6693 | $0.162\cdot10^{-6}$ | $0.121\cdot10^{-6}$ | $0.276\cdot10^{-6}$ |
| 2 | 0.68894 | 8609 | $-0.162\cdot10^{-6}$ | $-0.121\cdot10^{-6}$ | $-0.276\cdot10^{-6}$ |
| 3 | 0.21273 | 9475 | $0.484\cdot10^{-6}$ | $0.363\cdot10^{-6}$ | $0.551\cdot10^{-6}$ |
| 4 | 0.50730 | $1826\cdot10^{-1}$ | $-0.161\cdot10^{-5}$ | $-0.121\cdot10^{-5}$ | $-0.165\cdot10^{-5}$ |
| 5 | 0.98187 | $4502\cdot10^{-2}$ | $0.693\cdot10^{-5}$ | $0.520\cdot10^{-5}$ | $0.661\cdot10^{-5}$ |
| 6 | 0.16364 | $5750\cdot10^{-2}$ | $-0.363\cdot10^{-4}$ | $-0.272\cdot10^{-4}$ | $-0.331\cdot10^{-4}$ |

In the above, the approximate error (82)-(83) means that $G_{\nu+1}^{(N)}(z)$ is approximated by (83) with $m = 1$ and $O(n^{-1})$ neglected, and subsequent approximate values of the error are found by use of the recursion formula in (82). Use of the recurrence formula in this fashion is stable as the magnitude of the error in an increasing function in $m$. Also Eq. (84) means this equation with $O(m^{-1})$ and $O(n^{-1})$ neglected.

A measure of the accuracy of the three schemes treated can be had by use of normalization relations. Thus if the Case III procedure is employed, then (38) and (46) with $I_{k+\nu}(z)$ replaced by $i_{k+\nu}^{(N)}(z)$ are available as checks. Similarly, equations (46) and (38) are available as checks for the Case I and Case II techniques, respectively. For some other useful normalization relations, see [1, Vol. 2, pp. 45, 46].

Analyses of the error in the backward recursion process for the solution of a general second and higher order linear difference equation have been given by a number of authors. Some authors have studied the case of Bessel functions directly. We make no attempt to survey the various contributions here. Pertinent references are given by Wimp [4]. Suffice it to say, none of the analyses have the precision and simplicity of those developed in the present paper. We deliberately chose $N$ and as a consequence $n$ small ($N=5, n=3$) in our numerical examples to put our asymptotic estimates under a severe test. The efficiency and realism of our error formulas is manifest.

## CONCLUDING REMARKS

It appears that the techniques developed here for the Bessel function $I_\nu(z)$ can be extended to analyze more general second and higher order difference equations. In particular, it would be useful to have analogous results for ${}_2F_1(a,b;c;z)$ and its confluent forms. This we intend to do in future papers.

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