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WAKE COLLAPSE IN A STRATIFIED FLUID: LINEAR TREATMENT

Harold W. Lewis

September 1971

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13. ABSTRACT

The problem of wake collapse in an incompressible, linearly stratified fluid with no boundaries is solved in the linear Boussinesq approximation.

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I. INTRODUCTION

In recent years, a number of authors have considered the problem of wake collapse in an incompressible, stratified fluid. These treatments have usually involved a version of the linear Boussinesq approximation, though some numerical work has appeared. While there seems to be a consensus about the phenomenology in the case of fully mixed wake (for which the linear approximation is, of course, invalid), it has seemed to us useful to treat a particularly simple version of the linear problem, for which an exact solution is obtainable. The results contain some surprises, and provide some insight into the limitations of the linear treatment.

II. BASIC EQUATIONS

We study a linearly stratified fluid, with no boundaries, whose unperturbed density $\rho_o(z)$ in the vicinity of the region of interest varies only in the vertical z-direction, and gradually enough to justify a linear approximation. Thus

$$\rho_o(z) \approx \rho_{oo} - \beta z \qquad (1)$$

where ρ_{oo} is a "mean" density for the problem. The initial value problem is obtained by perturbing the density slope inside a cylinder of radius a

$$\begin{aligned} \delta\rho(t=0) &= \epsilon z \qquad r < a \\ &= 0 \qquad r > a \end{aligned} \qquad (2)$$

where the fully mixed case can be obtained formally by setting $\epsilon = \beta$. This state of affairs is illustrated in Fig. 1. We are working in a two-dimensional coordinate system with axes z (vertical) and x (horizontal), with everything assumed independent of y. We will also use polar coordinates (r,α) in the plane of the problem, with the polar axis vertical.

Newton's second law, linearized, is

$$\rho \frac{\partial \underset{\sim}{v}}{\partial t} = - \underset{\sim}{\nabla} p - \rho g \underset{\sim}{\hat{z}} \qquad (3)$$

where v is the fluid velocity (in the x-z plane), p is the pressure, g the acceleration of gravity, and $\underset{\sim}{\hat{z}}$ a vertical unit vector. Since the fluid is assumed incompressible, we have

$$\text{div}\ \underset{\sim}{v} = \frac{\partial \rho}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\rho = 0 \tag{4}$$

which allows us to introduce the stream function $\underset{\sim}{\psi}$, a vector in the y-direction, defined by

$$\underset{\sim}{v} = \text{curl}\ \underset{\sim}{\psi} \tag{5}$$

The curl of Eq. 3 yields

$$\rho_{oo} \frac{\partial}{\partial t} (\text{curl curl}\ \underset{\sim}{\psi}) = - g\ \underset{\sim}{\nabla}\rho \times \underset{\sim}{z} \tag{6}$$

where the insertion of ρ_{oo} is usually called the Boussinesq approximation. It amounts to taking into account any density gradient that leads to a force (is multiplied by g), but no other. Differentiating Eq. 6 with respect to time leads, in view of Eq. 4, to

$$\frac{\partial^2}{\partial t^2} \nabla^2 \underset{\sim}{\psi} = - N^2 \frac{\partial^2 \underset{\sim}{\psi}}{\partial x^2} \tag{7}$$

where we have used the linear approximation, and have defined the Väisälä frequency

$$N^2 \equiv - \frac{g}{\rho_o} \frac{d\rho_o}{dz} \approx g\beta/\rho_{oo} \tag{8}$$

which we will treat as a constant hereafter. (Obviously, we could have chosen ρ_o exponential, instead of as given in Eq. 1.)

Clearly, in view of Eq. 4, $\delta\rho$ can be obtained from $\underset{\sim}{\psi}$, so that Eq. 7, with the initial condition Eq. 2, is our problem.

It begs for Fourier analysis, and the (exact) answer is

$$\delta\rho = \epsilon z a^2 \int_0^\infty k\,dk\, J_2(ka) \frac{J_1\left(\sqrt{k^2z^2 + (Nt \pm kx)^2}\right)}{\sqrt{k^2z^2 + (Nt \pm kx)^2}}. \quad (9)$$

where the $\pm$ means that the integral should be evaluated with each of the two signs, and the results averaged. The corresponding expression for ψ is obtained from Eq. 9 by multiplying the integrand by $2N/k\beta$, and taking the difference of integrals for the two signs. (There are other simpler ways to get the velocity distribution.)

It is worth reemphasizing that Eq. 9 is the exact solution to a linear problem.

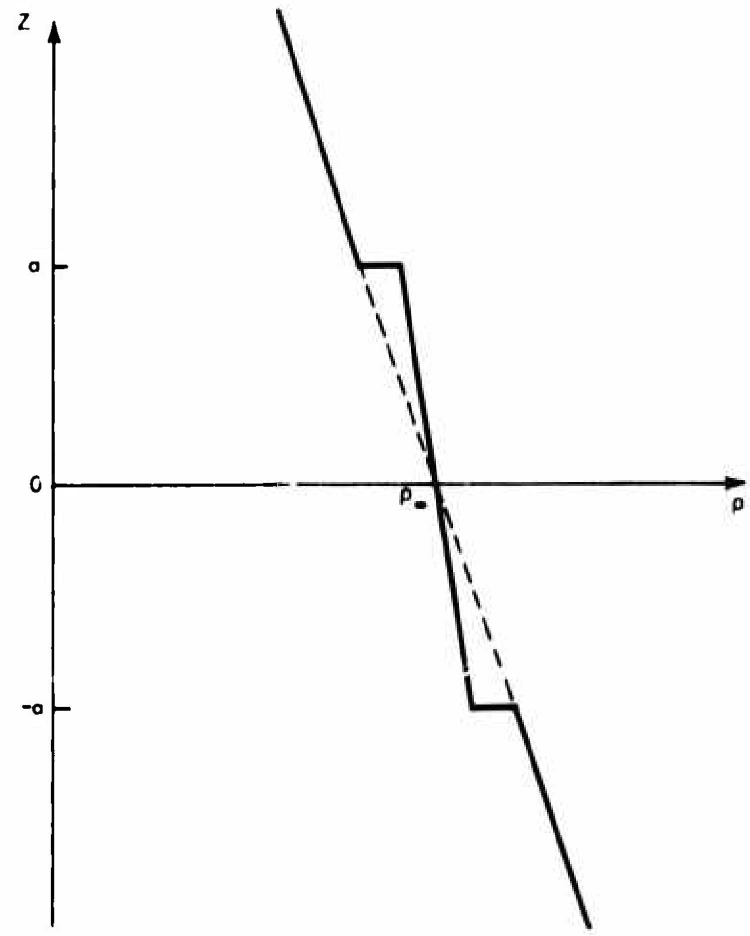

FIGURE 1. Initial Density Profile at x = 0

III. QUADRATURES

The evaluation of Eq. 9 is greatly aided by the observation that the expression in the integrand involving the square roots admits of an expansion in terms of Tschebysheff polynomials of the second kind. Explicitly, if $w^2 = u^2 + v^2 - 2uv \cos \gamma$,

then

$$\frac{J_1(w)}{w} = 2\sum_{m=0}^{\infty}(1 + m)\,\frac{J_{1+m}(u)}{u}\,\frac{J_{1+m}(v)}{v}\,A_m\,(\cos \gamma) \tag{10}$$

where

$$A_m(\cos \gamma) = \frac{\sin\left[(m+1)\gamma\right]}{\sin \gamma} \tag{11}$$

Thus, after a bit of arithmetic, we find

$$\delta\rho = \frac{2za^2\varepsilon}{r}\sum_{\ell=0}^{\infty}(-1)^{\ell}\,(2\ell+1)\,\frac{\cos\left[(2\ell+1)\alpha\right]}{\cos \alpha}\,\frac{J_{2\ell+1}(Nt)}{Nt}$$

$$\int_0^{\infty} J_2(ka)\,J_{2\ell+1}(kr)\,dk \tag{12}$$

Before proceeding to the final form, some observations about Eq. 12 are in order.

First and foremost, inside the original cylinder, where $r < a$, only the first term in Eq. 12 is non-vanishing. Thus, exactly,

$$\delta\rho = 2z\varepsilon \frac{J_1(Nt)}{Nt}; \quad r < a \tag{13}$$

The original linear (proportional to z) perturbation inside the cylinder remains linear, overshoots its correction, and finally damps out.

The fluid velocities for $r < a$ are most easily obtained from Eq. 13, by noting that $v_z = \frac{1}{\beta}\frac{\partial}{\partial t}(\delta\rho)$, so that

$$v_z = -\frac{2z\varepsilon}{\beta t} J_2(Nt) \tag{14a}$$

In view of the incompressibility condition (Eq. 4), we also have

$$v_x = \frac{2x\varepsilon}{\beta t} J_2(Nt) \tag{14b}$$

so that the fluid particles inside the original cylinder move on right hyperbolae, overshooting their ultimate positions on the first pass. The ultimate displacement of a fluid particle that starts inside the original cylinder is given by

$$\begin{aligned} \Delta x &= \frac{\varepsilon x}{\beta} \\ \Delta z &= -\frac{\varepsilon z}{\beta} \end{aligned} \tag{15}$$

so that the fluid particle moves just far enough down its own hyperbola to reach the level which is appropriate for it. In particular, the original circular cross section of the cylinder deforms into an ellipse of semi-major axis $a(1 + \frac{\varepsilon}{\beta})$ and semi-minor axis $a(1 - \frac{\varepsilon}{\beta})$, after oscillating around this shape.

We now turn to the behavior of the fluid for $r > a$, for which all terms <u>except</u> the first in Eq. 12 contribute. We need, for $r > a$, and for $\ell \geq 1$,

$$\int_0^\infty J_2(ka)J_{2\ell+1}(kr)\,dk = \frac{r}{4a^2}\left[P_{\ell-1} - 2P_\ell + P_{\ell+1} + \frac{P_{\ell-1} - P_{\ell+1}}{2\ell + 1}\right] \qquad (16)$$

where the P_ℓ are conventional Legendre polynomials, whose argument is $1 - \frac{2a^2}{r^2}$. The expression in square brackets can be written in other forms, using the many identities involving the Legendre polynomials. If we define

$$G_\ell(\xi) \equiv P_{\ell-1}(\xi) - 2P_\ell(\xi) + P_{\ell+1}(\xi) + \frac{P_{\ell-1}(\xi) - P_{\ell+1}(\xi)}{2\ell + 1}$$

$$= -2\left[(1 - \xi)\,P_\ell(\xi) - \frac{P_{\ell-1}(\xi) - P_{\ell+1}(\xi)}{2\ell + 1}\right] \qquad (17)$$

then the first two are given by

$$G_1\left(1 - \frac{2a^2}{r^2}\right) = \frac{4a^4}{r^4} \qquad (18a)$$

$$G_2\left(1 - \frac{2a^2}{r^2}\right) = \frac{12a^4}{r^4} - \frac{16a^6}{4^6}, \text{ etc.} \qquad (18b)$$

We have, then,

$$\delta\rho = \frac{z_\epsilon}{2}\sum_{\ell=1}^{\infty}(-1)^\ell(2\ell+1)\,\frac{\cos[(2\ell+1)\alpha]}{\cos\alpha}\,\frac{J_{2\ell+1}(Nt)}{Nt}\,G_\ell\left(1 - \frac{2a^2}{r^2}\right) \qquad (19)$$

$$= \frac{2z_\epsilon a^2}{r^2}\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\,\frac{\cos[(2\ell+1)\alpha]}{\cos\alpha}\,\frac{J_{2\ell+1}(Nt)}{Nt}\,H_\ell\left(1 - \frac{2a^2}{r^2}\right) \qquad (20)$$

$$r > a$$

where (again, with $\xi = 1 - \frac{2a^2}{r^2}$)

$$H_\ell(\xi) = -\frac{(2\ell+1)r^2}{4a^2} G_\ell(\xi)$$

$$= (2\ell+1)P_\ell(\xi) + \frac{r^2}{2a^2}\left(P_{\ell+1}(\xi) - P_{\ell-1}(\xi)\right) \tag{21}$$

It is now interesting to observe that, as $r \to a$, $H_\ell(\xi) \to (-1)^\ell(2\ell + 1)$, so that

$$\delta\rho(r\to a^+) = -2z\epsilon \sum_{\ell=1}^{\infty} \frac{\cos[(2\ell+1)\alpha]}{\cos\alpha} \frac{J_{2\ell+1}(Nt)}{Nt} (2\ell+1) \tag{22}$$

Comparing this with Eq. 13, we find an exact expression for the density discontinuity across the surface of the cylinder

$$\Delta\rho \equiv \delta\rho(r\to a^-) - \delta\rho(r\to a^+)$$

$$= 2z\epsilon \sum_{\ell=0}^{\infty} (2\ell+1) \frac{\cos[(2\ell+1)\alpha]}{\cos\alpha} \frac{J_{2\ell+1}(Nt)}{Nt} \tag{23}$$

This sum can be extended to $-\infty$, and carried out exactly. We find

$$\Delta\rho = \epsilon z \cos(Nt \sin\alpha) \tag{24}$$

an unexpected result. From Eqs. 24 and 13, we find

$$\delta\rho(r\to a^+) = \epsilon z \left[2\frac{J_1(Nt)}{Nt} - \cos(Nt \sin\alpha)\right] \tag{25}$$

which does not go to zero as $t \to \infty$, but which oscillates more and more rapidly as a function of α. Such behavior is indicative of an instability in a more realistic calculation, though the relation to the expected Rayleigh instability on the surface is unclear.

We turn finally to the behavior at large distances, for which we need the behavior of the H_ℓ for ξ near unity

$$H_\ell(\xi) \to -(2\ell+1)\, J_2\left[(2\ell+1)\frac{a}{r}\right] \quad ; \; r \gg a \tag{26}$$

in which we have also assumed $1 \ll \ell \ll \frac{r^2}{a^2}$ for convenience. It will be seen below that this is the range of interest for ℓ. This leads to

$$\delta\rho \to \frac{2\epsilon a^2}{r^2} \sum_{\ell=1}^{\infty} (-1)^\ell (2\ell+1) \frac{\cos[(2\ell+1)\alpha]}{\cos\alpha} \frac{J_{2\ell+1}(Nt)}{Nt} J_2\left[(2\ell+1)\frac{a}{r}\right] \tag{27}$$

(Recall, as always, that v_z can be obtained from $\delta\rho$ through $v_z = \frac{1}{\beta}\frac{\partial}{\partial t}(\delta\rho)$, and v_x from v_z through the incompressibility condition Eq. 4.)

For any given point (r,α) and time, the terms in the sum (Eq. 27) increase in magnitude with ℓ, and oscillate rapidly in sign, so it is appropriate to look for a "constant phase" value of ℓ as a means of estimating the sum. By either this method, or by a direct saddle-point integration in Eq. 9, we find that

$$\delta\rho \to \frac{2\epsilon z a^2}{r^2} \tan\alpha \sin(Nt\cos\alpha)\, J_2\left(\frac{Nta}{r}\sin\alpha\right) \tag{28}$$

for $r \gg a$. This represents a pulse of frequency $N\cos\alpha$, which passes a point (r, α) at a time $t \sim \frac{r}{Na\sin\alpha}$. The denominator is the group velocity of a wave of wave-number $\sim 1/a$, in the correct direction, where $N\cos\alpha$ is the frequency of such a wave. (Recall that the group and phase velocities of Väisälä waves are mutually orthogonal.)

IV. CONCLUSIONS AND CAVEATS

The behavior of the solution (Eq. 28) for large distances from the source contains no surprises, and represents the radiation from the source of the expected pulse of Väisälä waves, necessary to get rid of the energy stored in the initial perturbation.

On the other hand, the behavior near the perturbation has a number of unrealistic features. The rapidly oscillating (in space) behavior just outside the original cylinder at long times cannot appear in the solution of a realistic hydrodynamic problem. Since the solution is exact, this difficulty must be ascribed to the model. In addition, the velocity of the fluid displays an infinite shear on the surface of the cylinder, and this too, in a realistic problem, would lead to Rayleigh instability. It would seem foolhardy, therefore, to set $\epsilon = \beta$, and use a linear treatment to study the fully mixed wake collapse problem.