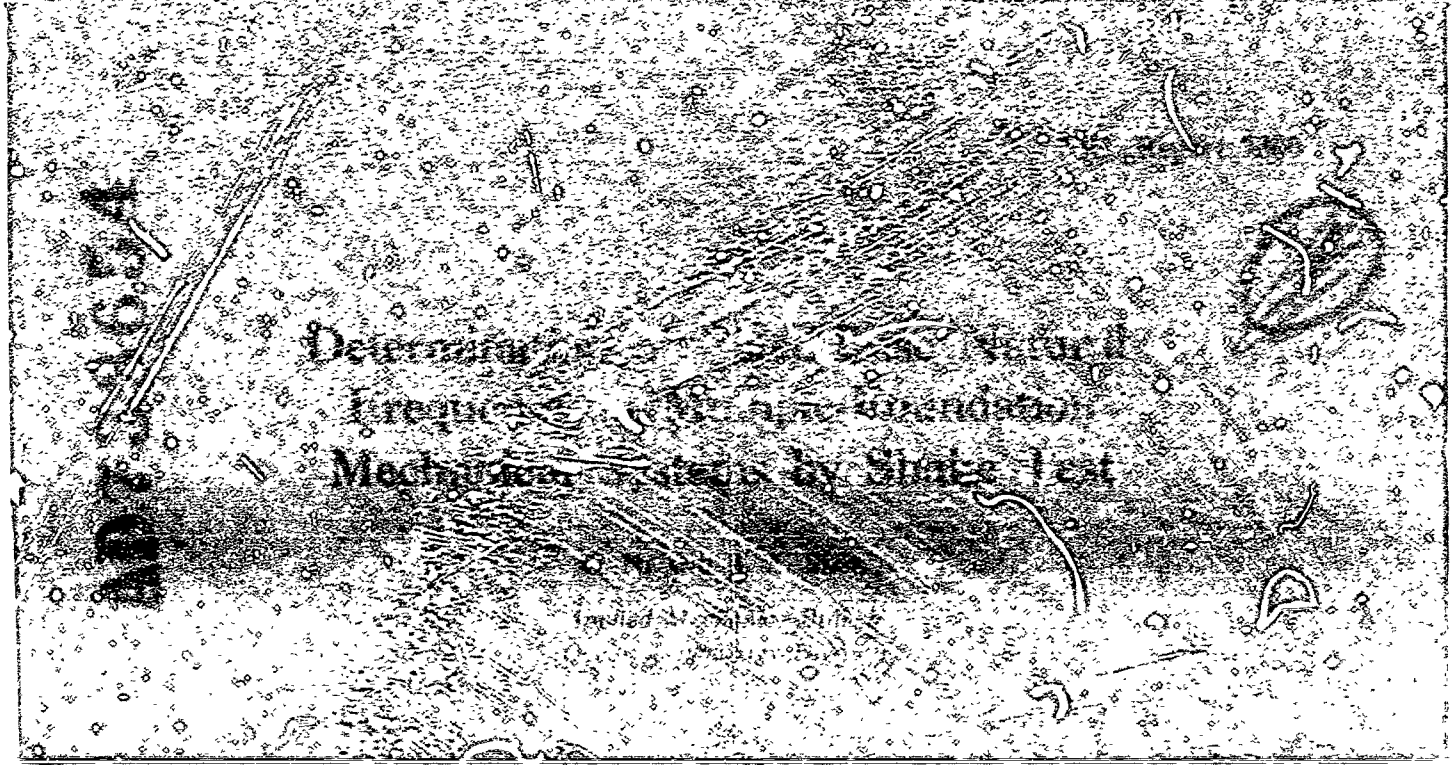
Determining [illegible] Yield Frequency [illegible] Foundation Mechanism Tested by Shake Test



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Shake Tests						
Mobility Measurements						
Shock and Vibration						
Normal Mode Analysis						

There are also disadvantages: (1) Tests will not reveal fixed-base natural frequencies if the system contains extensive viscous-type damping. This is not a serious limitation, however, since in the presence of large amounts of energy dissipation, natural freuqencies do not in any case represent the significant dynamic parameters. (2) Precise setting and alignment of the shaker is essential to avoid introducing unwanted complex responses.

In summary, this paper presents a practical method to determine the fixed-base natural frequencies of a subsystem. If such frequencies exist, they are significant in current methods of structure analysis and design criteria involving dynamic loading.

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13. ABSTRACT

The fixed-base natural frequencies of a system are those natural frequencies the system would have if it were mounted on a base of infinite mass and stiffness. The main purpose of this investigation is to find a generalized analytical method for determining the fixed-base natural frequencies of an in situ or laboratory mechanical subsystem. The analysis will be based on an arbitrary n-degree-of-freedom undamped linear time-invariant system. The application of this analytical method is extended to cases with small viscous damping. Only one shaker is needed to obtain the required response measurements necessary to calculate the fixed-base natural frequencies of the subsystem. Gages to obtain these measurements should be placed at all supporting points of the subsystem being tested and one additional point located any other place on that subsystem. The analytical work reveals that the following advantages of this method are realized. (1) No assumption other than linear time-invariant of the total system has to be imposed in the derivation. (2) No detailed physical properties of the whole system have to be known although, by its very nature, the mobility of any element within the system depends upon the physical properties of the whole system. In fact, acquiring detailed physical properties has always been the difficulty in calculating the fixed-base natural frequencies of a real complex system. (3) It is equally convenient to carry out the test in the field as well as in the laboratory, provided that the shaker is capable of generating enough mechanical energy to excite the subsystem. (4) When conditions allow, many subsystems can be measured simultaneously. Consequently, based on the liberal choice of force application points, savings in manpower, time, equipment, and money may be effected.

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ABSTRACT

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1. No assumption other than linear time-invariant of the total system has to be imposed in the derivation.

2. No detailed physical properties of the whole system have to be known although, by its very nature, the mobility of any element within the system depends upon the physical properties of the whole system. In fact, acquiring detailed physical properties has always been the difficulty in calculating the fixed-base natural frequencies of a real complex system.

3. It is equally convenient to carry out the test in the field as well as in the laboratory, provided that the shaker is capable of generating enough mechanical energy to excite the subsystem.

4. When conditions allow, many subsystems can be measured simultaneously. Consequently, based on the liberal choice of force application points, savings in manpower, time, equipment, and money may be effected.

There are also disadvantages:

1. Tests will not reveal fixed-base natural frequencies if the system contains extensive viscous-type damping. This is not a serious limitation, however, since in the presence of large amounts of energy dissipation, natural frequencies do not in any case represent the significant dynamic parameters.

2. Precise setting and alignment of the shaker is essential to avoid introducing unwanted complex responses.

In summary, this paper presents a practical method to determine the fixed-base natural frequencies of a subsystem. If such frequencies exist, they are significant in current methods of structure analysis and design criteria involving dynamic loading.

PROBLEM STATUS

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SYMBOLS

The symbols used in this report are defined as they appear within the text. The most important ones are listed below for reference.

A dot over a variable in the text represents the differentiation of that variable with respect to time.

A superscript T or -1 at the right corner of a matrix indicates the transpose or inverse of the matrix respectively.

$\lfloor\ \rfloor$	row matrix
$\{\ \}$	column matrix
$[\ \ ]$	square matrix
$\{\ \}$	non-square matrix
$[I]$	identity matrix
$\|\ \|$	determinant
$\|\|\ \|\|$	absolute value
a_{ij}	matrix element of $[A]$ at ith row and jth column
$L(q,\dot{q};t)$	Lagrangian
$T(\dot{q};t)$	kinetic energy
$V(q;t)$	potential
q_k	kth generalized coordinate
t	time
$[M]$	mass matrix
$[R]$	damping matrix
$[K]$	stiffness matrix

ω	frequency of excitation
ω_j^i	jth fixed-base natural frequency of structure or substructure i
$F_k(t)$	force at location k
$\overline{F}_k$	amplitude of the sinusoidal force $F_k(t)$
$\{F\}$	Spatial force column matrix (or vector)
ϕ_{kj}	relative phase angle of kth response with respect to the applied force $F_j(t)$ at location j
$\bar{q}_{kj}$	amplitude of the kth displacement response due to the applied force $F_j(t)$ at location j
$\bar{q}_k$	kth spatial displacement component
$\{q\}$	Spatial displacement column matrix (or vector)
$[Z]$	flexibility matrix
z_{ij}	flexibility matrix element at ith row and jth column
$[\mathfrak{M}]$	mobility matrix
m_{ij}	mobility matrix element at ith row and jth column
Z_{ij}	cofactor of the flexibility matrix element z_{ji}
$[Z^e]$	flexibility matrix of the equipment subsystem
$[Z^s]$	flexibility matrix of the support subsystem
$[Z^b]$	flexibility matrix of the base subsystem
$[C_\ell^k]$	coupling matrix between subsystem k and subsystem ℓ
$\psi(m_{ij};\omega)$	Resonance function

DETERMINATION OF FIXED-BASE NATURAL FREQUENCY OF MULTIPLE-FOUNDATION MECHANICAL SYSTEMS BY SHAKE TEST

INTRODUCTION

The theory explaining the small oscillation of a simple mechanical system has been experimentally validated for some time. The application of the theory to a real complex system, however, is limited, where precise information concerning the coupling mechanisms among the elements of the system is lacking. Even when the exact physical constants are given, the numerical calculations are very cumbersome. As a general practice, certain important constants can be extracted quantitatively from the experimental results. The shake test is often used to measure the vibrational responses. It was mentioned in the report by Petak and O'Hara (1) that knowledge of the fixed-base natural frequencies of a system is essential in current methods of treating dynamic problems. Since Ref. 1, the method of determination of fixed-base natural frequencies by shake test has been investigated at NRL. Simple cases were analyzed by Petak and Kaplan (2) for a one-dimensional linear chain, and by Petak and O'Hara (1) for a dual-foundation shipboard equipment. Experimental work on a single support beam was reported by Remmers (3). This analytical method is proved to be a generalization of the aforementioned special cases.

This report includes the analytical presentation of this method, the physical interpretation of the entities derived, the technical method suggested in taking and analyzing test data and drawing conclusions, and the extension of its application to damped systems, which is of importance in engineering practice. Various computer simulations of a problem designed to illustrate the developed method are provided. Experimental confirmation of this method on a three-support beam mounted on a truss-like frame will be presented in a separate report.*

The experimental techniques in principle are essentially the same as described in Ref. 2, except that phase differences (or the relative phase) between the applied force and corresponding responses at points of measurement are necessary for a subsystem having three or more supporting points. By use of the resonance condition, the fixed-base natural frequencies can be calculated from the experimental data. In order to eliminate the extraneous frequencies inherent in this method of measurement, one additional shaking point on the equipment subsystem itself is required. Contrary to the speculation that many shakers with variable forces (both magnitude and phase) ought to be used in a multiple-foundation system, this method requires only one shaker at a time. The magnitude of the forces applied is irrelevant to the result, as long as the system is being excited to a measurable level. However, the phase differences between the forces and the responses are important.

*NRL Report 7362

THEORETICAL ANALYSIS OF A LINEAR TIME-INVARIANT MECHANICAL SYSTEM

It is intended to develop a steady-state method for determining fixed-base natural frequencies of an in situ or laboratory mechanical subsystem. In this analytical treatment, generalized coordinates q_k in configuration space, Lagrangian mathematical formulation, and linear space matrix representation will be used to clarify the physical contents of the mathematical formulation. Since it is understood that the main goal of this investigation is to determine fixed-base natural frequencies, the result is confined to linear time-invariant systems, or to systems behaving approximately so.

Undamped System

Equation of Motion—The Lagrangian formulation describing an n-degree-of-freedom dynamical system can be found in many books on mechanics (4, 5). Since this formulation is not our main interest here, only the result will be introduced. Suppose $L(q,\dot{q};t)$ denotes the Lagrangian of the system, $T(\dot{q};t)$ the kinetic energy, and $V(q;t)$ the potential energy of the system; then by definition,

$$L = T(\dot{q};t) - V(q;t) \tag{1}$$

$$T(\dot{q};t) = \frac{1}{2}\,\underline{\dot{q}}\,[M]\{\dot{q}\} \tag{2}$$

$$V(q;t) = \frac{1}{2}\,\underline{q}\,[K]\{q\}\,. \tag{3}$$

From the variational principle, the equation of motion of a free system can be derived as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0\,. \tag{4}$$

When there are applied forces, the constitutive equation may be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = F_k(t) \tag{5}$$

The explicit expressions in matrix form of Eq. (4) and (5) are

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\} \tag{6}$$

$$[M]\{\ddot{q}\} + [K]\{q\} = \{F(t)\} \tag{7}$$

respectively.

For the particular purpose of natural frequency determination, the applied forces $F(t)$ on the right side of Eq. (7) have to be limited to being sinusoidal, so as to provide a single input with a well-defined driving frequency ω to the system. Let

$$F(t) = \overline{F} \sin \omega t \ . \tag{8}$$

Here the phase of the force $F(t)$ is assumed to be zero. This assumption will not lose its generality in the analysis, if a relative phase is assigned to the responses. Then the steady-state solution of the set of n simultaneous linear differential equations, Eq. (7), has the general form of

$$\{q_{kj}\} = \{ \bar{q}_{kj} \sin (\omega t + \phi_{kj})\} \tag{9}$$

where $\phi_{kj} = n\pi$, $n = 0, 1, 2, \cdots$ for undamped cases. Substituting Eq. (8) into Eq. (7), the set of n simultaneous linear differential equations is reduced to a set of n simultaneous algebraic equations, which has the following general form

$$\left(-\omega^2[M] + [K]\right)\{q\} = \{F\} \tag{10}$$

where

$$-\omega^2[M] + [K] = [Z] \tag{11}$$

is defined as the flexibility matrix of the system, and $(1/\omega)[Z]$ is normally defined as mechanical impedance matrix (1).

Linear Space—In order to simplify the mathematical manipulation and help to shed light on the physical insight of the formulation, Eq. (10) can be written in the following form:

$$[Z]\{q\} = \{F\}. \tag{12}$$

If one views Eq. (12) as a mapping or transformation between two linear spaces, then $[Z]$ is just an operator which maps the force space onto the coordinate space. Here $[Z]$ is assumed to be nonsingular; this is true for most real physical systems, and the mapping is in a one-to-one correspondence. When the inverse of $[Z]$ exists, one can apply operator $[Z]^{-1}$ to Eq. (12) to obtain

$$[Z]^{-1}[Z]\{q\} = [Z]^{-1}\{F\} \ , \tag{13}$$

or

$$\{q\} = [Z]^{-1}\{F\} \ ,$$

where the identity equation

$$[Z]^{-1}[Z] = [\mathrm{I}] \tag{14}$$

is used. The matrix $[Z]^{-1}$ is often defined as the influence matrix. The mobility matrix is defined in terms of the influence matrix as

$$[\mathcal{H}] = \omega \cdot [Z]^{-1} . \tag{15}$$

For the convenience of discussion, the linear spaces $\{q\}$ and $\{F\}$ are divided into subspaces. In doing so, a few definitions are necessary. Any physical system may be described as three main parts:

Equipment:	The subsystem under investigation
Support:	The subsystem supporting the equipment
Base:	The remaining part of the total system other than the equipment and support.

Correspondingly, these three parts can be defined as three subspaces $\{q^e\}$, $\{q^s\}$, $\{q^b\}$, in the coordinate space $\{q\}$, and three subspaces $\{F^e\}$, $\{F^s\}$, $\{F^b\}$, in the force space $\{F\}$. Then Eq. (12) can always be arranged to have the partitioned form

$$\left[\begin{array}{c|c|c} [Z^e] & \{C_s^e\} & \{0\} \\ \hline \{C_s^e\}^T & [Z^s] & \{C_b^s\} \\ \hline \{0\} & \{C_b^s\}^T & [Z^b] \end{array}\right] \left\{\begin{array}{c} \{q^e\} \\ \hline \{q^s\} \\ \hline \{q^b\} \end{array}\right\} = \left\{\begin{array}{c} \{F^e\} \\ \hline \{F^s\} \\ \hline \{F^b\} \end{array}\right\}, \tag{16}$$

or by matrix multiplication

$$[Z^e]\{q^e\} + \{C_s^e\}\{q^s\} = \{F^e\} , \tag{17}$$

$$\{C_s^e\}^T\{q^e\} + [Z^s]\{q^s\} + \{C_b^s\}\{q^b\} = \{F^s\} , \tag{18}$$

$$\{C_b^s\}^T\{q^s\} + [Z^b]\{q^b\} = \{F^b\} . \tag{19}$$

It is apparent that there is no direct coupling between the equipment and base subsystems because of the definition of the support subsystem. In the analysis as well as tests, the three subsystems (or subspaces) must be defined without ambiguity.

Resonance Condition—The resonance condition for the equipment subsystem requires

$$[Z^e]\{q^e\} = 0 . \tag{20}$$

Consequently, this requires

$$\{C_s^e\}\{q^s\} = \{F^e\} . \tag{21}$$

Theoretically, there is a definite solution for Eq. (21) if the rank of $\{C_s^e\}$ equals the dimension of $\{q^s\}$ for a given set of $\{F^e\}$. But this does not usually occur in practical applications. However, the applied forces are arbitrary and under our control. Therefore one may set

$\{F^e\} = \{0\}$; that is, there are no forces which can be applied to the equipment, and still have the resonance condition satisfied. Equation (21) becomes

$$\{C_s^e\}\{q^s\} = 0 \ . \tag{22}$$

Since $\{C_s^e\} \neq \{0\}$, the support-displacement response vector must vanish at the fixed-base natural frequencies of the equipment subsystem. Now let us examine the set of simultaneous equations, Eqs. (17), (18), and (19), under the resonance condition discussed previously. The system can be described by the following equations:

$$[Z^e]\{q^e\} = 0 \ , \tag{23}$$

$$\{C_s^e\}^T\{q^e\} + \{C_b^s\}\{q^b\} = \{F^s\} \ , \tag{24}$$

$$[Z^b]\{q^b\} = \{F^b\} \ . \tag{25}$$

It can be explicitly shown that the solution of Eqs. (23), (24), and (25) does contain the information we need to know about the fixed-base natural frequencies of the equipment subsystem. This information is in agreement with the computer simulation of a one-dimensional linear chain structure reported in Ref. 3. In the present report, we have shown analytically that it is true for all undamped linear time-invariant systems in general, no matter what the forces are.

Resonance Function—A resonance function is defined in this section because of its particular importance in fixed-base natural-frequency determination. It contains information about the resonance frequencies of the equipment subsystem. All extraneous natural frequencies of the base and support subsystems are excluded. The graphical representation of the function shows peaks at the fixed-base natural frequencies of the equipment subsystem. The resonance function is also a function of the mobility entities of the mobility matrix of the total system. These entities are the characteristic and invariant properties of the total physical system, and their measurability is of key importance in this investigation.

Before we proceed to derive an explicit expression of the resonance function, it seems essential to recall the linear space transformation in matrix form. The harmonic solution has shown that the mobility matrix is defined as in Eq. (15). The entities of the mobility matrix can be expressed as

$$m_{kj} = \omega \frac{Z_{jk}}{|Z|} \ , \tag{26}$$

where Z_{jk} is the adjoint of the flexibility matrix $[Z]$, which corresponds to the flexibility matrix element z_{kj}. Physically there is a difference in meaning between the mobility matrix and the inverse of the flexibility matrix; the mobility matrix is an operator to map the generalized velocity vector onto the generalized force space, while the inverse flexibility matrix is an operator to map the generalized coordinate vector onto the generalized force space. In our particular case, generalized coordinates and generalized velocities are related by a scalar multiplier ω, the frequency of excitation. It is immaterial which one is used in this analysis.

However, in order to avoid ever increasing confusion in terminology, the mobilities are used throughout the derivation.

We consider a system composed of an ℓ degree-of-freedom equipment subsystem, an m-ℓ degree-of-freedom support subsystem, and an n-m degree-of-freedom base subsystem. The explicit form of the set of simultaneous equations, Eq. (13), describing the total n degree-of-freedom dynamic system, is the following, when there is no force acting on the equipment subsystem:

$$
\begin{aligned}
\omega q_1 &= m_{1\,\ell+1}F_{\ell+1} + \cdots + m_{1\,m}F_m + \cdots + m_{1\,n}F_n \ , \\
\omega q_2 &= m_{2\,\ell+1}F_{\ell+1} + \cdots + m_{2\,m}F_m + \cdots + m_{2\,n}F_n \ , \\
&\vdots \\
\omega q_\ell &= m_{\ell\,\ell+1}F_{\ell+1} + \cdots + m_{\ell\,m}F_m + \cdots + m_{\ell\,n}F_n \ , \\
\omega q_{\ell+1} &= m_{\ell+1\,\ell+1}F_{\ell+1} + \cdots + m_{\ell+1\,m}F_m + \cdots + m_{\ell+1\,n}F_n \ , \\
&\vdots \\
\omega q_m &= m_{m\,\ell+1}F_{\ell+1} + \cdots + m_{m\,m}F_m + \cdots + m_{m\,n}F_n \ , \\
\omega q_{m+1} &= m_{m+1\,\ell+1}F_{\ell+1} + \cdots + m_{m+1\,m}F_m + \cdots + m_{m+1\,n}F_n \ , \\
&\vdots \\
\omega q_n &= m_{n\,\ell+1}F_{\ell+1} + \cdots + m_{n\,m}F_m + \cdots + m_{n\,n}F_n \ .
\end{aligned}
$$

In applying the resonance condition, we single out the part containing the generalized coordinates of the support subsystem from Eq. (27) and set them equal to zero; that is,

$$q_{\ell+1} = q_{\ell+2} = \cdots = q_m = 0 \ , \tag{28}$$

or

$$
\begin{aligned}
m_{\ell+1\,\ell+1}F_{\ell+1} + \cdots + m_{\ell+1\,m}F_m + \cdots + m_{\ell+1\,n}F_n &= 0 \ , \\
m_{\ell+2\,\ell+1}F_{\ell+1} + \cdots + m_{\ell+2\,m}F_m + \cdots + m_{\ell+2\,n}F_n &= 0 \ , \\
&\vdots \\
m_{m\,\ell+1}F_{\ell+1} + \cdots + m_{m\,m}F_m + \cdots + m_{m\,n}F_n &= 0 \ .
\end{aligned} \tag{29}
$$

Equation (29) is a set of simultaneous homogeneous algebriac equations. In general $n - \ell > m - \ell$; that means an infinite number of solutions can be developed for various given forces. In other words, the solution is indefinite. However, once again one may take advantage of

the fact that the applied forces are under our control during the measurement. If one limits the total number of forces applied, one at a time, to be equal to the degree of freedom of the support subsystem $m-\ell$, then a nontrivial solution exists. This solution requires that the determinant, consisting of the mobility entities associated with the applied forces, vanishes. For example, if all the forces are applied at the support points, then

$$\begin{vmatrix} m_{\ell+1\,\ell+1} & \cdots & m_{\ell+1\,m} \\ m_{\ell+2\,\ell+1} & \cdots & m_{\ell+2\,m} \\ & \vdots & \\ m_{m\,\ell+1} & \cdots & m_{m\,m} \end{vmatrix} = \Omega(\omega) \ . \tag{30}$$

It is equally true for any other combination of $m-\ell$ forces. In other words, the forces can be applied on the support subsystem, on the base subsystem, or on the combination of both. This liberal choice of application points offers great flexibility in application of this particular method.

It is important to show explicitly that Eq. (30) does contain information about the fixed-base natural frequencies of the equipment subsystem. Recall the matrix Eq. (14). The flexibility matrix and its inverse are commutative. The matrices in Eq. (14) may be partitioned and written in the following form:

$$\left[\begin{array}{c|c|c} [Z^e] & \{C_s^e\} & \{0\} \\ \hline \{C_s^e\}^T & [Z^s] & \{C_b^s\} \\ \hline \{0\} & \{C_b^s\}^T & [Z^b] \end{array}\right] \left[\begin{array}{c|c|c} m_{11/\omega} & \cdots & \cdots \\ \hline & \ddots & \\ \hline \cdots & \cdots & m_{nn/\omega} \end{array}\right] = \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & I \end{array}\right] .$$

If one replaces the inverse flexibility matrix by

$$\left|\begin{array}{c|c|c} [I] & m_{1\,\ell+1/\omega} \cdots & \{0\} \\ \hline \{0\} & \vdots & \{0\} \\ \hline \{0\} & \cdots\, m_{nm/\omega} & [I] \end{array}\right| ,$$

one has

$$\left[\begin{array}{c|c|c} [Z^e] & \{C_s^e\} & \{0\} \\ \hline \{C_s^e\}^T & [Z^s] & \{C_b^s\} \\ \hline \{0\} & \{C_b^s\}^T & [Z^b] \end{array}\right] \left[\begin{array}{c|c|c} [I] & m_{1\,\ell+1/\omega} \cdots & \{0\} \\ \hline \{0\} & \vdots & \{0\} \\ \hline \{0\} & \cdots\, m_{nm/\omega} & [I] \end{array}\right] = \left[\begin{array}{c|c|c} [Z^e] & \{0\} & \{0\} \\ \hline \{C_s^e\}^T & [I] & \{C_b^s\} \\ \hline \{0\} & \{0\} & [Z^b] \end{array}\right] . \tag{31}$$

Taking the determinant of both sides of Eq. (31) and applying the identical equation

$$|[A][B]| = |[A]|\,|[B]| \tag{32}$$

results in

$$|[Z]|\Omega(\omega) = [Z^e][Z^b]\omega^{m-\ell} ,$$

or

$$\Omega(\omega) = \frac{|[Z^e]|\ |[Z^b]|}{|[Z]|}\omega^{m-\ell} . \tag{33}$$

Equation (33) shows that $\Omega(\omega) = 0$ not only carries information about the fixed-base natural frequencies of the equipment subsystem, that $[Z^e] = 0$, but also the fixed-base natural frequencies of the base subsystem as well as the natural frequencies of the total system. These extraneous frequencies can be eliminated by constructing another determinant having an order of $m - \ell + 1$. It is done physically by adding another force applied at one of the equipment points. If we designate this determinant as $\Omega'(\omega)$, then

$$\begin{vmatrix} m_{\ell\,\ell} & \cdots & m_{\ell\,m} \\ \cdot & m_{\ell+1\,\ell+1} & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ m_{m\,\ell} & \cdots & m_{m\,m} \end{vmatrix} = \Omega'(\omega) .$$

Following the same procedure in deriving $\Omega(\omega)$, one has

$$\Omega'(\omega) = \frac{|[Z^{e'}]|\ |[Z^b]|}{|[Z]|}\omega^{m-\ell+1} . \tag{34}$$

In general, $|[Z^{e'}]|$ does not vanish at the fixed-base natural frequencies of the equipment subsystem unless the determinant $|[Z^{e'}]|$ and the determinant $|[Z^e]|$ do have common factors. This will be discussed later.

Now we are in position to construct the resonance function $\psi(m_{kj};\omega)$ in explicit form. This is defined as

$$\psi(m_{kj};\omega) = \left\| \frac{\Omega'(\omega)}{\Omega(\omega)} \right\| = \left\| \frac{|[Z^{e'}]|}{|[Z^e]|} \right\| \omega . \tag{35}$$

Here, the double line designates absolute value. The use of the absolute values in the resonance function evaluation essentially converts the zero-crossing points of function $|[Z^e]| = 0$

to minima. For a physical system, $|[Z^{e'}]|$ is finite for finite ω. The singularities of $\psi(m_{kj};\omega)$ corresponding to $|[Z^{e}]| = 0$ gives the fixed-base natural frequencies of the equipment subsystem.

As for the identification of the detected resonance frequencies concerned, the extraneous frequencies of the base subsystem and the total system are eliminated. As pointed out before, when there are decoupled subsystems within the equipment subsystem, ambiguity arises due to the fact that $|[Z^{e'}]|$ and $|[Z^{e}]|$ have common factors. In turn, the resonance function $\psi(m_{kj};\omega)$ does not reveal all the resonance frequencies as expected. The remedy for such ambiguities is to take one or two more force application points on the equipment subsystem for cross-check purposes.

Although the derivation of the resonance function is based on the assumption that the $m - \ell$ forces are applied to the supporting points, the actual locations of the application points of the forces are irrelevant as long as they are kept away from the equipment subsystem. The aforementioned fact is not obvious, because the extraneous frequencies induced in the derivation of the functions $\Omega(\omega)$ and $\Omega'(\omega)$ depend upon the force-application points. However, those induced extraneous frequencies are common factors of $\Omega(\omega)$ and $\Omega'(\omega)$. As a consequence, the resonance function remains unchanged. A mathematical proof for this liberal choice of force application points will be given in the next paragraph.

To prove the above statement of liberal choice of force-application points, we refer to Eq. (31), in which the matrix replaced the inverse flexibility matrix having its physical significance. In that matrix, one noticed that the mobility entities were replaced by unity or zero, except those lying in the $m - \ell$ columns. Those columns correspond to the locations where forces are applied and include the eigenvalue function $\Omega(\omega)$ for the resonance condition of the equipment subsystem under investigation. In this particular case, the resonance condition is

$$\bar{q}_{\ell+1} = \bar{q}_{\ell+2} = \cdots \bar{q}_m = 0$$

with forces applied at the supporting points $q_{\ell+1}, q_{\ell+2}, \cdots q_m$. Now suppose one of the application points, say q_w, is to be changed to a point on the base subsystem, say at q_v, where $\ell < w < m$, and $m < v < n$, the corresponding matrix change is as follows:

$$\left[\begin{array}{c|ccccc|c}
 & m_{1\,\ell+1/\omega} & \cdots & m_{1\,w/\omega} & \cdots & m_{1\,m/\omega} & \\
[I] & & & \cdots & & & \{0\} \\
\hline
\{0\} & & & \vdots & & & \{0\} \\
\hline
\{0\} & & & \cdots & & & [I] \\
 & m_{n\,\ell+1/\omega} & \cdots & m_{n\,m/\omega} & \cdots & m_{n\,w/\omega} &
\end{array}\right]$$

to

$$\left[\begin{array}{c|ccccccc|ccccc}
[I] & m_{1\,\varrho+1/\omega} & \cdots & m_{1\,w-1/\omega} & \cdots & m_{1\,w+1/\omega} & \cdots & m_{1\,m/\omega} & 0 & \cdots & m_{1\,v/\omega} & \cdots & 0 \\
 & \cdot & & \cdot & & \cdot & & \cdot & & & & & \\
\hline
\{0\} & & & & & \vdots & & & & & & & \\
\hline
\{0\} & & & & & & & & I & & & & \\
 & & & & & & & & & \ddots & & & \\
 & \cdot & & \cdot & & \cdot & & \cdot & & & 1 & & \\
 & & & & & & & & & & & m_{v\,v/\omega} & \\
 & m_{n\,\varrho+1/\omega} & \cdots & m_{n\,\omega-1/\omega} & \cdots & m_{n\,w+1/\omega} & \cdots & m_{n\,m/\omega} & & & & & I \\
 & & & & & & & & & & & & \ddots \\
 & & & & & & & & & & & & 1
\end{array}\right].$$

The determinant of this new matrix is also an eigenfunction $\Omega(\omega)$ corresponding to the same resonance condition as stated before, but with the difference that one of the applied forces is changed to a new location at q_v on the base subsystem instead of at q_w on the support subsystem. If one makes the corresponding change in Eq. (31) and designates the new matrix as $[V]$, one will have the following matrix equation:

$$[Z][V] = \left[\begin{array}{c|ccccccc|ccccc}
[Z^e] & & & & \{0\} & & & & & & \{0\} & & \\
\hline
\{C_s^e\}^T & 1 & 0 & \cdots & c_{\varrho+1\,v} & \cdots & 0 & & c_{\varrho+1\,\varrho+1} & \cdots & c_{\varrho+1\,v-1} & 0 & c_{\varrho+1\,v+1} \cdots \\
 & & 1 & & & & & & & & \vdots & & \\
 & & & \ddots & & & & & & & \cdot & & \\
 & & & & c_{w\,v} & & & & & & \cdot & & \\
 & & & & & \ddots & & & & & & & \\
 & & & & & & 1 & & & & & & \\
 & & & & & & & 1 & & & & & \\
\hline
\{0\} & 0 & \cdots & c_{m+1\,v} & \cdots & 0 & & & z_{m+1\,m+1} & \cdots & z^b_{m+1\,v-1} & 0 & z^b_{m+1\,v+1} \cdots \\
 & & & & & & & & \cdot & & & 1 & \\
 & & & & & & & & \cdot & & & & \\
 & & & & c_{n\,v} & & & & \cdot & & & 0 &
\end{array}\right]. \tag{36}$$

Taking the determinants on both sides of Eq. (36), one has

$$
\|[Z]\| \cdot \begin{vmatrix} m_{\varrho+1\ \varrho+1/\omega} & \cdots & m_{\varrho+1\ w-1/\omega} & m_{\varrho+1\ w+1/\omega} & \cdots & m_{\varrho+1\ m/\omega} & m_{\varrho+1\ v/\omega} \\ & & & \vdots & & & \\ m_{m\ \varrho+1/\omega} & \cdots & m_{m\ w-1/\omega} & m_{m\ w+1/\omega} & \cdots & m_{m\ m/\omega} & m_{m\ v/\omega} \end{vmatrix}
$$

$$
= \|[Z^e]\| \cdot \begin{vmatrix} c_{w\ v} & c_{w\ m+1} & \cdots c_{w\ v-1} & c_{w\ v+1} & \cdots c_{w\ n} \\ c_{m+1\ v} & c^b_{m+1\ m+1} & \cdots z^b_{m+1\ v-1} & z^b_{m+1\ v+1} & \cdots z^b_{m+1\ n} \\ & & \vdots & & \\ c_{v-1\ v} & z^b_{v-1\ m+1} & \cdots z^b_{v-1\ v-1} & z^b_{v-1\ v+1} & \cdots z^b_{v-1\ n} \\ c_{v+1\ v} & z^b_{v+1\ m+1} & \cdots z^b_{v+1\ v-1} & z^b_{v+1\ v+1} & \cdots z^b_{v+1\ n} \\ & & \vdots & & \\ c_{n\ v} & z^b_{n\ m+1} & \cdots z^b_{n\ v-1} & z^b_{n\ v+1} & \cdots z^b_{n\ n} \end{vmatrix} , \tag{37}
$$

or simply

$$
\Omega(\omega) = \frac{\|[Z^e]\|}{\|[Z]\|} \cdot \|[Z^{b'}]\|\, \omega^{m-\varrho} , \tag{38}
$$

where $[Z^{b'}]$ represents the matrix

$$
\begin{bmatrix} c_{w\ v} & c_{w\ m+1} & \cdots c_{w\ v-1} & c_{w\ v+1} & \cdots c_{w\ n} \\ c_{m+1\ v} & z^b_{m+1\ m+1} & \cdots z^b_{m+1\ v-1} & z^b_{m+1\ v+1} & \cdots z^b_{m+1\ n} \\ & & \vdots & & \\ c_{v-1\ v} & z^b_{v-1\ m+1} & \cdots z^b_{v-1\ v-1} & z^b_{v-1\ v+1} & \cdots z^b_{v-1\ n} \\ c_{v+1\ v} & z^b_{v+1\ m+1} & \cdots z^b_{v+1\ v-1} & z^b_{v+1\ v+1} & \cdots z^b_{v+1\ n} \\ & & \vdots & & \\ c_{n\ v} & z^b_{n\ m+1} & \cdots z^b_{n\ v-1} & z^b_{n\ v+1} & \cdots z^b_{n\ n} \end{bmatrix}
$$

The indices w and v are rather arbitrary within the limit of $\ell < w < m$, and $m < v < n$. This procedure can be repeated to interchange the force-application points between the support subsystem and the base subsystem as desired. When each additional application point of the forces is changed to another location, Eqs. (37) and (38) show that the difference in the eigenfunction $\Omega(\omega)$ is limited to the induced extraneous frequency function $[Z^b]$ alone, but not to the absolute value of the ratio of the determinant $|[Z^e]|$ and the determinant $|[Z]|$. This means that such exchange of force-application points will not alter the information of the fixed-base natural frequencies of the equipment subsystem. Nevertheless, the solutions of

$$|[Z^b]| = 0 \qquad \text{and} \qquad |[Z^{b'}]| = 0,$$

in general, will not be identical. Therefore the remaining task is to show that the extraneous frequency function, $|[Z^b]| = 0$, will remain unchanged if an additional force is applied to the equipment subsystem. Attention is called to the fact that the flexibility matrices $[Z^e]$ and $[Z^b]$ of the subsystems are merely artificial designations for the convenience of discussion. They are physically equivalent, in reality; choice of one or the other is dependent on which part of the system is of interest in the particular investigation. Because of this equivalence between the equipment and base subsystems, one may conclude, without repeating the mathematical manipulation, that one additional shaking point on the equipment subsystem will not change the determinant $|[Z^b]|$, but will change the determinant $|[Z^e]|$. This is the property being used in constructing the resonance function $\psi(m_{kj};\omega)$ to assure that all the induced extraneous frequencies are eliminated.

Damped System

The computer simulations in the "Example Problem" demonstrate the soundness of the developed fixed-base natural-frequency determination method for a one-dimensional multiple-foundation undamped system. It is of both theoretical and practical interest to ask whether this method can be used in studying damped systems. In order to answer this question, an analytical approach is necessary for two reasons: first, the extension of this method to apply to damped systems is not trivial; second, damping does exist in all physical systems. In other words, this analysis is intended to show the applicability and limitation of the fixed-base natural-frequency determination method on damped systems on the one hand, and the intrinsic properties of the damped system on the other. Either of the aforementioned interests requires proper understanding of the experimental results. Such understanding cannot be achieved merely from the measured dynamic responses. The exact manner of acquiring the test data sometimes is the determining factor to accomplish the purpose. It is known that the shake test has been used to measure impedance, mobility, and natural frequency of a mechanical system directly or indirectly through response measurements. Those quantities are immensely useful in structure analysis and design criteria where dynamic loadings are involved. The purpose of this analysis is to provide a better understanding of real complex systems, so that use of test data will be more confident.

Equation of Motion—A damped system involves dissipation of energy. In numerous experimentally justified cases, the dissipative forces in a mechanical system are indeed proportional to velocities and referred to as viscous damping. For such a system, one may use the Rayleigh dissipation function, which is defined as

$$R = \frac{1}{2}\underline{\dot{q}}[R]\{\dot{q}\} \tag{39}$$

where r_{ij} are the elements of the damping matrix $[R]$, which are real and positive.

The corresponding constitutive equation of motion in the Lagrangian formulation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = F_k(t) \tag{40}$$

or, written in explicit matrix representation of the governing differential equations,

$$[M]\{\ddot{q}\} + [R]\{\dot{q}\} + [K]\{q\} = \{F\} . \tag{41}$$

A great number of structures also exhibit energy loss during motion, with the dissipative forces proportional to displacements known as coulomb damping. It is equally possible that energy losses due to dissipative forces proportional to acceleration may have a sizable influence on the system in motion. However, these latter cases may be described by Eq. (41), if we regard the mass matrix [M] and the stiffness matrix $[K]$ as effective mass matrix and effective stiffness matrix accordingly. Therefore, Eq. (41) may be used to study damped linear time-invariant systems in general.

Closed Form Mathematical Solution—In an idealized shake test, a shaker providing a real sinusoidal force is assumed in a mathematical model. Such a force may be expressed as

$$F(t) = \overline{F} \sin \omega t.$$

The steady-state solution of the governing differential equation, Eq. (41), is supposed to exist, in which the whole system vibrates at the unique frequency ω of excitation. The responses can be expressed in the following general form:

$$\begin{aligned}\{q_k\} &= \{\bar{q}_k \sin(\omega t + \phi_{kj})\} \\ &= \{\bar{a}_k \cos \omega t + \bar{b}_k \sin \omega t\} ,\end{aligned} \tag{42}$$

where $\bar{a}_k = \bar{q}_k \cos \phi_{kj}$, $\bar{b}_k = \bar{q}_k \sin \phi_{kj}$, and in general $\phi_{kj} \neq n\pi$, $n = 0, 1, 2, \ldots$ for cases in which viscous damping is involved. It is evident that in a damped system, two components are needed to describe a single response, i.e., either $(\bar{q}_k, \phi_{kj})$ or $(\bar{a}_k, \bar{b}_k)$. Taking the time derivatives of Eq. (42) and substituting them into Eq. (41), one has

$$\begin{aligned}-\omega^2[M]\{\bar{a} \sin \omega t + \bar{b} \cos \omega t\} + \omega[R]\{\bar{a} \sin \omega t - \bar{b} \cos \omega t\} \\ + [K]\{\bar{a} \sin \omega t + \bar{b} \cos \omega t\} = \{\overline{F} \sin \omega t\} .\end{aligned} \tag{43}$$

Since sin ωt and cos ωt are independent, Eq. (43) may be written as

$$-\omega^2[M]\{\bar{a}\} - \omega[R]\{\bar{b}\} + [K]\{\bar{a}\} = \{\bar{F}\}, \tag{44}$$

$$-\omega^2[M]\{\bar{a}\} - \omega[R]\{\bar{b}\} + [K]\{\bar{a}\} = \{0\}. \tag{45}$$

By rearrangement of the terms in Eqs. (44) and (45), and writing them in a compact form, we have

$$\left[\begin{array}{c|c} [K] - \omega^2[M] & -\omega[R] \\ \hline \omega[R] & [K] - \omega^2[M] \end{array}\right] \begin{Bmatrix} \{\bar{a}\} \\ \{\bar{b}\} \end{Bmatrix} = \begin{Bmatrix} \{\bar{F}\} \\ \{0\} \end{Bmatrix}. \tag{46}$$

Theoretically, the problem is solved once a set of forces along with the physical properties $[M]$, $[K]$, and $[R]$ of the system are given. A closed-form solution can be worked out from Eq. (46) by any computer means. However, the application of Eq. (46) in practical measurement is not trivial. With this closed-form solution in mind, an approximate solution for a shake test will be developed in the next paragraph.

Approximate Solution For Shake Test—Equation (46) tells us: (a) when the system is undamped, i.e., when $[R] = [0]$, $\{\bar{a}\}$ and $\{\bar{b}\}$ are not coupled (the solution will be identical to the solution of Eq. (10) if $\{\bar{a}\}$ is replaced by $\{\bar{q}\}$; and (b) when the system is damped, or $[R] \neq [0]$, $\{\bar{a}\}$ and $\{\bar{b}\}$ are coupled. Then the phase factor becomes important, and a set of $2n$ linear simultaneous equations has to be solved. In general, the semi-analytical determination of fixed-base natural frequencies of such a system is complicated.

Since numerous small viscous damped mechanical structures do exist, it is well worth while to continue the investigation of these systems. In order to establish the justification to extend the developed method to such cases, we recall Eq. (46). In Eq. (46), one sees that when damping is small, the real part of the response $\{\bar{a}\}$ is weakly coupled with its corresponding imaginary part $\{\bar{b}\}$, and Eq. (44) approaches Eq. (10) as damping decreases towards zero. Consequently, the solution of the real part of Eq. (46), i.e., $\{\bar{a}\}$, also approaches the responses $\{\bar{q}\}$ of the undamped system in Eq. (10), which is real. Therefore for a slightly damped system, the solution may be approximated by using Eq. (46) for shake-test analysis. To be more specific, we rewrite Eq. (46) in the following form

$$\left[\begin{array}{c|c} [Z] & -\omega[R] \\ \hline \omega[R] & [Z] \end{array}\right] \begin{Bmatrix} \{\bar{a}\} \\ \{\bar{b}\} \end{Bmatrix} = \begin{Bmatrix} \{\bar{F}\} \\ \{0\} \end{Bmatrix}, \tag{47}$$

where $[Z]$ is the flexibility matrix of the very same system, but with no damping. Then the real part of the solution of Eq. (47) may be regarded as the approximate solution of a slightly damped system. To show this, we suppose that the system under study is not excessively damped; the real part of Eq. (47) may be approximated as the solution of the system. In solving for the responses in terms of applied forces

$$\begin{Bmatrix} \{\bar{a}\} \\ \{\bar{b}\} \end{Bmatrix} = [G] \begin{Bmatrix} \{\bar{F}\} \\ \{0\} \end{Bmatrix} , \tag{48}$$

where

$$[G] = \left[\begin{array}{c|c} [Z] & -\omega[R] \\ \hline \omega[R] & [Z] \end{array}\right]^{-1} . \tag{49}$$

The explicit expression of the real part is

$$\begin{aligned} \{\bar{a}\} &= \{\bar{q}\cos\phi\} \\ &= \left([I] + \omega^2[Z]^{-1}[R][Z]^{-1}[R]\right)^{-1}[Z]^{-1}\{\bar{F}\} \\ &= \left([I] + ([\mathfrak{M}][R])^2\right)^{-1}[Z]^{-1}\{\bar{F}\} \end{aligned} \tag{50}$$

where the matrix $\omega[Z]^{-1}$ is the mobility matrix $[\mathfrak{M}]$. When $[R]$ is small, the following approximation may be made:

$$\left([I] + ([\mathfrak{M}][R])^2\right)^{-1} \Rightarrow [I]^{-1} = [I] , \tag{51}$$

because the elements of the matrix $[\mathfrak{M}][R]$, $m_{ij}r_{jk}$, are much smaller than the damping matrix elements r_{ij}, which are in general small. Then Eq. (50) becomes

$$\{\bar{q}\cos\phi\} \cong [Z]^{-1}\{\bar{F}\} . \tag{52}$$

It is exactly the form of Eq. (14), except that the responses involve a phase factor $\cos\phi$. Therefore, the developed method can be applied to slightly damped system as well.

EXAMPLE PROBLEM AND DIGITAL COMPUTER SIMULATION

A specifically designed sample is in order to illustrate the applicability of the developed method. Also some comparisons were made to bring out the ambiguity which could arise due to negligence. A simple, but still general, undamped one-dimensional system of nine degrees of freedom as shown in Fig. 1 will be treated. The equations of motion for free vibration may be derived from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i} = 0 \qquad \text{where } L = T - V , \tag{E1}$$

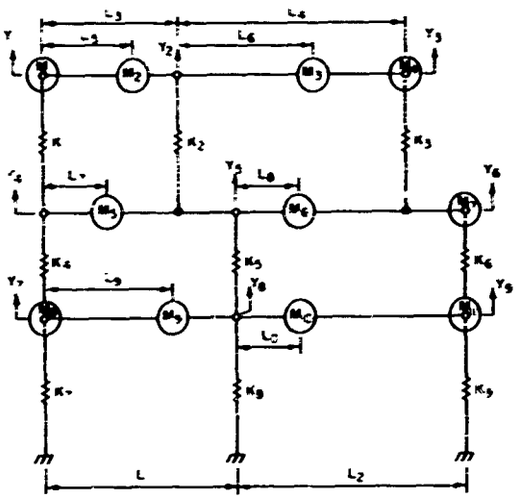

Fig. 1 - Configuration of the nine-degree-of-freedom, one-dimensional, undamped composite structure for digital computer simulation

$$T = \frac{1}{2}M_1\dot{y}_1^2 + \frac{1}{2}M_2\left[\dot{y}_2 + \frac{L_3 - L_5}{L_3}(\dot{y}_1 - \dot{y}_2)\right]^2 + \frac{1}{2}M_3\left[\dot{y}_3 + \frac{L_4 - L_6}{L_4}(\dot{y}_2 - \dot{y}_3)\right]^2$$

$$+ \frac{1}{2}M_4\dot{y}_3^2 + \frac{1}{2}M_5\left[\dot{y}_5 + \frac{L_1 - L_6}{L_1}(\dot{y}_4 - \dot{y}_5)\right]^2 + \frac{1}{2}M_6\left[\dot{y}_6 + \frac{L_2 - L_8}{L_2}(\dot{y}_5 - \dot{y}_6)\right]^2$$

$$+ \frac{1}{2}M_7\dot{y}_6^2 + \frac{1}{2}M_8\dot{y}_7^2 + \frac{1}{2}M_9\left[\dot{y}_8 + \frac{L_1 - L_9}{L_1}(\dot{y}_7 - \dot{y}_8)\right]^2$$

$$+ \frac{1}{2}M_{10}\left[\dot{y}_9 + \frac{L_2 - L_{10}}{L_2}(\dot{y}_8 - \dot{y}_9)\right]^2 + \frac{1}{2}M_{11}y_9^2 \ , \tag{E2}$$

$$V = \frac{1}{2}K_1(y_1 - y_4)^2 + \frac{1}{2}K_2\left[y_2 - \frac{L_1 - L_3}{L_1}(y_5 + y_4)\right]^2$$

$$+ \frac{1}{2}K_3\left[y_3 - \frac{L_2 - L_8}{L_2}(y_5 + y_6)\right]^2 + \frac{1}{2}K_4(y_4 - y_7)^2 + \frac{1}{2}K_5(y_5 - y_8)^2$$

$$+ \frac{1}{2}K_6(y_6 - y_9)^2 + \frac{1}{2}K_7y_7^2 + \frac{1}{2}K_8y_8^2 + \frac{1}{2}K_9y_9^2 \ . \tag{E3}$$

For this particular system, the kinetic energy T is coordinate independent and the potential energy is velocity independent; thus the Lagrangian equation has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial y_i}\right) + \frac{\partial V}{\partial y_i} = 0 \ . \tag{E4}$$

By differentiation of T and V, the equations of motion are:

$$\left[M_1 + M_2\left(\frac{L_3 - L_5}{L_3}\right)^2\right]\ddot{y}_1 + M_2\left(\frac{L_3 - L_5}{L_3}\right)\left(\frac{L_5}{L_3}\right)\ddot{y}_2 + K_1 y_1 - K_1 y_4 = 0,$$

$$M_2\left(\frac{L_3 - L_5}{L_3}\right)\left(\frac{L_5}{L_3}\right)\ddot{y}_1 + \left[M_2\left(\frac{L_5}{L_3}\right)^2 + M_3\left(\frac{L_4 - L_6}{L_4}\right)^2\right]\ddot{y}_2 + M_3\left(\frac{L_4 - L_6}{L_4}\right)\left(\frac{L_6}{L_4}\right)\ddot{y}_3$$

$$+ K_2 y_2 - K_2\left(\frac{L_1 - L_3}{L_1}\right)y_4 - K_2\left(\frac{L_1 - L_3}{L_1}\right)y_5 = 0,$$

$$M_3\left(\frac{L_4 - L_6}{L_4}\right)\left(\frac{L_6}{L_4}\right)\ddot{y}_2 + \left[M_3\left(\frac{L_6}{L_4}\right)^2 + M_4\right]\ddot{y}_3 + K_3 y_3 - K_3\left(\frac{L_2 - L_8}{L_2}\right)y_5$$

$$- K_3\left(\frac{L_2 - L_8}{L_2}\right)y_6 = 0,$$

$$M_5\left(\frac{L_1 - L_7}{L_1}\right)^2\ddot{y}_4 + M_5\left(\frac{L_1 - L_7}{L_1}\right)\left(\frac{L_7}{L_1}\right)\ddot{y}_5 - K_1 y_1 - K_2\left(\frac{L_1 - L_3}{L_1}\right)y_2$$

$$+ \left[K_1 + K_2\left(\frac{L_1 - L_3}{L_1}\right)^2 + K_4\right]y_4 + K_2\left(\frac{L_1 - L_3}{L_1}\right)^2 y_5 - K_4 y_7 = 0,$$

$$M_5\left(\frac{L_1 - L_7}{L_1}\right)\left(\frac{L_7}{L_1}\right)\ddot{y}_4 + \left[M_5\left(\frac{L_7}{L_1}\right)^2 + M_6\left(\frac{L_2 - L_8}{L_2}\right)^2\right]\ddot{y}_5 + M_6\left(\frac{L_2 - L_8}{L_2}\right)\left(\frac{L_8}{L_2}\right)\ddot{y}_6$$

$$- K_2\left(\frac{L_1 - L_3}{L_1}\right)y_2 - K_3\left(\frac{L_2 - L_8}{L_2}\right)y_3 + K_2\left(\frac{L_1 - L_3}{L_1}\right)^2 y_4$$

$$+ \left[K_2\left(\frac{L_1 - L_3}{L_1}\right)^2 + K_3\left(\frac{L_2 - L_8}{L_2}\right)^2 + K_5\right]y_5 + K_3\left(\frac{L_2 - L_8}{L_2}\right)^2 y_6 - K_5 y_8 = 0,$$

$$M_6\left(\frac{L_2 - L_8}{L_2}\right)\left(\frac{L_8}{L_2}\right)\ddot{y}_5 + \left[M_6\left(\frac{L_8}{L_2}\right)^2 + M_7\right]\ddot{y}_6 - K_3\left(\frac{L_2 - L_8}{L_2}\right)y_3$$

$$+ K_3\left(\frac{L_2 - L_8}{L_2}\right)^2 y_5 + \left[K_3\left(\frac{L_2 - L_8}{L_2}\right)^2 + K_6\right]y_6 - K_6 y_9 = 0,$$

$$\left[M_8 + M_9\left(\frac{L_1 - L_9}{L_1}\right)^2\right]\ddot{y}_7 + M_9\left(\frac{L_1 - L_9}{L_1}\right)\left(\frac{L_9}{L_1}\right)\ddot{y}_8 - K_4 y_4 + (K_4 + K_7)y_7 = 0,$$

$$M_9\left(\frac{L_1 - L_9}{L_1}\right)\left(\frac{L_9}{L_1}\right)\ddot{y}_7 + \left[M_9\left(\frac{L_9}{L_1}\right)^2 + M_{10}\left(\frac{L_2 - L_0}{L_2}\right)^2\right]\ddot{y}_8 + M_{10}\left(\frac{L_2 - L_0}{L_2}\right)\left(\frac{L_0}{L_2}\right)\ddot{y}_9$$

$$- K_5 y_5 + (K_5 + K_8)y_8 = 0,$$

(E5 Con't.)

$$M_{10}\left(\frac{L_2 - L_0}{L_2}\right)\left(\frac{L_0}{L_2}\right)\bar{y}_8 + \left[M_{10}\left(\frac{L_0}{L_2}\right)^2 + M_{11}\right]\bar{y}_9 - K_6 y_6 + (K_6 + K_9) y_9 = 0. \quad \text{(E5)}$$

We try a solution as the real part of

$$y_i = \bar{y}_i e^{i\omega t}, \qquad \text{where } \bar{y}_i = |\bar{y}_i| e^{i\phi}. \quad \text{(E6)}$$

From straightforward substitution of Eq. (E6) into Eq. (E5), the influence matrix $[Z]$ can be determined.

$$[Z] = \begin{pmatrix} Z_{11} & \cdots & Z_{13} \\ & \vdots & \\ Z_{91} & \cdots & Z_{99} \end{pmatrix}.$$

The elements of the $[Z]$ matrix are real, in this case, and symmetric.

$$Z_{11} = K_1 - \omega^2\left[M_1 + \left(\frac{L_3 - L_5}{L_3}\right)M_2\right],$$

$$Z_{12} = Z_{21} = -\omega^2 M_2\left(\frac{L_3 - L_5}{L_3}\right)\left(\frac{L_5}{L_3}\right),$$

$$Z_{13} = Z_{31} = Z_{15} = Z_{51} = Z_{16} = Z_{61} = Z_{17} = Z_{71} = Z_{18} = Z_{81} = Z_{19} = Z_{91} = 0,$$

$$Z_{14} = Z_{41} = -K_1,$$

$$Z_{22} = K_2 - \omega^2\left[M_2\left(\frac{L_5}{L_3}\right)^2 + M_3\left(\frac{L_4 - L_6}{L_4}\right)^2\right],$$

$$Z_{23} = Z_{32} = -\omega^2 M_3\left(\frac{L_4 - L_6}{L_4}\right)\left(\frac{L_6}{L_4}\right),$$

$$Z_{24} = Z_{42} = -K_2\left(\frac{L_1 - L_3}{L_1}\right),$$

$$Z_{25} = Z_{52} = -K_2\left(\frac{L_1 - L_3}{L_1}\right),$$

$$Z_{26} = Z_{62} = Z_{27} = Z_{72} = Z_{28} = Z_{82} = Z_{29} = Z_{92} = 0,$$

$$Z_{33} = K_3 - \omega^2\left[M_3\left(\frac{L_6}{L_4}\right)^2 + M_4\right],$$

$$Z_{34} = Z_{43} = Z_{37} = Z_{73} = Z_{38} = Z_{83} = Z_{39} = Z_{93} = 0,$$

$$Z_{35} = Z_{53} = -K_3\left(\frac{L_2 - L_8}{L_2}\right) ,$$

$$Z_{36} = Z_{63} = -K_3\left(\frac{L_2 - L_8}{L_2}\right) ,$$

$$Z_{44} = K_1 + K_2\left(\frac{L_1 - L_3}{L_1}\right)^2 + K_4 - \omega^2 M_5\left(\frac{L_1 - L_7}{L_1}\right)^2 ,$$

$$Z_{45} = Z_{54} = K_2\left(\frac{L_1 - L_3}{L_1}\right)^2 - \omega^2 M_5\left(\frac{L_1 - L_7}{L_1}\right)\left(\frac{L_7}{L_1}\right) .$$

$$Z_{46} = Z_{64} = Z_{48} = Z_{84} = Z_{49} = Z_{94} = 0 ,$$

$$Z_{47} = Z_{74} = -K_4 ,$$

$$Z_{55} = K_2\left(\frac{L_1 - L_3}{L_1}\right)^2 + K_3\left(\frac{L_2 - L_8}{L_2}\right)^2 + K_5 - \omega^2\left[M_5\left(\frac{L_7}{L_1}\right)^2 + M_6\left(\frac{L_2 - L_8}{L_2}\right)^2\right] ,$$

$$Z_{56} = Z_{65} = K_3\left(\frac{L_2 - L_8}{L_2}\right)^2 - \omega^2 M_6\left(\frac{L_2 - L_8}{L_2}\right)\left(\frac{L_8}{L_2}\right) ,$$

$$Z_{57} = Z_{75} = Z_{59} = Z_{95} = 0 ,$$

$$Z_{58} = Z_{85} = -K_5 ,$$

$$Z_{66} = K_3\left(\frac{L_2 - L_8}{L_2}\right)^2 + K_6 - \omega^2\left[M_6\left(\frac{L_8}{L_2}\right)^2 + M_7\right] ,$$

$$Z_{67} = Z_{76} = Z_{68} = Z_{86} = 0 ,$$

$$Z_{69} = Z_{96} = -K_6 ,$$

$$Z_{77} = K_4 + K_7 - \omega^2\left[M_8 + M_9\left(\frac{L_1 - L_9}{L_1}\right)^2\right] ,$$

$$Z_{78} = Z_{87} = -\omega^2 M_9\left(\frac{L_1 - L_9}{L_1}\right)\left(\frac{L_9}{L_1}\right) ,$$

$$Z_{79} = Z_{97} = 0 ,$$

$$Z_{88} = K_5 + K_8 - \omega^2\left[M_9\left(\frac{L_9}{L_1}\right)^2 + M_{10}\left(\frac{L_2 - L_0}{L_2}\right)^2\right] ,$$

$$Z_{89} = Z_{98} = -\omega^2 M_{10}\left(\frac{L_2 - L_0}{L_2}\right)\left(\frac{L_0}{L_2}\right) ,$$

$$Z_{99} = K_6 + K_9 - \omega^2\left[M_{10}\left(\frac{L_0}{L_2}\right)^2 + M_{11}\right] . \qquad \text{(E7)}$$

In calculating the natural frequencies, the numerical values of the physical constants are assigned:

$$M_1 = M_2 = M_3 = 1 ,$$

$$M_4 = M_5 = M_6 = 2 ,$$

$$M_7 = 1 ,$$

$$M_8 = M_9 = M_{10} = 3 ,$$

$$M_{11} = 4 ,$$

$$L_1 = 3 ,$$

$$L_2 = L_4 = 4 ,$$

$$L_3 = L_9 = 2 ,$$

$$L_5 = L_7 = L_8 = L_0 = 1 ,$$

$$L_6 = 2.5 ,$$

$$K_1 = K_2 = 1 ,$$

$$K_3 = K_4 = 1.5 ,$$

$$K_5 = K_6 = 1 ,$$

$$K_7 = 2 ,$$

$$K_8 = K_9 = 3 .$$

The natural frequencies of the total system are calculated by setting

$$|[Z]| = 0.$$

The results are:

$$\omega_1^T = 0.3631 ,$$

$$\omega_2^T = 0.4802 ,$$

$$\omega_3^T = 0.7801 ,$$

$$\omega_4^T = 0.9196 ,$$

$$\omega_5^T = 1.0266 .$$

$$\omega_6^T = 1.2656 \ ,$$

$$\omega_7^T = 1.4065 \ ,$$

$$\omega_8^T = 1.4890 \ ,$$

$$\omega_9^T = 2.1373 \ .$$

Separate cases will be taken into consideration.

Case 1 — General Illustration of the Applicability of the Proposed Method with Forces Applied at the Supporting Points

Define the subsystems:

y_1, y_2, and y_3 span the subspace of the equipment subsystem (Fig. 2),

y_4, y_5, and y_6 span the subspace of the support system,

y_7, y_8, and y_9 span the subspace of the base subsystem.

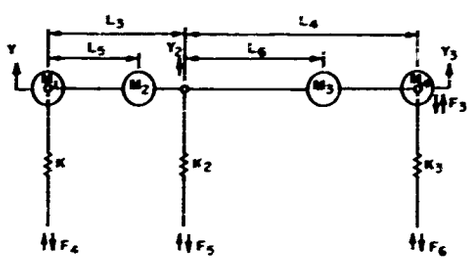

Fig. 2 - Configuration of the equipment substructure for illustration case 1, with forces applied at its supporting points y_4, y_5, and y_6 and at point y_3 on the equipment

In order to test the method, the fixed-base natural frequencies of the equipment subsystem are first calculated by setting:

$$|[Z^e]| = 0 = \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix} .$$

The results are:

$$\omega_1^e = 0.7816 \ ,$$

$$\omega_2^e = 0.8774,$$

$$\omega_3^e = 1.2794 \ .$$

Then apply the resonance condition

$$y_4 = y_5 = y_6 = 0,$$

with forces f_3, f_4, f_5, and f_6 applied, one at a time, at the equipment and supporting points y_3, y_4, y_5, and y_6 respectively. The resonance function $\psi(m_{ij};\omega)$ may be written as:

$$\psi(m_{ij};\omega) = \frac{\left\| \begin{vmatrix} m_{11} & m_{14} & m_{15} & m_{16} \\ m_{41} & m_{44} & m_{45} & m_{46} \\ m_{51} & m_{54} & m_{55} & m_{57} \\ m_{61} & m_{64} & m_{65} & m_{66} \end{vmatrix} \right\|}{\left\| \begin{vmatrix} m_{44} & m_{45} & m_{46} \\ m_{54} & m_{55} & m_{56} \\ m_{64} & m_{65} & m_{66} \end{vmatrix} \right\|}$$

where

$$m_{ij} = \omega \frac{Z_{ji}}{|[Z]|} = \frac{\omega y_{ij}}{f_j}$$

and Z_{ji} is cofector of z_{ij} in Z. The result of the digital computer simulation is shown in Fig. 3. The peaks occur right at the frequencies 0.786, 0.8774, and 1.2794.

Case 2 — Illustration of the Actual Physical Location of the Equipment Subsystem Being Immaterial

In this case, the attempt is made to show that the actual physical location of the equipment subsystem is immaterial, wherever response devices can be positively secured, and measurement can be taken. The systems in this case are redefined as:

y_4, y_5, and y_6 span the subspace of the equipment subsystem (Fig. 4),

y_1, y_2, y_3, y_7, y_8, and y_9 span the subspace of the support subsystem.

The rest belong to the base subsystem.

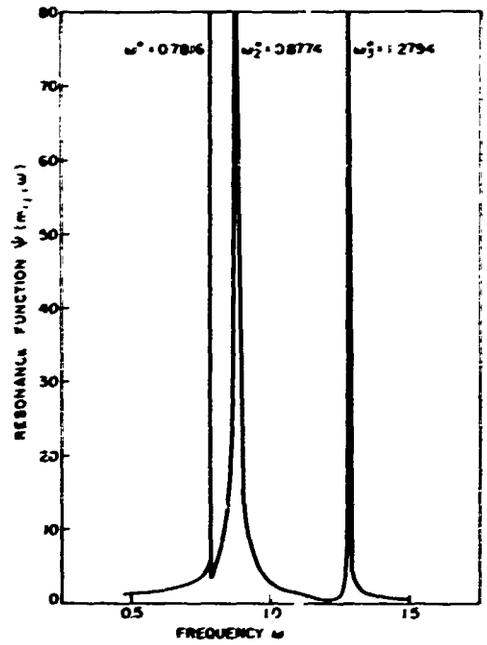

Fig. 3 - Resonance function—frequency plot for digital computer simulation of illustration case 1

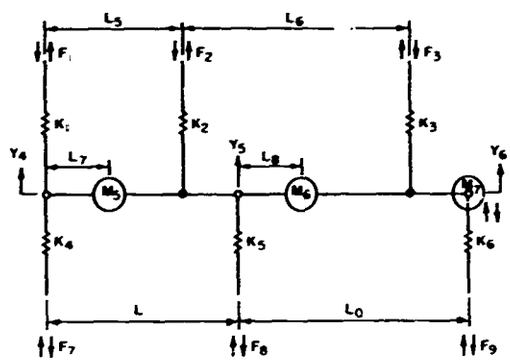

Fig. 4 - Configuration of the equipment substructure for illustration case 2, with forces applied at its supporting points y_1, y_2, y_3, y_7, y_8, and y_9, and at point y_6 on the equipment

Again solve for the fixed-base natural frequencies by:

$$|[Z^e]| = 0 = \begin{vmatrix} Z_{44} & Z_{45} & Z_{46} \\ Z_{54} & Z_{55} & Z_{56} \\ Z_{64} & Z_{65} & Z_{66} \end{vmatrix} .$$

The following listed are the calculated frequencies:

$$\omega_1^e = 1.0712 ,$$
$$\omega_2^e = 1.3015 ,$$
$$\omega_3^e = 1.9682 .$$

The resonance conditions are:

$$\bar{y}_1 = \bar{y}_2 = \bar{y}_3 = \bar{y}_7 = \bar{y}_8 = \bar{y}_9 = 0 \ ,$$

with forces f_1, f_2, f_3, f_7, f_8, f_9, and f_6 applied, each at a time, at the points y_1, y_2, y_3, y_7, y_8, y_9, and y_6 respectively.

The resonance function is:

$$\psi(m_{ij};\omega) = \frac{\begin{Vmatrix} m_{66} & m_{61} & m_{62} & m_{63} & m_{67} & m_{68} & m_{69} \\ m_{16} & m_{11} & m_{12} & m_{13} & m_{17} & m_{18} & m_{19} \\ m_{26} & m_{21} & m_{22} & m_{23} & m_{27} & m_{28} & m_{29} \\ m_{36} & m_{31} & m_{32} & m_{33} & m_{37} & m_{38} & m_{39} \\ m_{76} & m_{71} & m_{72} & m_{73} & m_{77} & m_{78} & m_{79} \\ m_{86} & m_{81} & m_{82} & m_{83} & m_{87} & m_{88} & m_{89} \\ m_{96} & m_{91} & m_{92} & m_{93} & m_{97} & m_{98} & m_{99} \end{Vmatrix}}{\begin{Vmatrix} m_{11} & m_{12} & m_{13} & m_{17} & m_{18} & m_{19} \\ m_{21} & m_{22} & m_{23} & m_{27} & m_{28} & m_{29} \\ m_{31} & m_{32} & m_{33} & m_{37} & m_{38} & m_{39} \\ m_{71} & m_{72} & m_{73} & m_{77} & m_{78} & m_{79} \\ m_{81} & m_{82} & m_{83} & m_{87} & m_{88} & m_{89} \\ m_{91} & m_{92} & m_{93} & m_{97} & m_{98} & m_{99} \end{Vmatrix}} \ .$$

The mobility elements are defined as before. The result of the digital computer simulation shows that the peaks occurring in the graphical representation truly coincide with the calculated values, which are shown in Fig. 5.

Case 3 — Illustrations of Frequency Shift Due to Change of Loading Condition and of Possible Ambiguity Due to Coupled (or uncoupled) Subsystems within the Equipment Subsystem

This case is designed with the intention of showing that any detachment, addition, or rearrangement of loading condition within the equipment subsystem could cause a large shift in the fixed-base natural frequencies of that subsystem. (Actually it will effect all the natural

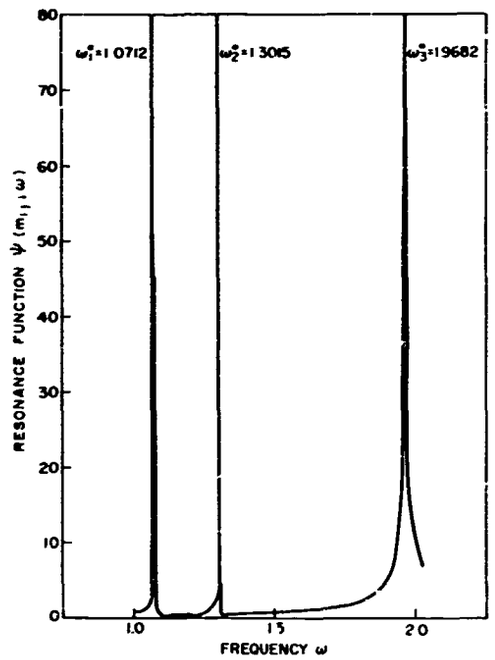

Fig. 5 - Resonance function—frequency plot for digital computer simulation of illustration case 2

frequencies of the total system. Here emphasis has been put on the equipment subsystem, because it is the main concern in this case.)

Everything remains the same as in Case 1, except that the mass M_2 which couples y_1 and y_2 is removed (Fig. 6). The fixed-base natural frequencies calculated by setting $M_2 = 0$ are:

$$\omega_1^e = 1.000 \ ,$$

$$\omega_2^e = 0.786 \ ,$$

$$\omega_3^e = 2.938 \ .$$

When M_2 is set equal to zero, it physically decouples y_1 and y_2, which causes the existence of uncoupled subsystems within the equipment subsystem. Caution must be exercised to clear the ambiguity expected by the theoretical analysis. In order to reveal all the fixed-base natural frequencies of this partially decoupled equipment subsystem, more than one point on the equipment must be shaken. More specifically, we start with the influence matrix of the equipment. When $M_2 = 0, Z_{12} = Z_{21} = 0$,

$$[Z^e] = \begin{pmatrix} Z_{11} & 0 & 0 \\ 0 & Z_{22} & Z_{23} \\ 0 & Z_{23} & Z_{33} \end{pmatrix} ,$$

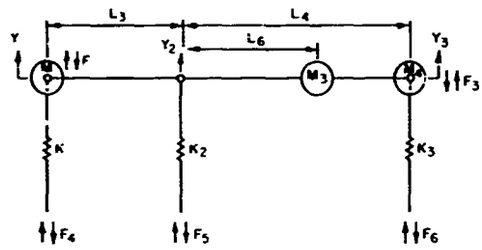

Fig. 6 - Configuration of the equipment substructure for illustration case 3, with forces applied at its supporting points y_4, y_5, and y_6, and at either point y_3 or point y_1 on the equipment

where

$$Z_{11} = K_1 - \omega^2 M_1 ,$$

$$Z_{13} = Z_{31} = 0 ,$$

$$Z_{22} = K_2 - \omega^2 M_3 \left(\frac{L_4 - L_6}{L_4}\right)^2 ,$$

$$Z_{23} = Z_{32} = -\omega^2 M_3 \left(\frac{L_4 - L_6}{L_4}\right)\left(\frac{L_6}{L_4}\right) ,$$

$$Z_{33} = K_3 - \omega^2 \left[M_3 \left(\frac{L_6}{L_4}\right)^2 + M_4\right] .$$

When the shaking point on the equipment is at y_3,

$$[Z^{e'}_{(y_3)}] = \begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix}$$

the resonance function becomes

$$\psi(m_{ij};\omega) = \left\| \frac{\Omega'(\omega)}{\Omega(\omega)} \right\| = \frac{\| \, |[Z'^{e}_{(y_3)}]| \, \|}{\| \, |[Z^e]| \, \|} \cdot \omega$$

$$= \frac{\|Z_{11} Z_{22}\| \cdot \omega}{\|Z_{11}(Z_{22}Z_{33} - Z_{23}^2)\|} = \frac{\|Z_{22}\|}{\|Z_{22}Z_{33} - Z_{23}^2\|} \cdot \omega$$

which will show peaks at the frequencies

$$\omega_2^e = 0.786 ,$$

$$\omega_3^e = 2.938 ,$$

and miss the one $\omega_1^e = 1.000$ (Fig. 7).

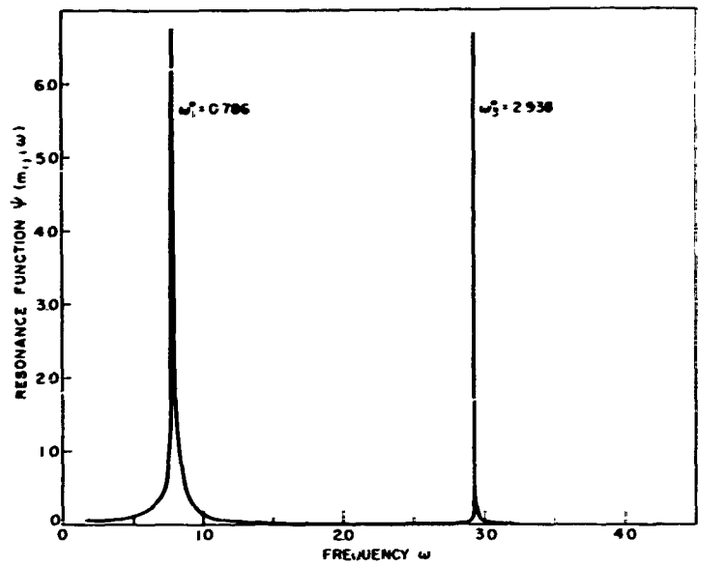

Fig. 7 - Resonance function—frequency plot for digital computer simulation of illustration case 3—no force applied at y_1

When the shaking point on the equipment is at y_2, the result will be the same as shaking at y_3, because in the resonance function, only the numerator changes from Z_{22} to Z_{33}, and it will not affect the frequency values corresponding to the peaks. However, if the shaking point on the equipment is at y_1, then

$$[Z'^{e}_{(y_1)}] = \begin{pmatrix} Z_{22} & Z_{23} \\ Z_{32} & Z_{33} \end{pmatrix} .$$

The resonance function in this case is:

$$\psi(m_{ij};\omega) = \frac{\omega \| \, \| [Z'^{e}_{(y_1)}] \| \, \|}{\| \, \| [Z^{e}] \| \, \|} = \frac{\omega \, \| Z_{22}Z_{33} - Z_{23}^2 \|}{\| Z_{11}(Z_{22}Z_{33} - Z_{23}^2) \|} = \frac{\omega}{\| Z_{11} \|} .$$

The corresponding computer simulation is shown in Fig. 8, which reveals only one of the fixed-base natural frequencies ω^e and loses the other two. The complete information has to be the combination of these two. The important points illustrated here are the following: first, the frequency shift due to change of loading condition (Fig. 9); and second, the possible ambiguity which may arise from a decoupled equipment subsystem. To the second point, it appears to be a good practice to take one or two more shaking points on the equipment for checking purposes.

Case 4 — Illustration of the Irrelevancy of the Physical Location of the Shaking Points and its Key Rule

This case is designed to show that the shaking points can be on the base points. With the same definition of the subsystems (or subspaces) in Case 1, the influence matrix of the

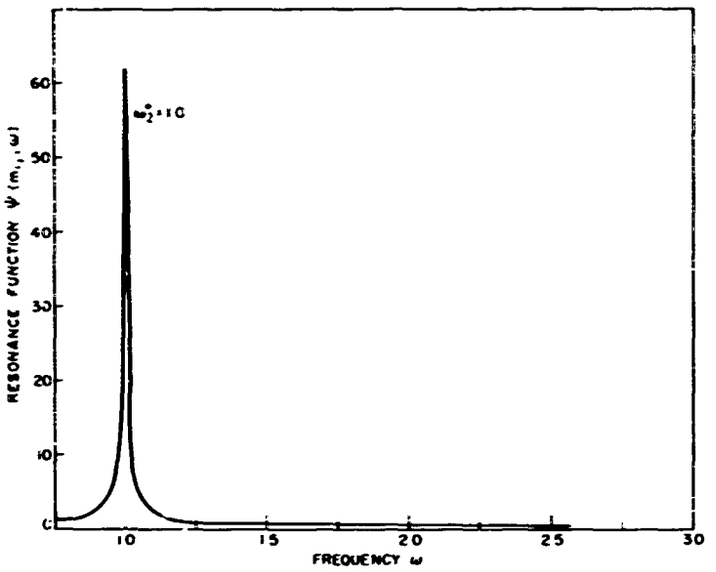

Fig. 8 - Resonance function—frequency plot for digital computer simulation of illustration case 3—no force applied at y_3

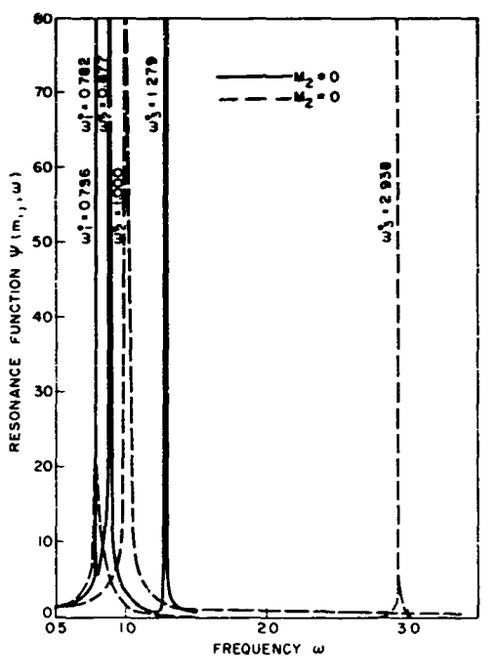

Fig. 9 - Resonance function—frequency plot for comparison of frequency shift—case 1 versus case 3

equipment subsystem remains the same. The theoretical analysis predicts that the resonance function is independent of application points of the forces although, in practice, one measures different mobility entities, i.e.,

$$\psi(m_{ij};\omega) = \frac{\left\| \begin{matrix} m_{11} & m_{17} & m_{18} & m_{19} \\ m_{41} & m_{47} & m_{48} & m_{49} \\ m_{51} & m_{57} & m_{58} & m_{59} \\ m_{61} & m_{67} & m_{68} & m_{69} \end{matrix} \right\|}{\left\| \begin{matrix} m_{47} & m_{48} & m_{49} \\ m_{57} & m_{58} & m_{59} \\ m_{67} & m_{68} & m_{69} \end{matrix} \right\|}.$$

This time the shaking points are at y_7, y_8, and y_9. The computer simulation of this case is shown in Fig. 10. One notices that they exactly coincide with the result obtained by applying forces at y_4, y_5, and y_6. The same result will be achieved by applying forces at the combination of support and base points. Duplication of another computer simulation will not be necessary. The key rules for the force application points are:

1. Keep the shaking points away from the equipment subsystem.

2. The number of shaking points is equal to the number of supporting points plus one.

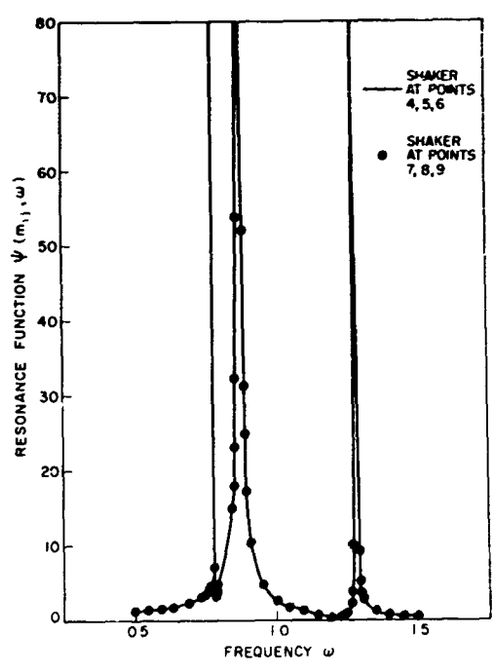

Fig. 0 - Resonance function—frequency plot for digital computer simulation of illustration case 4—irrelevancy of the physical locations of the shaking points

DISCUSSION

The theoretical analysis of this problem is rather general, as long as the dynamic system under consideration is undamped or slightly damped linear time-invariant. It should be understood that a reliable result in actual measurement can be expected only if the test is being carried out according to this condition. The proposed mechanical-resonance frequency-measurement method applied to a multiple-support equipment system is quite powerful, because the current difficulties—for instance, the requirement of force ratio of the shakers (both magnitude and phase), physical location of the shakers, and size of the object to be tested—are removed, or their limitations are lifted partially or entirely. The method suggests indirect mobility measurements through response measurements by shake test. The necessary measuring points are limited to only a few, namely the support points and one or two points on the equipment subsystem, depending on each individual case. Furthermore, devices and techniques to perform such tests are currently available, and results can be acquired in a rather routine manner.

The theoretical analysis has been proved correct by computer simulation. A vast group of mechanical systems and structures in reality can be treated by this analysis as if they are linear time-invariant. Therefore, there is no reason why it should not be feasible to determine the fixed-base natural frequencies semi-analytically. We have been talking about resonances, but in practice, we use the resonance properties only to deduce the fixed-base natural frequencies without actually exciting the resonance modes. It is a great advantage in determining the fixed-base natural frequencies, because in reality only in scarce cases may one excite a resonance mode of a complex subsystem unless "inner resonance" or "beating" exists. It is simple to illustrate: in the analysis of the text, we define the superstructure, base structure, and substructure on an arbitrary basis. There is no difference in reality between the superstructure and the substructure, both physically and mathematically. The so-called superstructure requires the base points to be stationary while it vibrates at its fixed-base natural frequency, but the substructure demands otherwise; therefore they interfere with each other. That is, the substructure prevents the superstructure from vibrating at its own fixed-base natural frequency. There is only one condition under which both the superstructure and the substructure have the common fixed-base natural frequency. This condition is what is called "inner resonance," or "beat" phenomena. It is also said that the system is degenerate.

SUMMARY AND CONCLUSION

It is realized that fixed-base natural-frequency analysis of a mechanical system is of importance in the dynamic design and dynamic behavior study. A resonance function has been derived for an n degree-of-freedom system to measure the fixed-base natural frequencies of the equipment subsystem. The analysis is based on a linear time-invariant system.

The resonance function is an explicit function of mobility entities, which is composed of intrinsic properties of the total system with vibrational frequency as a parameter. It is clear that as long as the dynamic distortion, or disturbance, remains within the linear limit, this function remains invariant with respect to its input, output, and time as well. This invariant feature, besides the advantageous facts pointed out in the discussion, makes the proposed method more desirable than currently available methods.

ACKNOWLEDGMENTS

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