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STRESSES, DEFORMATIONS AND PROGRESSIVE FAILURE OF NON-HOMOGENEOUS FISSURED ROCK

Semi-annual Technical Report: August 30, 1971

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The objective of this research program is to develop computer programs, using the finite element method, to predict stresses and deformations in the vicinity of underground excavations. The computer programs should allow for arbitrary initial stresses in rock, arbitrary shape and size of the opening, any given sequence of construction, non-homogeneous material properties, interaction of rock with supporting structures, progressive damage and time dependent deformation and load development on supporting structure. Limited experimental work to verify key points in the theory is planned.

Research during the first year is directed towards literature survey, selection of mathematical models for behavior of rock and development of computer programs for elastic-plastic, elastic-brittle rock and for progressive failure of rock around underground openings.

At this reporting, selection of mathematical models has been completed. The report presents finite element computer programs for i) Plane Strain Analysis of Elastic-Plastic Mohr-Coulomb Materials and ii) Two-Dimensional Analysis of a No Tension System. Plans for further requirements of these programs in line with research objectives are discussed.

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Technical Report Summary

The objective of this research program is to develop computer programs, using the finite element method, to predict stresses and deformations in the vicinity of underground excavations. The computer programs will have the capability to allow for arbitrary initial stresses in rock, arbitrary shape and size of the opening, any given sequence of construction, nonhomogeneous material properties, interaction of rock with supporting structures, progressive damage, and time dependent deformation and load development on supporting structure. Limited experimental work to verify key points in the theory is planned.

Research during the first year is directed towards survey of literature on the subject, selection of mathematical models for mechanical behavior of rock, and development of computer programs for elastic-plastic Mohr-Coulomb materials, for brittle rock following Griffith's theory, and for progressive deformation and fracture of rock around underground openings under stress changes associated with excavation.

At this reporting, selection of mathematical models for elastic-plastic Mohr-Coulomb materials and for elastic-brittle materials failing according to Griffith theory has been completed. Chapter 1 of the report describes the theoretical considerations leading to the model selected. The stress-strain relations for incremental or rate type theory of plasticity are generally based on the normality rule and convexity and regularity of the yield surface in a 'stress-space'. Using these concepts, various investigators have proposed conflicting constitutive equations. In this report the elastic-plastic behavior of materials has been re-examined as a mathematical generalization of observations on a one-dimensional test. The role of kinematic constraints upon yield conditions has been studied and adequacy of certain postulates examined. Current theories of elastic-plastic behavior are found to be inadequate as it is, in general, not possible to satisfy the 'normality' rule as well as continuity of stress path under plane strain conditions. Further research into this aspect of material behavior is needed to clear the air. Experimental phase of the research program is being planned with this requirement in view. In the mathematical model selected as the basis for development of computer programs, a modification of the yield surface is introduced to eliminate discontinuity in stress paths. For behavior of elastic-brittle rock, the model selected assumes elements of rock to be incapable of supporting tensile and shearing forces across a crack. A review of literature showed errors in similar formulations by other investigators. These have been corrected in the present development. The mathematical models of elastic-plastic and elastic-brittle rock have been incorporated into finite element computer programs for analyses of stresses and deformations of plane strain systems. Chapters III and IV of the report present two computer codes along with relevant description, instructions for usage and illustrative examples for:

i. Plane Strain Analysis of Elastic-Plastic Mohr-Coulomb Materials

ii. Two-Dimensional Analysis of a Non-Tension System

Further work on these computer programs is continuing. However, even in their present form, program capabilities include consideration of arbitrary initial stresses, arbitrary shape of openings with or without linings, and considerable variation in material properties. These computer programs should be of immediate application to a variety of problems.

Adequate mathematical models of rock behavior have been chosen. The finite element method has been used successfully to develop computer codes for analysis of complex problems of stresses, deformations and fracture in rock. The method appears to be suitable for further development to realize the objectives of the current research program.

Experimental work so far has been directed towards development of suitable laboratory material (exhibiting elastic-plastic behavior). No equipment has so far been purchased under the contract. However, procurement of a plane strain testing machine has been initiated. It is expected to be received in September 1971.

PREFACE

The terrestrial crust is in a complex state of stress. Underground excavations in this stressed medium profoundly influence the distribution of stress which in turn determines the stability of the opening and of the rock in the vicinity. Traditional methods based upon the theory of linear elastic solids are inadequate. It is necessary that the sequence of construction and realistic material properties be taken into account in calculation of stresses and deformations in rock.

The objective of the present research program is development of finite element techniques to predict stresses and deformations in the vicinity of underground excavations allowing for arbitrary initial stresses in the rock, arbitrary shape and size of the opening and any given sequence of construction. The procedures will allow for nonhomogeneous material properties, interaction of rock with supporting structures, progressive damage, time dependent deformation and load development on supporting structures. Limited experimental work to verify key points in the theory is also planned. The entire program is expected to extend over three years.

This is the first semi-annual progress report covering the period 2/1/71 to 7/31/71. The main activity in this period has been a review of the work done by other investigators in mathematical simulation of mechanical behavior of elastic-plastic solids and of jointed rock. Models of stress-strain behavior have been proposed and work on development of relevant computer programs started. A sequential approach has been followed whereby a basic program is coded and then modified to include all the ramifications of material behavior and actual loading sequences. Two computer programs, viz.

i. Plane Strain Analysis of Elastic-Plastic Mohr-Coulomb Materials

ii. Two-Dimensional Analysis of a No-Tension System

are included. The present capabilities of each program are indicated in the program descriptions. Further development on all these is continuing and will be included in future reports.

The work is supported by the U.S. Government through the Advanced Research Projects Agency, ARPA, and its agent the Bureau of Mines, Department of the Interior. At the Ohio State University the work is under direct supervision of Professors T.H. Wu, R.S. Sandhu, and J.R. Hooper. Messrs. S.W. Huang, R.D. Singh, C.W. Chang and T. Chang, graduate students in the Department of Civil Engineering, have contributed to the research reported. Dr. William Karwoski of the Spokane Mining Research Center, Spokane, Washington is the Project Officer designated by the sponsor. In early stages Dr. Syd Peng of Twin Cities Mining Research Center, Twin Cities, Minnesota acted as the Project Officer.

The opinions, findings and conclusions expressed in the report are those of the authors and not necessarily those of the U.S. Bureau of Mines, Department of the Interior or the Advanced Research Project Agency.

R. S. Sandhu
Project Supervisor

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CHAPTER I

THEORETICAL CONSIDERATIONS

Chapter I. Theoretical Considerations

1.1. Mechanical Behavior of Rock

Figs. 1-1 and 1-2 show, respectively, typical stress-strain plots for a granite and a marble (Swanson 1970). Upon loading the stress-strain curve is almost linear and reversible over a short portion. Unloading from higher loads does not coincide with initial loading. This characteristic along with rate independence distinguishes elastic-plastic behavior. Reloading closely follows unloading until the previous maximum is reached; whereupon the original curve is followed. This leads to some simplifying assumptions.

i. A yield point exists below which the material is linear elastic.

ii. The yield point corresponds to the maximum stress level previously attained.

iii. Unloading and reloading paths are linear, coincident and parallel to the initial elastic loading curve.

Fig. 1-3 shows this simplification. Clearly the yield point can be described by the permanent or irrecoverable strain or the area bounded by the loading curve, the unloading curve and the horizontal axis. Whereas in generalization to the three-dimensional case, the stresses and strains become second rank tensors and are, therefore, unordered, the area is still a scalar product and retains its ordering characteristics. To this extent, it is often preferred as a measure of the elastic limit.

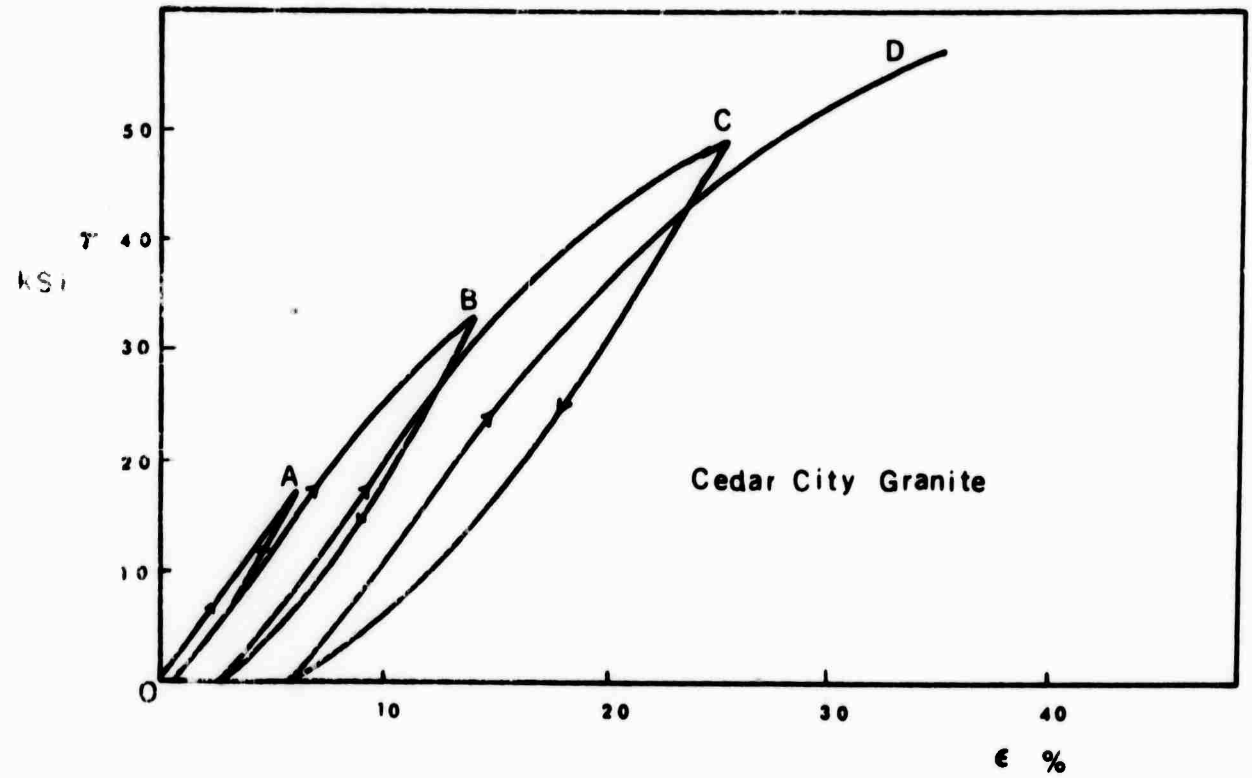


Figure 1-1. Stress-Strain Curves for Cedar City Granite

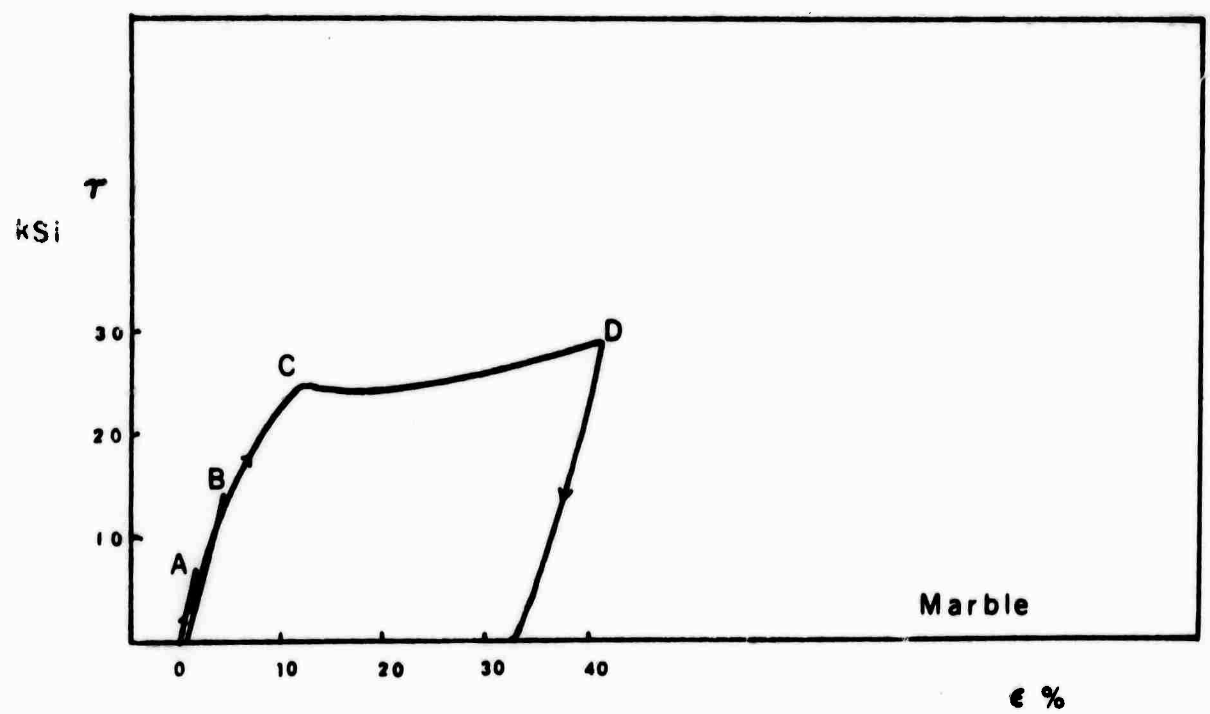


Figure 1-2. Stress-Strain Curves for Marble

Mechanical behavior of rock under polyaxial state of stress has been examined in the light of brittle failure theories (Brace, 1964; Bieniawski, 1967, 1969; Brady, 1969, 1970). Four regions of behavior are identified in Figure 1-4. The first region corresponds to closure of pre-existing open cracks and is peculiar to compressional loading. In region II material behavior is linear elastic. Fracture initiation occurs near the end of this region in accordance with Griffith or modified Griffith Theory. This stage also corresponds to onset of nonlinearity in the stress to volumetric strain curve (Brace, 1966). Stable fracture propagation characterizes region III. In region IV, unstable fracture propagation results in strength failure and rupture. Differences in loading and unloading behavior are observed (Walsh, 1965).

We have, thus, two general approaches to the characterization of stress-strain behavior of rock. One follows the theory of elastic-plastic solids without consideration of micro-mechanics of the system. The other uses Griffith theory or modifed Griffith theory to relate deformation and failure to initiation and propagation of fracture. It has been observed (Swanson 1970) that Mohr-Coulomb failure law applies for moderate values of confining pressure and that at low confining pressures, failure is by rupture. Contrary to plastic behavior, strength of material drops to almost zero in the direction normal to the crack if rupture theory is followed. Figs. 1-5 and 1-6 depict typical relationships of failure strength and post failure behavior in relation to confining pressures. It is reasonable to assume that the material is linear elastic upto yield or rupture , as the case may

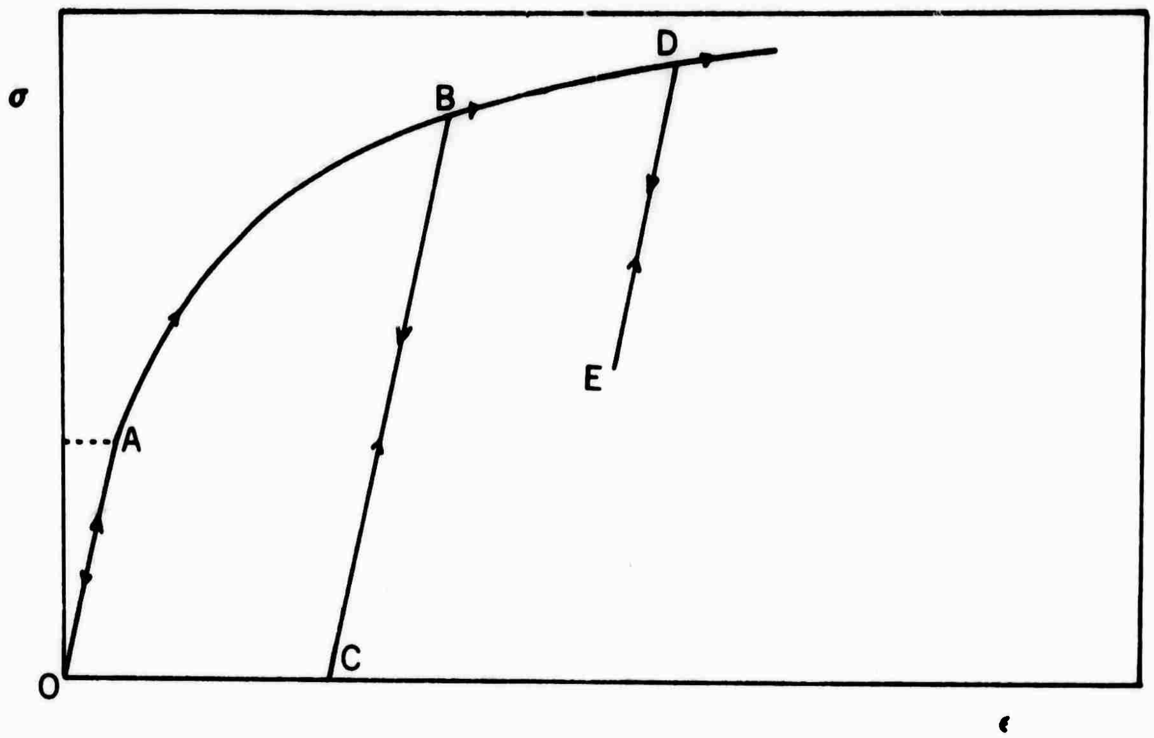


Figure 1-3. Idealization of Elastic-Plastic Stress-Strain Behavior for Rocks

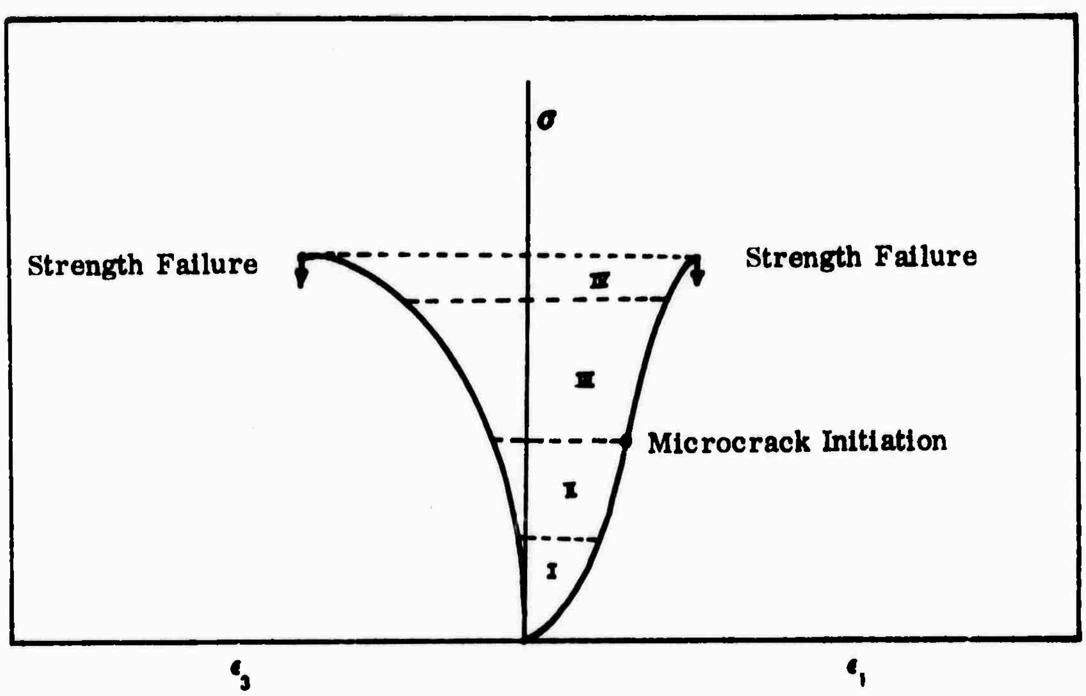
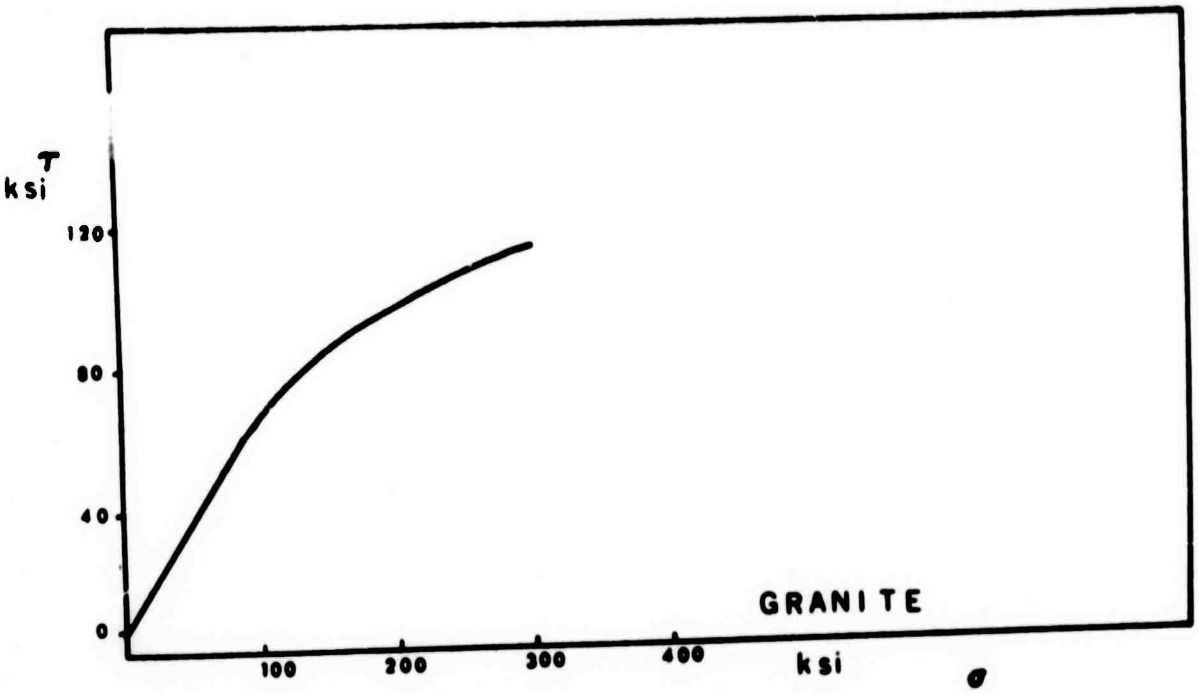


Figure 1-4. Typical Axial and Lateral Stress-Strain Behavior of Brittle Rock



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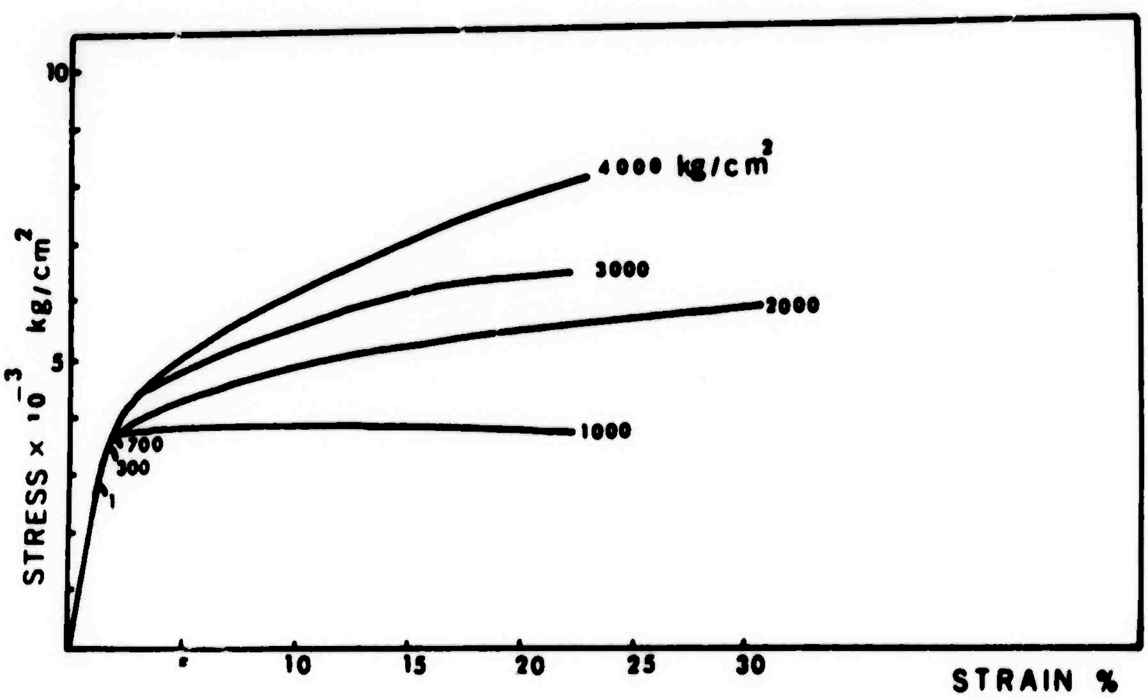


Figure 1-6. Effect of Confining Pressure on Stress-Strain Behavior of Limestone

be, and that the post-failure behavior only is governed by the theory used to define failure. Both the elastic-plastic Mohr-Coulomb failure theory and the Griffith theory have been used in the course of present research to develop computer programs for analysis of stress and deformation in rock.

1.2 Stress-Strain Relations for Elastic-Plastic Solids

Several approaches have been used for formulation of elastic-plastic behavior. Excellent presentations of the theory are available in literature (Drucker, 1951; Naghdi, 1960; Green and Naghdi, 1965; Koiter, 1953; Hill, 1950). Specializations to Mohr-Coulomb solids under plane strain (Drucker and Prager, 1952; Drucker, Gibson and Henkel, 1955; Reyes, 1965; Reyes & Deere, 1966) have been presented. Not intending to survey the entire range of different philosophies, we present briefly a discussion of elastic-plastic behavior under kinematic constraints leading to the plane strain formulation for Mohr-Coulomb materials used in the computer program in Chapter III. We present a generalization of observations on the conventional uniaxial test to the case of three-dimensions before discussing the role of constraints.

An isothermal system undergoing infinitesimal deformations is of interest to the present report. Extension to more comprehensive situations is direct.

1.2.1. A Generalization of One-Dimensional Test Results to Three-Dimensional Theory

Analogous to the case of one-dimensional test, we assume the existence of a set D consisting of all admissible states of stress. D is convex,* has boundary B and includes the original or reference state. Clearly D is six-dimensional (assuming symmetric stress tensors, i.e. absence of body couples) and is contained in the six-dimensional linear vector space V spanned by the six components σ_{kl} of the stress tensor. The stress in uniaxial test is given by a real number and in this case D is ordered and convex. To introduce an ordering in the six-dimensional stress space so that 'increase' and 'decrease' of stress are meaningful, a mapping g is defined

$$g: \ V \longrightarrow R \tag{I-1}$$

with the following properties

i. The image of D under g is a positive interval $I \subset R$.

ii. Image of complement of D in V is the complement of I in R and g maps the interior D_I of D to the interior of I. (I-2)

iii. g maps the 'original' or 'reference' state to zero $\in I$.

Then D is ordered by its image i.e. for $\nu_1, \nu_2 \in D$, if $g(\nu_1) = g_1$, $g(\nu_2) = g_2$, ν_2 is greater than, equivalent**to, or less than ν_1 depending upon g_2 being greater than,

---

*Convexity of D is the property that $\nu_1, \nu_2 \in D \Rightarrow \alpha\nu_1 + (1-\alpha)\nu_2 \in D \ \forall \alpha \in [0,1]$. In literature, there are frequent references to a convex yield surface. This is inaccurate. It is easy to see that convexity of D does not imply convexity of its boundary B. Indeed, B is in general not convex.

**The term equivalent is used because g, in general, is not one to one. Thus, ν_1, ν_2 may be distinct while their images coincide.

equal to or less than g_1. The mapping preserves convexity of D. The interval $I = [o,f]$ where f is the image of boundary B of D.

In one-dimensional tests, the set of admissible stress states may include negative points. The limiting stress states in tension and compression provide a positive supremum and a negative infimum to this set. The boundary B of this set is clearly discontinuous. In a multi-dimensional stress space, B may be continuous. To ensure correspondence between the one-dimensional and multi-dimensional cases, g is a two to one mapping in the case of one-dimensional loading. Thus the image of B is in all cases the supremum of non-negative interval I.

The boundary B of D and hence its image f under g are defined by prior deformation and load history. Considering components ϵ_{kl} of the strain tensor, on the analogy of the results of one-dimensional test, a plastic strain tensor with components ϵ''_{kl} is defined such that

i. For a given ϵ''_{kl}, there is one to one correspondence between elements of D and a set of points in the six-dimensional space spanned by $\epsilon'_{kl} = \epsilon_{kl} - \epsilon''_{kl}$. ϵ'_{kl} are identified as components of the elastic strain tensor.

(I-3)

ii. If a generalization of Prandtl's simplifying assumption is admitted, the one to one correspondence between $\sigma_{kl} \in D$ and ϵ'_{kl} is independent of prior deformation history, and for stress states defined by interior D_I of D, ϵ''_{kl} vanish.

A positive measure of history of deformation can be defined in various ways. If plastic strain is used as representative of deformation history, a mapping χ on H the six-dimensional linear vector space spanned by components ε''_{kl} of the plastic strain tensor is introduced

$$\chi : \quad H \longrightarrow P \tag{I-4}$$

where P is the positive class of real numbers. Other measures using bilinear or nonlinear maps involving both the stress and plastic strain components are in use. An example is*

$$\chi(\sigma_{kl}, \varepsilon''_{kl}) = \int_{\tau=-\infty}^{t} \sigma_{kl}(\tau)\, d\varepsilon''_{kl}(\tau) \tag{I-5}$$

In all cases the objective is to define a positive number k such that it equals the image f under g of boundary B of D. For elastic-perfectly plastic solids, k is constant but, in general, for stable material, k is a monotone increasing function of history of deformation and stress. In certain cases, the mapping g may also vary with plastic deformation. This happens when kinematic constraints are present. Theory of kinematic hardening is an instance in which the reference point in D, having image zero in I, depends upon stress and deformation history.

Considering, for the present discussion,** $\chi(\varepsilon''_{ij}) = k$,

$$g(\sigma_{kl}) \leq f = k = \chi(\varepsilon''_{kl}) \tag{I-6}$$

---

*Here and in subsequent work, standard indicial notation is used. Summation on repeated indices is implied unless otherwise indicated.

**A more general assumption uses $\chi = \chi(\kappa, \varepsilon''_{ij})$ where $\kappa = \int_{\tau=-\infty}^{t} \sigma_{ij}(\tau)\, d\varepsilon''_{ij}(\tau)$. In that case $\dot{\chi} = \frac{\partial \chi}{\partial \kappa}\dot{\kappa} + \frac{\partial \chi}{\partial \varepsilon''_{ij}} \dot{\varepsilon}''_{ij}$.

In the interior of D, $\epsilon''_{kl} = 0$, $g(\sigma_{kl}) < f = k$ and

$$\sigma_{kl} = E_{kl}(\epsilon'_{mn})$$

$$= E_{kl}(\epsilon_{mn}) \qquad \text{(I-7)}$$

For differential changes in stress and strain components, using a superposed dot to indicate differential quantities,

$$\dot{\sigma}_{kl} = E_{klmn}\,\dot{\epsilon}'_{mn}$$

$$= E_{klmn}\,\dot{\epsilon}_{mn} \qquad \text{(I-8)}$$

assuming that E_{kl} is sufficiently smooth and its derivative E_{klmn}, a tensor of fourth rank, exists.

On the boundary B of D,

$$g(\sigma_{kl}) = f = k = \chi(\epsilon''_{kl}) \qquad \text{(I-9)}$$

For g, χ sufficiently smooth in their arguments

$$\dot{\chi} = \dot{\chi}(\epsilon''_{kl}) = h_{kl}\,\dot{\epsilon}''_{kl} \qquad \text{(I-10)}$$

$$\dot{g} = \dot{g}(\sigma_{kl}) = q_{kl}\,\dot{\sigma}_{kl} \qquad \text{(I-11)}$$

In the case of elastio-perfectly plastic solids, $h_{kl} = 0$ and arbitrary plastic straining can occur for $\dot{\chi} = 0$ i.e. $\chi = k$, a positive constant. Also $g = f = k$ requires $\dot{g} = 0$ leading to the relationship

$$q_{kl}\dot{\sigma}_{kl} = 0 \qquad \text{(I-12)}$$

Equation I-12 requires the stress changes to be in a plane tangent to the hyperplane defined by $g(\sigma_{kl}) = k$. However, for $h_{kl} \neq 0$, for nonvanishing $\dot{\epsilon}''_{kl}$, $\dot{\chi} > 0$ and $\dot{g} = \dot{f} > 0$. This is termed loading and $\dot{g} = \dot{f} = \dot{k} = \dot{\chi}$. For vanishing $\dot{\epsilon}''_{kl}$, $\dot{\chi} = 0$ and once again equation I-12 applies. This is the case of neutral loading. In all cases $\dot{g} < 0$ implies decreasing load. This is the case of unloading and equation I-8 applies with $\varepsilon''_{kl} = 0$.

1.2.2 Evaluation of Incremental Plastic Strain in Loading

Equations I-10 and I-11 suggest a relationship of the type

$$\dot{\varepsilon}''_{kl} = S_{klmn}\,\dot{\sigma}_{mn} \tag{I-13}$$

where S_{klmn} may depend upon ε''_{mn}, σ_{mn}. Resolving $\dot{\sigma}_{mn}$ into components along the boundary B and normal to it, the plastic strain is due only to the normal component. Prager (1949) showed that

$$S_{klmn} = S_{klij}\,\frac{\left(\frac{\partial g}{\partial \sigma_{ij}}\right)}{\left(\frac{\partial g}{\partial \sigma_{pq}}\right)\left(\frac{\partial g}{\partial \sigma_{pq}}\right)} \cdot \left(\frac{\partial g}{\partial \sigma_{mn}}\right)$$

$$= B_{kl}\,\frac{\partial g}{\partial \sigma_{mn}} \tag{I-14}$$

independent of $\dot{\sigma}_{mn}$. Hence direction of ε''_{ij} is independent of the direction of stress change given by $\dot{\sigma}_{mn}$. Other relations for plastic strain increments have been proposed. Using a thermo-dynamic postulate, Drucker (1951) obtained the normality rule

$$\varepsilon''_{ij} = \lambda\,\frac{\partial g}{\partial \sigma_{ij}} \quad \text{at } g = f \tag{I-15}$$

where λ is a positive scalar which for rate independence must be homogeneous of order one in $\dot{\sigma}_{kl}$.

Drucker used the normality rule to evaluate λ. Defining

$$\dot{\epsilon}''_e = \left[\tfrac{1}{2}\,\dot{\epsilon}''_{ij}\,\dot{\epsilon}''_{ij}\right]^{\frac{1}{2}} \tag{I-16}$$

$$\lambda = \frac{\dot{\epsilon}''_e}{\left[\tfrac{1}{2}\,\dfrac{\partial g}{\partial \sigma_{ij}}\,\dfrac{\partial g}{\partial \sigma_{ij}}\right]^{\frac{1}{2}}} \qquad \text{at } g = f \tag{I-17}$$

Now defining

$$\sigma_e = \left[\tfrac{1}{2}\,S_{ij}\,S_{ij}\right]^{\frac{1}{2}} \tag{I-18}$$

where

$$S_{ij} = \sigma_{ij} - \delta_{ij}\,\frac{\sigma_{kk}}{3} \tag{I-19}$$

and writing $\dfrac{\dot{\sigma}_e}{\dot{\epsilon}''_e}$, the slope of the σ_e, ϵ''_e curve as H, equation I-17 gives, with the normality rule

$$\dot{\epsilon}''_{ij} = \frac{\dfrac{\partial g}{\partial \sigma_{ij}}}{H\left[\tfrac{1}{2}\,\dfrac{\partial g}{\partial \sigma_{mn}}\,\dfrac{\partial g}{\partial \sigma_{mn}}\right]^{\frac{1}{2}}}\,\dot{\sigma}_e$$

$$= \frac{\dfrac{\partial g}{\partial \sigma_{ij}}\;S_{kl}\;\dot{S}_{kl}}{2\,\sigma_e\,H\left[\tfrac{1}{2}\,\dfrac{\partial g}{\partial \sigma_{mn}}\,\dfrac{\partial g}{\partial \sigma_{mn}}\right]^{\frac{1}{2}}} \tag{I-20}$$

This formulation was used in the so-called tangent modulus methods e.g. Swedlow and Yang (1965).

Hill used the normality rule assuming λ to be a fourth rank tensor linear in $\dot{\sigma}_{kl}$ and introduced a plastic potential. Using normality as well as the condition $\dot{g} = \dot{f} = \dot{k} = \dot{\chi}$ on the boundary B of D, Prager (1949) obtained

$$\lambda = \frac{\dfrac{\partial g}{\partial \sigma_{ij}}\;\dot{\sigma}_{ij}}{\dfrac{\partial \chi}{\partial \epsilon''_{mn}}\;\dfrac{\partial g}{\partial \sigma_{mn}}} \tag{I-21}$$

This formulation breaks down for elastic-perfectly solids where χ is independent of ϵ''_{mn}. Felippa (1966) obtained λ in terms of $\dot{\epsilon}_{ij}$, increment in the total strain tensor. In this procedure

$$\dot{\sigma}_{ij} = \left[E_{ijkl}\, \delta_{km}\, \delta_{ln} - E_{ijkl}\, L_{klmn} \right] \dot{\epsilon}_{mn} \tag{I-22}$$

where

$$L_{klmn} = \left[\frac{\partial \chi}{\partial \epsilon''_{ij}} \frac{\partial g}{\partial \sigma_{ij}} + \frac{\partial g}{\partial \sigma_{ij}} E_{ijpq} \frac{\partial g}{\partial \sigma_{pq}} \right]^{-1} \frac{\partial g}{\partial \sigma_{rs}} E_{rskl} \frac{\partial g}{\partial \sigma_{mn}} \tag{I-23}$$

This approach is valid for all cases including perfect plasticity and was used by Zienkiewicz, Valliappan and King (1969) in developing finite element procedures.

Using rate of work equations, it is possible to evaluate λ in terms of stress rates for materials of von Mises or Mohr-Coulomb type. Yamada (1968) used this approach for finite element analysis of von Mises solids. Using Drucker and Prager's generalization of Mohr-Coulomb law, Reyes (1965) developed the stress-strain equation for generalized Mohr-Coulomb elastic-perfectly plastic solids under plane strain conditions. The finite element procedures presented by Reyes and Deere (1966), Baker, Sandhu and Shieh (1969) and those included in Chapter III of this report were based on these equations. For plane strain

$$\begin{Bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{12} \end{Bmatrix} = 2G \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{Bmatrix} \dot{\epsilon}_{11} \\ \dot{\epsilon}_{22} \\ \dot{\gamma}_{12} \end{Bmatrix} \tag{I-24}$$

where

$$D_{11} = 1 - h_2 - 2\, h_1\, \sigma_{11} - h_3\, \sigma_{11}^2$$

$$D_{22} = 1 - h_2 - 2\, h_1\, \sigma_{22} - h_3\, \sigma_{22}^2$$

$$D_{33} = \tfrac{1}{2} - h_3 \sigma_{12}$$

$$D_{12} = D_{21} = - h_2 - h_1 \sigma_{22} - h_1 \sigma_{11} - h_3 \sigma_{22} \sigma_{11} \qquad \text{(I-25)}$$

$$D_{13} = D_{31} = -h_1 \sigma_{12} - h_3 \sigma_{12} \sigma_{11}$$

$$D_{23} = D_{32} = - h_1 \sigma_{12} - h_3 \sigma_{12} \sigma_{22}$$

and

$$h_1 = \frac{\frac{3}{2} \quad \alpha \quad \frac{K}{G} - \frac{\sigma_{kk}}{6 J_2^{\frac{1}{2}}}}{J_2^{\frac{1}{2}} \; (1 + 9 \alpha^2 \frac{K}{G})}$$

$$h_2 = \frac{\alpha - \frac{\sigma_{kk}}{6 J_2^{\frac{1}{2}}} \quad 3\alpha \frac{K}{G} - \frac{\sigma_{kk}}{3 J_2^{\frac{1}{2}}}}{(1 + 9\alpha^2 \frac{K}{G})} - \frac{3\nu f K}{J_2^{\frac{1}{2}}} \quad \frac{1}{E(1 + 9\alpha^2 \frac{K}{G})}$$

$$h_3 = \frac{1}{2 J_2^{\frac{1}{2}} (1 + 9\alpha^2 \frac{K}{G})}$$

$$J_2 = \tfrac{1}{2} S_{ij} S_{ij}$$

E, K, G = elastic Young's modulus, bulk modulus and shear modulus, respectively.

$$\alpha = \frac{\tan \phi}{\sqrt{9 + 12 \tan^2 \phi}}$$

ϕ = the angle of internal friction

$$f = \alpha \frac{\sigma_{kk}}{3} + J_2^{\frac{1}{2}}$$

1.2.3. Kinematic Constraints

Plane strain conditions impose a kinematic constraint upon the deforming solid. In relation to elastic-plastic behavior, a consequence is that the yield sur-

surface from the elastic side and plastic side do not, in general, coincide (Baker et.al.1969). Consider the deformation of a body undergoing deformation. F, the set of all admissible deformation, is contained in the six-dimensional vector space S spanned by components ϵ_{kl} of the strain tensor. A kinematic constraint can be written as

$$C\,(\epsilon_{kl}) = 0 \qquad \text{(I-26)}$$

and the admissible deformation is restricted to the intersection of F with the hyperplane in S defined by equation I-26. If several constraints are present, the admissible deformation is restricted to

$$\bigcap_{i=1}^{n} \left[C_i\,(\epsilon_{kl}) = 0 \right] \qquad \text{(I-27)}$$

As the multiple intersection reduces the dimension of the vector space by n, it is clear that n cannot exceed six.

Consider a single constraint. In differential form the equation is

$$C_{kl}\,\dot{\epsilon}_{kl} = 0 \qquad \text{(I-28)}$$

where coefficients C_{kl} depend upon ϵ_{mn}.

As elastic-plastic behavior is studied with reference to loading paths in the stress space V, it is necessary that kinematic constraints be rewritten as constraints on stress. Here, for no plastic strain, we simply use the inverse of equation I-8 to write equation I-27 as

$$C_{kl}\,C_{klmn}\,\dot{\sigma}_{mn} = 0 \qquad \text{(I-29)}$$

or

$$G_{mn}\,\dot{\sigma}_{mn} = 0 \qquad \text{(I-30)}$$

where $C_{klmn} = \left[E_{klmn} \right]^{-1}$ (I-31)

and $G_{mn} = C_{kl}\ C_{klmn}$ (I-32)

For the case of not all of $\dot{\epsilon}''_{kl}$ vanishing, two alternative procedures are available. Using definition of elastic strain tensor,

$$\dot{\epsilon}_{kl} = \dot{\epsilon}'_{kl} + \dot{\epsilon}''_{kl} \tag{I-33}$$

If T_{klmn} is the inverse of S_{klmn} in equation I-13

$$\dot{\epsilon}_{kl} = C_{klmn}\,\dot{\sigma}_{mn} + T_{klmn}\,\dot{\sigma}_{mn} \tag{I-34}$$

$$= \left[I_{klpq} + T_{klmn}\ E_{mnpq} \right] \dot{\epsilon}'_{pq} \tag{I-35}$$

$$= K_{klpq}\ \dot{\epsilon}'_{pq} \tag{I-36}$$

where I_{klpq} is a fourth rank identity tensor. Thus the constraint is expressed by

$$C_{kl}\,\dot{\epsilon}_{kl} = C_{kl}\ K_{klpq}\ \dot{\epsilon}'_{pq} \tag{I-37}$$

$$= C_{kl}\ K_{klpq}\ C_{pqmn}\,\dot{\sigma}_{mn} \tag{I-38}$$

$$= L_{klmn}\,\dot{\sigma}_{mn} = 0 \tag{I-39}$$

where $L_{klmn} = C_{kl} \left[I_{klpq} + T_{klrs}\ E_{rspq} \right] C_{pqmn}$ (I-40)

An alternative procedure is to use the normality rule and to satisfy the constraint both upon loading and unloading i.e. $C_{kl}\,\dot{\epsilon}'_{kl} = 0 = C_{kl}\,\dot{\epsilon}''_{kl}$. Then the first equation is identical with equation I-29 but the second equation gives

$$C_{kl}\ \lambda \frac{\partial g}{\partial \sigma_{kl}} = 0 \tag{I-41}$$

Equation I-41 may or may not coincide with equation I-39. Equations I-29 and I-39 have linear relationship between incremental stresses and describe hyperplanes tangent to any loading path in the stress space V. As the two equations are in general different, there is a slope discontinuity in the stress path as plastic straining begins upon reaching the boundary B of D. We note in particular that proportional stress paths in V may not be possible in the presence of kinematic constraints. In the case of linear elasticity, let equation I-30 define a plane passing through the origin in V. A proportional loading path lying in this plane is possible upto the point of intersection with boundary B. Beyond that, upon loading, stress path has to be in the surface determined by Equation I-39 and this will in general be non-planar. If loading is continued to a certain point along this surface, unloading therefore will be along a path lying in a plane parallel to the original loading plane but different from it and not passing through the origin. Thus unloading to initial state is impossible. This corresponds to setting up of residual stresses corresponding to kinematic constraints.

Specifically considering plane strain conditions and elastic-perfectly plastic Mohr-Coulomb material, the mapping g from the set of all admissible stress states to the positive interval $[o, f]$ is

$$g(\sigma_{ij}) = \alpha \frac{\sigma_{kk}}{3} + J_2^{\frac{1}{2}} \qquad \text{(I-42)}$$

$$J_2 = \frac{1}{2} \sigma_{ij} \sigma_{ij} - \frac{\sigma_{kk}^2}{6} \qquad \text{(I-43)}$$

Linear isotropic elasticity implies, for $\dot{\varepsilon}''_{ij} = 0$

$$\varepsilon_{ij} = \varepsilon'_{ij} = \frac{1}{2G} \sigma_{ij} - \frac{\nu \delta_{ij}}{E} \sigma_{kk} \qquad \text{(I-44)}$$

Plane strain condition implies

$$\varepsilon_{13} = \varepsilon_{23} = \dot{\varepsilon}_{33} = 0 \qquad \text{(I-45)}$$

From equations I-44 and I-45

$$\sigma_{13} = \sigma_{23} = 0 \tag{I-46}$$

$$\sigma_{33} = \nu (\sigma_{11} + \sigma_{22}) \tag{I-47}$$

For $\dot{\epsilon}''_{ij}$, not all vanishing, using normality rule

$$\dot{\epsilon}''_{ij} = \lambda \left[\frac{\alpha}{3} \delta_{ij} + \frac{\sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}}{2 J_2^{\frac{1}{2}}} \right]$$

$$\frac{\alpha}{3} + \frac{\sigma_{33} - \frac{\sigma_{kk}}{3}}{2 J_2^{\frac{1}{2}}} = 0$$

$$\frac{2}{3} \sigma_{33} = - \frac{2 \alpha J_2^{\frac{1}{2}}}{3} + \frac{\sigma_{11} + \sigma_{22}}{3}$$

or

$$\sigma_{33} = -\alpha J_2^{\frac{1}{2}} + \tfrac{1}{2} (\sigma_{11} + \sigma_{22}) \tag{I-48}$$

Equations I-47 and I-48 define different surfaces in a three-dimensional space spanned by σ_{11}, σ_{22}, σ_{33}. Let their intersections with B be respectively, P and Q. The stress path is constrained to lie in the plane defined by equation I-47 for stress states in the interior of D and for neutral loading. For plastic deformation to occur the stress path must lie in the surface defined by equation I-48. For a continuous stress path to be possible, P and Q must coincide. In general this is not the case. Figure I-7 illustrates the difference between the surfaces P and Q for Mohr-Coulomb plane strain case.

Considering that the stresses σ_{33} do not contribute to energy/work of the system, it appears reasonable to assume that progress from P to Q is possible with gradually increasing the value of σ_{33}. This would amount to following the

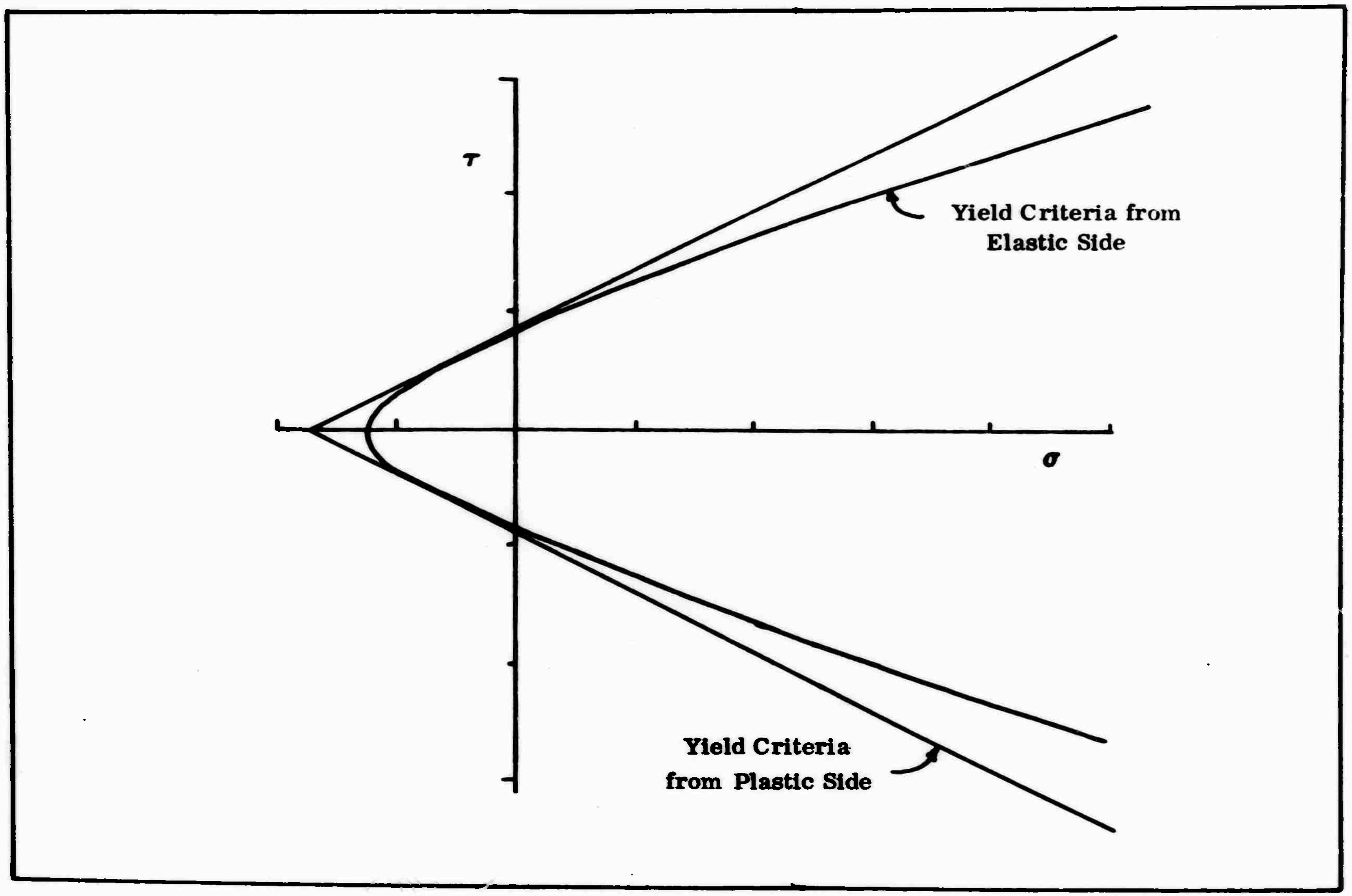


Figure 1-7. Plane Strain Mohr-Coulomb Criteria

boundary B. Growth of σ_{33} in progress from P to Q and the behavior upon unloading are not clearly understood. Later investigations may throw light on this aspect of material behavior. For the purpose of the computer program in Chapter III, it is assumed that elastic loading in plane strain can be continued upto a point from which the transition to plastic plane strain loading is possible merely by adding a residual value of σ_{33}. Referring to Drucker and Prager (1952), this is given by

$$\begin{aligned} k = f &= g(\sigma_{11}, \sigma_{22}, \tau_{12}) \\ &= \frac{3}{2} \alpha (\sigma_{11} + \sigma_{22}) + \sqrt{1 - 3\alpha^2} \left[\frac{\sigma_{11} - \sigma_{22}}{2}^{2} + \tau_{12}^{2} \right]^{\frac{1}{2}} \end{aligned} \tag{I-49}$$

1.3. Stress-Strain Behavior of Jointed/Cracked Rock

Mathematical simulation of behavior of jointed rock must allow for closing of pre-existing open joints under compressive loads followed by linear elastic behavior upto initiation of fracture. After fracture occurs, the material cannot take any tension locally in the direction normal to the plane of crack. Non-monotonic loads may involve closing-cracking-closing cycles.

The finite element method has been applied to jointed rock (Anderson and Dodd, 1966; Goodman et al, 1968; Duncan and Goodman, 1968; Malina, 1971). Anderson and Dodd used pin ended one-dimensional elements across a fault to allow compressive stresses to be transferred in the direction normal to the fault. The fault plane was assumed to have no resistance against shear or tensile loads. This capability is now routinely incorporated in most finite element programs. A two-dimensional 'soft' material element has long been used to represent weak joint planes in rock. Duncan and Goodman (1968) object to this on the basis of large number of elements needed to ensure a reasonable 'aspect ratio' in shape of elements. This becomes a problem for elements representing very thin joints. A one-dimensional element with shear and normal stiffness characteristics was proposed by Goodman et al (1968) to eliminate this objection. Recently (1971), the same investigators have introduced nonlinear properties in this type of element. This approach is quite effective for the case of pre-existing joints in rock. For well defined orthogonal joint systems, an orthotropic continuum approach was suggested by Duncan and Goodman (1968). Christian is credited (Einstein, Bruhn and Hirschfeld, 1970) with development of an element capable of simulating constant shear and residual shear characteristics.

In all these investigations, a distinct set of elements is used to represent the joint. This is alright for pre-existing joints but is impracticable for discontinuities arising as a result of fracture under applied load. To use the same procedure both for pre-existing joints as well as post failure cracks, it is necessary to allow cracks and joints within elements. Then, the mesh layout is more flexible and arbitrary failure laws can be used. Malina (1971) used this approach to study failure along joint planes and then went on to compute the amount of slip and accompanying stress redistribution on the basis of deformation or slip theory of plasticity.

Apparently, a bimodular analysis procedure (Sandhu and Wilson, 1970) can be used to represent pre-existing joints as well as fractures. Bimodularity would be dependent upon the joint opening. However, noting that fractured or open jointed rock has no resistance to tension in the direction normal to that of fracture, a simple approach following the procedure introduced by Zienkiewicz et al (1968) is more convenient. The 'no tension' method of Zienkiewicz consists of first obtaining a solution assuming the system to be linear elastic. Then the elements in tension are relieved of the tensile stresses by application of self-equilibrating forces in elements and at nodal points. This gives an iterative scheme for redistribution of loads to surrounding rock and a lower bound to the exact solution. This approach is essentially an orthotropic continuum approach with the orthotropy being applied to individual elements depending upon the orientation of the fracture plane. The fracture plane defines also a plane of material orthotropy. The relationship between principal stresses and strains can be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \end{Bmatrix} \tag{I-50}$$

or symbolically

$$\sigma_p = C_p \, \epsilon_p \tag{I-51}$$

The laws of transformation of stress and strain give,

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{I-52}$$

and

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} \qquad \text{(I-53)}$$

where σ_x, σ_y, τ_{xy} are components of stress and ϵ_x, ϵ_y, γ_{xy} are components of strain in x, y coordinate system and θ is the angle between the principal direction 1 and x-axis. Symbolically, the above equations are

$$\sigma = J^T \sigma_p \qquad \text{(I-54)}$$

$$\epsilon_p = J \epsilon \qquad \text{(I-55)}$$

Substitution in I-51 gives

$$\sigma = J^T C_p J \epsilon \qquad \text{(I-56)}$$

$$= C \epsilon \qquad \text{(I-57)}$$

where $C = J^T C_p J$ (I-58)

Equation I-58 gives the transformation for stress-strain relation for principal direction to any arbitrary choice of coordinates. The matrix C is singular only for $\theta = 0$ or 90°. It is thus possible to use the relationship in principal stresses and principal strains as the starting point.

In finite element analysis procedures, the stiffness mxtrix for the system is the sum of element stiffnesses.

$$K = \sum_{m=1}^{M} k^m \qquad \text{(I-59)}$$

where K is the system stiffness, k^m is the stiffness of the nth element, and Σ is viewed as a direct stiffness summation operator. Further, element stiffness is related to constitutive relationship through the equation

$$k^m = \int_{V_m} \left[b^T \ c \ b \right] \, dV \tag{I-60}$$

where b is the matrix relating strains to nodal point displacements, and V is the volume of the element. Using Equation I-58, the integrand in I-60 can be written

$$b^T c \ b = b^T \ J^T \ c_p \ J \ b \tag{I-61}$$

$$= b^T \ J^T \ \sigma_p \tag{I-62}$$

$$= B^T \ \sigma_p \tag{I-63}$$

where $B^T = b^T \ J^T$ relates principal stresses to nodal point forces.

Occurrence of fracture in an element reduces its ability to take tensile stresses normal to the fracture plane. Also there can be no shear transmitted across a crack which therefore is a principal direction. Thus, it is reasonable to reanalyze the system assigning an orthotropic constitutive relationship and a presecribed principal direction to the element containing a fracture. The procedure is to be repeated until no further fracturing occurs under a given load. To allow for nonlinearity introduced by progressive cracking, incremental procedures are required.

In Zienkiewicz et al (1968), a change in element stiffness was considered equivalent to a pseudo-load. Thus an iterative solution scheme was set up in which each iteration only involved a back-substitution operation. The pseudo-loads were computed as equivalent to tensile principal stresses. This is satisfactory when both principal stresses are tensile. However, when only one of the principal stresses is tensile, use of pseudo-load corresponding to one principal stress introduces a non-symmetric constitutive law. Actually if the physical concept of 'unloading without any displacements' be followed, a change in the second principal stress corresponding to Poission's effect due to the first stress must be included. This modification is included in the computer program in Chapter IV.

CHAPTER II

THE FINITE ELEMENT METHOD

Chapter II. The Finite Element Method

2.1. Basic Concepts

A boundary value problem can be stated in the form

$$A u = f \text{ on } F \tag{II-1}$$

where u is the unknown function to be determined, A is an operator, and f is the 'forcing' function. F is the domain of interest and may be an open, connected, bounded spatial region embedded in R^3 or in a cartesian product, $R^3 \times [0, \infty)$ where $[0, \infty)$ is the non-negative time interval. In addition to the field equation II-1, there will be some conditions to be satsifed on boundary S of F. For A linear positive, it can be shown that equation II-1 has a unique solution. Necessarily, any approximate solution will in general not coincide with the unique solution of II-1 and consequently no approximate solution is expected to satisfy the field equation as well as the boundary conditions completely.

Solutions to engineering problems as well as the forcing functions are in general bounded and therefore belong to L_2, the space of square integrable function. However u may be contained in a subset D of L_2 such that A is defined on D. We assume that D is dense in L_2. If the set of functions $\{\phi_k, k = 1, 2, .. \infty\}$ is an orthonormal basis in D, then any function u can be expressed as an infinite sum:

$$u = \sum_{k=1}^{N} a_k \phi_k \tag{II-2}$$

A scheme to generate approximate solutions is to use a finite set of terms in the infinite sum above. Thus, we use

$$\bar{u} = \sum_{k=1}^{N} a_k \phi_k \qquad \text{(II-3)}$$

as an approximation. The approximation process then consists of appropriate choice of N, ϕ_k and the coefficients a_k. Several alternative procedures are available. The finite element method is a special process of selection of a finite subset of the basis $\{\phi_k\}$. The coefficients a_k are evaluated by an extension of Ritz method or other standard procedures.

The finite element method is well documented in literature (Zienkiewicz, 1967; Bell and Holand, WPAFB Conference, 1965, 1968; Felippa, 1966; Clough, 1960, 1965). Its theoretical basis (Oden, 1969; de Arantes e Oliveira, 1968, Zlamal, 1968; Melkes, 1970) and relationship to variational principles (Melosh, 1963, Pian and Tong, 1969) have been examined. Essentially, a finite element idealization partitions the spatial region F into a finite number of nontrivial discrete elements or subregions. The geometry of the elements is defined by a set of points in space called the nodal points of the system.

Over an element e let an approximation to u be

$$u^e = \sum_{k=1}^{N^e} a_k^e \bar{\phi}_k^e \qquad \text{(II-3)}$$

or in matrix form

$$u^e = \left[\bar{\phi}^e\right]^T \left\{a^e\right\} \qquad \text{(II-4)}$$

where $\{\bar{\phi}^e\}^T$ is a row vector consisting of $\bar{\phi}_k^e$ as its elements and $\{a^e\}$ is a column vector of coefficients a_k^e. Evaluating the function at nodal points

$$\left\{ u_i^e \right\} = \left\{ \bar{\phi}_i^e \right\}^T \left\{ a^e \right\} \tag{II-5}$$

where $\left\{ u_i^e \right\}$ is the vector of nodal point values of the function and $\left\{ \bar{\phi}_i^e \right\}^T$ is the matrix of base functions evaluated at each nodal point. Rows and columns of $\left[\bar{\phi}_i^e \right]^T$ are linearly independent. If square, the matrix is invertible. If the number of nodal points is not equal to the number of independent base functions, a least square procedure can be used for inversion. Hence, we can write

$$\begin{aligned} \left\{ a^e \right\} &= \left[\left[\bar{\phi}^e \right]^T \right]^{-1} \left\{ u_i^e \right\} \\ &= \left[A \right]^{-1} \left\{ u_i^e \right\} \end{aligned} \tag{II-6}$$

where $\qquad A = \left[\bar{\phi}_i^e \right]^T$

Substituting II-6 in II-5

$$u^e = \left\{ \bar{\phi}^e \right\}^T \left[A \right]^{-1} \left\{ u_i^e \right\} \tag{II-7}$$

$$= \left\{ \phi^e \right\}^T \left\{ u_i^e \right\} \tag{II-8}$$

where $\left\{ \phi^e \right\}$ can now be regarded as a set of interpolating functions relating nodal point values of a function to the value of an arbitrary point within the element.

2.2. A Potential Energy Formulation

We assume the rock continuum or 'discontinuum' to be stepwise linear for sufficiently small steps in loading. For such a case the governing equations are

$$\sigma_{ki,k} + \pi_{,k}\, \delta_{ik} + \rho F_i = 0 \tag{II-9}$$

$$\sigma_{ij} = E_{ijkl}\, \epsilon_{kl} + \bar{\sigma}_{ij} \tag{II-10}$$

$$\epsilon_{kl} = u_{(k,l)} \tag{II-11}$$

where σ_{ij}, ϵ_{ij}, E_{ijkl}, F_i, u_i are components, respectively, of the symmetric stress tensor, the symmetric strain tensor, the isothermal elasticity tensor, the body forces vector per unit mass, and the displacement vector. ρ is the mass density and δ_{ik} is the kronecker delta. $\bar{\sigma}_{ij}$ are components of initial stress corresponding to zero displacement and π is the pore pressure. Potential energy formulation uses the functional

$$\Omega = \int_F \Big[\epsilon_{ij} E_{ijkl} \epsilon_{kl} - u_i \sigma_{ij,j} - 2 \epsilon_{ij} \sigma_{ij} + u_{i,} \sigma_{ij} - 2 u_i \rho F_i - 2 u_i \pi_{,i} + 2 \epsilon_{ij} \bar{\sigma}_{ij}\Big] dF + \int_{s_1} u_i (\sigma_{ij} n_j - 2 \hat{t}_i) \, ds - \int_{s_2} (u_i - 2 \hat{u}_i) \, \sigma_{ij} n_j \, ds \quad \text{(II-12)}$$

where we have included the boundary condition

$$\sigma_{ij} n_j = \hat{t}_i \text{ on } s_1 \quad \text{(II-13)}$$

$$u_i = \hat{u}_i \text{ on } s_2 \quad \text{(II-14)}$$

s_1, s_2 are complementary subsets of S the boundary of F and n_j are components of unit vector normal to surface.

Symmetry of the field equations leads to the functional

$$\Omega = \int_F \Big[\epsilon_{ij} E_{ijkl} \epsilon_{kl} - 2 \epsilon_{ij} \sigma_{ij} + 2 u_{i,j} \sigma_{ij} - 2 u_i \rho F_i - 2 u_i \pi_{,i} + 2 \epsilon_{ij} \bar{\sigma}_{ij}\Big] dF - 2 \int_{s_1} u_i \hat{t}_i \, ds - 2 \int_{s_2} (u_i - \hat{u}_i) \, \sigma_{ij} n_j \, ds \quad \text{(II-15)}$$

Further assuming that we restrict our choice of ϵ_{ij}, u_i such that II-11 and II-14 are identically satisfied, the functional in II-15 reduces to

$$\Omega = \int_F \left[u_{i,j}\, E_{ijkl}\, u_{k,l} - 2 u_i \rho F_i - 2 u_i \pi_{,i} + 2 u_{i,j} \bar{\sigma}_{ij} \right] dF - 2 \int_{s_1} u_i \hat{t}_i \, ds \tag{II-16}$$

Replacing $\int_F$ by $\sum_{m=1}^{m} \int_{F^m}$ where F^m represents the subregion or element m, and using suitable interpolation scheme to express the integrandin terms of nodal point vectors of displacement, vanishing of variation of the functional yields the matrix equation

$$\left[K\right] \{u\} = \{R\} \tag{II-17}$$

where

$$\left[K\right] = \sum_{m=1}^{m} \left[k^m\right] \tag{II-18}$$

$$\{R\} = \sum_{m=1}^{m} \left[\{L^m\} + \{P^m\} - \{Q^m\} + \{T^m\} \right] \tag{II-19}$$

Components of element stiffness matrix and load vectors are:

$$k_{ij}^m = \int_{F^m} \phi_{im,n}^m \, E_{mnpq} \, \phi_{jp,q}^m \tag{II-20}$$

$$L_i^m = \int_{F^m} \rho \, \phi_{ij}^m \, F_j \tag{II-21}$$

$$P_i^m = \int_{F^m} \phi_{ij}^m \, \pi_{,j} \tag{II-22}$$

$$Q_i^m = \int_{F^m} \phi_{im,n}^m \, \bar{\sigma}_{mn} \tag{II-23}$$

$$T_i^m = \int_{F^m} \phi_{ij}^m \, \hat{t}_j \tag{II-24}$$

and ϕ_{ij} are components of a matrix formed by the row vectors $\{\phi^e\}^T$ corresponding to each degree of displacement freedom. The vectors $\{L^m\}$, $\{P^m\}$, $\{Q^m\}$, $\{T^m\}$ represent the contribution to the load vector made respectively by the body forces, the pore pressure gradients, the initial stresses and the boundary loads in the element m.

2.3 Incremental Analysis

In case of incremental construction and incremental application of loads, the loads, stresses and displacements for any incremental step can be written as $\{R_n\}$, $\{\sigma_n\}$ and $\{u_n\}$. Then for the next stage, $\{\sigma_n\}$ and $\{u_n\}$ can be regarded as the initial stresses and the initial displacements for the structural system. Thus the matrix equations are

$$\left[K_{n+1}\right]\left\{u_{n+1} - u_n\right\} = \left[K_{n+1}\right]\left\{\Delta u_n\right\} = \left\{\Delta R_n\right\} \qquad (\text{II-25})$$

where

$$\Delta R_n = \sum_{m=1}^{m}\left[\left\{\Delta L^m\right\} + \left\{\Delta P^m\right\} - \left\{\Delta Q^m\right\} + \left\{\Delta T^m\right\}\right] \qquad (\text{II-26})$$

and $\{\Delta L^m\}$, $\{\Delta P^m\}$, $\{\Delta Q^m\}$, $\{\Delta T^m\}$ are increments in the respective quantities.

For elastic-plastic analysis, the stiffness depends upon stress and has to be re-evaluated at small increments of load. To ensure manageable computation, the increments are kept at the largest practicable without loss of accuracy.

CHAPTER III

COMPUTER PROGRAM FOR PLANE STRAIN ANALYSIS OF ELASTIC-PLASTIC MOHR-COULOMB MATERIALS

Chapter III. Computer Program for Plane Strain Analysis of Elastic-Plastic Mohr-Coulomb Materials

3.1. Organization

Computer program described here is based on the theory presented earlier in this report. The program is written in Fortran IV language.

The program is intended to calculate stresses and strains for a plane strain problem in rock mechanics. Mohr-Coulomb yield criterion has been used. It makes allowance for the boundary conditions, residual stresses, stresses due to temperature change, and varying pressure boundaries. The structure may consist of different materials. It uses Wilson's (1965) quadrilateral elements and generates stiffness in line with the integration procedures discussed by Felippa (1966).

The principal program called MAIN controls all the data input and control information. It does the basic system initialization and prints the control data and material and geometrical properties of the structure. Stiffness formulation, equation solving and stress calculations are done by the subroutines called by MAIN.

3.12. Stiffness and Load Matrices

Stiffness matrix for each analysis is computed in blocks by the subroutine STIFF. For the element stiffness it calls QUAD for triangular and quadrilateral elements which have been allowed by this program. The element stiffness is added to the total stiffness using the direct stiffness technique. Concentrated loads are included in the load matrix. Equations are modified for the displacement boundary conditions by calling subroutine MODIFY. For the stress-strain matrix QUAD calls STRSTR. With the constitutive law being available, stiffness of the system and load matrix are computed in the STIFF subroutine.

3.13. Calculations of Displacements

After the stiffness and load matrices for a stage have been computed, the resulting equations are solved by calling subroutine BANSOL. This uses Gaussian elimination technique for banded equations by Wilson (1963). In this the triangularization of stiffness matrix is done. Back substitution through the triangulized matrix gives the solution.

3.14. Calculations of Stresses

In the first cycle a purely elastic solution is obtained for the problem by solving

$$[K^e]\{r\} = \{R\} \qquad \text{(III-1)}$$

where $[K^e]$ is the elastic stiffness

$\{r\}$ is the displacement vector

$\{R\}$ is the load vector.

This can be done easily by assuming all the elements to be elastic to begin with. As the problem is a nonlinear one, therefore this solution will not be correct. In our analysis the system is assumed to be stepwise linear between the yielding of one element to the other. This is assumed not to cause any significant error. In Fig. 3-1 point A represents the initial stresses and C the final stresses in an element. The curve f = k represents the yield surface. For those elements which become plastic under this loading $\overrightarrow{AC}$ meets the surface f = k at pt. B. It is seen easily that for the element it is not possible to be loaded to point C but it can be loaded to point B only, assuming proportional loading.

Let $S_r = \dfrac{|\overrightarrow{AB}|}{|\overrightarrow{AC}|}$. S_r is called the stress ratio. To calculate S_r we know that

$$\overrightarrow{AC} = (\sigma_{ij})_f - (\sigma_{ij})_i$$

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$= \overrightarrow{OA} + S_r \cdot \overrightarrow{AC}$$

$$= (\sigma_{ij})_i + S_r \quad (\sigma_{ij})_f - (\sigma_{ij})_i$$

$$\sigma_{ij} = (\sigma_{ij})_i \, (1 - S_r) + S_r \, (\sigma_{ij})_f$$

As the point B lies on the yield surface $f = k$

$$f \left[(\sigma_{ij})_i \, (1 - S_r) + S_r \, (\sigma_{ij})_f \right] = k \qquad \text{(III-2)}$$

From equation (III-2) the value of S_r can be calculated. This stress ratio represents the fraction by which the increment in stress is to be scaled to bring the final load on the yield surface.

Value of S_r is calculated for all the elements. The element in which the final stress state is farthest from the yield surface will have the minimum stress ratio. If we scale down the displacements and stresses in this ratio, we shall have the stresses and strains precisely at the point when the system has its first element just going into the plastic region from purely elastic system.

In the next step the element having stress ratio equal to the minimum value is assumed to be plastic. To economize on computer time all such elements which have their value of stress ratio in the vicinity of minimum were also allowed to go plastic. As the stress-strain matrix is known the stiffness is calculated again and equation

$$\left[K_i \right] \left\{ r_i \right\} = \left\{ R_i \right\} \qquad \text{(III-3)}$$

is solved. This procedure is repeated until the whole load has been applied to the system and cumulative stresses and displacements calculated. The stresses in (i-1)th step become the initial stresses for ith step.

3.2. Input Data Preparation

1. Control Card (A6). This card will carry the characters START in columns 1-5. This will start the processing of the data deck which consists of the following set of cards.

2. Job Title (72H). This card will give the descriptive identification for the job.

3. Control Information (4I5, 2F10.2, I5)

Information	Columns
Total number of nodal points	1 - 5
Total number of elements	6 - 10
Number of different materials	11 - 15
Number of pressure boundary cards	16 - 20
Body Force in x-direction	21 - 30
Body Force in y-direction	31 - 40
Number of Approximations	41 - 45

4. Material Property Cards. One set of 2 cards is provided for each material. In each set:

   a. first card (1I5, F10.0) will give the following information

Material identification number	1 - 5
Mass density of the material	6 - 15

   b. The second card will carry the following information (4F10.0)

Information	Columns
Elastic Modulus	1 - 10
Poisson's Ratio	11 - 20
Cohesion	21 - 30
Friction Angle in Degrees	31 - 40

5. Nodal Point Cards (I5, F5.0, 5F 10.0).

One card for each nodal point with the following information:

Nodal Point number	1 - 5
Type of Nodal point	6 - 10
X-ordinate	11 - 20
Y-ordinate	21 - 30
XR	31 - 40
XZ	41 - 50

If the number in columns 6 - 10 is

Zero	XR is the specified X-load and XZ is the specified Y-load
1	XR is the specified X-displacement and XZ is the specified Y-load
2	XR is the specified X-load and XZ is the specified Y-displacement
3	XR is the specified X-displacement and XZ is the specified Y-displaceme

All loads are considered to be total forces acting on an element of unit thickness.

Nodal point cards must be in numerical sequence. If cards are omitted, the omitted nodal points are generated at equal intervals along a straight line between the defined nodal points. The type of the nodal point, as well as XR, XZ, are set equal to zero.

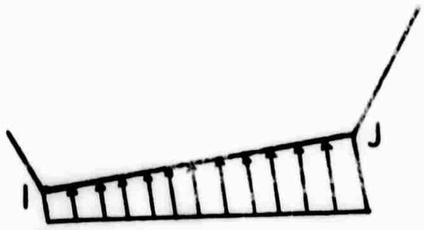
6. Element Cards (6I5). One card for each element will provide the following data.

Information	Columns
Number of element	1 - 5
Nodal point I	6 - 10
Nodal point J	11 - 15
Nodal point K	16 - 20
Nodal point L	21 - 25
Material type	26 - 30

Nodal points I, J, K, L are corners of each individual element in a counter-clockwise order for a right handed system of coordinates. For triangular elements set nodal point L same as nodal point K. The element cards must be in the numerical sequence. Any cards that are omitted will be automatically generated in the program by incrementing each of the I, J, K, and L nodal points by one. The material type will be taken the same as for the last element defined.

7. Pressure Boundary Cards (2I5, 2F10.0). One card for each boundary element which is subjected to a normal pressure will carry the following information:

Information	Columns
Nodal Point I	1 - 5
Nodal Point J	6 - 10
Normal Pressure at I	11 - 20
Normal Pressure at J	21 - 30



As shown in the sketch, the boundary element must be on the left as one progresses from I to J. Surface tensile forces is input as a negative pressure.

Output Information:

The following information is developed and printed by the program:

1. Reprint of input data

2. Nodal point displacements

3. Stresses at the center of each element

3.3 Program Listing

```
C        PLANE STRAIN ANALYSIS OF ELASTIC-PLASTIC MOHR-COULOMB MATERIALS
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON ACELR,ACELZ,VOL,TEMP, SIGI(400,7),HED(18),E(4,12),RO(12),
     *R(500),Z(500),UR(500),UZ(500),CODE(500),T(500),PR(100,2),
     *NUMMAT,NUMPC,N,MTYPE,KKK,NUMNP,NUMEL,NNN,
     *IBC(100),JBC(100),MTAG(400)
      COMMON/ARG/ RR(5),ZZ(5),S(10,10),P(8),ST(3,10),C(3,3),SIG(7),EE(4)
     *,BO(1000),SR1,SR2,
     *RATIO(400),LM(4),IX(400,5),XC,YC
      COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
      DIMENSION WORD(2)
C
      DATA WORD/ 6HSTART ,6HSTOP  /
      CALL ERRSET(207,256,-1,1)
      CALL ERRSET(208,256,-1,1)
C
C
    5 READ (5,1006) WORD1
      IF (WORD1.EQ.WORD(1)) GO TO 50
      IF (WORD1.EQ.WORD(2))  STOP
      GO TO 5
   50 READ (5,1000) HED,NUMNP,NUMEL,NUMMAT,NUMPC,ACELR,ACELZ,NP
      WRITE(6,2000) HED,NUMNP,NUMEL,NUMMAT,NUMPC,ACELR,ACELZ,NP
      DO 55 M=1,NUMMAT
      READ (5,1001) MTYPE,RO(MTYPE)
      WRITE(6,2001) MTYPE,RO(MTYPE)
      READ (5,1002) (E(J,MTYPE),J=1,4)
      WRITE(6,2002) (E(J,MTYPE),J=1,4)
   55 CONTINUE
C
      WRITE (6,2003)
      L=0
C
   60 READ  (5,1003) N,CODE(N),R(N),Z(N),UR(N),UZ(N),T(N)
      NL=L+1
      ZX=N-L
      DR=(R(N)-R(L))/ZX
      DZ=(Z(N)-Z(L))/ZX
      DT=(T(N)-T(L))/ZX
   70 L=L+1
      IF(N-L) 100,90,80
   80 CODE(L)=0.0
      R(L)=R(L-1)+DR
      Z(L)=Z(L-1)+DZ
      T(L)=T(L-1)+DT
      UR(L)=0.0
      UZ(L)=0.0
```

```
      GO TO 70
   90 WRITE (6,2004) (K,CODE(K),R(K),Z(K),UR(K),UZ(K),T(K),K=NL,N)
      IF(NUMNP-N) 100,110,60
  100 WRITE (6,2005) N
      CALL EXIT
  110 CONTINUE
C
      WRITE (6,2006)
      N=0
  130 READ (5,1004) M,(IX(M,I),I=1,5),(SIGI(M,I),I=1,4)
      ZX=M-N
      DO 135 I=1,4
  135 SIG(I)=(SIGI(M,I)-SIGI(N,I))/ZX
  140 N=N+1
      IF (M-N) 170,170,150
  150 IX(N,1)=IX(N-1,1)+1
      IX(N,2)=IX(N-1,2)+1
      IX(N,3)=IX(N-1,3)+1
      IX(N,4)=IX(N-1,4)+1
      IX(N,5)=IX(N-1,5)
      DO 160 I=1,4
  160 SIGI(N,I)=SIGI(N-1,I)+SIG(I)
  170 WRITE(6,2007) N,(IX(N,I),I=1,5),(SIGI(N,I),I=1,4)
      IF (M-N) 180,180,140
  180 IF (NUMEL-N) 190,190,130
  190 CONTINUE
C
      IF (NUMPC) 290,310,290
  290 WRITE (6,2008)
      DO 300 L=1,NUMPC
      READ  (5,1005) IBC(L),JBC(L),PR(L,1),PR(L,2)
  300 WRITE (6,2009) IBC(L),JBC(L),PR(L,1),PR(L,2)
  310 CONTINUE
C
      J=0
      DO 340 N=1,NUMEL
      MTAG(N)=0
      SIGI(N,5)=0.
      SIGI(N,6)=0.
      SIGI(N,7)=0.
      DO 340 I=1,4
      DO 325 L=1,4
      KK=IABS(IX(N,I)-IX(N,L))
      IF (KK-J) 325,325,320
  320 J=KK
  325 CONTINUE
  340 CONTINUE
      MBAND=2*J+2
```

```
      WRITE(6,1007) MBAND
      DO 350 N=1,NUMNP
      BO(2*N-1)=0.
  350 BO(2*N)=0.
C
      SR1=1.0
      SR2=0.0
      DO 500 NNN=1,NP
      KKK=0
      CALL STIFF
      CALL BANSOL
      CALL STRESS
      DO 400 N=1,NUMNP
      NN=2*N
      BO(NN-1)=BO(NN-1)+B(NN-1)
      BO(NN)=BO(NN)+B(NN)
  400 CONTINUE
      WRITE(6,2010) (N,BO(2*N-1),BO(2*N),N=1,NUMNP)
      IF(KKK.EQ.0) CALL EXIT
  500 CONTINUE
      GO TO 5
C
 1000 FORMAT (18A4/4I5,2F10.2,2I5)
 1001 FORMAT (1I5,1F10.0)
 1002 FORMAT (6F10.0)
 1003 FORMAT (I5,F5.0,5F10.0)
 1004 FORMAT(6I5,4F10.0)
 1005 FORMAT (2I5,2F10.0)
 1006 FORMAT(A6)
 1007 FORMAT( ' BAND WIDTH FOR THIS DATA =  '  , I5 )
 2000 FORMAT (1H1 18A4/
     1 30H0 NUMBER OF NODAL POINTS------ I3 /
     2 30H0 NUMBER OF ELEMENTS---------- I3 /
     3 30H0 NUMBER OF DIFF. MATERIALS--- I3 /
     4 30H0 NUMBER OF PRESSURE CARDS---- I3 /
     5 30H0 X-ACCELERATION-------------- E12.4/
     6 30H0 Y-ACCELERATION-------------- E12.4/
     7 30H0 NUMBER OF APPROXIMATIONS---- I12)
 2001 FORMAT (17H0MATERIAL NUMBER# I3, 15H, MASS DENSITY# E12.4)
 2002 FORMAT(16H0ELASTIC MODULUS 14X 2HNU 8X 8HCOHESION 2X 14HFRICTION A
     *NGLE/(2E16.5,2F16.5))
 2003 FORMAT (111H1NODAL POINT        TYPE  X ORDINATE  Y ORDINATE  X LO
     1AD OR DISPLACEMENT  Y LOAD OR DISPLACEMENT  PORE PRESSURE )
 2004 FORMAT (I12,F12.2,2F12.3,2E24.7,F15.3)
 2005 FORMAT (26H0NODAL POINT CARD ERROR N# I5)
 2006 FORMAT(96H1ELEMENT NO.     I     J     K     L    MATERIAL    X-ST
     *RESS    Y-STRESS    XY-STRESS    Z-STRESS)
 2007 FORMAT(I12,4I6,1I12,4F12.3)
```

```
2008 FORMAT (29HOPRESSURE BOUNDARY CONDITIONS/ 40H     I      J     PRESS
    *URE I     PRESSURE J                                              )
2009 FORMAT (2I6,2F12.3)
2010 FORMAT (12H1N.P. NUMBER 18X 2HUX 18X 2HUY / (1I12,2E20.7))
     END
```

```
      SUBROUTINE STIFF
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON ACELR,ACELZ,VOL,TEMP, SIGI(400,7),HED(18),E(4,12),RO(12),
     *R(500),Z(500),UR(500),UZ(500),CODE(500),T(500),PR(100,2),
     *NUMMAT,NUMPC,N,MTYPE,KKK,NUMNP,NUMEL,NNN,
     *IBC(100),JBC(100),MTAG(400)
      COMMON/ARG/ RR(5),ZZ(5),S(10,10),P(8),ST(3,10),C(3,3),SIG(7),EE(4)
     *,BO(1000),SR1,SR2,
     *RATIO(400),LM(4),IX(400,5),XC,YC
      COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
C
C
      REWIND 2
      NB=27
      ND=2*NB
      ND2=2*ND
      STOP=0.0
      NUMBLK=0
      DO 50 N=1,ND2
      B(N)=0.0
      DO 50 M=1,ND
   50 A(N,M)=0.0
C
   60 NUMBLK=NUMBLK+1
      NH=NB*(NUMBLK+1)
      NM=NH-NB
      NL=NM-NB+1
      KSHIFT=2*NL-2
      DO 210 N=1,NUMEL
      IF (IX(N,5)) 210,210,65
   65 DO 80 I=1,4
      IF (IX(N,I)-NL) 80,70,70
   70 IF (IX(N,I)-NM) 90,90,80
   80 CONTINUE
      GO TO 210
   90 CALL QUAD
      IX(N,5)=-IX(N,5)
      IF(VOL) 100,100,110
  100 WRITE(6,2000) N
      STOP=1.0
C
  110 MM=4
      IF(IX(N,3)-IX(N,4)) 130,120,130
  120 MM=3
  130 DO 140 I=1,MM
  140 LM(I)=2*IX(N,I)-2
      DO 200 I=1,MM
```

```
      DO 200 K=1,2
      II=LM(I)+K-KSHIFT
      KK=2*I-2+K
      B(II)=B(II)+P(KK)
      DO 200 J=1,MM
      DO 200 L=1,2
      JJ=LM(J)+L-II+1-KSHIFT
      LL=2*J-2+L
      IF(JJ) 200,200,175
  175 IF(ND-JJ) 180,195,195
  180 WRITE (6,2001) N
      STOP=1.0
      GO TO 210
  195 A(II,JJ)=A(II,JJ)+S(KK,LL)
  200 CONTINUE
  210 CONTINUE
C
      DO 220 N=NL,NM
      K=2*N-KSHIFT
      B(K)=B(K)+UZ(N)
      B(K-1)=B(K-1)+UR(N)
      T(N)=0.
      UZ(N)=0.
  220 UR(N)=0.
C
      IF (NUMPC) 225,310,225
  225 DO 300 L=1,NUMPC
      I=IRC(L)
      J=JRC(L)
      DR=Z(I)-Z(J)
      DZ=R(J)-R(I)
      PP2=(PR(L,2)+PR(L,1))/6.
      PP1=PP2+PR(L,1)/6.
      PP2=PP2+PR(L,2)/6.
      II=2*I-KSHIFT
      JJ=2*J-KSHIFT
      IF(II) 265,265,235
  235 IF(II-ND) 240,240,265
  240 B(II-1)=B(II-1)+PP1*DR
      B(II)=B(II)+PP1*DZ
  265 IF(JJ) 300,300,270
  270 IF(JJ-ND) 275,275,300
  275 B(JJ-1)=B(JJ-1)+PP2*DR
      B(JJ)=B(JJ)+PP2*DZ
  300 CONTINUE
C
  310 DO 400 M=NL,NM
      IF (M-NUMNP) 315,315,400
```

```
  315 U=UR(M)
      N=2*M-1-KSHIFT
      IF (CODE(M)) 390,400,316
  316 IF (CODE(M)-1.) 317,370,317
  317 IF (CODE(M)-2.) 318,390,318
  318 IF (CODE(M)-3.) 390,380,390
  370 CALL MODIFY(A,B,ND2,MBAND,N,U)
      GO TO 400
  380 CALL MODIFY(A,B,ND2,MBAND,N,U)
  390 U=UZ(M)
      N=N+1
      CALL MODIFY(A,B,ND2,MBAND,N,U)
  400 CONTINUE
C
      WRITE (2) (B(N),(A(N,M),M=1,MBAND),N=1,ND)
C
      DO 420 N=1,ND
      K=N+ND
      B(N)=B(K)
      B(K)=0.0
      DO 420 M=1,ND
      A(N,M)=A(K,M)
  420 A(K,M)=0.0
C
      IF (NM-NUMNP) 60,480,480
  480 CONTINUE
      ACELR=0.
      ACELZ=0.
      NUMPC=0
C
      IF(STOP) 490,500,490
  490 CALL EXIT
  500 RETURN
C
 2000 FORMAT (26HONEGATIVE AREA ELEMENT NO. I4)
 2001 FORMAT (29HOBAND WIDTH EXCEEDS ALLOWABLE I4)
      END
```

```
      SUBROUTINE QUAD
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON ACELR,ACELZ,VOL,TEMP, SIGI(400,7),HED(18),E(4,12),RO(12),
     *R(500),Z(500),UR(500),UZ(500),CODE(500),T(500),PR(100,2),
     *NUMMAT,NUMPC,N,MTYPE,KKK,NUMNP,NUMEL,NNN,
     *IBC(100),JBC(100),MTAG(400)
      COMMON/ARG/ RR(5),ZZ(5),S(10,10),P(8),ST(3,10),C(3,3),SIG(7),EE(4)
     *,BO(1000),SR1,SR2,
     *RATIO(400),LM(4),IX(400,5),XC,YC
      COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
      DIMENSION U(3),V(3)
C
      CALL STRSTR
      DO 130 J=1,10
      DO 120 I=1,3
  120 ST(I,J)=0.
      DO 130 I=1,10
  130 S(I,J)=0.
      DO 140 I=1,4
      NPP=IX(N,I)
      RR(I)=R(NPP)
  140 ZZ(I)=Z(NPP)
      XC=(RR(1)+RR(2)+RR(3)+RR(4))/4.
      YC=(ZZ(1)+ZZ(2)+ZZ(3)+ZZ(4))/4.
      RR(5)=XC
      ZZ(5)=YC
      K=5
      J=1
      I=4
      LM(3)=9
      NT=4
      IF(IX(N,3)-IX(N,4)) 160,150,160
  150 NT=1
      LM(3)=5
      I=1
      K=3
      J=2
      XC=(RR(1)+RR(2)+RR(3))/3.
      YC=(ZZ(1)+ZZ(2)+ZZ(3))/3.
      RR(5)=RR(3)
      ZZ(5)=ZZ(3)
  160 DO 200 NN=1,NT
      LM(1)=2*I-1
      LM(2)=2*J-1
      U(1)=ZZ(J)-ZZ(K)
      U(2)=ZZ(K)-ZZ(I)
      U(3)=ZZ(I)-ZZ(J)
```

```
      V(1)=RR(K)-RR(J)
      V(2)=RR(I)-RR(K)
      V(3)=RR(J)-RR(I)
      AREA=(RR(J)*U(2)+RR(I)*U(1)+RR(5)*U(3))/2.
      VOL=VOL+AREA
      COMM=.25/AREA
      XNT=NT
      COM=2.0/XNT
      COM=COM*COMM
C
      DO 180 L=1,3
      II=LM(L)
      ST(1,II)=ST(1,II)+U(L)*COM
      ST(2,II+1)=ST(2,II+1)+V(L)*COM
      ST(3,II)=ST(3,II)+V(L)*COM
      ST(3,II+1)=ST(3,II+1)+U(L)*COM
      DO 180 M=1,3
      JJ=LM(M)
      S(II,JJ)=S(II,JJ)+(U(L)*C(1,1)*U(M)+V(L)*C(3,3)*V(M)+V(L)*C(1,3)*U
     1(M)+U(L)*C(1,3)*V(M))*COMM
      S(II,JJ+1)=S(II,JJ+1)+(U(L)*C(1,2)*V(M)+V(L)*C(3,3)*U(M)+V(L)*C(2,
     13)*V(M)+U(L)*C(1,3)*U(M))*COMM
      S(II+1,JJ+1)=S(II+1,JJ+1)+(V(L)*C(2,2)*V(M)+U(L)*C(3,3)*U(M)+U(L)*
     1C(2,3)*V(M)+V(L)*C(2,3)*U(M))*COMM
      S(JJ+1,II)=S(II,JJ+1)
  180 CONTINUE
      I=J
      J=J+1
  200 CONTINUE
C
      IF(IX(N,3)-IX(N,4)) 220,250,220
  220 DO 240 I=1,2
      KK=10-I
      DO 240 K=1,KK
      CC=S(KK+1,K)/S(KK+1,KK+1)
      DO 230 J=1,3
  230 ST(J,K)=ST(J,K)-CC*ST(J,KK+1)
      DO 240 J=1,KK
  240 S(J,K)=S(J,K)-CC*S(J,KK+1)
  250 CONTINUE
C
      II=C
      IF(NNN.EQ.1) GO TO 260
      II=4
  260 SIG(1)=-SIGI(N,II+1)+TEMP
      SIG(2)=-SIGI(N,II+2)+TEMP
      SIG(3)=-SIGI(N,II+3)
      DO 520 I=1,8
```

```
      P(I)=0.0
      DO 510 J=1,3
  510 P(I)=P(I)+ST(J,I)*SIG(J)
  520 P(I)=P(I)*VOL
C
      DX=VOL*ACELR*RO(MTYPE)/4.
      DY=VOL*ACELZ*RO(MTYPE)/4.
      DO 530 I=1,4
      P(2*I)=P(2*I)+DY
  530 P(2*I-1)=P(2*I-1)+DX
      RETURN
      END
```

```
      SUBROUTINE STRSTR
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON ACELR,ACELZ,VOL,TEMP, SIGI(400,7),HED(18),E(4,12),RO(12),
     *R(500),Z(500),UR(500),UZ(500),CODE(500),T(500),PR(100,2),
     *NUMMAT,NUMPC,N,MTYPE,KKK,NUMNP,NUMEL,NNN,
     *IBC(100),JBC(100),MTAG(400)
      COMMON/ARG/ RR(5),ZZ(5),S(10,10),P(8),ST(3,10),C(3,3),SIG(7),EE(4)
     *,BO(1000),SR1,SR2,
     *RATIO(400),LM(4),IX(400,5),XC,YC
      COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
C
      I=IX(N,1)
      J=IX(N,2)
      K=IX(N,3)
      L=IX(N,4)
      MTYPE=IX(N,5)
      VOL=0.
C
      TEMP=(T(I)+T(J)+T(K)+T(L))/4.0
      DO 50 KK=1,4
   50 EE(KK)=E(KK,MTYPE)
      IF(MTAG(N)) 60,60,70
   60 CC=EE(2)/(1.-EE(2))
      BB=EE(1)/(1.-EE(2)**2)
      COMM=BB/(1.-CC**2)
      C(1,1)=COMM
      C(1,2)=COMM*CC
      C(1,3)=0.
      C(2,1)=C(1,2)
      C(2,2)=C(1,1)
      C(2,3)=0.
      C(3,1)=0.
      C(3,2)=0.
      C(3,3)=.5*COMM*(1.-CC)
      CC=DTAN(EE(4)/57.296)
      BB=DSQRT(9.0+12.0*CC*CC)
      EE(4)=CC/BB
      EE(3)=3.*EE(3)/BB
      GO TO 500
   70 CC=DTAN(EE(4)/57.296)
      BB=DSQRT(9.0+12.0*CC*CC)
      EE(4)=CC/BB
      EE(3)=3.*EE(3)/BB
      CC=2.*(1.+EE(2))/(3.-6.*EE(2))
      DD=(SIGI(N,1)-SIGI(N,2))/2.
      BJ2=(DD*DD+SIGI(N,3)**2)/(1.-3.*(EE(4)**2))
      BJ2=DSQRT(BJ2)
```

```
      BJ1=1.5*(SIGI(N,1)+SIGI(N,2))-3.*EE(4)*BJ2
      DD=BJ1/BJ2
      BB=1.+9.*(EE(4)**2)*CC
      CC=3.*EE(4)*CC-DD/3.
      DD=EE(4)-DD/6.
      H1=.5*CC/(BB*BJ2)
      H2=DD*CC/BB-EE(2)*EE(3)/(BB*BJ2*(1.-2.*EE(2)))
      H3=.5/(BB*BJ2*BJ2)
C
      BB=EE(1)/(1.+EE(2))
      C(1,1)=BB*(1.-H2-2.*H1*SIGI(N,1)-H3*(SIGI(N,1)**2))
      C(1,2)=-BB*(H2+H1*(SIGI(N,1)+SIGI(N,2))+H3*SIGI(N,1)*SIGI(N,2))
      C(1,3)=-BB*(H1*SIGI(N,3)+H3*SIGI(N,1)*SIGI(N,3))
      C(2,1)=C(1,2)
      C(2,2)=BB*(1.-H2-2.*H1*SIGI(N,2)-H3*(SIGI(N,2)**2))
      C(2,3)=-BB*(H1*SIGI(N,3)+H3*SIGI(N,2)*SIGI(N,3))
      C(3,1)=C(1,3)
      C(3,2)=C(2,3)
      C(3,3)=BB*(.5-H3*(SIGI(N,3)**2))
  500 RETURN
      END
```

```
      SUBROUTINE MODIFY(A,B,NEQ,MBAND,N,U)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(108,54),B(108)
      DO 250 M=2,MBAND
      K=N-M+1
      IF(K) 235,235,230
  230 B(K)=B(K)-A(K,M)*U
      A(K,M)=0.0
  235 K=N+M-1
      IF(NEQ-K) 250,240,240
  240 B(K)=B(K)-A(N,M)*U
      A(N,M)=0.0
  250 CONTINUE
      A(N,1)=1.0
      B(N)=U
      RETURN
      END
```

```
      SUBROUTINE BANSOL
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /BANARG/ MM,NUMBLK,B(108),A(108,54)
C
      NN=54
      NL=NN+1
      NH=NN+NN
      REWIND 1
      REWIND 2
      NB=0
      GO TO 150
C
  100 NB=NB+1
      DO 125 N=1,NN
      NM=NN+N
      B(N)=B(NM)
      B(NM)=0.0
      DO 125 M=1,MM
      A(N,M)=A(NM,M)
  125 A(NM,M)=0.0
C
      IF (NUMBLK-NB) 150,200,150
  150 READ (2) (B(N),(A(N,M),M=1,MM),N=NL,NH)
      IF (NB) 200,100,200
C
  200 DO 300 N=1,NN
      IF (A(N,1)) 225,300,225
  225 B(N)=B(N)/A(N,1)
      DO 275 L=2,MM
      IF (A(N,L)) 230,275,230
  230 C=A(N,L)/A(N,1)
      I=N+L-1
      J=0
      DO 250 K=L,MM
      J=J+1
  250 A(I,J)=A(I,J)-C*A(N,K)
      B(I)=B(I)-A(N,L)*B(N)
      A(N,L)=C
  275 CONTINUE
  300 CONTINUE
C
      IF (NUMBLK-NB) 375,400,375
  375 WRITE (1) (B(N),(A(N,M),M=2,MM),N=1,NN)
      GO TO 100
C
  400 DO 450 M=1,NN
      N=NN+1-M
      DO 425 K=2,MM
```

```
      L=N+K-1
  425 B(N)=B(N)-A(N,K)*B(L)
      NM=N+NN
      B(NM)=B(N)
  450 A(NM,NB)=B(N)
      NB=NB-1
      IF (NB) 475,500,475
  475 BACKSPACE 1
      READ (1) (B(N),(A(N,M),M=2,MM),N=1,NN)
      BACKSPACE 1
      GO TO 400
C
  500 K=0
      DO 600 NB=1,NUMBLK
      DO 600 N=1,NN
      K=K+1
      NM=N+NN
  600 B(K)=A(NM,NB)
C
      RETURN
      END
```

```
      SUBROUTINE STRESS
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON ACELR,ACELZ,VOL,TEMP, SIG(400,7),HED(18),E(4,12),RO(12),
     *R(500),Z(500),UR(500),UZ(500),CODE(500),T(500),PR(100,2),
     *NUMMAT,NUMPC,N,MTYPE,KKK,NUMNP,NUMEL,NNN,
     *IBC(100),JBC(100),MTAG(400)
      COMMON/ARG/ RR(5),ZZ(5),S(10,10),P(8),ST(3,10),C(3,3),SIG(7),EE(4)
     *,RO(1000),SR1,SR2,
     *RATIO(400),LM(4),IX(400,5),XC,YC
      COMMON /BANARG/ MBAND,NUMBLK,B(108),A(108,54)
C
      TOL=0.
      SR=1.
      NUMR=0
      MPRINT=0
      KK=0
      DO 300 N=1,NUMEL
      RATIO(N)=1.
      IX(N,5)=IABS(IX(N,5))
      MTYPE=IX(N,5)
      CALL QUAD
      MM=4
      IF(IX(N,3)-IX(N,4)) 170,160,170
  160 MM=3
  170 DO 180 I=1,3
      RR(I)=0.
      DO 180 J=1,MM
      II=2*J
      JJ=2*IX(N,J)
  180 RR(I)=RR(I)+ST(I,II)*B(JJ)+ST(I,II-1)*B(JJ-1)
      DO 190 I=1,3
      SIG(I)=0.
      DO 190 J=1,3
  190 SIG(I)=SIG(I)+C(I,J)*RR(J)
      DO 195 I=1,3
      II=I+4
  195 SIG(N,II)=SIG(N,I)+SIG(I)
      DD=(SIG(N,5)-SIG(N,6))/2.
      AJ2=(DD*DD+SIG(N,7)**2)/(1.-3.*(EE(4)**2))
      AJ2=DSQRT(AJ2)
      AJ1=1.5*(SIG(N,5)+SIG(N,6))-3.*EE(4)*AJ2
      FAIL=AJ2+EE(4)*AJ1
      IF(MTAG(N).EQ.0) GO TO 200
      IF(MTAG(N).EQ.2) GO TO 300
      DD=DABS(FAIL-EE(3))
      CHECK=.02*EE(3)
      IF(DD-CHECK) 300,300,196
  196 KKK=1
```

```
      CR=CHECK/DD
      IF(CR-SR) 197,300,300
  197 SR=CR
      GO TO 300
  200 CONTINUE
      IF(FAIL.LT.EE(3)) GO TO 300
      KK=KK+1
      DD=(SIGI(N,1)-SIGI(N,2))/2.
      BJ2=(DD*DD+SIGI(N,3)**2)/(1.-3.*(EE(4)**2))
      BJ2=DSQRT(BJ2)
      BJ1=1.5*(SIGI(N,1)+SIGI(N,2))-3.*EE(4)*BJ2
      AAA=BJ2*BJ2
      BBB=AJ2*AJ2
      FFF=1.-3.*EE(4)*EE(4)
      CCC= (SIGI(N,1)-SIGI(N,2))*(SIGI(N,5)-SIGI(N,6))/4.+SIGI(N,3)*SIGI
     *(N,7)
      DDD=SIGI(N,1)+SIGI(N,2)
      GGG=SIGI(N,5)+SIGI(N,6)
      FF=FFF*FFF
      GG=GGG*GGG
      DD=DDD*DDD
      EF=2.25*EE(4)*EE(4)
      AA=AAA*FF-EF*DD
      BB=BBB*FF-EF*GG
      CC=CCC*FFF-EF*DDD*GGG
      DD=1.5*EE(4)*EE(3)*DDD
      FF=1.5*EE(4)*EE(3)*GGG
      GG=EE(3)*EE(3)
      AAA=AA+BB-2.*CC
      BBB=AA-CC+DD-FF
      CCC=2.*DD-GG+AA
      GGG=BBB*BBB-AAA*CCC
      GGG=DSQRT(GGG)
      IF(AAA) 220,210,220
  210 RATIO(N)=.5*CCC/BBB
      GO TO 300
  220 AA=BBB/AAA
      BB=DABS(GGG/AAA)
      RATIO(N)=AA-BB
      IF(RATIO(N).LT.0.) RATIO(N)=AA+BB
      IF(RATIO(N).GE.1.) RATIO(N)=.99999
      IF(RATIO(N).LT.0.) RATIO(N)=0.
  300 CONTINUE
C
      IF(KK.EQ.0) GO TO 420
      DO 350 N=1,NUMEL
      IF(MTAG(N).GT.0) GO TO 350
      DD=RATIO(N)
```

```
      IF(DD-SR) 305,350,350
  305 SR=DD
      NUMR=N
      KKK=1
  350 CONTINUE
      IF(SR.LT.0.1) SR=0.1
      IF(NUMR.EQ.0) GO TO 420
C
  260 DO 370 N=1,NUMEL
      IF(MTAG(N).GT.0) GO TO 370
      IF(RATIO(N).LE.SR) GO TO 355
      DD=RATIO(N)-SR
      IF(DD-.05) 355,355,370
  355 MTAG(N)=1
  370 CONTINUE
C
  420 CONTINUE
      DO 410 N=1,NUMNP
      II=2*N-1
      B(II)=B(II)*SR
  410 B(II+1)=B(II+1)*SR
      DO 600 N=1,NUMEL
      I=IX(N,1)
      J=IX(N,2)
      K=IX(N,3)
      L=IX(N,4)
      MTYPE=IX(N,5)
      IF(K.EQ.L) GO TO 460
      XC=(R(I)+R(J)+R(K)+R(L))/4.
      YC=(Z(I)+Z(J)+Z(K)+Z(L))/4.
      GO TO 470
  460 XC=(R(I)+R(J)+R(K))/3.
      YC=(Z(I)+Z(J)+Z(K))/3.
  470 CONTINUE
      DO 450 I=1,3
      II=I+4
      SIG(I)=SIGI(N,II)-SIGI(N,I)
      SIGI(N,II)=SIG(I)*(1.-SR)
      SIGI(N,II)=-SIGI(N,II)
      SIG(I)=SIG(I)*SR+SIGI(N,I)
  450 SIGI(N,I)=SIG(I)
      SIG(7)=EE(2)*(SIG(1)+SIG(2))
      DO 455 I=1,4
  455 EE(I)=E(I,MTYPE)
      IF(MTAG(N).EQ.0) GO TO 500
      CC=DTAN(EE(4)/57.296)
      BB=DSQRT(9.0+12.0*CC*CC)
      EE(4)=CC/BB
```

```
      DD=(SIG(1)-SIG(2))/2.
      BJ2=(DD*DD+SIG(3)**2)/(1.-3.*(EE(4)**2))
      BJ2=DSQRT(BJ2)
      SIG(7)=.5*(SIG(1)+SIG(2))-3.*EE(4)*BJ2
  500 SIGI(N,4)=SIG(7)
      CC=(SIG(1)+SIG(2))/2.
      BB=(SIG(1)-SIG(2))/2.
      CR=DSQRT(BB**2+SIG(3)**2)
      SIG(4)=CC+CR
      SIG(5)=CC-CR
      SIG(6)=0.0
      IF ((BB.EQ.0.0).AND.(SIG(3).EQ.0.0)) GO TO 510
      SIG(6)=28.698*DATAN2(SIG(3),BB)
  510 CONTINUE
      IF(MPRINT) 520,520,550
  520 WRITE(6,2000) NNN
      MPRINT=50
  550 MPRINT=MPRINT-1
      WRITE(6,2001) N,XC,YC,(SIG(I),I=1,7),MTAG(N)
      IF(MTAG(N).EQ.0) GO TO 560
      DD=(SIGI(N,5)-SIGI(N,6))/2.
      BJ2=(DD*DD+SIGI(N,7)**2)/(1.-3.*(EE(4)**2))
      BJ2=DSQRT(BJ2)
      TOL=TOL+BJ2
      GO TO 600
  560 SIG(7)=EE(2)*(SIGI(N,5)+SIGI(N,6))
      BJ2= DSQRT(((SIGI(N,5)-SIGI(N,6))**2+(SIGI(N,6)-SIG(7))**2+(SIG(7)
     *-SIGI(N,5))**2)/6.0+SIGI(N,7)**2)
      TOL=TOL+BJ2
  600 CONTINUE
      SR2= (SR1*SR) + SR2
      SR1= (1.0-SR) * SR1
      WRITE(6,2002) TOL,SR ,NUMR,KK,SR2
      IF(TOL-1.) 660,660,650
  650 KKK=1
      GO TO 700
  660 KKK=0
  700 CONTINUE
      II=0
      DO 710 I=1,NUMEL
  710 IF(MTAG(I).GE.1)  II=II+1
      IF(KK.GE.NUMEL) CALL EXIT
  800 RETURN
C
 2000 FORMAT(1H1/
     *36H STRESSES AFTER APPROXIMATION NUMBER I4////
     *7H EL.NO. 7X 1HX 7X 1HY 4X 8HX-STRESS 4X 8HY-STRESS 3X 9HXY-STRESS
     * 2X 10HMAX-STRESS 2X 10HMIN-STRESS 7H  ANGLE 4X 8HZ-STRESS 5X 7HPL
```

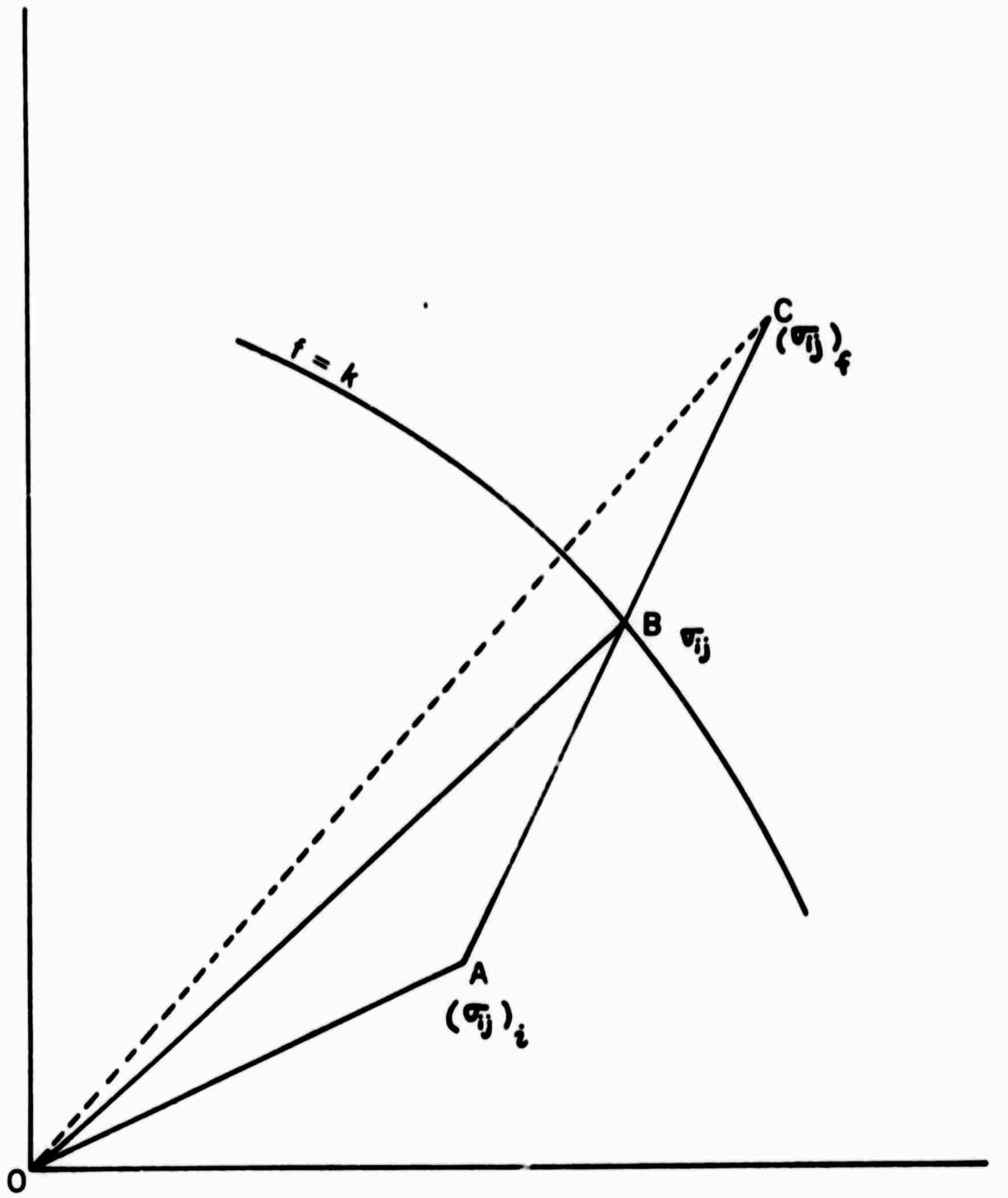
```
     *ASTIC)
 2001 FORMAT (I7,2F8.2,1P5E12.4,0P1F7.2,1PE12.4,I12)
 2002 FORMAT(39H0THE UNBALANCED LOAD AT THIS STAGE IS  E14.5//
     *47H THE RATIO FOR CORRECTION OF STORED STRESSES IS F10.4//
     *31H THE NEXT ELEMENT YIELDING IS     I4/
     *91H AND THE TOTAL NUMBER OF ELEMENTS THAT CAN YIELD WITH THE LINEA
     *R ADDITION OF TOTAL LOAD IS       I4/
     *50H LOAD UP TO THIS STAGE AS A FRACTION OF TOTAL IS       F20.5   )
C
      END
```

3.4 Example

Naghdi (1957) solved the problem of an elastic-perfectly plastic wedge under uniform loading on one face (Fig. 3.2). Plane strain conditions were considered. The wedge material was assumed to yield according to Von Mises' yield criterion. This type of material is a special case of Mohr-Coulomb material having the angle of internal friction $\Phi = 0$.

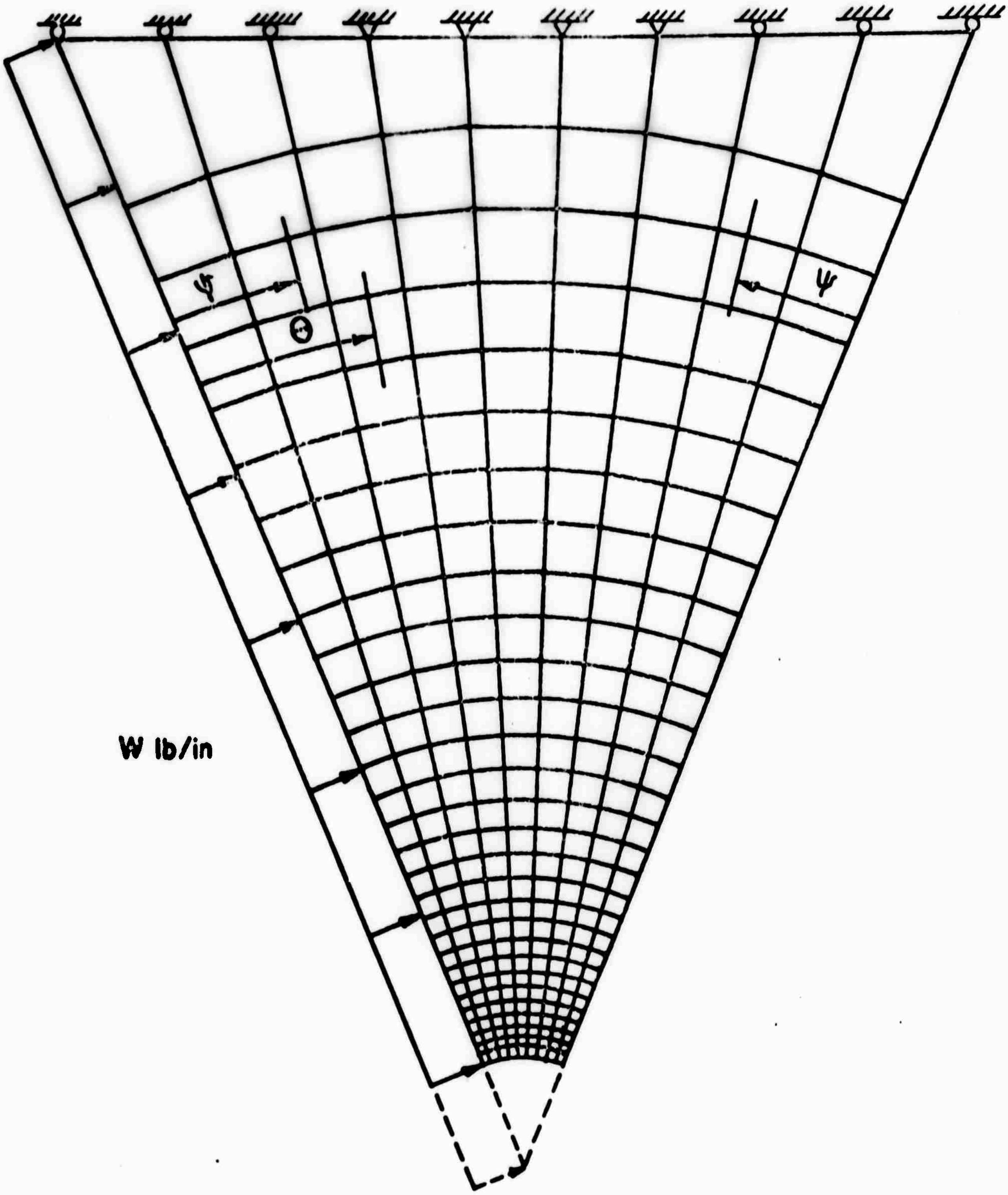
Figures 3.3 and 3.4 show the theoretical and computed results for the distribution of radial and circumferential stress at various stages of loadings. The angle ψ denotes the angle upto which the yielding has progressed from the boundaries. Fig. 3.5 shows the radial strain distribution at various stages.

Generally the agreement between results computed by the method outlined and the exact analysis were found to be good.



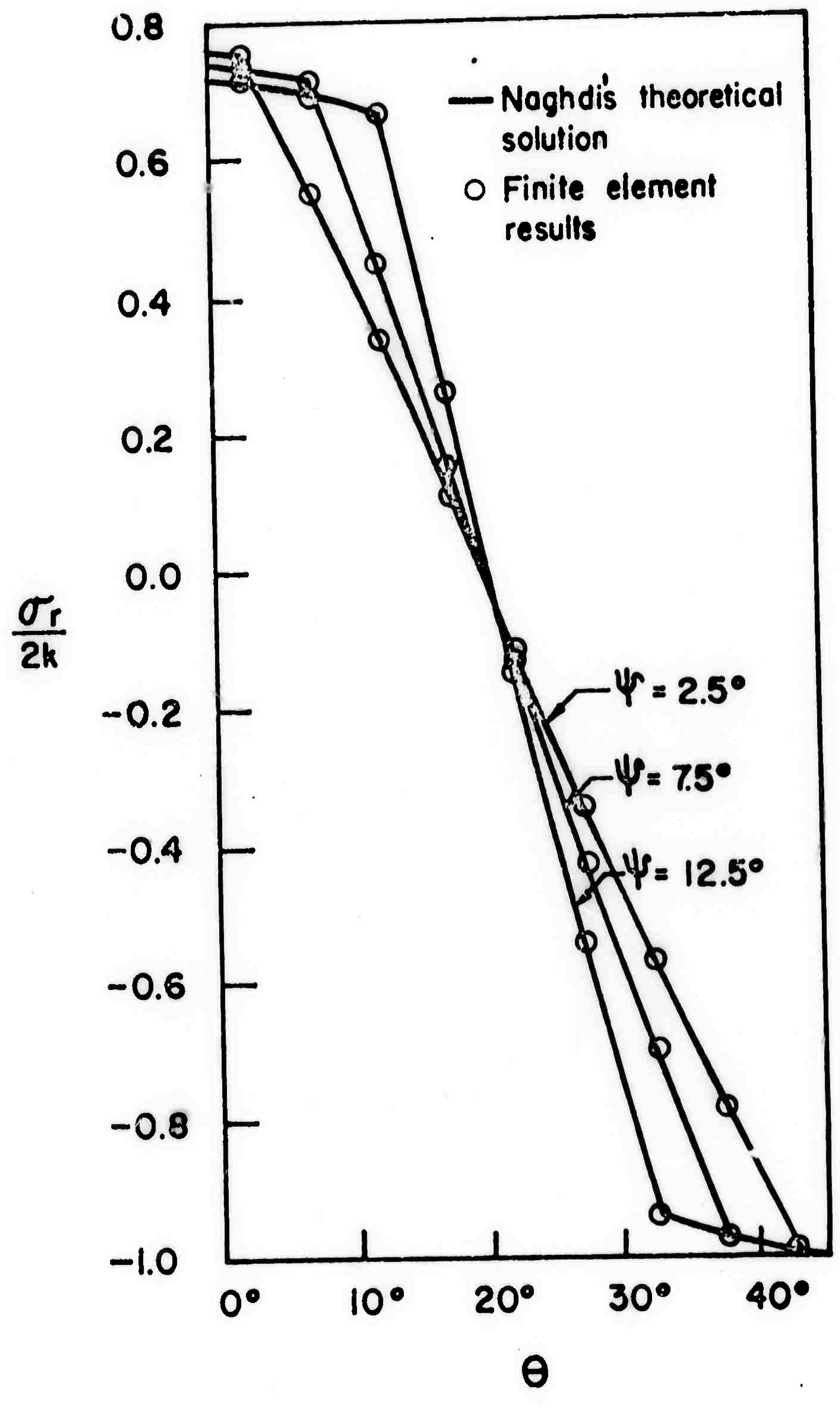
CALCULATION OF STRESS RATIO

FIG 3.1



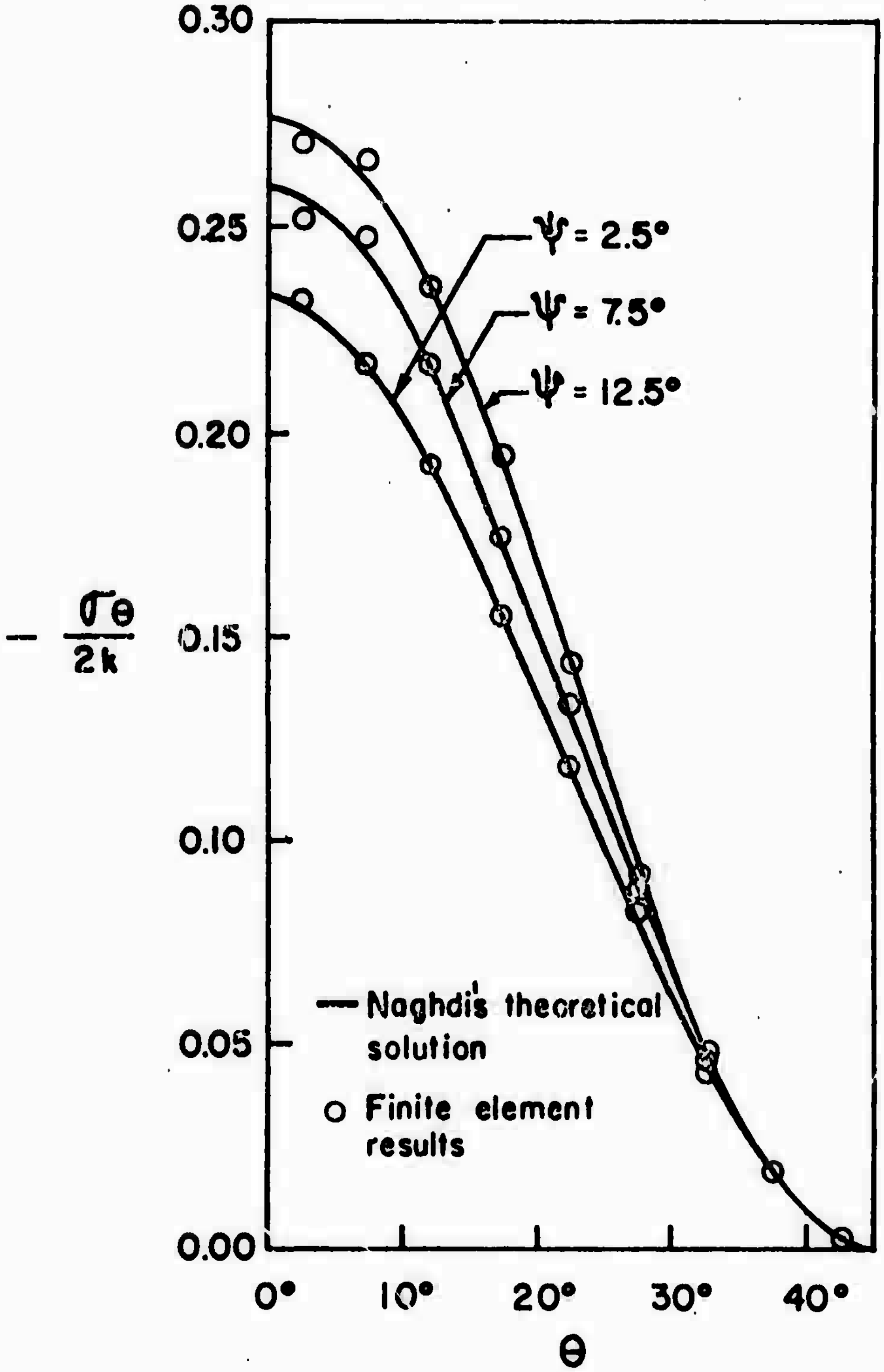
FINITE ELEMENT IDEALIZATION FOR ELASTIC-PLASTIC WEDGE

FIG 3.2



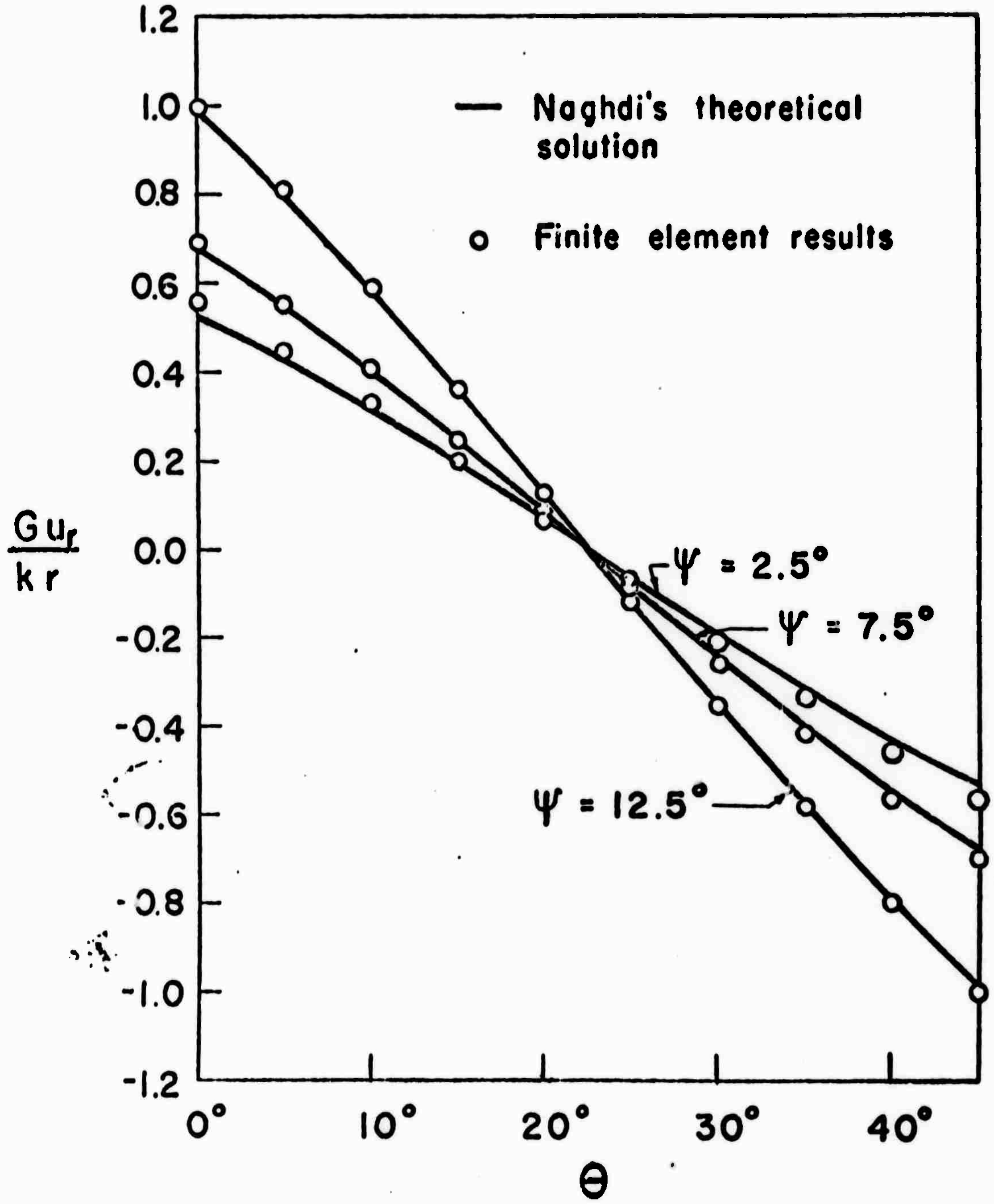
DISTRIBUTION OF RADIAL STRESS FOR WEDGE AT VARIOUS LOADS

FIG 3.3



DISTRIBUTION OF CIRCUMFERENTIAL STRESS FOR WEDGE AT VARIOUS LOADS

FIG 3.4



DISTRIBUTION OF RADIAL DISPLACEMENTS

FIG 3.5

CHAPTER IV

COMPUTER PROGRAM FOR ANALYSIS OF JOINTED ROCK

Chapter IV. Computer Program for Analysis of Jointed Rock

4.1 Organization

The computer program described here is based on the theory described in Chapter I and II. The rock mass is considered as a linear elastic material in the direction of compressive stresses and is assumed to have no resistance to deformation in the direction of principal stresses. The program corrects the discrepancy in the method presented by Zienkiewicz et al (1968). This was pointed out towards the end of Chapter I. It makes allowance for the boundary conditions, residual stresses, stresses due to temperature change, and varying pressure boundaries. This program also uses the quadrilateral elements and generates stiffness the same way as that described in Chapter III.

The principal program called MAIN controls all the data input and control information. It does the system initialization, prints the control data, geometrical and material properties. MAIN calls the subroutines for stiffness, solution of equations and stress calculations.

4.12. Stiffness Matrix

Stiffness matrix for the analysis is computed in blocks by the subroutine STIFF. For element stiffness, it calls QUAD for triangular and quadrilateral elements and ONED for bar elements. Direct stiffness technique was used to get the total stiffness. Equations are modified for displacement boundary conditions.

4.13. Load Matrix

Load matrix for the analysis is computed in LOAD subroutine. The varying pressure boundary is taken into account. The load matrix is modified in the LOAD subroutine for each iteration performed. This accounts for the nonlinearity introduced by progressive cracking by considering the change in element stiffness as a pseudo load.

4.14. Calculations of Displacements

After the stiffness and load matrices for a stage have been computed, the resulting equations are solved by calling subroutine SYMBAN. This subroutine was developed to take the advantage of the fact that stiffness matrix remains the same throughout. Gaussian elimination technique is used to solve the equations. The elimination is done once for all and the reduced matrix stored on auxiliary units. Solution for each iteration consists of back-substitution only. This approach results in considerable economy in machine time.

4.15. Calculations of Stresses

After the displacements have been computed, the stresses are computed using the constitutive law. A change in the minor principal stress corresponding to Poisson's effect has been introduced when one of the principal stresses was compressive and others tensile.

4.2. Input Data Preparation

1. Control Card (A6). This card will carry the characters START in columns 1-5. This will start the processing of the data deck which consists of the following set of cards.

2. Job Title (72H). This card will give the descriptive identification for the job.

3. Control Information (4I5, 3F10.2, I5, E15.4)

Information	Columns
Total number of nodal points	1 - 5
Total number of elements	6 - 10
Number of different materials	11 - 15
Number of pressure boundary cards	16 - 20
Body Force in X-direction	21 - 30
Body Force in Y-direction	31 - 40
Reference (stress-free) temperature	41 - 50
Number of Iterations	51 - 55
Tolerance to Convergence	56 - 70

4. Material Property Cards. One set of cards must be provided for each material. In each set:

a. First card (2I5, F10.3, I5) will give the following information:

Material identification number	1 - 5
Number of temperature cards (8 maximum)	6 - 10
Mass density of the material	11 - 20

Material code to designate materials which cannot take tension	21 - 26

code = 1 for materials which cannot take tension
0 for materials which can take tension.

b. Subsequent cards, one for each temperature, the number being defined in columns 6-10 of the first card, will carry the following information (4F10.0):

Information	Columns
Temperature	1 - 10
Elastic modulus	11 - 20
Poisson's ratio	21 - 30
Coefficient of thermal expansion	31 - 40

5. Nodal Point Cards (I5, F5.0, 5F10.0)

One card for each nodal point with the following information:

Nodal point number	1 - 5
Type of nodal point	6 - 10
X-ordinate	11 - 20
Y-ordinate	21 - 30
XR	31 - 40
XZ	41 - 50
Temperature	51 - 60

If the number in columns 6-10 is

Zero XR is the specified X-load and XZ is the specified Y-load

1 XR is the specified X-displacement and XZ is the specified Y-load

2 XR is the specified X-load and XZ is the specified Y-displacement

3 XR is the specified X-displacement and XZ is the specified Y-displacement

All loads are considered to be total forces acting on an element of unit thickness. Nodal point cards must be in numerical sequence. If cards are omitted, the omitted nodal points are generated at equal intervals along a straight line between the defined nodal points. The necessary temperatures are determined by linear interpolation. The type of the nodal point, as well as XR, XZ, are set equal to zero.

6. Element Material Cards (12I5)

These cards shall carry the material type of all the elements. Each card shall have material types for 12 elements in sequence. The material type for each element must be read in as no interpolation has been provided for.

7. Element Cards (5I5, 5X, 3 F10.0)

One card for each element will provide the following data.

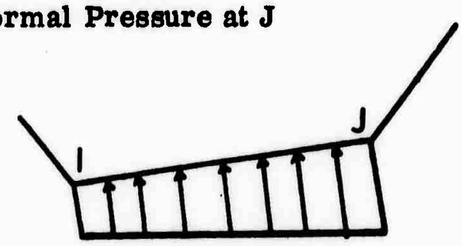
Information	Columns
Number of element	1 - 5
Nodal point I	6 - 10
Nodal point J	11 - 15
Nodal point K	16 - 20
Nodal point L	21 - 25
Initial stresses:	
(i) component in x-direction	31 - 40
(ii) component in y-direction	41 - 50
(iii) shearing stress on x-y planes	51 - 60

Nodal points I, J, K, L are corners of each individual element in a counter-clockwise order for a right handed system of coordinates. For triangular elements set nodal point L same as nodal point K. The element cards must be in the numerical sequence. Any cards that are omitted will be automatically generated in the program by incrementing each of the I, J, K, and L nodal points by one.

8. Pressure Boundary Cards (2I5, F10.0)

One card for each boundary element which is subjected to a normal pressure will carry the following information:

Information	Columns
Nodal Point I	1 - 5
Nodal Point J	6 - 10
Normal Pressure at I	11 - 20
Normal Pressure at J	21 - 30



As shown in the sketch, the boundary element must be on the left as one progresses from I to J. Surface tensile force is input as a negative pressure.

Output Information:

The following information is developed and printed by the program:

1. Reprint of input data
2. Nodal point displacements
3. Stresses at the center of each element

```
C       4.3 Program Listing
C       ******************************************************************
C       * TWO DIMENSIONAL ANALYSIS OF A NO TENSION SYSTEM                *
C       ******************************************************************
        DOUBLE PRECISION S,C,B,A,P,ST,SIG,U,V,CC,BB,CR,AREA,COMM,DU,DV,DL,
       1DX,DY,XL,RR,FOR,TOL,UR,UZ ,SIGI,VOL,COM,WORD(2),WORD1,E,COSA,SINA
       2,CU,ZZ,EL,XC,YC,PR,ACELR,ACELZ,TEMP,T, R,Q,RO,Z
       3,DR,DZ,DT,ZX
        COMMON ACELR,ACELZ,TEMP,Q,RO(12),R(900),Z(900),T(900),PR(200,2),
       1NUMNP,NUMEL,NUMMAT,NUMPC,MTYPE,LLL,N,HED(18),NTC(10),CODE(900),
       2MTC(10),IBC(200),JBC(200),NCHECK
        COMMON/SYSARG/UR(900),UZ(900),SIGI(900,6),CU(1800),TOL,VOL
       1,E(8,4,12)
        COMMON /ARG/C(3,3),S(10,10),SIG(6),P(8),ST(3,10),RR(5),ZZ(5),
       1XC,YC,FE(3),LM(4),IX(800,5)
        COMMON /BANARG/ MBAND,NUMBLK,B(1800),A(108,54),KKK,JA
        DATA WORD/6HSTART ,6HSTOP  /
        DEFINE FILE 1(50,1500,U,NBK),2(800,256,U,ID)
        CALL FRRSET(208,256,-1,1)
     10 READ(5,1006)   WORD1
        IF(WORD1.EQ.WORD(1))  GO TO 30
        IF(WORD1.EQ.WORD(2))   STOP
        GO TO 10
     30 READ (5,1000) HED,NUMNP,NUMEL,NUMMAT,NUMPC,ACELR,ACELZ,Q,NP,TOL
        WRITE(6,2000) HED,NUMNP,NUMEL,NUMMAT,NUMPC,ACELR,ACELZ,Q,NP,TOL
     40 DO 50 M=1,NUMMAT
        READ (5,1001) MTYPE,NTC(MTYPE),RO(MTYPE),MTC(MTYPE)
        WRITE(6,2001) MTYPE,NTC(MTYPE),RO(MTYPE),MTC(MTYPE)
        NUMTC=NTC(MTYPE)
        READ (5,1002) ((E(I,J,MTYPE),J=1,4),I=1,NUMTC)
        WRITE (6,2002) ((E(I,J,MTYPE),J=1,4),I=1,NUMTC)
     50 CONTINUE
        WRITE (6,2003)
        L=0
        R(L)=0.
        Z(L)=0.
        T(L)=0.
     60 READ  (5,1003) N,CODE(N),R(N),Z(N),UR(N),UZ(N),T(N)
        NL=L+1
        ZX=N-L
        DR=(R(N)-R(L))/ZX
        DZ=(Z(N)-Z(L))/ZX
        DT=(T(N)-T(L))/ZX
     70 L=L+1
        IF(N-L) 100,90,80
     80 CODE(L)=0.0
        R(L)=R(L-1)+DR
        Z(L)=Z(L-1)+DZ
```

```
      T(L)=T(L-1)+DT
      UR(L)=C.0
      UZ(L)=C.0
      GO TO 70
   90 WRITE (6,2004) (K,CODE(K),R(K),Z(K),UR(K),UZ(K),T(K),K=NL,N)
      IF(NUMNP-N) 100,110,60
  100 WRITE (6,2005) N
      CALL EXIT
  110 CONTINUE
      READ(5,1007) (IX(N,5),N=1,NUMEL)
      WRITE (6,2006)
      N=0
  130 READ(5,1C04) M,(IX(M,I),I=1,4),(SIGI(M,I),I=1,3)
      IF(M.EQ.1) GO TO 140
      ZX=M-N
      DO 135 I=1,3
  135 SIG(I)=(SIGI(M,I)-SIGI(N,I))/ZX
  140 N=N+1
      IF (M-N) 170,170,150
  150 IX(N,1)=IX(N-1,1)+1
      IX(N,2)=IX(N-1,2)+1
      IX(N,3)=IX(N-1,3)+1
      IX(N,4)=IX(N-1,4)+1
      DO 160 I=1,3
  160 SIGI(N,I)=SIGI(N-1,I)+SIG(I)
  170 WRITE(6,2007) N,(IX(N,I),I=1,5),(SIGI(N,I),I=1,3)
      IF (M-N) 180,180,140
  180 IF (NUMEL-N) 190,190,130
  190 CONTINUE
      IF (NUMPC) 290,31C,290
  290 WRITE (6,2008)
      DO 300 L=1,NUMPC
      READ(5,1CC5) IBC(L),JBC(L),PR(L,1),PR(L,2)
  300 WRITE(6,2009) IBC(L),JBC(L),PR(L,1),PR(L,2)
  310 CONTINUE
      DO 440 N=1,NUMNP
      NN=2*N
      CU(NN-1)=0.
  440 CU(NN)=0.
      J=0
      DO 340 N=1,NUMEL
      DO 340 I=1,4
      DO 325 L=1,4
      KK=IABS(IX(N,I)-IX(N,L))
      IF (KK-J) 325,325,320
  320 J=KK
  325 CONTINUE
  340 CONTINUE
```

```
      MBAND=2*J+2
      WRITE(6,2012) MBAND
  350 CALL STIFF
      NCHECK=1
      KKK=1
      CALL SYMBAN
      KKK=2
      DO 500 LLL=1,NP
      CALL LOAD
      CALL SYMBAN
      DO 400 N=1,NUMNP
      NN=2*N
      CU(NN-1)=CU(NN-1)+B(NN-1)
  400 CU(NN)=CU(NN)+B(NN)
      WRITE(6,2013) LLL
      WRITE(6,2010) (N,B(2*N-1),B(2*N),CU(2*N-1),CU(2*N), N = 1,NUMNP)
      CALL STRESS
      IF(NCHECK.EQ.0) GO TO 600
  500 CONTINUE
      GO TO 700
  600 WRITE(6,2011) LLL
  700 GO TO 10
  999 FORMAT(I5)
 1000 FORMAT (18A4/4I5,3F10.2,I5,E15.4,I5)
 1001 FORMAT (2I5,1F10.3,I5)
 1002 FORMAT (5F10.3)
 1003 FORMAT (I5,F5.1,5F10.4)
 1004 FORMAT (5I5,5X,3F10.4)
 1005 FORMAT(2I5,2F10.3)
 1006 FORMAT(A6)
 1007 FORMAT(12I5)
 2000 FORMAT (1H1 18A4/
     1 30H0 NUMBER OF NODAL POINTS------ I3 /
     2 30H0 NUMBER OF ELEMENTS---------- I3 /
     3 30H0 NUMBER OF DIFF. MATERIALS--- I3 /
     4 30H0 NUMBER OF PRESSURE CARDS---- I3 /
     5 30H0 X-ACCELERATION-------------- E12.4/
     6 30H0 Y-ACCELERATION-------------- E12.4/
     7 30H0 REFERENCE TEMPERATURE------- E12.4/
     8 30H0 NO. OF APPROXIMATIONS------- I5/
     9 30H0 TOLERANCE FOR CONVERGENCE--- E12.4)
 2001 FORMAT (17H0MATERIAL NUMBER= I3, 30H, NUMBER OF TEMPERATURE CARDS=
     1 I3, 15H, MASS DENSITY= E12.4,16H, MATERIAL CODE= I5)
 2002 FORMAT (15H0  TEMPERATURE 10X 5HEC     9X 6HNU      10X 5HALPHA/
     1(F15.2,3E15.5))
 2003 FORMAT (108H1NODAL POINT         TYPE  X ORDINATE  Y ORDINATE  X LO
     1AD OR DISPLACEMENT  Y LOAD OR DISPLACEMENT TEMPERATURE )
 2004 FORMAT (I12,F12.2,2F12.5,2E24.7,F12.3)
```

```
2005 FORMAT (26HONODAL POINT CARD ERROR N= I5)
2006 FORMAT (99H1ELEMENT NO.        I      J      K      L     MATERIAL
    1SIGIXX       SIGIYY       SIGIXY                                    )
2007 FORMAT (1I13,4I6,1I12,3F12.3)
2008 FORMAT(29HOPRESSURE BOUNDARY CONDITIONS/40H      I       J     PRESSUR
    1E I     PRESSURE J)
2009 FORMAT(2I6,2F14.3)
2010 FORMAT(12HON.P.NUMBER 17X 3HDUX 17X 3HDUY 18X 2HUX 18X 2HUY/
    1(1I12,4F20.7))
2011 FORMAT(35HO NUMBER OF CYCLES TO CONVERGENCE =    I5)
2012 FORMAT(28H BAND WIDTH FOR THIS DATA =     I5)
2013 FORMAT(30H1 RESULTS OF ITERATION  NO. =      I5///)
     END
```

```
      SUBROUTINE STIFF
C
      DOUBLE PRECISION S,C,B,A,P,S1,SIG,U,V,CC,BB,CR,AREA,COMM,DU,DV,DL,
     1DX,DY,XL,RR,FOR,TOL,UR,UZ  ,SIGI,VOL,COM,E,COSA,SINA
     2,CU,ZZ,EE,XC,YC,PR,ACELR,ACELZ,TEMP,T,  R,Q,RO,Z
      COMMON ACELR,ACELZ,TEMP,Q,RO(12),R(900),Z(900),T(900),PR(200,2),
     1NUMNP,NUMEL,NUMMAT,NUMPC,MTYPE,LLL,N,HED(18),NTC(10),CODE(900),
     2MTC(10),IBC(200),JBC(200),NCHECK
      COMMON/SYSARG/UR(900),UZ(900),SIGI(900,6),CU(1800),TOL,VOL
     1,E(8,4,12)
      COMMON /ARG/C(3,3),S(10,10),SIG(6),P(8),ST(3,10),RR(5),ZZ(5),
     1XC,YC,EE(3),LM(4),IX(800,5)
      COMMON /BANARG/ MBAND,NUMBLK,B(1800),A(108,54),KKK,JA
      DEFINE FILE 1(50,1500,U,NBK),2(800,256,U,ID)
      NB=27
      ND=2*NB
      ND2=2*ND
      STOP=0.0
      NUMBLK=0
      JA=ND2*(MBAND+1)/1500+1
      NBK=1
      DO 50 N=1,ND2
      DO 50 M=1,ND
   50 A(N,M)=0.0
   60 NUMBLK=NUMBLK+1
      NH=NB*(NUMBLK+1)
      NM=NH-NB
      NL=NM-NB+1
      KSHIFT=2*NL-2
      DO 210 N=1,NUMEL
      IF (IX(N,5)) 210,210,65
   65 DO 80 I=1,4
      IF (IX(N,I)-NL) 80,70,70
   70 IF (IX(N,I)-NM) 90,90,80
   80 CONTINUE
      GO TO 210
   90 IF(IX(N,3)-IX(N,2)) 95,85,95
   85 CALL ONED
      IX(N,5)=-IX(N,5)
      MM=2
      GO TO 130
   95 CALL QUAD
      IX(N,5)=-IX(N,5)
      ID=N
      WRITE(2'ID) ((C(IIK,JJK),JJK=1,3),EE(IIK),IIK=1,3)
     1 ,((S(JJI,KKI),KKI=1,8),JJI=1,8),((ST(IKK,JKK),JKK=1,8),IKK=1,3)
     2 ,(RR(JII),ZZ(JII),JII=1,4),XC,YC,TEMP,VOL
      IF(VOL) 100,100,110
```

```
100 WRITE(6,2000) N
    STOP=1.0
110 MM=4
    IF(IX(N,3)-IX(N,4)) 130,120,130
120 MM=3
130 DO 140 I=1,MM
140 LM(I)=2*IX(N,I)-2
    DO 200 I=1,MM
    DO 200 K=1,2
    II=LM(I)+K-KSHIFT
    KK=2*I-2+K
    DO 200 J=1,MM
    DO 200 L=1,2
    JJ=LM(J)+L-II+1-KSHIFT
    LL=2*J-2+L
    IF(JJ) 200,200,175
175 IF(ND-JJ) 180,195,195
180 WRITE (6,2001) N
    STOP=1.0
    GO TO 210
195 A(II,JJ)=A(II,JJ)+S(KK,LL)
200 CONTINUE
210 CONTINUE
    DO 400 M=NL,NH
    IF (M-NUMNP) 315,315,400
315 U=UR(M)
    N=2*M-1-KSHIFT
    IF(CODE(M)) 390,400,316
316 IF(CODE(M)-1.) 317,370,317
317 IF(CODE(M)-2.) 318,390,318
318 IF(CODE(M)-3.) 390,380,390
370 CALL MODIFY(A,ND2,MBAND,N)
    GO TO 400
380 CALL MODIFY(A,ND2,MBAND,N)
390 U=UZ(M)
    N=N+1
    CALL MODIFY(A,ND2,MBAND,N)
400 CONTINUE
    WRITE(1'NBK) ((A(N,M),M=1,MBAND),N=1,ND)
    NBK=NBK+JA
    DO 420 N=1,ND
    K=N+ND
    DO 420 M=1,ND
    A(N,M)=A(K,M)
420 A(K,M)=0.0
    IF(NM.LT.NUMNP) GO TO 60
    DO 480 N=1,NUMEL
    IX(N,5)=IABS(IX(N,5))
```

```
 480 CONTINUE
     IF(STOP) 490,500,490
 490 CALL EXIT
 500 RETURN
2000 FORMAT (26HONEGATIVE AREA ELEMENT NO. I4)
2001 FORMAT (29HOBAND WIDTH EXCEEDS ALLOWABLE I4)
     END
```

```
      SUBROUTINE ONED
C
      DOUBLE PRECISION S,C,B,A,P,ST,SIG,U,V,CC,BB,CR,AREA,COMM,DU,DV,DL,
     1DX,DY,XL,RR,FOR,TOL,UR,UZ  ,SIGI,VOL,COM,E,COSA,SINA
     2,CU,ZZ,EE,XC,YC,PR,ACELR,ACELZ,TEMP,T,  R,Q,RO,Z
      COMMON ACELR,ACELZ,TEMP,Q,RO(12),R(900),Z(900),T(900),PR(200,2),
     1NUMNP,NUMEL,NUMMAT,NUMPC,MTYPE,LLL,N,HED(18),NTC(10),CODE(900),
     2MTC(10),IBC(200),JBC(200),NCHECK
      COMMON/SYSARG/UR(900),UZ(900),SIGI(900,6),CU(1800),TOL,VOL
     1,E(8,4,12)
      COMMON /ARG/C(3,3),S(10,10),SIG(6),P(8),ST(3,10),RR(5),ZZ(5),
     1XC,YC,EE(3),LM(4),IX(800,5)
      COMMON /BANARG/ MBAND,NUMBLK,B(1800),A(108,54),KKK,JA
C
      DO 100 I=1,8
      P(I)=0.0
      DO 100 J=1,8
  100 S(I,J)=0.0
      MTYPE=IX(N,5)
      I=IX(N,1)
      J=IX(N,2)
      DX=R(J)-R(I)
      DY=Z(J)-Z(I)
      XL=DSQRT(DX**2+DY**2)
      COSA=DX/XL
      SINA=DY/XL
      COMM=E(1,2,MTYPE)*E(1,4,MTYPE)/XL
C
      S(1,1)=COSA*COSA*COMM
      S(1,2)=COSA*SINA*COMM
      S(1,3)=-S(1,1)
      S(1,4)=-S(1,2)
      S(2,1)=S(1,2)
      S(2,2)=SINA*SINA*COMM
      S(2,3)=-S(1,2)
      S(2,4)=-S(2,2)
      S(3,1)=S(1,3)
      S(3,2)=S(2,3)
      S(3,3)=S(1,1)
      S(3,4)=S(1,2)
      S(4,1)=S(1,4)
      S(4,2)=S(2,4)
      S(4,3)=S(3,4)
      S(4,4)=S(2,2)
C
C
      RETURN
C
```

```
      SUBROUTINE QUAD
      DOUBLE PRECISION S,C,B,A,P,ST,SIG,U,V,CC,BB,CR,AREA,COMM,DU,DV,DL,
     1DX,DY,XL,RR,FOR,TOL,UR,UZ ,SIGI,VOL,COM,E,COSA,SINA
     2,CU,ZZ,EE,XC,YC,PR,ACELR,ACELZ,TEMP,T, R,Q,RO,Z
     3,U,V,XT,XS,RATIO,DEN,XNT
      COMMON ACELR,ACELZ,TEMP,Q,RO(12),R(900),Z(900),T(900),PR(200,2),
     1NUMNP,NUMEL,NUMMAT,NUMPC,MTYPE,LLL,N,HED(18),NTC(10),CODE(900),
     2MTC(10),IBC(200),JBC(200),NCHECK
      COMMON/SYSARG/UR(900),UZ(900),SIGI(900,6),CU(1800),TOL,VOL
     1,E(8,4,12)
      COMMON /ARG/C(3,3),S(10,10),SIG(6),P(8),ST(3,10),RR(5),ZZ(5),
     1XC,YC,EE(3),LM(4),IX(800,5)
      COMMON /BANARG/ MBAND,NUMBLK,B(1800),A(108,54),KKK,JA
C
      DIMENSION U(3),V(3)
      I=IX(N,1)
      J=IX(N,2)
      K=IX(N,3)
      L=IX(N,4)
      MTYPE=IX(N,5)
      VOL=0.
      TEMP=(T(I)+T(J)+T(K)+T(L))/4.0
      RATIO=0.0
      NUMTC=NTC(MTYPE)
      IF (NUMTC.EQ.1) GO TO 100
      DO 50 M=2,NUMTC
      IF (E(M,1,MTYPE)-TEMP) 50,60,60
   50 CONTINUE
   60 DEN=E(M,1,MTYPE)-E(M-1,1,MTYPE)
      IF(DEN) 70,80,70
   70 RATIO=(TEMP-E(M-1,1,MTYPE))/DEN
   80 DO 90 KK=1,3
   90 EE(KK)=E(M-1,KK+1,MTYPE)+RATIO*(E(M,KK+1,MTYPE)-E(M-1,KK+1,MTYPE))
      GO TO 110
  100 DO 105 KK=1,3
  105 EE(KK)=E(1,KK+1,MTYPE)
  110 CONTINUE
      COMM=EE(1)/(1.-EE(2)**2)
      C(1,1)=COMM
      C(1,2)=COMM*EE(2)
      C(1,3)=0.
      C(2,1)=C(1,2)
      C(2,2)=C(1,1)
      C(2,3)=0.
      C(3,1)=0.
      C(3,2)=0.
      C(3,3)=.5*COMM*(1.-EE(2))
      DO 130 J=1,10
```

```
      DO 120 I=1,3
  120 ST(I,J)=0.
      DO 130 I=1,10
  130 S(I,J)=0.
      DO 140 I=1,4
      NPP=IX(N,I)
      RR(I)=R(NPP)
  140 ZZ(I)=Z(NPP)
      IF(IX(N,3)-IX(N,4)) 145,150,145
  145 XC=(RR(1)+RR(2)+RR(3)+RR(4))/4.
      YC=(ZZ(1)+ZZ(2)+ZZ(3)+ZZ(4))/4.
      RR(5)=XC
      ZZ(5)=YC
      K=5
      J=1
      I=4
      LM(3)=9
      NT=4
      GO TO 160
  150 NT=1
      LM(3)=5
      I=1
      K=3
      J=2
      XC=(RR(1)+RR(2)+RR(3))/3.
      YC=(ZZ(1)+ZZ(2)+ZZ(3))/3.
      RR(5)=RR(3)
      ZZ(5)=ZZ(3)
  160 DO 200 NN=1,NT
      LM(1)=2*I-1
      LM(2)=2*J-1
      U(1)=ZZ(J)-ZZ(K)
      U(2)=ZZ(K)-ZZ(I)
      U(3)=ZZ(I)-ZZ(J)
      V(1)=RR(K)-RR(J)
      V(2)=RR(I)-RR(K)
      V(3)=RR(J)-RR(I)
      AREA=(RR(J)*U(2)+RR(I)*U(1)+RR(5)*U(3))/2.
      VOL=VOL+AREA
      COMM=.25/AREA
      XNT=NT
      COM=2./XNT
      COM=COM*COMM
      DO 180 L=1,3
      II=LM(L)
      ST(1,II)=ST(1,II)+U(L)*COM
      ST(2,II+1)=ST(2,II+1)+V(L)*COM
      ST(3,II)=ST(3,II)+V(L)*COM
```

```
      ST(3,II+1)=ST(3,II+1)+U(L)*COM
      DO 180 M=1,3
      JJ=LM(M)
      S(II,JJ)=S(II,JJ)+(U(L)*C(1,1)*U(M)+V(L)*C(3,3)*V(M))*COMM
      S(II,JJ+1)=S(II,JJ+1)+(U(L)*C(1,2)*V(M)+V(L)*C(3,3)*U(M))*COMM
      S(II+1,JJ+1)=S(II+1,JJ+1)+(V(L)*C(1,1)*V(M)+U(L)*C(3,3)*U(M))*COMM
      S(JJ+1,II)=S(II,JJ+1)
180   CONTINUE
      I=J
      J=J+1
200   CONTINUE
      IF(IX(N,3)-IX(N,4)) 220,250,220
220   DO 240 I=1,2
      KK=10-I
      DO 240 K=1,KK
      CC=S(KK+1,K)/S(KK+1,KK+1)
      DO 230 J=1,3
230   ST(J,K)=ST(J,K)-CC*ST(J,KK+1)
      DO 240 J=1,KK
240   S(J,K)=S(J,K)-CC*S(J,KK+1)
250   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE STRESS
C
      DOUBLE PRECISION S,C,B,A,P,ST,SIG,U,V,CC,BB,CR,AREA,COMM,DU,DV,DL,
     1DX,DY,XL,RR,FOR,TOL,UR,UZ ,SIGI,VOL,COM,E,COSA,SINA
     2,CU,ZZ,EE,XC,YC,PR,ACELR,ACELZ,TEMP,T, R,Q,RO,Z
     3,CC,BB,CR,SS,SC,S2,C2,EPS,DT
      COMMON ACELR,ACELZ,TEMP,Q,RO(12),R(900),Z(900),T(900),PR(200,2),
     1NUMNP,NUMEL,NUMMAT,NUMPC,MTYPE,LLL,N,HED(18),NTC(10),CODE(900),
     2MTC(10),IBC(200),JBC(200),NCHECK
      COMMON/SYSARG/UR(900),UZ(900),SIGI(900,6),CU(1800),TOL,VOL
     1,E(8,4,12)
      COMMON /ARG/C(3,3),S(10,10),SIG(6),P(8),ST(3,10),RR(5),ZZ(5),
     1XC,YC,EE(3),LM(4),IX(800,5)
      COMMON /BANARG/ MBAND,NUMBLK,B(1800),A(108,54),KKK,JA
      DEFINE FILE 1(50,1500,U,NBK),2(800,256,U,ID)
      FOR = 0.0
      MPRINT=0
      DO 600 M=1,NUMEL
      ID=M
      FIND(2'ID)
      N=M
      IX(N,5)=IABS(IX(N,5))
      MTYPE=IX(N,5)
      SIGI(N,4)=0.
      SIGI(N,5)=0.
      SIGI(N,6)=0.
      IF(IX(N,3)-IX(N,2)) 90,60,90
   60 I=IX(N,1)
      J=IX(N,2)
      XC=(R(I)+R(J))/2.0
      YC=(Z(I)+Z(J))/2.0
      DX=R(J)-R(I)
      DY=Z(J)-Z(I)
      XL=DSQRT(DX**2+DY**2)
      DU=B(2*J-1)-B(2*I-1)
      DV=B(2*J)-B(2*I)
      DL =DV*DY/XL+DU*DX/XL
      SIG(1)=E(1,4,MTYPE)*DL*E(1,2,MTYPE)/XL+SIGI(N,1)*E(1,4,MTYPE)
      IF(SIG(1).GT.0.) GO TO 100
      SIGI(N,1)=SIG(1)
      GO TO 500
  100 SIGI(N,4)=E(1,2,MTYPE)*DL/XL+SIGI(N,1)
      SIGI(N,1)=0.
      GO TO 420
   90 READ(2'ID)  ((C(IIK,JJK),JJK=1,3),EE(IIK),IIK=1,3)
     1 ,((S(JJI,KKI),KKI=1,8),JJI=1,8),((ST(IKK,JKK),JKK=1,8),IKK=1,3)
     2 ,(RR(JII),ZZ(JII),JII=1,4),XC,YC,TEMP,VOL
      MM=4
```

```
      IF(IX(N,3)-IX(N,4)) 170,160,170
  160 MM=3
  170 DO 180 I=1,3
      RR(I)=0.
      DO 180 J=1,MM
      II=2*J
      JJ=2*IX(N,J)
  180 RR(I)=RR(I)+ST(I,II)*B(JJ)+ST(I,II-1)*B(JJ-1)
      IF(LLL.GT.1) GO TO 182
      DT=TEMP-Q
      DX=EE(3)*DT
      DY=DX
      SIG(1)=-C(1,1)*DX-C(1,2)*DY +SIGI(N,1)
      SIG(2)=-C(2,1)*DX-C(2,2)*DY +SIGI(N,2)
      SIG(3)=SIGI(N,3)
      GO TO 184
  182 DO 183 I=1,3
  183 SIG(I)=0.0
  184 CONTINUE
      DO 190 I=1,3
      DO 185 J=1,3
  185 SIG(I)=SIG(I)+C(I,J)*RR(J)
  190 CONTINUE
      IF(LLL.EQ.1) GO TO 195
      DO 192 I=1,3
  192 SIG(I)= SIG(I)+SIGI(N,I)
  195 CONTINUE
      CC=(SIG(1)+SIG(2))/2.0
      BB=(SIG(1)-SIG(2))/2.
      CR=DSQRT(BB**2+SIG(3)**2)
      SIG(4)=CC+CR
      SIG(5)=CC-CR
      SIG(6)=0.0
      IF((BB.EQ.0.).AND.(SIG(3).EQ.0.)) GO TO 200
      SIG(6)=28.648*DATAN2(SIG(3),BB)
      DX=0.0
  200 SIGI(N,1)=SIG(1)
      SIGI(N,2)=SIG(2)
      SIGI(N,3)=SIG(3)
      IF((SIG(4).LE.0.00).OR.(MTC(MTYPE).EQ.0)) GO TO 500
      IF(SIG(5).GE.0.00001) GO TO 370
      EPS=SIG(6)/57.296
      CC=DCOS(EPS)
      SS=DSIN(EPS)
      C2=CC*CC
      S2=SS*SS
      SC=SS*CC
C
```

```
      DX= EF(2)*SIG(4)
      SIGI(N,4)= SIG(4)*C2+DX*S2
      SIGI(N,5)= SIG(4)*S2+DX*C2
      SIGI(N,6)=SIG(4)*SC-DX*SC
C
C
      GO TO 400
  370 SIGI(N,4)=SIG(1)
      SIGI(N,5)=SIG(2)
      SIGI(N,6)=SIG(3)
  400 SIGI(N,1)=SIG(1)-SIGI(N,4)
      SIGI(N,2)=SIG(2)-SIGI(N,5)
      SIGI(N,3)=SIG(3)-SIGI(N,6)
  420 DX=SIGI(N,4)**2+SIGI(N,5)**2+SIGI(N,6)**2
      DX=DSQRT(DX)
      IF(DX.LE.FOR) GO TO 450
      IJK=N
      FOR=DX
  450 CONTINUE
  500 IF(MPRINT) 550,520,550
  520 WRITE(6,2000)
      MPRINT=50
  550 MPRINT=MPRINT-1
      WRITE(6,2001)N,XC,YC,(SIG(I),I=1,6),DX
  600 CONTINUE
      WRITE(6,2002)FOR,IJK
      IF(FOR.LE.TOL) NCHECK = 0
      RETURN
 2000 FORMAT (7H1EL.NO.  7X 1HX  7X 1HY 4X 8HX-STRESS 4X 8HY-STRESS 3X
     1 9HXY-STRESS 2X 10HMAX-STRESS 2X 10HMIN-STRESS 7H  ANGLE 2X 17HUNB
     2ALANCED FORCE    )
 2001 FORMAT (I7,2F8.2,1P5E12.4,0P1F7.2,1PE20.4)
 2002 FORMAT(30H0MAXIMUM UNBALANCED FORCE =    E12.5,16H IN ELEMENT NO.
     1 I5)
      END
```

```
      SUBROUTINE MODIFY(A,NEQ,MBAND,N)
      DOUBLE PRECISION A
      DIMENSION A(108,54)
      DO 250 M=2,MBAND
      K=N-M+1
      IF(K.LE.0) GO TO 235
      A(K,M)=0.0
  235 K=N+M-1
      IF(NEQ.LT.K) GO TO 250
      A(N,M)=0.0
  250 CONTINUE
      A(N,1)=1.0
      RETURN
      END
```

```
      SUBROUTINE  LOAD
      DOUBLE PRECISION S,C,B,A,P,ST,SIG,U,V,CC,BB,CR,AREA,COMM,DU,DV,DL,
     1DX,DY,XL,RR,FOR,TOL,UR,UZ  ,SIGI,VOL,COM,E,COSA,SINA
     2,CU,ZZ,EE,XC,YC,PR,ACELR,ACELZ,TEMP,T,  R,Q,RO,Z
     3,PP1,PP2,DR,DZ,EP,XMM,DT
      COMMON ACELR,ACELZ,TEMP,Q,RO(12),R(900),Z(900),T(900),PR(200,2),
     1NUMNP,NUMEL,NUMMAT,NUMPC,MTYPE,LLL,N,HED(18),NTC(10),CODE(900),
     2MTC(10),IBC(200),JBC(200),NCHECK
      COMMON/SYSARG/UR(900),UZ(900),SIGI(900,6),CU(1800),TOL,VOL
     1,E(8,4,12)
      COMMON /ARG/C(3,3),S(10,10),SIG(6),P(8),ST(3,10),RR(5),ZZ(5),
     1XC,YC,EE(3),LM(4),IX(800,5)
      COMMON /BANARG/ MBAND,NUMBLK,B(1800),A(108,54),KKK,JA
      DO 50 N=1,NUMNP
      B(2*N-1)=UR(N)
      B(2*N)=UZ(N)
      UR(N)=0.
      UZ(N)=0.
   50 CONTINUE
      IF((NUMPC.EQ.0).OR.(LLL.GT.1)) GO TO 300
      DO 200 L=1,NUMPC
      I=IBC(L)
      J=JBC(L)
      DR=Z(I)-Z(J)
      DZ=R(J)-R(I)
      PP2=(PR(L,2)+PR(L,1))/6.
      PP1=PP2+PR(L,1)/6.
      PP2=PP2+PR(L,2)/6.
      II=2*I
      JJ=2*J
      B(II-1)=B(II-1)+PP1*DR
      B(II)=B(II)+PP1*DZ
      B(JJ-1)=B(JJ-1)+PP2*DR
      B(JJ)=B(JJ)+PP2*DZ
  200 CONTINUE
  300 DO 700 N=1,NUMEL
      I=IX(N,1)
      J=IX(N,2)
      K=IX(N,3)
      L=IX(N,4)
      MTYPE=IX(N,5)
      IF(LLL.EQ.1) GO TO 330
      IF(MTC(MTYPE).EQ.0) GO TO 700
      IF(SIGI(N,4).NE.0.) GO TO 320
      IF(SIGI(N,5).NE.0.) GO TO 320
      IF(SIGI(N,6).NE.0.) GO TO 320
      GO TO 700
  320 CONTINUE
```

```
330 IF(J.EQ.K) GO TO 500
    ID=N
    READ(2'ID)  ((C(IIK,JJK),JJK=1,3),EE(IIK),IIK=1,3)
   1 ,((S(JJI,KKI),KKI=1,8),JJI=1,8),((ST(IKK,JKK),JKK=1,8),IKK=1,3)
   2 ,(RR(JII),ZZ(JII),JII=1,4),XC,YC,TEMP,VOL
    IF(LLL.EQ.1) GO TO 400
    SIG(1)=-SIGI(N,4)
    SIG(2)=-SIGI(N,5)
    SIG(3)=-SIGI(N,6)
    GO TO 450
400 DT=TEMP-Q
    DX=EE(3)*DT
    DY=EE(3)*DT
    SIG(1)=-C(1,1)*DX-C(1,2)*DY +SIGI(N,1)
    SIG(2)=-C(2,1)*DX-C(2,2)*DY +SIGI(N,2)
    SIG(3)=SIGI(N,3)
450 DO 520 I=1,8
    P(I)=0.0
    DO 510 J=1,3
510 P(I)=P(I)-ST(J,I)*SIG(J)
520 P(I)=P(I)*VOL
    IF(LLL.EQ.1) GO TO 540
    DO 530 I=1,3
530 SIG(I)=0.0
    GO TO 600
540 MM=4
    IF(IX(N,3).EQ.IX(N,4)) MM=3
    XMM=MM
    DY=VOL*ACELZ*RO(MTYPE)/XMM
    DX=VOL*ACELR*RO(MTYPE)/XMM
    DO 550 I=1,MM
    P(2*I)=P(2*I)+DY
550 P(2*I-1)=P(2*I-1)+DX
    GO TO 600
500 CALL ONED
    DX=R(J)-R(I)
    DY=Z(J)-Z(I)
    EP=-SIGI(N,4)/E(1,2,MTYPE)
    DX=DX*EP
    DY=DY*EP
    P(1)=S(1,1)*DX+S(1,2)*DY
    P(2)=S(2,1)*DX+S(2,2)*DY
    P(3)=-P(1)
    P(4)=-P(2)
600 DO 620 II=1,4
620 LM(II)=2*IX(N,II)-1
    DO 650 JJ=1,4
    II=LM(JJ)
```

```
      B(II)=B(II)+P(2*JJ-1)
  650 B(II+1)=B(II+1)+P(2*JJ)
  700 CONTINUE
      DO 750 N=1,NUMNP
      IF(CODE(N).EQ.0.) GO TO 750
      IF((CODE(N).EQ.1.).OR.(CODE(N).EQ.3.)) B(2*N-1)=0.
      IF((CODE(N).EQ.2.).OR.(CODE(N).EQ.3.)) B(2*N)=0.0
  750 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SYMBAN
      DOUBLE PRECISION S,C,B,A,P,ST,SIG,U,V,CC,BB,CR,AREA,COMM,DU,DV,DL,
     1DX,DY,XL,RR,FUR,TOL,UR,UZ  ,SIGI,VOL,COM,E,COSA,SINA
     2,CU,ZZ,EE,XC,YC,PR,ACELR,ACELZ,TEMP,T,DT,Q,RO
      COMMON /BANARG/ MBAND,NUMBLK,B(1800),A(108,54),KKK,JA
      DEFINE FILE 1(50,1500,U,NBK),2(800,256,U,ID)
C
C
      NN=54
      NL=NN+1
      NH=NN+NN
      NB=0
      NNK=0
      NBK=1
      FIND(1'1)
      IF(KKK.GT.1) GO TO 2000
      GO TO 150
  100 NBK=NNK+JA
      FIND(1'NBK)
      NB=NB+1
      DO 125 N=1,NN
      NM=NN+N
      DO 125 M=1,MBAND
      A(N,M)=A(NM,M)
  125 A(NM,M)=0.
C
      IF(NUMBLK-NB) 150,200,150
  150 READ(1'NBK)((A(N,M),M=1,MBAND),N=NL,NH)
      NNK=NBK
      IF(NB) 200,100,200
  200 DO 300 N=1,NN
      IF(A(N,1)) 225,300,225
  225 DO 275 L=2,MBAND
      IF(A(N,L)) 230,275,230
  230 C=A(N,L)/A(N,1)
      I=N+L-1
      J=0
      DO 250 K=L,MBAND
      J=J+1
  250 A(I,J)=A(I,J)-C*A(N,K)
      A(N,L)=C
  275 CONTINUE
  300 CONTINUE
      NBK=NBK-JA
      WRITE(1'NBK) ((A(N,M),M=1,MBAND),N=1,NN)
      IF(NUMBLK.EQ.NB) RETURN
      GO TO 100
 2000 NQ=0
```

```
      NEQ=1800
      GO TO 450
 400  NB=NB+1
      NBK=NNK+JA
      FIND(1'NBK)
      DO 425 N=1,NN
      NM=NN+N
      DO 425 M=1,MBAND
      A(N,M)=A(NM,M)
 425  A(NM,M)=0.
      IF(NUMBLK-NB) 450,500,450
 450  READ(1'NBK)((A(N,M),M=1,MBAND),N=NL,NH)
      NNK=NBK
      IF(NB) 500,400,500
 500  DO 550 N=1,NN
      J=NQ+N
      DO 540 L=2,MBAND
      I=J+L-1
      IF(NEQ-I) 545,540,540
 540  B(I)=B(I)-A(N,L)*B(J)
 545  IF(A(N,1).EQ.0.) A(N,1)=1.
 550  B(J)=B(J)/A(N,1)
      NBK=NNK-JA
      IF(NUMBLK.EQ.NB) GO TO 700
 600  NQ=NQ+NN
      GO TO 400
 700  DO 750 M=1,NN
      N=NN+1-M
      J=NQ+N
      DO 750 L=2,MBAND
      IF(A(N,L)) 740,750,740
 740  I=J+L-1
      IF(NEQ-I) 750,745,745
 745  B(J)=B(J)-A(N,L)*B(I)
 750  CONTINUE
      NB=NB-1
      IF(NB.EQ.0) RETURN
      FIND(1'NBK)
      DO 800 N=1,NN
      NM=NN+N
      DO 800 M=1,MBAND
      A(NM,M)=A(N,M)
 800  A(N,M)=0.
      READ (1'NBK) ((A(N,M),M=1,MBAND),N=1,NN)
      NBK=NBK-JA
      NQ=NQ-NN
      GO TO 700
      END
```

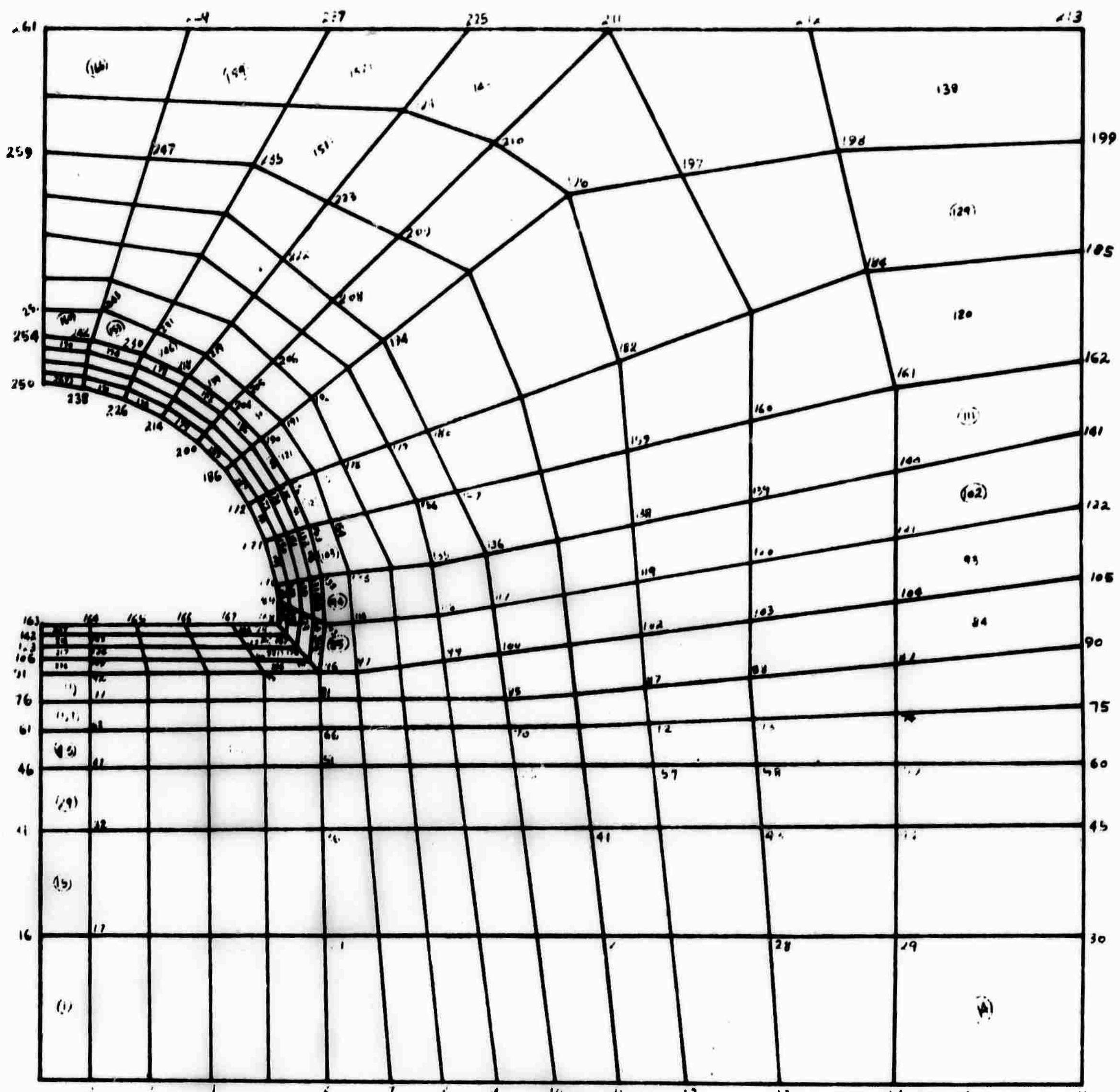
4.4 Example

A tunnel whose configuration and finite element idealization is shown in Fig. 4-1 was solved assuming rock to crack under tension. Lining was assumed to be capable of resisting tensile loads. Figs. 4-2 and 4-3 show the initial elastic and final 'no tension' solutions. The tensile stresses are indicated by arrows. Comparison of Fig. 4-2 and 4-3 shows the redistribution of stresses caused by the inability of rock to withstand tension. In the example, the excavation and lining of tunnel is assumed to be a single step ignoring the effect of sequential operations. This is unrealistic and further development will allow for actual sequence of construction. However, the example illustrates use of the computer program and is similar to the one used by Zienkiewicz, Valliappan and King (1968).

The following material constants were used for the solution:

Lining: $E = 2 \times 10^6$ psi
$\nu = 0.15$
$\gamma = 150$ lbs./ft.^{3}

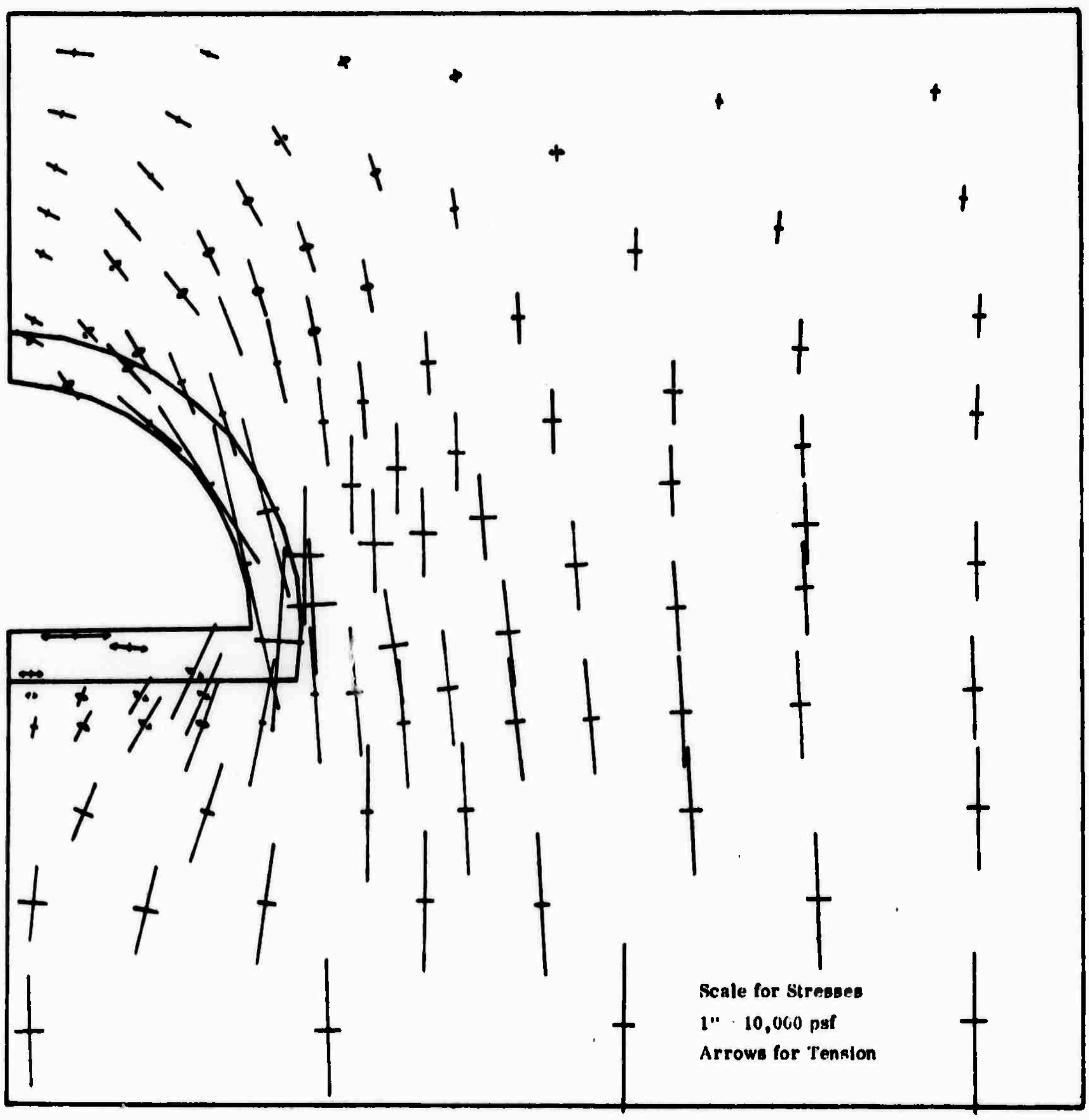
Rock: $E = 1 \times 10^6$ psi
$\nu = 0.2$
$\gamma = 150$ lbs./ft.^{3}



Scale 1" = 10'

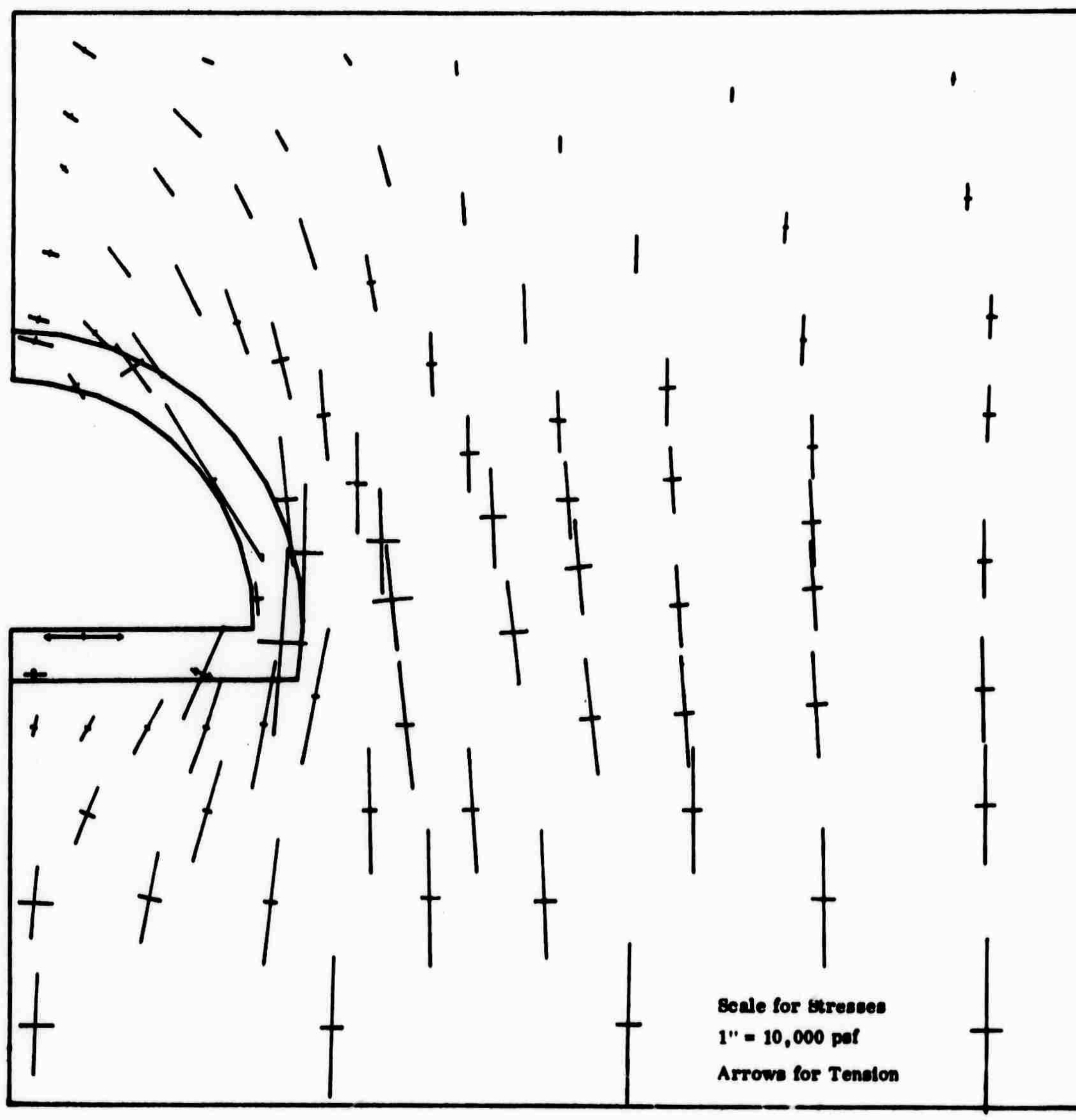
FINITE ELEMENT IDEALIZATION OF LINED TUNNEL

FIG. 4.1



ELASTIC SOLUTION FOR THE LINED TUNNEL

FIG. 4.2



FINAL NO-TENSION SOLUTION FOR THE LINED TUNNEL

FIG. 4.3

CHAPTER V

ADDITIONAL COMMENTS

Chapter V. Additional Comments

The work reported is continuing and significant changes may be made before the finite element computer programs reach their final form. In the case of elastic-plastic materials following Mohr-Coulomb Theory non-monotonic loads have to be allowed for and also alternative numerical solution procedures have to be examined. For the case of jointed rock the program included in the report represents a necessary first step. Modification to include the Griffith failure criterion is being done. For Griffith rupture caused by a tensile stress field acting on pre-existing flaws, the solution scheme appears to be fairly straight forward. However, the case of fracture under compressive stress fields may require development of new methods.

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