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Impedance Circuits Imbedding an LC-Lattice Two-Port

KURT H. HAASE

AIR FORCE SYSTEMS COMMAND

United States Air Force

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1. INTRODUCTION

In 1963, Fusachika Miyata (1963) showed that a positive real driving-point function $F(s) = N(s)/D(s)$, where $D(s)$ is of the degree 5 and $N(s)$ of the degree 4, could be realized by the circuit shown in Figure 1. This is possible, provided that $N(s)$ and $D(s)$ satisfy some conditions beyond the mere necessity of making up a positive real function. The circuit shown needs only a few elements and contains no transformers.

This paper originates from Miyata's, but it's aspect is quite different. We realized that the heart of Miyata's circuit was the lattice structure that is box-framed in Figure 1. This lattice structure derives from a driving-point impedance $F(s) = N(s)/D(s)$ that must satisfy certain conditions. Augmenting the lattice two-port by some elements allowed us to design similar circuits for a family of driving-point functions $F(s)$ in which Miyata's circuit is one member. The design procedure outlined in our discussion is extremely simple and uses straightforward formulas.

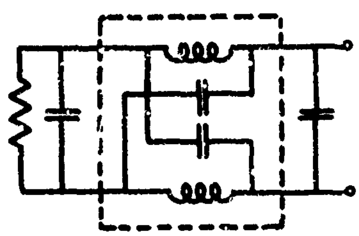


Figure 1. Miyata Circuit

(Received for publication 19 May 1971)

2. A CODE NOTATION FOR POSITIVE REAL FUNCTIONS

A positive real function F(s) is the quotient of a numerator polynomial N(s) and a denominator polynomial D(s):

$$F(s) = \frac{N(s)}{D(s)} = \frac{N_\mu s^\mu + N_{\mu-1}s^{\mu-1} + \ldots + N_1 s + N_0}{s^\nu + D_{\nu-1}s^{\nu-1} + \ldots + D_1 s + D_0} . \tag{1}$$

The necessary and sufficient conditions for F(s) to be positive real (pr) were established by Otto Brune (1930). They are: The zeros of N(s) and D(s) cannot be located in the right half of the complex s-plane. Any zeros on the imaginary $j\omega$ axis must be simple and have positive residues. The real component of the complex function $F(j\omega)$ must be Re $F(j\omega)$ nonnegative for all $\pm\omega$.

It can easily be shown that if F(s) is pr the following statements must hold:

(1) The coefficients of N(s) and of D(s) must be positive.

(2) The degrees μ and ν in Eq. (1) are either equal or $|\mu - \nu| = 1$.

(3a) With the exception of N_0 and D_0, none of the coefficients up to N_μ and D_ν is missing; either N_0 or D_0 may be missing, but not both, since we assume that N(s) and D(s) have no common factor.

(3b) All coefficients with an even subindex in the numerator and all coefficients with an odd subindex in the denominator, or vice versa, may be missing.

Without causing any limitations we shall agree that:

(1) The polynomial D(s) is always assumed to be a normalized polynomial by the fact that $D_\nu = 1$.

(2) The polynomial N(s) is not assumed to be normalized if $\nu = \mu \pm 1$. Thus N_μ is a positive coefficient of any magnitude, including 1 of course. But when $\nu = \mu$, and neither N_0 nor D_0 are zero, N(s) is also considered to be normalized and we express the function as KF(s) with K a positive constant.

(3) We express the degrees of both polynomials by ν and $\nu \pm 1$.

In a paper that the author presented at the Third Hawaii International Conference on System Sciences 1970 (Haase, 1970a), it was shown that conveniently coded notations can be used to express a pr function by a capital letter, a numerical subindex, and eventually the exponent -1. The letter P is used for functions where N(s) and D(s) are of the same degree, and the letter Q is used when the degrees differ by one. In this paper we deal only with P and Q functions. The subindex is either an even or an odd integer and is related to the degrees of the polynomials. The code notations for the functions of interest are listed in Table 1.

Table 1. Impedances $F(s) = N(s)/D(s)$ Expressed by Function Codes

Code Code	Degree of (Ns)	D(s)	Zero Coefficients	Special Relation of Coefficients
P_7	4	4	None	$N_0 > D_0$, $N_4 = 1$
P_7^{-1}	4	4	None	$N_0 < D_0$, $N_4 = 1$
P_{10}	5	5	$D_0 = 0$	
P_{10}^{-1}	5	5	$N_0 = 0$	
Q_{10}	5	4	None	
Q_{10}^{-1}	4	5	None	
Q_{11}	6	5	$N_0 = 0$	
Q_{11}^{-1}	5	6	$D_0 = 0$	

3. THE BIQUARTIC DRIVING-POINT FUNCTIONS OF TYPES P_7 AND P_7^{-1}

Consider the function

$$F(s) = \frac{\bar{N}(s)}{\bar{D}(s)} = \frac{s^4 + \bar{N}_3 s^3 + \bar{N}_2 s^2 + \bar{N}_1 s + \bar{N}_0}{s^4 + \bar{D}_3 s^3 + \bar{D}_2 s^2 + \bar{D}_1 s + \bar{D}_0} . \tag{2}$$

This function is of the type P_7 if $\bar{N}_0 > \bar{D}_0$, or $\bar{N}_0/\bar{D}_0 > 1$. It is of the type P_7^{-1} if, inversely, $\bar{N}_0 < \bar{D}_0$, or $\bar{N}_0/\bar{D}_0 < 1$. We have added a bar over the capital letters in order to enhance the fact that not only $\bar{D}(s)$ but also $\bar{N}(s)$ is a normalized polynomial, and the whole function is normalized by the fact that $\bar{F}(\infty) = 1$. No matter what other relations exist between the positive coefficients $\bar{N}_0, \ldots, \bar{N}_3, \bar{N}_4 = 1$ and $\bar{D}_0, \ldots, \bar{D}_3, \bar{D}_4 = 1$, it is evident that $\bar{F}(0) = \mathrm{Re}\,\bar{F}(j0) = \bar{N}_0/\bar{D}_0$ and $\bar{F}(\infty) = \mathrm{Re}\,\bar{F}(j\infty) = 1$. Since, due to the positive realness of $\bar{F}(s)$, $\mathrm{Re}\,\bar{F}(j\omega)$ must be ≥ 0 for all positive and negative ω, the curve representing $F(j\omega)$ over the abscissa scaled in $\Omega = \omega^2$ can never tresspass the Ω-axis between $\Omega = 0$ and $\Omega = +\infty$. In our discussions we are especially interested in the case where the curve $\mathrm{Re}\,\bar{F}(j\omega)$ vs Ω has a minimum appearing at Ω_0 and a magnitude $\mathrm{Re}\,F(j\omega_0)$, where

$$\Omega_0 = \omega_0^{\,2} . \tag{3}$$

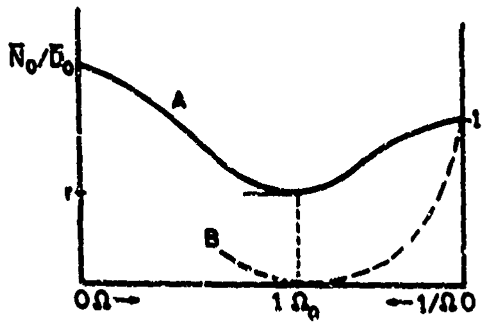


Figure 2. Real Component of F(jω) vs $L\Omega = \omega^2$ of a P_7-type Function

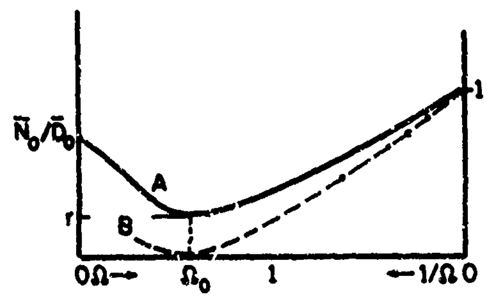


Figure 3. Real Component F(jω) vs Ω of a P_7^{-1}-type Function

This case is shown in Figure 2 by curve (A), assuming that F(s) is of the type P_7. Likewise, Figure 3 shows the situation for a P_7^{-1} type function. Let the minimum be of the magnitude r. Then evidently the function

$$\bar{F}(s) - r = (1 - r)\,\frac{[\bar{N}(s) - r\bar{D}(s)]/(1 - r)}{\bar{D}(s)} \tag{4}$$

is still pr. This function is represented by curve (B) in Figures 2 and 3. The minimum is now located on the abscissa at Ω_0. A function with the minimum of Re F(jω) on the abscissa is referred to in the literature as a "Minimum Resistance Function". Note that by extracting the factor (1-r) in Eq. (3) the numerator in the fraction becomes a normalized polynomial. It has the same degree as $\bar{D}(s)$. It is necessary to extract the factor, since we agreed to assume that in a P_7 or P_7^{-1} function the numerator is a normalized polynomial. In the next Section we discuss the minimum function of types P_7 and P_7^{-1}.

3.1 The Minimum Functions of Types P_7 and P_7^{-1} and Their Brune Realization

A recent paper of mine (Haase, 1970b) was extensively devoted to the computational technique of the design of driving-point impedances according to Brune (1970). Applying this technique to a minimum function of the type P_7 (or P_7^{-1}, it does not matter) has the result shown in Figure 4. The realized circuit consists of a Brune section in T form, terminated with an impedance $z'F'(s)$, where z' is a positive constant and $F'(s)$ is a function of the type P_3(or P_3^{-1}). The latter function is biquadratic, the quotient of two

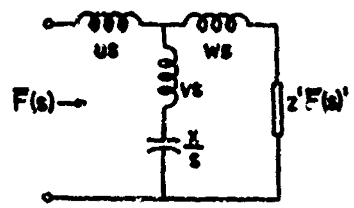


Figure 4. Brune Section Terminated with an Impedance Function

normalized quadratic polynomials. There is of course no resistance at the input, since $\bar{F}(s)$ is assumed to be a minimum function (for such a resistance we used the letter symbol k in a previous paper (Haase, 1970b); in the present paper we used a different letter in order to save k for another purpose (in Eq. (4) we used the letter r).

The T of the Brune section in Figure 5 consists of the inductive impedances us and ws and a shunt branch with the impedance $vs + x/s$. Between the constants u, v, and w the equation

$$1/u + 1/v + 1/w = 0 \tag{5}$$

must hold. This is the case when

$$u = -w/n = v(n-1)\,, \tag{6}$$

with v and n positive constants. The shunt branch has the impedance

$$vs + x/s = v\,\frac{s^2 + \Omega_0}{s} \tag{7}$$

when we define

$$\Omega_0 = \frac{x}{v}\,. \tag{8}$$

Note that since x and v are positive constants,

$$s^2 + \Omega_0 = 0 \text{ is identical with } s = \pm j\omega_0\,. \tag{9}$$

In Haase (1970b) we defined

$$\bar{N}(j\omega_0) = R_{\bar{N}} + j\Omega_0 S_{\bar{N}} \tag{10}$$

and

$$\bar{D}(j\omega_0) = R_{\bar{D}} + j\Omega_0 S_{\bar{D}}\,, \tag{11}$$

and we devised a computational routine that yields the real constants $R_{\bar{N}}$, $S_{\bar{N}}$, $R_{\bar{D}}$, and $S_{\bar{D}}$ from the coefficients of $\bar{N}(s)$ and $\bar{D}(s)$ respectively. We also showed that

$$\text{Re } \overline{F}(j\omega_0) = \frac{R_{\overline{N}} + j\omega_0 S_{\overline{N}}}{R_{\overline{D}} + j\omega_0 S_{\overline{D}}} \cdot \frac{R_{\overline{D}} - j\omega_0 S_{\overline{D}}}{R_{\overline{D}} - j\omega_0 S_{\overline{D}}}$$

$$= \frac{R_{\overline{N}} R_{\overline{D}} + \Omega_0 S_{\overline{N}} S_{\overline{D}}}{R_{\overline{D}}^2 + \Omega_0 S_{\overline{D}}^2} . \tag{12}$$

Since $\overline{F}(s)$ is a minimum function, the right side in Eq. 12 must be zero, that is,

$$R_{\overline{N}} R_{\overline{D}} + \Omega_0 S_{\overline{N}} S_{\overline{D}} = 0 . \tag{13}$$

Evaluated at $j\omega_0$, it is physically necessary that

$$u = \text{Im } \overline{F}(j\omega_0) = \frac{R_{\overline{D}} S_{\overline{N}} - R_{\overline{N}} S_{\overline{D}}}{R_{\overline{D}}^2 + \Omega_0 S_{\overline{D}}^2} . \tag{14}$$

We found further that with

$$R_g = S_D - \Omega_0 S_d - R_n/u \tag{15}$$

and

$$s_g = R_d - S_n/u \tag{16}$$

the constant n in Eq. 6) is

$$n = 1 + \frac{(R_D/S_D - R_g/S_g)S_D}{(R_g{}^2/S_g{}^2 + \Omega_0)S_g} , \tag{17}$$

with

$$v = \frac{u}{n - 1} \tag{18}$$

following from Eq. (6). The constant z' is

$$z' = 1/n^2. \tag{19}$$

In Eqs. (15) and (16), R_n, S_n, R_d, and S_d are the second-order evaluation coefficients obtained when the first-order evaluation coefficients, $R_{\bar{N}}$, $S_{\bar{N}}$, $R_{\bar{D}}$, and $S_{\bar{D}}$ have been computed and the routine computation is repeated and applied to the remainder polynomials $\bar{N}(s)/(s^2 + \Omega_0)$ and $\bar{D}(s)/(s^2 + \Omega_0)$. Thus, the constants n, v, x, and z' can be obtained by the straight-forward formulas of Eqs. (17), (18), (19), and (8).

The next step we have to perform is the realization of the terminating driving-point function $z'\bar{F}'(s)$. This is discussed below.

3.2 The Realization of the Driving-Point Function $z'\bar{F}(s)'$

As has already been mentioned, $\bar{F}'(s)$ is a biquadratic function of type P_3 or P_3^{-1}. Generally, Re $\bar{F}'(s)$ may have a minimum at a certain location on the Ω-axis. For the subclass of functions of types P_7 and P_7^{-1}, it would not have such a minimum. But let us first deviate for a moment from our subject and consider the realization of a minimum function P_3 (or P_3^{-1}), and let us suppose that the minimum is at Ω_0 again. Such a function would be

$$F(s) = \frac{s^2 + N_1 s + N_0}{s^2 + D_1 s + D_0} \tag{20}$$

with

$$N_1 D_1 = (\sqrt{N_0} - \sqrt{D_0})^2 . \tag{21}$$

It would have the realization shown in Figure 5. The constants n and v would be positive, the terminating constant, realized by a resistor, would be

$$z = 1/n^2 = N_0/D_0 . \tag{22}$$

Eqs. (5) and (6) would hold.

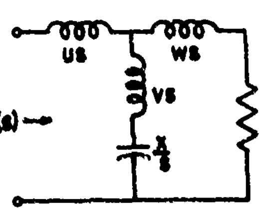


Figure 5. Resistively Terminated Brune Section

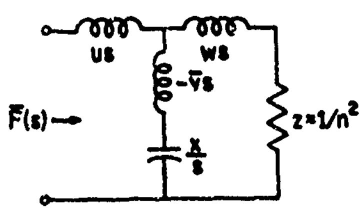


Figure 6. Resistively Terminated Brune Section with Negative Mutual Inductance and Turn Ratio

Consider now the very similar circuit shown in Figure 6. Here we assume that the inductance v is negative. Eq. (5) postulates that one of the three inductances u, v, and w is negative. Previously we assumed that either u or w was negative (depending on the magnitude of the positive constant n). Why shouldn't v be the negative inductance? We recognize immediately that if v is negative, then n must also be negative to satisfy Eq. (6). Introducing

$$v = -\bar{v} \tag{23a}$$

and

$$n = -\bar{n}\,, \tag{23b}$$

$$F(s) = \frac{s^2 + x(\bar{n}+1)^2 + x/\bar{v}\bar{n}}{s^2 x\, s/\bar{v}\bar{n} + x\bar{n}/\bar{v}}\,. \tag{24}$$

A comparison with Eq. (20) then shows that

$$N_1 D_1 > (\sqrt{N_0} - \sqrt{D_0})^2\,, \tag{25}$$

which proves that F(s) is not a minimum function.

Let us now consider the shunt branch in the circuit in Figure 6. It has the impedance

$$-\bar{v}s + x/s = -\frac{s^2 - \Omega_0}{s}\,, \tag{26}$$

and its "resonance frequency" is

$$s = +\sqrt{\Omega_0} = \omega_0\,, \tag{27}$$

whereas for the circuit in Figure 6

$$s = j\omega_0\,. \tag{28}$$

It was Brune's idea to recognize that an inductance star built by the inductances u, v, and w is equivalent to a perfectly coupled transformer having the turn ratio n and the mutual inductance v. Thus, the circuit in Figure 5 is equivalent to the circuit shown in Figure 7. We obtain the picture of a circuit implying a negative inductance v and a negative turn ratio n simply by interchanging the transformer terminals on one side, as it is shown in Figure 8 where the circuit is equivalent to that in Figure 6.

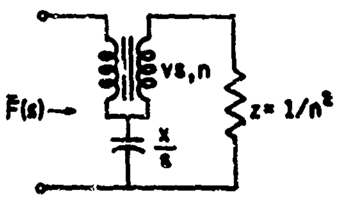


Figure 7. Circuit Equivalent to that in Figure 5

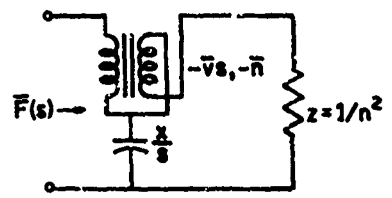


Figure 8. Circuit Equivalent to that in Figure 6

Let us now go back to our problem of realizing a driving-point impedance of the type P_7 (or P_7^{-1}). We are interested in a special class of functions $\bar{F}(s)$ of this type: We want the driving-point impedance $\bar{F}(s)$ to be realizable as shown in Figure 9, a tandem of two Brune sections with a resistive termination. The function $\bar{F}(s)$ of Eq. (2) would have the termination

$$z = \bar{N}_0/\bar{D}_0 \,. \tag{29}$$

In the first section of Figure 9, n_1 and v_1 are positive; x_1 is also positive and

$$\Omega_0 = x_1/v_1 \,. \tag{30}$$

In the second section, v_2 and n_2 are negative. The constant x_2 is positive and

$$\Omega_0 = -x_2/v_2 \,. \tag{31}$$

It is evident that $\bar{F}(s)$ as presented in Eq. (2) cannot have coefficients of $\bar{N}(s)$ and $\bar{D}(s)$ at random. A certain relationship between these coefficients is necessary, which will be the subject of our discussions in Section 4. The realization of $\bar{F}(s)$ is special, insofar as there is no resistance between the two Brune sections and Ω_0 obtained in Eqs. (30) and (31) is the same. There is, we may say, "a very special" function $\bar{F}(s)$ that, in addition to the aforementioned properties of its realization, also has the property that $-n_1 n_2 = 1$. If this is the case, then the circuit in Figure 9 is equivalent to the circuit in Figure 10; the two Brune sections together then become equivalent to a lattice two-port consisting of two inductances v_a and v_b and two capacitances $1/x_a$ and $1/x_b$. The tandem-lattice equivalence will be discussed in Section 5.

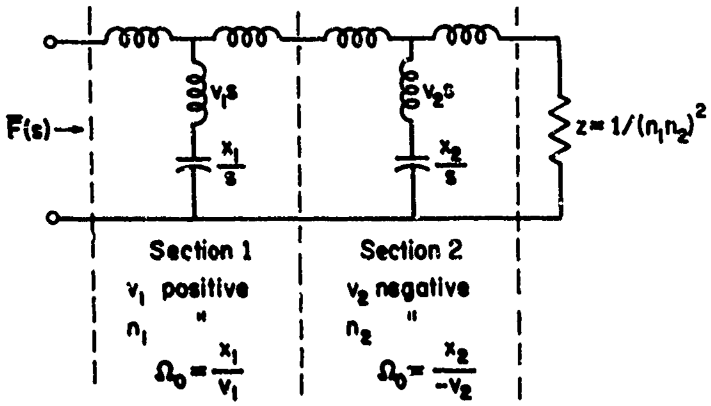


Figure 9. Tandem of Two Brune Sections Realizing a Driving-Point Function of the Type P_7 or P_7^{-1}

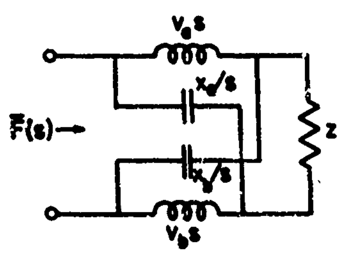


Figure 10. Lattice Equivalent to the Circuit in Figure 9

4. THE REALIZATION OF THE SPECIAL P_7 TYPE FUNCTION $\bar{F}(s)$

Assume that a driving-point impedance function is expressed in the form of Eq. (2) and should be realized as shown in Figure 9. From physically investigating the circuit in this figure, let us now find the relation between the coefficients of $\bar{N}(s)$ and $\bar{D}(s)$.

First assume that $s = j\omega_0$. Then the shunt impedance of the first Brune section becomes zero, since $x_1/v_1 = \Omega_0 = \omega_0^2$. Since there is no resistance at the input,

$$\mathrm{Re}\ \bar{F}(j\omega_0) = \frac{R_{\bar{N}}R_{\bar{D}} + \Omega_0 S_{\bar{N}}S_{\bar{D}}}{R_{\bar{D}}^2 + \Omega_0 S_{\bar{D}}^2} = 0$$

according to Eq. (12).

As explained by Haase (1970b),

$$R_{\bar{N}} = \Omega_0^2 - \Omega_0\bar{N}_2 + \bar{N}_0 \tag{32}$$

$$R_{\bar{D}} = \Omega_0^2 - \Omega_0\bar{D}_2 + \bar{D}_0 \tag{33}$$

$$S_{\bar{N}} = -\Omega^0\bar{N}_3 + \bar{N}_1 \tag{34}$$

$$S_{\bar{D}} = -\Omega_0\bar{D}_3 + \bar{D}_1\,. \tag{35}$$

Therefore,

$$\begin{aligned} R_{\bar{N}}R_{\bar{D}} + \Omega_0 S_{\bar{N}}S_{\bar{D}} = L\Omega_0^4 &+ \Omega_0^2[\bar{N}_0 + \bar{D}_0 + \bar{N}_2\bar{D}_2 - \bar{N}_3\bar{D}_1 - \bar{N}_1\bar{D}_3] + \bar{N}_0\bar{D}_0 \\ &+ \Omega_0^3[\bar{N}_3\bar{D}_3 - \bar{N}_2 - \bar{D}_2] \\ &+ \Omega_0[\bar{N}_1\bar{D}_1 - \bar{N}_2\bar{D}_0 - \bar{N}_0\bar{D}_2] = 0\,. \end{aligned} \tag{36}$$

We are also able to cascade the circuit in Figure 9 in such a way that Brune section 2 is to the left of section 1. Assume we have done this, and we now let $s = \omega_0$. Let us denote

$$\bar{N}(\omega_0) = R_{\bar{N}}^* + \omega_0 S_{\bar{N}}^* \tag{37}$$

$$\bar{D}(\omega_0) = R_{\bar{D}}^* + \omega_0 S_{\bar{D}}^*\,. \tag{38}$$

Since the evaluation program computes the constants $R_{\bar{N}}^*$, $S_{\bar{N}}^*$, $R_{\bar{D}}^*$, and $S_{\bar{D}}^*$ by feeding in $-\Omega_0$ instead of $+\Omega_0$, as discussed by Haase (1970b), we may formally use the script

$$\bar{F}(\omega_0) = \mathrm{Re}\ \bar{F}(\omega_0) + \mathrm{Im}\ \bar{F}(\omega_0)\,, \tag{39}$$

although interpreting "Re" as "real component" and "Im" as "imaginary component" does not make much sense at the moment. Thus

$$F(\omega_0) = \frac{R^*_{\bar{N}} + \omega_0 S^*_{\bar{N}}}{R^*_{\bar{D}} + \omega_0 S^*_{\bar{D}}} = \frac{R^*_{\bar{N}} + \omega_0 S^*_{\bar{N}}}{R^*_{\bar{D}} + \omega_0 S^*_{\bar{D}}} \cdot \frac{R^*_{\bar{D}} - \omega_0 S^*_{\bar{D}}}{R^*_{\bar{D}} - \omega_0 S^*_{\bar{D}}}$$

$$= \frac{R^*_{\bar{N}} R^*_{\bar{D}} - \Omega_0 S^*_{\bar{N}} S^*_{\bar{D}} + \omega_0 [R^*_{\bar{D}} S_{\bar{N}} - R^*_{\bar{N}} S^*_{\bar{D}}]}{R^{*2}_{\bar{D}} - \Omega_0 S^{*2}_{\bar{D}}} . \tag{40}$$

Therefore

$$\mathrm{Re}\bar{F}(\omega_0) = \frac{R^*_{\bar{N}} R^*_{\bar{D}} - \Omega_0 S^*_{\bar{N}} S^*_{\bar{D}}}{R^{*2}_{\bar{D}} - \Omega_0 S^{*2}_{\bar{D}}} \text{ must be} = 0 . \tag{41}$$

Since

$$R^*_{\bar{N}} = \Omega_0^{\ 2} + \Omega_0 \bar{N}_2 + \bar{N}_0 \tag{42}$$

$$R^*_{\bar{D}} = \Omega_0^{\ 2} + \Omega_0 \bar{D}_2 + \bar{D}_0 \tag{43}$$

$$S^*_{\bar{N}} = \Omega_0 \bar{N}_3 + \bar{N}_1 \tag{44}$$

$$S^*_{\bar{D}} = \Omega_0 \bar{D}_3 + \bar{D}_1 . \tag{45}$$

we obtain

$$\begin{aligned} R^*_N R^*_{\bar{D}} - \Omega_0 S^*_{\bar{N}} S^*_{\bar{D}} = \Omega_0^{\ 4} &+ \Omega_0^{\ 2}[\bar{N}_0 + \bar{D}_0 + \bar{N}_2\bar{D}_2 - \bar{N}_3\bar{D}_1 - \bar{N}_1\bar{D}_3] + \bar{N}_0\bar{D}_0 \\ &- \Omega_0^{\ 3}[\bar{N}_3\bar{D}_3 - \bar{N}_2 - \bar{D}_2] \\ &- \Omega_0(\bar{N}_1\bar{D}_1 - \bar{N}_2\bar{D}_0 - \bar{N}_0\bar{D}_2] = 0 . \end{aligned} \tag{46}$$

The condition that Re $\bar{F}(j\omega_0)$ and Re $\bar{F}(\omega_0)$ are both zero is satisfied when

$$c_1 = \bar{N}_1\bar{D}_1 - \bar{N}_0\bar{D}_2 - \bar{N}_2\bar{D}_0 = 0 \tag{47}$$

and

$$c_3 = \bar{N}_3\bar{D}_3 - \bar{N}_2 - \bar{D}_2 = 0 \tag{48}$$

and when Ω_0 is a duplex root of the quartic equation

$$\Omega^4 + \Omega^2(\bar{N}_2\bar{D}_2 + \bar{N}_0 + \bar{D}_0 - \bar{N}_3\bar{D}_1 - \bar{N}_1\bar{D}_3) + \bar{N}_0\bar{D}_0 = 0 . \tag{49}$$

Therefore, the left side of Eq. (49) must be identical with

$$(\Omega + \Omega_0)^2(\Omega - \Omega_0)^2 = \Omega^4 - 2\Omega_0^{\ 2} + \Omega_0^{\ 4} . \tag{50}$$

By comparison

$$\Omega_0 = + 4\sqrt{\bar{N}_0\bar{D}_0} . \tag{51}$$

But then,

$$c_2 = \bar{N}_2\bar{D}_2 - \bar{N}_3\bar{D}_1 - \bar{N}_1\bar{D}_3 + (\bar{N}_0 + \bar{D}_0 \pm 2\sqrt{\bar{N}_0\bar{D}_0}) = 0 \tag{52}$$

with either the + or the - sign.

Note that in Eqs. (47), (48), and (52) the letters $\bar{N}$ can be interchanged with the letters $\bar{D}$. Therefore, when these zero identities hold for $\bar{F}(s)$, they also hold for $1/\bar{F}(s)$.

We are now able to state:

A function $\bar{F}(s)$ presented in Eq. (2) is a special function of either the type P_7 or the type P_7^{-1}, if its coefficients $\bar{N}_i$ and $\bar{D}_i$ satisfy Eqs. (47), (48), and (52).

Here is a numerical example:

Assume coefficients listed in the following table:

i	$\bar{N}_i$	$\bar{D}_i$
0	0.36	2 7/9
1	5.12	2 1/9
2	2.48	10 8/9
3	2.56	5 2/9
4	1.00	1

By Eq. (47) $c_1 = 0.0000003$

By Eq. (48) $c_3 = 0.0000000$

By Eq. (51) $\Omega_0 = 1.0$

By Eq. (52) $C_2 = -0.0000003$ (with + sign in Eq. (52)).

Later we shall also need the result

$$-n_1n_2 = \sqrt{\bar{D}_0/\bar{N}_0}$$

for which we obtain in this example $n_1n_2 = -2\,\frac{7}{9}$.

It can easily be shown that n_1 is the ratio u_1/w_1 in the first and n_2 is the ratio u_2/w_2 in the second Brune section in Figure 9. Since $n_1 > 0$ and $n_2 < 0$, the minus sign in Eq. (53) becomes evident.

If we aim to the design of the circuit in Figure 9, it is necessary that we start with a special function. Therefore, consider the content of the following Section as a test.

4.1 Test Routine T

The purpose of the following routine computation is to test whether or not a given function is a special one, and to compute Ω_0 and the product n_1n_2 (that must be negative).

Given: the coefficients $\bar{N}_i$ and $\bar{D}_i$, $0 \leq i \leq 3$, $\bar{N}_4 = \bar{D}_4 = 1$

(1) Compute c_1 Eq. (47)
(2) Compute c_2 Eq. (52)
(3) Compute c_3 Eq. (48)
(4) Compute Ω_0 Eq. (51)
(5) Compute n_1n_2 Eq. (53).

4.2 Realization of the Circuit in Figure 9

We could design our circuit by applying twice the Brune procedure to the function $\bar{F}(s)$ known by its coefficients and proved to be a special function. This, however, would be inefficient. Instead let us synthesize the function $\bar{F}(s)$, starting from the circuit.

First of all, since the shunt impedance of the first Brune section in Figure 9 is

$$v_1s + x_1/s = v_1(s^2 + \Omega_0)/s \,,$$

and the shunt impedance of the second section is

$$v_2s + x_2/s = -v_2(s^2 - \Omega_0)/s \,,$$

we can express v_2 and x_2 in terms of v_1 and x_1, using a positive constant k_1. Then

$$v_2 = -v_1/k_1 \tag{54}$$

$$x_2 = x_1/k_1 \; . \tag{55}$$

For convenience, we have redrawn the circuit, and present it with these notations in Figure 11.

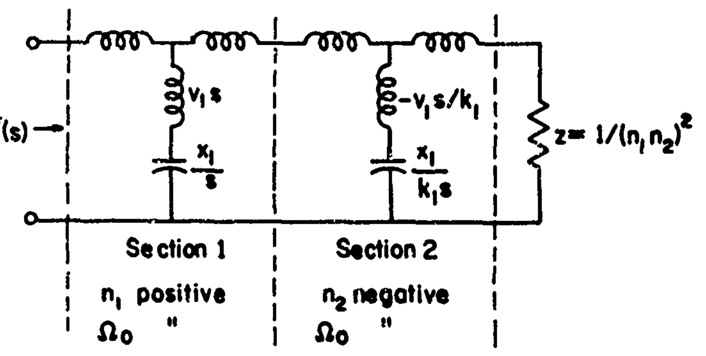


Figure 11. Tandem of Two Brune Sections Realizing a Driving-Point Function of the Type P_7 or $P_7{}^{-1}$

Analyzing the circuit yields the following results:

$$\bar{N}_0 = -\frac{\Omega_0{}^2}{n_1 n_2} \tag{56}$$

$$\bar{N}_1 = -x_1(n_1-1)^2 \Omega_0 n_2 + \Omega_0 x_1 (n_2-1)^2 \frac{n_1}{k_1}$$

$$= \frac{x_1 \Omega_0}{k_1} \left[n_1(n_2-1)^2 - k_1 n_2 (n_1-1)^2 \right] \tag{57}$$

$$N_2 = -\frac{\Omega_0}{n_2} + \frac{\Omega_0}{n_1} - \Omega_0 k_1 \frac{(n_1-1)^2}{n_1{}^2 n_2}$$

$$= \Omega_0 \frac{n_1(n_2-n_1) - k_1(n_1-1)^2}{n_1{}^2 n_2} \tag{58}$$

$$\bar N_3 = \frac{x_1 n_1^2 (n_2-1)^2}{k_1} + x_1(n_1-1)^2$$

$$= \frac{x_1 [n_1^2 (n_2-1)^2 + k_1(n_1-1)^2]}{k_1} \tag{59}$$

$$\bar D_0 = -\Omega_0^2 n_1 n_2 \tag{60}$$

$$\bar D_1 = -\frac{\Omega_0}{v_1 n_1 n_2} - \frac{k_1 \Omega_0}{v_1 n_1 n_2} = -\frac{\Omega_0}{v_1 n_1 n_2}(k_1+1) \tag{61}$$

$$\bar D_2 = -\Omega_0 n_2 + \Omega_0 n_1 + \frac{n_1 \Omega_0 (n_2-1)^2}{k_1}$$

$$= \frac{\Omega_0}{k_1}\; n_1(n_2-1)^2 + k_1(n_1-n_2) \tag{62}$$

$$\bar D_3 = -\frac{k_1}{v_1 n_1^2 n_2} + \frac{1}{v_1 n_1} = \frac{n_1 n_2 - k_1}{v_1 n_1^2 n_2}\,. \tag{63}$$

Equations (56) and (60) show that Eqs. (51) and (53) are true. In Eqs. (56) to (63) we know the coefficients on the left side and we know the product $n_1 n_2$ and Ω_0 on the right side. It would be tremendously complicated to solve this system of equations, since it is non-linear. Thus, we have to look for another way, but we can easily check our results with the system of these equations.

In Section 3.1 we presented the formulas of Eqs. (14) to (18) by which the constants v_1 and n_1 can be determined. We are only intermediately interested in the value of $u_1 = v_1(n_1-1)$. The constant x_1 is

$$x_1 = v_1 \Omega_0\,. \tag{64}$$

Similarly to Eq. (14),

$$u^* = \frac{R_D^* S_N^* - R_N^* S_D^*}{R_D^{*2} - \Omega_0 S_D^{*2}} = \frac{S_N^* [R_D^*/S_D^* - R_N^*/S_N^*]}{S_D^* [R_D^{*2}/S_D^{*2} - \Omega_0]}. \tag{65}$$

But by $k_1 = x_1/x_2$,

$$k_1 = n_1 \;\frac{n_2-1}{n_1-1} \cdot \frac{(n_1-1)u^* - u(n_1+1)}{(n_1+1)u^* - u(n_1-1)}\,. \tag{66}$$

All constants in the circuit Figure 11 are now known. We refer to this design procedure as the "Realization Procedure R_1" and present it in compact form below.

4.3 Realization Procedure R_1

Prerequisit:

The "Test Procedure T" (Section 4.1) showed that $c_1=c_2=c_3=0$ and, therefore, that the $\bar{F}(s)$ under investigation is a special function. The procedure also presented Ω_0 and the product n_1n_2:

(1) Compute the evaluation coefficients $R_{\bar{N}}$, $S_{\bar{N}}$, $R_{\bar{D}}$, $S_{\bar{D}}$, $R_{\bar{n}}$, $S_{\bar{n}}$, $R_{\bar{d}}$, and $S_{\bar{d}}$ for the N(s) and D(s) evaluated with $-\Omega_0$ (see Chapter 3 in Haase (1970b) for computational routine)

(2) Compute the evaluation coefficients $R^*_{\bar{N}}$, $S^*_{\bar{N}}$, $R^*_{\bar{D}}$, and $S^*_{\bar{D}}$ for the N(s) and D(s) evaluated with $+\Omega_0$

(3) Compute u_1 according to Eq. (15)

(4) Compute n_1 according to Eq. (17)

(5) Compute v_1 according to Eq. (18)

(6) Compute x_1 according to Eq. (64)

(7) Compute k_1 according to Eq. (66).

4.4 Numerical Examples

Included in most of the main sections of this paper are numerical examples in which we show the application of the theory discussed in this Section. The examples are treated only insofar as the content of the Section deals with the matter. All examples were computed on the desk-top computer Programma 101 of the Olivetti Underwood Corporation. The pertinent programs are available on request from the author.

The numerical values used in the examples are chosen to show the numerical procedure rather than to represent technically reasonable circuits. For this reason the reader should not be concerned when the sizes of the circuit elements obtained are in some instances awkward.

Example 4.4.1

Let a function $\bar{F}(s)$ have the coefficients that are listed in Storages 164 A and B of the following program that computes the test values c_1, c_2, and c_3, the value Ω_0, and the product n_1n_2. The computational program of Cards 164 A and B assumes that in Eq. (52) the + sign holds. Since all three test values are almost zero, $\bar{F}(s)$ can be considered as a "special function."

Stored on Card 164A

	Coefficients $\bar{N}_i$
i = 0	0•1066666 d0
1	20•3520000 D0
2	1•1333333 •0
3	34•0800000 E0
4	1•0000000 f0

Stored on Card 164B

	Coefficients $\bar{D}_i$
i = 0	3•8400000 d0
1	0•2539682 D0
2	7•6571428 •0
3	0•2579365 E0
4	1•0000000 f0

V
V

c_1 =	-0•0000003 A0
c_2 =	0•0000011 A0
c_3 =	-0•0000002 A0
Ω_0 =	0•7999998 A0
n_1n_2 =	-6•0000018 A0

Test with + sign in Eq. (52)

Store on Card 165A

	Coefficients $\bar{N}_i$
i = 0	0•1066666 d0
1	20•3520000 D0
2	1•1333333 •0
3	34•0800000 E0
4	1•0000000 f0

Store on Card 165B

	Coefficients $\bar{D}_i$
i = 0	3•8400000 d0
1	0•2539682 D0
2	7•6571428 •0
3	0•2579365 E0
4	1•0000000 f0

V
V

c_1 =	-0•0000003 A0
c_2 =	-2•5599977 A0
c_3 =	-0•0000002 A0
Ω_0 =	0•7999998 A0
n_1n_2 =	-6•0000018 A0

Test with - sign in Eq. (52)

Test Procedure T Applied to $\bar{F}(s)$ in Example 4.4.1

We have also devised a program that holds with the - sign in Eq. (52). Testing the function with this program on Cards 165 A and B yields the same values for c_1 and c_3, but here $c_2 = -2.5599977$ due to the wrong polarity of the sign in Eq. (52).

Since we have shown that a Type P_7^{-1} function $\bar{F}(s)$ (N(s) and D(s) of degree 4 each, $\bar{N}_0 < \bar{D}_0$) is a special function, we can continue with the realization procedure R_1. For this purpose we have stored the coefficients of $\bar{N}(s)$ and $\bar{D}(s)$ on Evaluation Cards 171 AA and 171 BB. The evaluation program yields evaluation coefficients R_N, S_N, R_n, S_n and R_D, S_D, R_d, S_d when N(s) and D(s) are evaluated for $-\Omega_0 = -0.8$, and evaluation coefficients R_N^*, S_N^* and R_D^*, S_D^* when N(s) and D(s) are evaluated for $+\Omega_0 = 0.8$. The evaluation coefficients have been printed out on the following tapes for the reader's convenience. The continuation of the program following the

Store on Card 171AA	Coefficients $\bar{N}_i$
i = 0	0·1066666 e◊
1	20·3520000 E◊
2	1·1333333 f◊
3	34·0800000 F◊
4	1·0000000 e◊

Store on Card 171BB	Coefficients $\bar{D}_i$
i = 0	3·8400000 e◊
1	0·2539682 E◊
2	7·6571428 f◊
3	0·2579365 F◊
4	1·0000000 e◊

```
-Ω0 =  -0·8        b f
                   R S
                   V
                   V
RN  = -0·1600000   e◊
SN  = -6·9120000   E◊
RN  = -0·4666667   f◊
SN  = 34·0800000   F◊
                   R S
                   V
                   V
RD  = -1·6457142   e◊
SD  =  0·0476190   E◊
RD  =  6·0571428   f◊
SD  =  0·2579365   F◊
```

```
+Ω0 =   0·8        b f
                   R S
                   V
                   V
RN* =  1·6533332   e◊
SN* = 47·6160000   E◊
                   R S
                   V
                   V
RD* = 10·6057142   e◊
SD* =  0·4603174   E◊
-------------------------
                   W
                   W
                   Y
                   Y
                   Y
                   W
                   W
n1n2 = -6·0000018  c f
                   V
n1  =  2·0000001   b◊
n2  = -3·0000007   B◊

v1  =  4·1999997   c◊
x1  =  3·3599997   C◊

k1  =  7·0000004   A◊
```

Example 4.4.1. Realization Procedure R_1

evaluations computes the ratios n_1 and n_2, the inductance v_1, the inverse capacitance x_1 of the first shunt section, and the factor k_1. These values determine the circuit in Figure 11 where the terminating resistance is $z = \bar{N}_0/\bar{D}_0 = 1/(n_1 n_2)^2 = 1/36$.

Example 4.4.2

In this example we interchange the polynomials N(s) and D(s) with the polynomials of example 4.4.1. In the test procedure this exchange has the effect that the product $n_1 n_2$ becomes inverse. All other results are unchanged.

		Coefficients $\bar{N}_i$
Store on Card 164A	i = 0	3·8400000 d0
	1	0·2539682 D0
	2	7·6571428 e0
	3	0·2579365 E0
	4	1·0000000 f0

		Coefficients $\bar{D}_i$
Store on Card 164B	i = 0	0·1066666 d0
	1	20·3520000 D0
	2	1·1333333 e0
	3	34·0800000 E0
	4	1·0000000 f0

V
V

c_1 =	-0·0000003 A0
c_2 =	0·0000011 A0
c_3 =	-0·0000002 A0
Ω_0 =	0·7999998 A0
n_1n_2 =	-0·1666664 A0

Test with + sign in Eq. (52)

		Coefficients $\bar{N}_i$
Store on Card 165A	i = 0	3·8400000 d0
	1	0·2539682 D0
	2	7·6571428 e0
	3	0·2579365 E0
	4	1·0000000 f0

		Coefficients $\bar{D}_i$
Store on Card 165B	i = 0	0·1066666 d0
	1	20·3520000 D0
	2	1·1333333 e0
	3	34·0800000 E0
	4	1·0000000 f0

V
V

c_1 =	-0·0000003 A0
c_2 =	-2·5599977 A0
c_3 =	-0·0000002 A0
Ω_0 =	0·7999998 A0
n_1n_2 =	-0·1666664 A0

Test with - sign in Eq. (52)

Test Procedure T Applied to $\bar{F}(s)$ in Example 4.4.2

As the tape representing the realization procedure R_1 shows, the evaluation coefficients for evaluating N(s) are exchanged with the coefficients for evaluating D(s) in example 4.4.1 and vice versa. The result is evident. There is, however, no simple relation of the previous results to the v_1, x_1, and k_1 obtained in the present example.

The circuit realizing the driving-point impedance $\bar{F}(s)$ in example 4.4.2 (type P_7, $N_0 > D_0$, N(s) and D(s) both of degree 4) is the same as pictured in Figure 11. The terminating resistance is $z = \bar{N}_0/\bar{D}_0 = 1/(n_1n_2)^2 = 36$.

Store on Card 171AA	Coefficients $\bar{N}_i$	
i = 0	3•8400000	e0
1	0•2539682	F0
2	7•6571428	f0
3	0•2579365	F0
4	1•0000000	e0

Store on Card 171BB	Coefficients $\bar{D}_i$	
i = 0	0•1066666	e0
1	20•3520000	F0
2	1•1333333	f0
3	34•0800000	F0
4	1•0000000	e0

$-\Omega_0$ = -0•8 b1
RS
V
V
R_N = -1•6457142 e0
S_N = 0•0476190 E0
R_n = 6•0571428 f0
S_n = 0•2579365 F0
RS
V
V
R_D = -0•1600000 e0
S_D = -6•9120000 F0
R_d = -0•4666667 f0
S_d = 34•0800000 F0

$+\Omega_0$ = 0•8 b1
RS
V
V
R_N^* = 10•6057142 e0
S_N^* = 0•4603174 F0
RS
V
V
R_D^* = 1•6533332 e0
S_D^* = 47•6160000 E0
W
W
Y
Y
Y
d
d
n_1n_2 = -0•1666664 c1
V
n_1 = 0•4999999 b0
n_2 = -0•3333328 B0
v_1 = 0•5952378 c0
x_1 = 0•4761902 C0
k_1 = 1•5238088 A0

Example 4.4.2. Realization Procedure R_1

5. THE CIRCUIT EQUIVALENCE FOR A "VERY SPECIAL FUNCTION $\bar{F}(s)$"

A very special function $\bar{F}(s)$ of the type P_7 or P_7^{-1} has the tandem realization shown in Figure 12. Compared with the circuit in Figure 11, the ratios are:

$$n_1 = n \tag{67}$$

$$n_2 = -1/n \,. \tag{68}$$

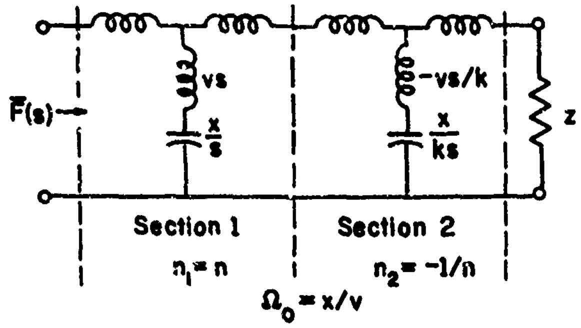


Figure 12. Brune Tandem of a "Very Special Function $\bar{F}(s)$"

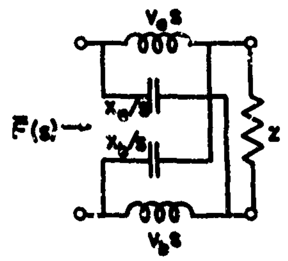


Figure 13. Lattice Equivalent Circuit to Figure 12

We will now show that the two sequential Brune sections are identical with the lattice section in Figure 13. If this is so, then both circuits have the same driving-point function and the same termination. Two-ports are equivalent when they have the same chain matrix.

The chain matrix derives from the primary-secondary two-port equations

$$E_1 = E_2\left(\frac{A}{E}\right) - I_2\left(\frac{B}{E}\right), \tag{69}$$

$$I_1 = E_2\left(\frac{C}{E}\right) - I_2\left(\frac{D}{E}\right). \tag{70}$$

In the familiar matrix notation

$$K = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix}, \tag{71}$$

we introduce a common denominator E. Then each of the symbols A, B, C, D, and E represents a polynomial in the matrix

$$K = \frac{1}{E}\begin{Vmatrix} A & B \\ C & D \end{Vmatrix}. \tag{72}$$

For the sake of briefness, we also refer to the matrix in Eq. (72) as the ABCD matrix.

The constants in the circuit of Figure 12 are:

First Section	Second Section
$v,\ n$	$v/k,\ -1/n$
$u_1 = -nw_1 = v(n-1)$	$u_2 = w_2/n = v(n+1)/nk$
$x_1 = v\Omega_0 = x$	$x_2 = x/k$.

With these constants, the matrix elements are:

First Section		Second Section	
$A_1 = s^2 n + \Omega_0$	(73)	$A_2 = s^2/n + \Omega_0$	(78)
$B_1 = sv\Omega_0 (n-1)^2/n$	(74)	$B_2 = sv\Omega_0 (n+1)^2/nk$	(79)
$C_1 = s/v$	(75)	$C_2 = ks/v$	(80)
$D_1 = s^2/n + \Omega_0$	(76)	$D_2 = s^2 n + \Omega_0$	(81)
$E_1 = \Omega_0 + s^2$	(77)	$E_2 = \Omega_0 - s^2$	(82)

The matrix of the circuit in Figure 12 is obtained as the product

$$\frac{1}{E_T}\left\| \begin{matrix} A_T & B_T \\ C_T & D_T \end{matrix} \right\| = \frac{1}{E_1 E_2}\left\| \begin{matrix} A_1 & B_1 \\ C_1 & D_1 \end{matrix} \right\| \left\| \begin{matrix} A_2 & B_2 \\ C_2 & D_2 \end{matrix} \right\| = \frac{1}{E_T}\left\| \begin{matrix} A_T & B_T \\ C_T & D_T \end{matrix} \right\| . \tag{83}$$

Therefore, the elements of the product matrix are:

$$A_T = s^4 + s^2 \frac{\Omega_0}{n}\left[(n^2+1) + k(n-1)^2\right] + \Omega_0^{\,2} \tag{84}$$

$$B_T = s(s^2 + \Omega_0/n)\ \frac{v\Omega_0}{k}\left[(n+1)^2 + k(n-1)^2\right] \tag{85}$$

$$C_T = s(s^2 + n\Omega_0)\ \frac{1+k}{vn} \tag{86}$$

$$D_T = s^4 + s^2 \frac{\Omega_0}{kn}\left[(n+1)^2 + k(n^2+1)\right] + \Omega_0^{\,2} \tag{87}$$

$$E_T = \Omega_0^{\,2} - s^4 \,. \tag{88}$$

The lattice in Figure 13 has the impedances $v_a s$, $v_b s$, x_a/s, and x_b/s in its branches. The elements of the ABCD matrix of the lattice are:

$$A_L = s^4 + s^2\left[\frac{x_a}{v_b} + \frac{x_b}{v_a}\right] + \frac{x_a x_b}{v_a v_b} \tag{89}$$

$$B_L = s\left[s^2 + \frac{(v_a + v_b)x_a x_b}{(x_a + x_b)v_a v_b}\right](x_a + x_b) \tag{90}$$

$$C_L = s\left[s^2 + \frac{x_a + x_b}{v_a + v_b}\right]\frac{v_a + v_b}{v_a v_b} \tag{91}$$

$$D_L = s^4 + s^2\left[\frac{x_a}{v_a} + \frac{x_b}{v_b}\right] + \frac{x_a x_b}{v_a v_b} \tag{92}$$

$$E_L = \frac{x_a x_b}{v_a v_b} - s^4 \,. \tag{93}$$

The circuits in Figures 12 and 13 are supposed to be equivalent. Therefore, $A_T = A_L$, $B_T = B_L$, ... The comparison of the elements in Eqs. (84) to (88) and (89) to (92) yields the following set of equations that has the constants of the circuit in Figure 12 on the left side and the impedance constants of the lattice on the right side:

$$\Omega_0^{\,2} = \frac{x_a x_b}{v_a v_b} \tag{94}$$

$$n\Omega_0 = \frac{x_a + x_b}{v_a + v_b} \tag{95}$$

$$\frac{1 + k}{vn} = \frac{v_a + v_b}{v_a v_b} \tag{96}$$

$$\frac{\Omega_0}{n}\left[(n^2+1) + k(n-1)^2\right] = \frac{v_a x_a + v_b x_b}{v_a v_b} \tag{97}$$

$$\frac{v\Omega_0}{k}\left[(n+1)^2 + k(n-1)^2\right] = x_a + x_b \tag{98}$$

$$\frac{\Omega_0}{kn}\left[(n+1)^2 + k(n^2+1)\right] = \frac{v_b x_a + v_a x_b}{v_a v_b} \tag{99}$$

Our next problem is to express each of the unknowns v_a, v_b, x_a, and x_b in terms of the known constants. For this purpose we introduce two terms, P and Q, where each again can be expressed either in terms of the unknowns or in terms of the constants:

$$P = \frac{v}{k}\left[(n+1)^2 + k(n-1)^2\right] = (x_a + x_b)\sqrt{\frac{v_a v_b}{x_a x_b}} \tag{100}$$

$$Q = \frac{1+k}{4v\Omega_0} = \frac{x_a + x_b}{4x_a x_b} . \tag{101}$$

By introducing the third term,

$$K = \sqrt{1 - \frac{1}{PQ\Omega_0}} , \tag{102}$$

We obtain

$$v_a = \frac{Pn}{2}(1 - K)\ \frac{K}{1 - (1-K)(n + P/2v)} \tag{103}$$

$$v_b = \frac{-Pn}{2}(1 + K)\ \frac{K}{1 - (1+K)(n + P/2v)} \tag{104}$$

$$x_a = \frac{P\Omega_0}{2}(1 - K) \tag{105}$$

$$x_b = \frac{P\Omega_0}{2}(1 + K) . \tag{106}$$

Although they are not important, we present below the reverse formulas:

$$\Omega_0 = +\sqrt{\frac{x_a x_b}{v_a v_b}} \tag{107}$$

$$n = \frac{1}{\Omega_0} \cdot \frac{x_a + x_b}{v_a + v_b} \tag{108}$$

$$k = \left[\frac{\sqrt{v_a x_a} + \sqrt{v_b x_b}}{\sqrt{v_b x_b} - \sqrt{v_a x_b}}\right]^2 \tag{109}$$

$$v = \frac{(1+k)\sqrt{v_a v_b x_a x_b}}{x_a + x_b} \quad . \tag{110}$$

Note that when in Eqs. (89) to (93) the subindexes a and b are interchanged, the formulas do not change; however, when subscripts a and b are interchanged in v_a and v_b only, or in x_a and x_b only, then A_L and D_L become exchanged whereas B_L and C_L remain unchanged. This means that the lattice two-port is turned by 180 deg (input and output are interchanged) and is thus equivalent to the tandem in which Section 2 is followed by Section 1. Therefore, it would be completely unimportant if in realizing a special function $\bar{F}(s)$ we first designed the T-section with the negative constants n and v.

So far we did not further discuss the "very special function $\bar{F}(s)$", but in Section 6 we will show how such a function can be derived from a special function.

We refer to the design of the lattice network as "Realization Procedure R_2", and present compact instructions below.

5.1 Realization Procedure R_2

Given: The constants n, v, x, and k of the circuit in Figure 12.

(1) Compute P according to Eq. (100)
(2) Compute Q according to Eq. (101)
(3) Compute K according to Eq. (102)
(4) Compute v_a and v_b according to Eqs. (103) and (104)
(5) Compute x_a and x_b according to Eqs. (105) and (106).

5.2 Numerical Examples

Presented below are four numerical examples. In each example the constants n, k, v, and x are known. With the Programma 101, constants v_a, v_b, x_a, and x_b of the circuit shown in Figure 13 (which is equivalent to the circuit in Figure 12) are computed. We also present the intermediate results of P and K that are not printed by the program.

Store on Cards 156 A & B

```
n = 1•2307690  d◊
k = 1•1666663  D◊
v = 6•8249999  •◊
x = 5•4599998  E◊
-------------------
                V
P =29•4750009  b◊
K = 0•7566499  B◊

                V
va =19•0830544 b◊
vb = 4•8653741 B◊

xa = 2•8690976 c◊
xb =20•7109023 C◊
```

Example 5.2.1

Store on Cards 156 A & B

```
n =  0•8125003 d◊
k = 80•0955910 D◊
v =  4•2328149 •◊
x =  3•3862516 E◊
-------------------
                V
P =  0•3224201 b◊
K =  0•5936792 B◊

                V
va = 0•0482836 b◊
vb = 0•3485408 B◊

xa = 0•0524023 c◊
xb = 0•2055336 C◊
```

Example 5.2.2

Store on Cards 156 A & B

```
n = 0•7464789  d◊
k = 1•5238088  D◊
v = 0•5952378  •◊
x = 0•4761902  E◊
-------------------
                V
P = 1•2297374  b◊
K = 0•4825428  B◊

                V
va = 1•4469391 b◊
vb = 0•2004454 B◊

xa = 0•2545345 c◊
xb = 0•7292550 C◊
```

Example 5.2.3

Store on Cards 156 A & B

```
n =  1•3396226 d◊
k = 31•1435267 D◊
v = 24•3898237 •◊
x = 19•5118586 E◊
-------------------
                V
P =  7•1000002 b◊
K =  0•7565499 B◊

                V
va = 1•3712615 b◊
vb = 3•9287389 B◊

xa = 0•6911142 c◊
xb = 4•9888857 C◊
```

Example 5.2.4

6. THE VERY SPECIAL DRIVING-POINT IMPEDANCE CIRCUIT

The circuit pictured in Figure 12 has the very special driving-point impedance $\bar{F}(s)$, since the ratio n_2 of the second section is the negative inverse of the ratio of the first section. At first glance, this class of circuits looks very limited. However, we shall show in the following sections that a circuit realizing the very special function can be obtained under certain circumstances from the more general special class of special functions by transposing a capacitance or an inductance from the circuit input to its output. By this transposition both ratios u_1 and u_2 are changed, and when the transposed element has the correct magnitude the two ratios become negative inverse. The element to be transposed can be either a series or a shunt element. In Tables 5 and 6 of a previous paper, Haase (1970b) presented the formulas for computing the change of constants n, v, x, and z to constants n', v', x', and z', after the transposition.

6.1 The Transposition of a Series Capacitance

Consider the circuit in Figure 14, part (a). The first section in this circuit is determined by constants v_1, x_1, and $n_1 > 0$, and the second section by constants v_2, x_2, and $n_2 < 0$; all constants in the first section are positive, but with n_2 negative in the second section, v_2 must also be negative. Due to the normalization of the special function $\bar{F}(s)$, the termination resistance must be $z = \bar{N}_0/\bar{D}_0 = 1/(n_1n_2)^2$. The circuit has the series impedance x_0/s at its input. This impedance consists of a capacitance $1/x_0$. The driving-point impedance of the circuit implements $\bar{F}(s)$, it is not $F(s)$ itself. Since $\bar{F}(s) = \bar{N}(s)/\bar{D}(s)$, with $\bar{N}(s)$ and $\bar{D}(s)$ normalized quartic polynomials, the driving-point impedance

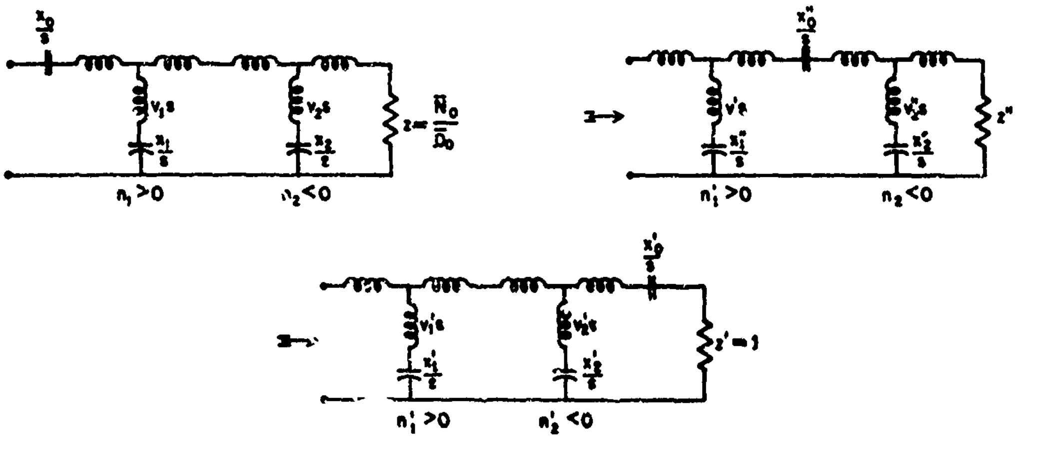


Figure 14. Stepwise Transposition of a Series Capacitance x_0

$$x_0/s + \frac{\bar{N}(s)}{\bar{D}(s)} = \frac{s\bar{D}(s) + x_0\bar{N}(s)}{s\bar{D}(s)} \text{ is of the type } P_{10}.$$

Transposing the capacitance over the first section, we obtain the circuit shown in part (b) of Figure 14. According to the formulas in Table 6 (upper part) of Haase (1970b), we find that the constants n_1, v_1, and x_1 change to

$$n_1' = n_1 \frac{x_1}{x_1 + x_0} , \tag{111}$$

$$v_1' = v_1(1 + \frac{x_0}{x_1}) , \tag{112}$$

and

$$x_1' = x_1(1 + \frac{x_0}{x_1}) . \tag{113}$$

Note that

$$x_1/v_1 = x_1'/v_1' = \Omega_0 . \tag{114}$$

The transposed inverse capacitance that now appears between the two sections becomes

$$x_0'' = x_0 \left(1 + \frac{x_0}{x_1}\right) . \tag{115}$$

The transposition also influences constants v_2 and x_2 of the second section and the termination. In part (b) of Figure 14

$$v_2'' = v_2 \left[\frac{n_1}{n_1'}\right]^2 , \tag{116}$$

$$x_2'' = x_2 \left[\frac{n_1}{n_1'}\right]^2 , \tag{117}$$

and

$$z'' = z \left[\frac{n_1}{n_1'}\right]^2 . \tag{118}$$

The ratio n_2 as well as $-\Omega_0 = x_2/v_2 = x_2''/v_2''$ remain unchanged.

We now transpose the inverse capacitance $1/x_0''$ over the second section, thus obtaining the circuit in part (c) of Figure 14. According to the same formulas as applied before, the constants n_2'', v_2'', and x_2'' and the termination z'' change to

$$n_2' = \frac{n_2}{1 + A} , \tag{119}$$

$$v_2' = v_2'' (1 + A), \tag{120}$$

$$x_2' = x_2''(1 + A) , \tag{121}$$

and

$$z' = z\left[\frac{n_1 n_2}{n_1' n_2'}\right]^{-2} . \tag{122}$$

In Eqs. (119), (120), and (121),

$$A = \frac{x_0''}{x_2}\left[\frac{n_1'}{n_1}\right]^2 . \tag{123}$$

Note also that

$$x_2/v_2 = x_2'/v_2' = -x_1/v_1 = -\Omega_0 . \tag{124}$$

The transposed inverse capacitance becomes

$$x_0' = x_0'' (1 + A) . \tag{125}$$

Equations (114) and (124) suggest the introduction of the positive constants

$$k_1 = x_1/x_2 = -v_1/v_2 \tag{126}$$

and

$$k = x_1'/x_2' = -v_1'/v_2' , \tag{127}$$

and we can write:

$$v = v_1 \tag{128}$$

and

$$x = x_1 . \tag{129}$$

Then, by Eqs. (126) to (129),

$$x_0 = -x_1 \frac{1 + n_1 n_2}{k_1 + 1} \tag{130}$$

causes

$$n_1 n_2 = -1 , \tag{131}$$

$$x_0' = - x_0 n_1 n_2 , \tag{132}$$

$$n = n_1' = n_1 \frac{k_1 + 1}{k_1 - n_1 n_2} , \tag{133}$$

$$k = -k_1 / n_1 n_2 , \tag{134}$$

$$v = v_1 \frac{k_1 - n_1 n_2}{k_1 + 1} \tag{135}$$

$$x = v \Omega_0 , \tag{136}$$

and

$$z' = 1 . \tag{137}$$

Equation (130) presents the magnitude of the capacitance that is necessary to obtain the implied very special function $F(s)$, for which $n_1' n_2' = n(-1/n) = -1$. Since by definition the product $n_1 n_2$ is negative, the transposed capacitance according to Eq. (132) has the same polarity as x_0. Since both have to be positive, it is necessary that, according to the numerator in Eq. (130),

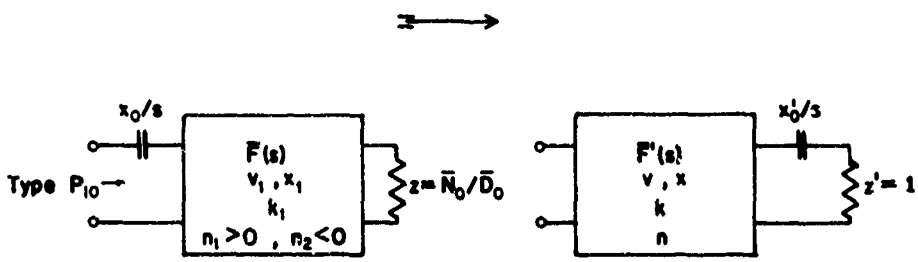
$$- n_1 n_2 > 1 . \tag{138}$$

Equation (138) is a necessary condition for transposing a series capacitance from the input of a circuit to its output. There is, however, no restriction imposed on the other constants.

We said that the driving-point impedance of the circuit in Figure 14 is of the type P_{10}. In order to be able to transpose the impedance x_0/s with its magnitude given by Eq. (130), x_0/s must be subtracted from the total impedance x_t/s available in a certain driving-point impedance of the type P_{10}. This is a second necessary condition. We have compiled the formulas for the transposition of a series capacitance in Table 2.

Table 2. Formulas for the Transposition of a Series Capacitance $1/x_0$

Necessary Condition: $-n_1n_2 > 1 \equiv \bar{N}_0/\bar{D}_0 < 1$



$$x_0 = -x_1 \frac{1 + n_1 n_2}{k_1 + 1}$$

$$x_0' = -x_0 n_1 n_2$$

$$n = n_1 \frac{k_1 + 1}{k_1 - n_1 n_2}$$

$$k = -\frac{k_1}{n_1 n_2}$$

$$v = v_1 \frac{k_1 - n_1 n_2}{k_1 + 1}$$

$$x = v\Omega_0$$

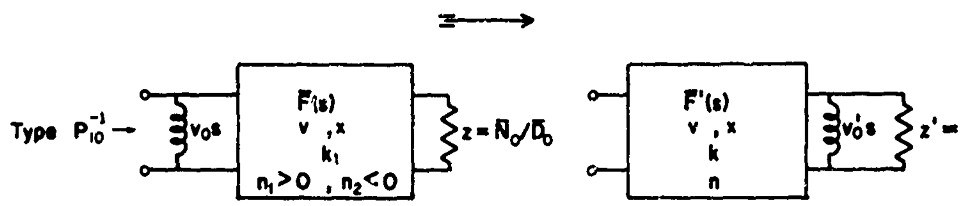
$$z' = 1$$

6.2 The Transposition of a Shunt Inductance (see also Figure 16)

Next suppose that we transpose a shunt inductance v_0 instead of a series capacitance $1/x_0$ over the two sections. To do this we have to use the formulas presented in Table 5 (lower part) of Haase (1970b). The transposition procedure is very similar to that described in Section 6. 1, and we can immediately go to the results presented in Table 3.

Table 3. Formulas for the Transposition of a Shunt Inductance v_0

Necessary Condition: $-n_1 n_2 < 1 \equiv \bar{N}_0/\bar{D}_0 > 1$



$$v_0 = \frac{v_1}{k_1} \frac{n_1(n_2-1)^2 - k_1 n_2 (n_1-1)^2}{1 + n_1 n_2}$$

$$v_0' = -v_0 n_1 n_2$$

$$n = \frac{n_1^2(n_2-1)^2 + k_1(n_1-1)^2}{n_1(n_2-1)^2 - k_1 n_2 (n_1-1)^2}$$

$$k = \frac{k_1}{n_1^2}\left[\frac{n_1(n_1+1)(n_2-1) - k_1(n_1-1)^2}{n_1(n_2-1)^2 + k_1(n_1-1)(n_2+1)}\right]^2$$

$$v = v_1\left[\frac{n_1(n_2-1)^2 - k_1 n_2 (n_1-1)^2}{n_1(n_2-1)^2 + k_1(n_1-1)(n_2+1)}\right]^2$$

$$x = v\Omega_0$$

$$z' = 1$$

The magnitude of the inductance to be transposed according to this table is

$$v_0 = \frac{v_1}{k_1} \cdot \frac{n_1(n_2-1)^2 - k_1 n_2 (n_1-1)^2}{1 + n_1 n_2} , \tag{139}$$

and the transposed inductance is

$$v_0' = - v_0 n_1 n_2 . \tag{140}$$

Since v_0' must be positive, v_0 must be positive. The numerator in Eq. (139) is certainly positive since n_2 is negative. The denominator is positive and with it also v_0 if

$$-n_1 n_2 < 1 . \tag{141}$$

This is the necessary first condition for the transposition of a shunt inductance.

The second necessary condition is that the admittance s/v_0 be available at the input for the transposition. Adding admittance s/v_0 to admittance $\bar{E}(s)/\bar{N}(s)$ yields a driving-point function of the type P_{10}^{-1}. Thus it is necessary that admittance s/v_t in such a function is at least equal to the admittance s/v_0 to be transposed.

6.3 The Transposition of a Series Inductance

Consider the circuit in Figure 15 where we twice transpose the series impedance $v_0 s$ from the input in part (a): first over the first section, obtaining the

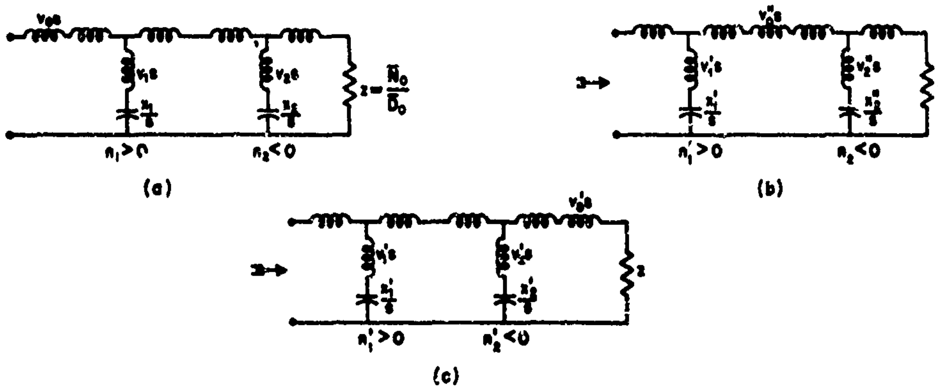


Figure 15. Stepwise Transposition of a Series Inductance v_0

circuit in part (b), and then over the second section, obtaining the circuit in part (c). We apply the formulas in Table 5 (lower part) of Haase (1970b).

Transposing v_0 over the first section yields the constants

$$n_1' = n_1 + \frac{v_0}{v_1} , \tag{142}$$

$$v_1' = v = v_1 , \tag{143}$$

and

$$x_1' = x_1 , \tag{144}$$

and the transposed inductance becomes

$$v_0'' = \frac{v_0}{n_1 n_1'} . \tag{145}$$

In contrast to the discussions in Section 6.1, constants v_2 and x_2 and the termination z remain unchanged.

Transposition of v_0'' yields

$$n_2' = n_2 + \frac{v_0''}{v_2} , \tag{146}$$

$$v_2' = -v/k = v_2 , \tag{147}$$

$$x_2' = x/k = x_2 , \tag{148}$$

and, therefore,

$$k = k_1 . \tag{149}$$

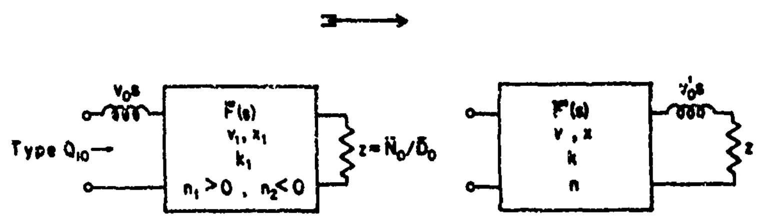
Also, the termination z remains unchanged. The formulas for the series inductance transposition are compiled in Table 4.

The inductance to be transposed and causing $n_1'n_2' = -1$ is

$$v_0 = v_1 n_1 \frac{1 + n_1 n_2}{k_1 - n_1 n_2} , \tag{150}$$

Table 4. Formulas for the Transposition of a Series Inductance v_0

Necessary Condition: $-n_1 n_2 < 1 \equiv \bar{N}_0/\bar{D}_0 > 1$



$$v_0 = v_1 n_1 \frac{1 + n_1 n_2}{k_1 - n_1 n_2}$$

$$v_0' = -\frac{v_0}{n_1 n_2}$$

$$n = n_1 \frac{k_1 + 1}{k_1 - n_1 n_2}$$

$$k = k_1$$

$$v = v_1$$

$$x = v\Omega$$

$$z = 1/(n_1 n_2)^2$$

and the transposed inductance is

$$v_0' = -\frac{v_0}{n_1 n_2} . \tag{151}$$

Therefore, the first necessary condition is

$$-n_1 n_2 < 1 , \tag{152}$$

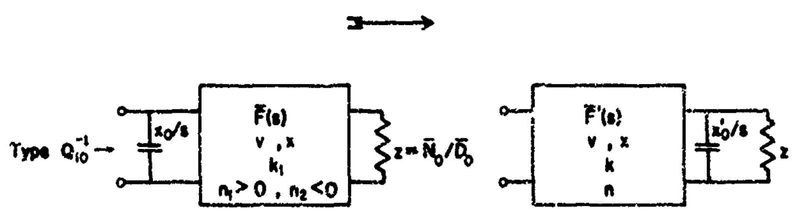
and the second necessary condition is that the total inductance v_t available at the input for the transposition is at least v_0. Adding impedance $v_0 s$ to impedance $\bar{F}(s) = \bar{N}(s)/\bar{D}(s)$ yields a driving-point impedance of the type Q_{10}.

6.4 The Transposition of a Shunt Capacitance (see also Figure 17)

To transpose a shunt capacitance over the circuit, we have to apply the formulas presented in Table 5 (upper part) of Haase (1970b). The computational procedure is similar to that discussed in Section 6.3, so we can go immediately to the presentation of Table 5.

Table 5. Formulas for the Transposition of a Shunt Capacitance $1/x_0$

Necessary Condition: $-n_1 n_2 > 1 \equiv \bar{N}_0/\bar{D}_0 < 1$



$$x_0 = -\frac{x_1}{k_1} \frac{n_1^2(n_2-1)^2 + k_1(n_1-1)^2}{1 + n_1 n_2}$$

$$x_0' = -\frac{x_0}{n_1 n_2}.$$

$$n = \frac{n_1^2(n_2-1)^2 + k_1(n_1-1)^2}{n_1(n_2-1)^2 - k_1 n_2(n_1-1)^2}$$

$$k = -k_1 \frac{n_2}{n_1}\left[\frac{n+1}{n}\right]^2 \left[\frac{n_1-1}{n_2-1}\right]^2$$

$$v = v_1 \frac{n}{n_1}\left[\frac{n_1-1}{n-1}\right]^2$$

$$x = v\Omega_0$$

$$z = 1/(n_1 n_2)^2$$

The inverse capacitance to be transposed is

$$x_0 = -\frac{x_1}{k_1} \cdot \frac{n_1^2(n_2-1)^2 + k_1(n_1-1)^2}{1 + n_1 n_2} \tag{153}$$

and the transposed inverse capacitance is

$$x_0' = -\frac{x_0}{n_1 n_2}. \tag{154}$$

The results in Eqs. (153) and (154) are positive if

$$-n_1 n_2 > 1 . \tag{155}$$

This is the first necessary condition. The second necessary condition is that the available admittance s/x_t at the input is at least x_0/s. Adding the admittance x_0/s to the admittance $\bar{D}(s)/\bar{N}(s)$ yields a driving-point impedance of the type Q_{10}^{-1}.

6.5 The Realization Procedures $A_1, \ldots, A_4$

In this Section we compile the instructions for obtaining constants n, v, x, k, and z' of the tandem circuit, implying the very special function $\bar{F}(s)$ when constants $n_1 > 0$, v_1, and x_1 and $n_2 < 0$, k_1, and z are known. Subsequently, this circuit will be transformed into the lattice circuit, according to the instructions presented in Section 5.1.

6.5.1 PROCEDURE A_1

Known: $n_1 > 0$, $n_2 < 0$, v_1, x_1, and k_1 of a circuit realizing a special function $\bar{F}(s)$ in which $\bar{N}_0/\bar{D}_0 < 1$.

Requested: The transposition of a series capacitance $1/x_0$.

Procedure: According to the formulas on Table 2, compute the constants x_0, x_0', n, k, v, and x in sequence.

6.5.2 PROCEDURE A_2

Known: $n_1 > 0$, $n_2 < 0$, v_1, x_1, and k_1 of a circuit realizing a special function $\bar{F}(s)$ in which $\bar{N}_0/\bar{D}_0 > 1$.

Requested: The transposition of a shunt inductance v_0.

Procedure: According to the formulas in Table 3, compute the constants v_0, v_0', n, k, v, and x in sequence.

6.5.3 PROCEDURE A_3

Known: $n_1 > 0$, $n_2 < 0$, v_1, x_1, k_1 of a circuit realizing a special function $\bar{F}(s)$ in which $\bar{N}_0/\bar{D}_0 > 1$.

Requested: The transposition of a series inductance v_0.

Procedure: According to the formulas in Table 4, compute the constants v_0, v_0', n, k, v, and x in sequence.

6.5.4 PROCEDURE A_4

Known: $n_1 > 0$, $n_2 < 0$, v_1, x_1, k_1 of a circuit realizing a special function F(s) in which $\bar{N}_0/\bar{D}_0 < 1$.

Requested: The transposition of a shunt capacitance $1/x_0$.

Procedure: According to the formulas in Table 5, compute the constants x_0, x_0', n, k, v, and x in sequence.

Numerical Examples

Below are four numerical examples in which we assume that constants $n_1 > 0$, $n_2 < 0$, v_1, x_1, and k_1 are known. They are the constants obtained in example 4.4.1 for examples 6.5.5.1 and 6.5.5.4, and those obtained in example 4.4.2 for examples 6.5.5.2 and 6.5.5.3 as listed below.

First we shall answer the questions:

(1) Can a capacitance be transposed ?

(2) Can an inductance be transposed ?

The affirmative answer listed in the table depends on whether $-n_1n_2$ is greater or smaller than 1.

Examples	6.5.5.1 and 6.5.5.4	6.5.5.2 and 6.5.5.3
Transposed Element	Capacitance	Inductance
n_1	2.0000001	0.4999999
n_2	-3.0000007	-0.3333328
$-n_1n_2$	6.0000018	0.1666664
v_1	4.1999997	0.5952378
x_1	3.3599997	0.4761902
k_1	7.0000004	1.5238088

Using programs designed for the Programma 101 computer, we apply:

Procedure A_1 to the constants of Example 6.5.5.1,
Procedure A_2 to the constants of Example 6.5.5.2,
Procedure A_3 to the constants of Example 6.5.5.3,
Procedure A_4 to the constants of Example 6.5.5.4.

The procedures yield the constants n, k, v, and x of Figure 12, and inverse capacitance x_0 that is to be transposed as a series capacitance in example 6.5.5.1 and as a shunt capacitance in example 6.5.5.4. The x_0' is the transposed capacitance. In example 6.5.5.2, v_0 is the shunt inductance to be transposed, and in example 6.5.5.3, v_0 is the series inductance to be transposed. v_0' is the transposed inductance.

```
Store on      n1  =   2•0000001  d◊
Card 153A     n2  =  -3•0000007  D◊
              v1  =   4•1999997  •◊
              x1  =   3•3599997  E◊
              k1  =   7•0000004  f◊
-----------------------------------
                                 V
              x0  =   2•1000001  A◊
              x0' =  12•6000041  A◊
              n   =   1•2307690  b◊
              x   =   1•1666663  B◊
              v   =   6•0249999  c◊
              x   =   5•1599998  C◊
```

Example 6.5.5.1 Procedure A_1
See Figure 14

```
Store on      n1  =   0•4999999  d◊
Cards 154A,B  n2  =  -0•3333328  D◊
              v1  =   0•5952378  •◊
              x1  =   0•4761902  E◊
              k1  =   1•5238088  f◊
-----------------------------------
                                 V
              v0  =   0•4761895  A◊
              v0' =   0•0793647  A◊
              k   =   0•8125063  b◊
                                 V
              n   =  80•0955910  B◊
              v   =   4•2328149  c◊
              x   =   3•3862516  C◊
```

Example 6.5.5.2 Procedure A_2
See Figure 16

```
Store on      n1  =   0•4999999  d◊
Card 153 B    n2  =  -0•3333328  D◊
              v1  =   0•5952378  •◊
              x1  =   0•4761902  E◊
              k1  =   1•5238088  f◊
-----------------------------------
                                 V
              v0  =   0•1467136  A◊
              v0' =   0•3292830  A◊
              n   =   0•7464789  b◊
              k   =   1•5238088  B◊
              v   =   0•5952378  c◊
              x   =   0•4761902  C◊
```

Example 6.5.5.3 Procedure A_3
See Figure 15

```
Store on      n1  =   2•0000001  d◊
Cards 155A,B  n2  =  -3•0000007  D◊
              v1  =   4•1999997  •◊
              x1  =   3•3599997  E◊
              k1  =   7•0000004  f◊
-----------------------------------
                                 V
              x0  =   6•6159995  A◊
              x0' =   1•1359995  A◊
                                 V
              n   =   1•3396226  b◊
              k   =  31•1435267  B◊
              v   =  24•3398237  c◊
              x   =  19•5118586  C◊
```

Example 6.5.5.4 Procedure A_4
See Figure 17

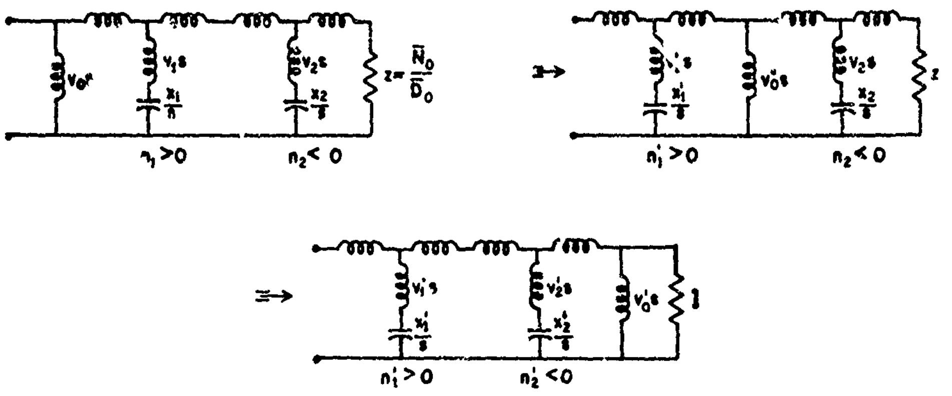


Figure 16. Stepwise Transposition of a Shunt Inductance v_0

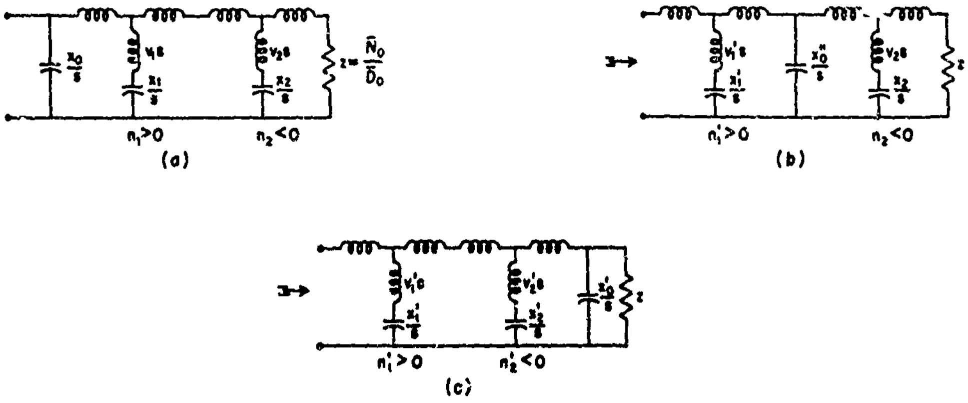


Figure 17. Stepwise Transposition of a Shunt Capacitance x_0

7. THE DECOMPOSITIONS OF THE IMPEDANCE FUNCTIONS OF THE TYPES P_{10}, P_{10}^{-1}, Q_{10}, AND Q_{10}^{-1}

The Decomposition of F(s) of the Type P_{10}

The polynomials of F(s) are

$$N(s) = \sum_{i=0}^{5} N_i s^i \tag{156}$$

and

$$D(s) = \sum_{i=1}^{5} D_i s^i,\ D_5 = 1 . \tag{157}$$

F(s) can be decomposed as

$$F(s) = K\bar{F}(s) + x_t/s \,, \tag{158}$$

where

$$x_t = N_0/D_1, \tag{159}$$

$$K = N_5 \,, \tag{160}$$

and $\bar{F}(s)$ is a function of the type P_7^{-1}. (161)

The coefficients of $\bar{F}(s)$ are

$$\bar{N}_i = \frac{N_{i+1} - N_0 D_{i+2}}{N_5 D_1} \tag{162}$$

and

$$\bar{D}_1 = D_{i+1} \,. \tag{163}$$

With the coefficients of F(s) known, the coefficients of $\bar{F}(s)$ and x_t and K can be computed. A circuit representation of the decomposition, to which we shall refer as decomposition procedure De1, is shown in Figure 18. In this circuit the factor K is presented as an ideal transformer with the turn ratio $\sqrt{K:1}$. This transformer will not appear in the realization. It has the meaning that all impedances on its right side must be multiplied by K when the transformer is omitted.

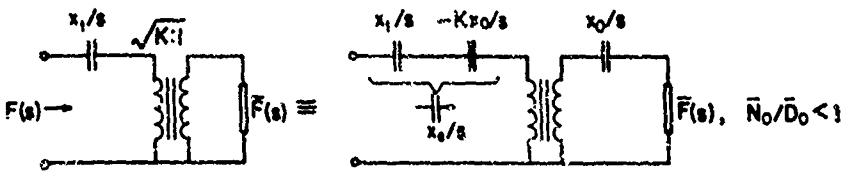


Figure 18. Transposition in a P_{10}-type Function

The Decomposition of F(s) of the Type Q_{10}

The polynomials of F(s) are

$$N(s) = \sum_{i=0}^{5} N_i s^i \tag{164}$$

and

$$D(s) = \sum_{i=0}^{4} D_i s^i, \; D_4 = 1 . \tag{165}$$

F(s) can be decomposed as

$$F(s) = K\bar{F}(s) + v_t s , \tag{166}$$

where

$$v_t = N_5 , \tag{167}$$

$$K = N_4 - N_5 D_3 , \tag{168}$$

and $\bar{F}(s)$ is a function of the type P_7 . (169)

The coefficients of $\bar{F}(s)$ are

$$\bar{N}_i = \frac{N_i - N_5 D_{i-1}}{N_4 - N_5 D_3} \tag{170}$$

and

$$\bar{D}_i = D_i . \tag{171}$$

The circuit representing the decomposition, which we refer to as decomposition procedure De3, is shown in Figure 19.

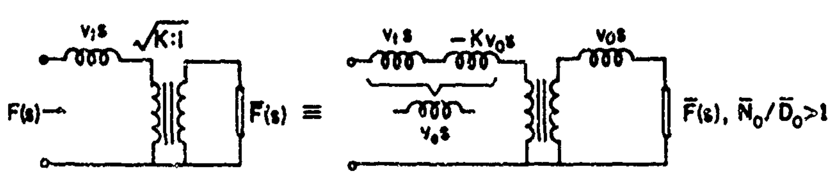


Figure 19. Transposition in a Q_{10}-type Function

The Decomposition of F(s) of the Type P_{10}^{-1}

The polynomials of F(s) are

$$N(s) = \sum_{i=1}^{5} N_i s^i \tag{172}$$

and

$$D(s) = \sum_{i=0}^{5} D_i s^i, \; D_5 = 1. \tag{173}$$

F(s) can be decomposed as

$F(s) = v_t s \oplus K\bar{F}(s)$, with is equivalent to the script

$$1/F(s) = 1/v_t s + 1/K\bar{F}(s) \; . \tag{174}$$

In Eq. (173),

$$v_t = N_1/D_0 \; , \tag{175}$$

$$K = N_5 \; , \tag{176}$$

and $\bar{F}(s)$ is a function of the type P_7 . (177)

The coefficients of $\bar{F}(s)$ are

$$\bar{N}_i = \frac{N_{i+1}}{N_5} \tag{178}$$

and

$$\bar{D}_i = \frac{N_1 D_{i+1} - N_{i+2} D_0}{N_1} \; . \tag{179}$$

The circuit representing the decomposition, which we refer to as decomposition procedure De2, is shown in Figure 20.

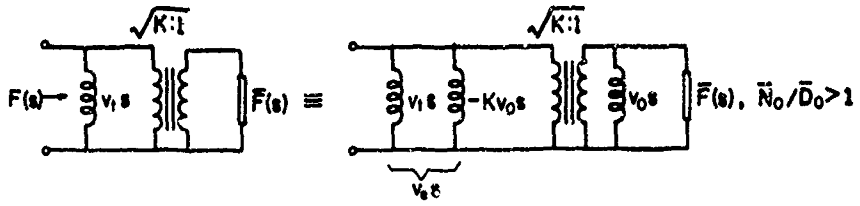


Figure 20. Transposition in a P_{10}^{-1}-type Function

The Decomposition of F(s) of the Type Q_{10}^{-1}

The polynomials of F(s) are

$$N(s) = \sum_{i=0}^{4} N_i s^i \tag{180}$$

and

$$D(s) = \sum_{i=0}^{5} D_i s^i, \ D_5 = 1 . \tag{181}$$

F(s) can be decomposed as

$F(s) = x_t/s \oplus K\bar{F}(s)$, which is equivalent to the script

$$1/F(s) = s/x_t + 1/K\bar{F}(s) \tag{182}$$

where

$$x_t = N_4^{\bullet}, \tag{183}$$

$$K = \frac{N_4^2}{N_4 D_4 - N_3} , \tag{184}$$

and $\bar{F}(s)$ is a function of the type P_7^{-1}. (185)

The coefficients of $\bar{F}(s)$ are

$$\bar{N}_i = \frac{N_5 D_i - N_{i-1}}{N_4 D_4 - N_3} \tag{186}$$

and

$$\bar{D}_i = \frac{N_i}{N_4}. \tag{187}$$

The circuit for the decomposition, which we refer to as decomposition procedure De4, is shown in Figure 21.

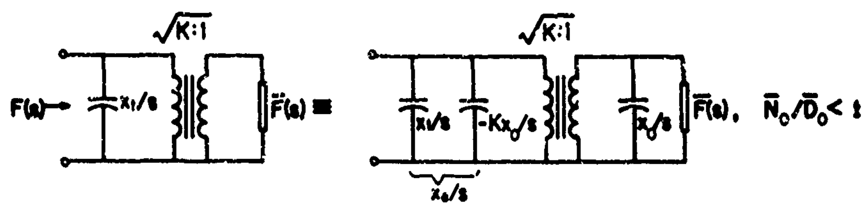


Figure 21. Transposition in a Q_{10}^{-1}-type Function

In our attempted circuit realization, a part of impedance x_t/s in the decomposition De1 and a part of impedance v_ts in decomposition De3 had to be transposed over the circuit in order to make $\bar{F}(s)$ a very special function. In decompositions De2 and De4, a part of admittance $1/v_ts$ and admittance s/x_t, respectfully, had to be transposed for the same reason. These parts must be of such magnitude that the remaining impedance or admittance, respectively, remains positive. Also, in decompositions De1 and De4, where the transposed impedance is of a capacitive nature, function $\bar{F}(s)$ must be type P_7^{-1}; in decompositions De2 and De3, where the transposed impedance is of an inductive nature, function $\bar{F}(s)$ must be type P_7.

We are able, according to our discussions in Section 6, to determine the magnitude of the element to be transposed. The element is x_0/s in decompositions De1 and De4 and it is v_0s in decompositions De2 and De3. In general the element has to be taken to the left side of the ideal transformer in Figures 18, 19, 20, and 21. Therefore, the remaining element is:

in decomposition De1, $x_e/s = x_t/s - Kx_0/s$ (188)

in decomposition De2, $s/v_e = s/v_t - s/Kv_0$ (189)

in decomposition De3, $v_es = v_ts - Kv_0s$ (190)

in decomposition De4, $s/x_e = s/x_t - s/Kx_0$. (191)

All differences have to be positive.

7.1 Decomposition Procedures De1, ..., De4

The results of the foregoing are compiled in Table 6. The split of a function F(s), known by the coefficients of N(s) and D(s) and being of the types P_{10}, P_{10}^{-1}, Q_{10}, or Q_{10}^{-1}, into an inductive or capacitive component and a special function $\bar{F}(s)$ multiplied with an impedance factor K is a routine procedure. We refer to these procedure as De1, . . . , De4, depending on what type of the aforementioned sequential functions is applied. Table 6 presents the formulas to compute the magnitude of $v_t s$ or x_t/s, the positive impedance factor K, and the coefficients of $\bar{F}(s)$. The table also presents the necessary condition for $-n_1 n_2$, which can also be expressed by the ratio $\bar{N}_0/\bar{D}_0$. In the last column of Table 6 is the magnitude of the input element that is left when the element x_0/s or $v_0 s$ to be transposed is subtracted from the available element x_t/s or $v_t s$. Procedure De1 is followed by procedure A_1, De2 by A_2, and so forth. In more detail the instructions of the procedure are as follows:

7.1.1 DECOMPOSITION PROCEDURE De1 APPLIED TO A FUNCTION OF THE TYPE P_{10}

Known: The coefficients N_0, . . . , N_5 and D_1, . . . , D_5. Make sure that $D_5 = 1$. If not, divide all coefficients of N(s) and D(s) by D_5.

Compute: x_t, K, $\bar{N}_i$, and $\bar{D}_i$ according to formulas in Table 6, first row.

Test: The necessary condition $\bar{N}_0/\bar{D}_0 < 1$.

Continue with procedure A_1, Section 6.5.1.

Compute $x_e = x_t - Kx_0$.

For Circuit Realization, see circuit in Figure 18.

Table 6. Decomposition Components

Function Type	Decomposition		$v_t = 1/x_t$	Impedance Factor	Coefficients $\bar{N}_i$	Coefficients $\bar{D}_i$	$-n_1 n_2$	$\frac{\bar{N}_0}{\bar{D}_0}$	
P_{10}	De1	$\frac{x_t}{s} + K\bar{F}(s)$	$x_t = \frac{N_0}{D_1}$	$K = N_5$	$\frac{N_{i+1} - N_0 D_{i+2}}{N_5 D_1}$	D_{i+1}	>1	<1	$x_e = x_t - Kx_0$
P_{10}^{-1}	De2	$v_t s \oplus K\bar{F}(s)$	$v_t = \frac{N_1}{D_0}$	$K = N_5$	$\frac{N_{i+1}}{N_5}$	$\frac{N_i D_{i+1} - N_{i+2} D_0}{N_1}$	<1	>1	$v_e = \frac{K v_t v_0}{K v_0 - v_t}$
Q_{10}	De3	$v_t s + K\bar{F}(s)$	$v_t = N_3$	$K = N_4 - N_5 D_3$	$\frac{N_i - N_5 D_{i-1}}{N_4 - N_5 D_3}$	D_i	<1	>1	$v_e = v_t - Kv_0$
Q_{10}^{-1}	De4	$\frac{x_t}{s} \oplus K\bar{F}(s)$	$x_t = N_4$	$K = \frac{N_4^2}{N_4 D_4 - N_3}$	$\frac{N_5 D_i - N_{i-1}}{N_4 D_4 - N_3}$	$\frac{N_i}{N_4}$	>1	<1	$x_e = \frac{K x_t x_0}{K x_0 - x_t}$

7.1.2 DECOMPOSITION PROCEDURE De2 APPLIED TO A FUNCTION OF THE TYPE P_{10}^{-1}

Known: The coefficients $N_1, \ldots, N_5$, and $D_0, \ldots, D_5$. Be sure that $D_5 = 1$. If not, divide all coefficients of N(s) and D(s) by D_5.

Compute: v_t, K, $\bar{N}_i$, and $\bar{D}_i$ according to the formulas in Table 6, second row.

Test: The necessary condition $\bar{N}_0/\bar{D}_0 < 1$

Continue with procedure A_2, Section 6.5.2.

Compute $v_e = Kv_t v_0/(Kv_0 - v_t)$.

For Circuit Realization see Figure 20.

7.1.3 DECOMPOSITION PROCEDURE De3 APPLIED TO A FUNCTION OF THE TYPE Q_{10}

Known: The coefficients $N_0, \ldots, N_5$, and $D_0, \ldots, D_4$. Be sure that $D_4 = 1$. If not, divide all coefficients of N(s) and D(s) by D_4.

Compute: v_t, K, $\bar{N}_i$, and $\bar{D}_i$ according to the formulas in the third row of Table 6.

Test: The necessary condition $\bar{N}_0/\bar{D}_0 > 1$.

Continue with procedure A_3, Section 6.5.3.

Compute: $v_e = v_t - Kv_0$.

For Circuit Realization see Figure 19.

7.1.4 DECOMPOSITION PROCEDURE De4 APPLIED TO A FUNCTION OF THE TYPE Q_{10}^{-1}

Known: The coefficients $N_0, \ldots, N_4$, and $D_0, \ldots, D_5$. Be sure that $D_5 = 1$. If not, divide all coefficients of N(s) and D(s) by D_5.

Compute: x_t, K, $\bar{N}_i$, and $\bar{D}_i$ according to the formulas in the fourth row of Table 6.

Test: The necessary condition $\bar{N}_0/\bar{D}_0 > 1$.

Continue with procedure A_4, Section 6.5.4.

Compute: $x_e = Kx_t x_0/(Kx_0 - x_t)$.

For Circuit Realization see Figure 21.

7.2 Numerical Examples

Following are four numerical examples where the driving-point impedance F(s) is given by the coefficients of N(s) and D(s). In these examples,

Example 7.2.1 F(s) is of the Type P_{10},

Example 7.2.2 F(s) is of the Type P_{10}^{-1}, Store on Card 160A

	Coefficients N_i
$i = 0$	14·8992000 d◊
1	1·0920632 D◊
2	50·0617140 e◊
3	2·1341269 E◊
4	37·9600000 f◊
5	1·0000000 F◊

Store on Card 160B

	Coefficients D_i
$i = 0$	0·0000000 d◊
1	3·8400000 D◊
2	0·2539682 e◊
3	7·6571428 E◊
4	0·2579365 f◊
5	1·0000000 F◊

V

V

x_t = 3·8800000 F◊

W

K = 1·0000000 A◊

	Coefficients $\bar{N}_i$
$i = 0$	0·1066666 b◊
1	20·3520000 B◊
2	1·1333333 c◊
3	34·0800000 C◊
4	1·0000000 d◊

	Coefficients $\bar{D}_W$
$i = 0$	3·8400000 b◊
1	0·2539682 B◊
2	7·6571428 c◊
3	0·2579365 C◊
4	1·0000000 d◊

Example 7.2.1
Circuit Figure 18

Store on Card 161 A

	Coefficients N_i
$i = 0$	0·0000000 d◊
1	3·8400000 D◊
2	0·2539682 e◊
3	7·6571428 E◊
4	0·2579365 f◊
5	1·0000000 F◊

Store on Card 161 B

	Coefficients D_i
$i = 0$	20·7360000 d◊
1	1·4780948 D◊
2	61·7005711 e◊
3	2·5261901 E◊
4	39·4800000 f◊
5	1·0000000 F◊

V

W

v_t = 0·1851851 A◊

	Coefficients $\bar{D}_i$
$i = 0$	0·1066659 b◊
1	20·3519810 B◊
2	1·1333324 c◊
3	34·0799976 C◊
4	1·0000000 d◊

W

K = 1·0000000 F◊

	Coefficients $\bar{N}_i$
$i = 0$	3·8400000 b◊
1	0·2539682 B◊
2	7·6571428 c◊
3	0·2579365 C◊
4	1·0000000 d◊

Example 7.2.2
Circuit Figure 20

Store on Card 162A

	Coefficients N_i
i = 0	3·2640000 d0
1	0·3225395 D0
2	26·8605713 e0
3	1·3525793 E0
4	34·9300000 f0
5	1·0000000 F0

Store on Card 162B

	Coefficients D_i
i = 0	0·1066666 d0
1	20·3520000 D0
2	1·1333333 e0
3	34·0800000 E0
4	1·0000000 f0
5	0·0000000 F0

V

V

v_t = 1·0000000 B0

W

K = 0·8500000 A0

	Coefficients $\bar{N}_i$
i = 0	3·8400000 b0
1	0·2539681 B0
2	7·6571427 c0
3	0·2579364 C0
4	1·0000000 d0

	Coefficients $\bar{D}_i$
	W
i = 0	0·1066666 b0
1	20·3520000 B0
2	1·1333333 c0
3	34·0800000 C0
4	1·0000000 d0

Example 7.2.3
Circuit Figure 19

Store on Card 163A

	Coefficients N_i
i = 0	0·1066666 d0
1	20·3520000 D0
2	1·1333333 e0
3	34·0800000 E0
4	1·0000000 f0
5	0·0000000 F0

Store on Card 163B

	Coefficients D_i
i = 0	9·6000000 d0
1	0·7415821 D0
2	39·4948570 e0
3	1·7781745 E0
4	36·5800000 f0
5	1·0000000 F0

V

W

x_t = 1·0000000 A0

	Coefficients $\bar{N}_i$
i = 0	3·8400000 b0
1	0·2539662 B0
2	7·6571428 c0
3	0·2579364 C0
4	1·0000000 d0

W

K = 0·4000000 A0

	Coefficients $\bar{D}_i$
i = 0	0·1066666 b0
1	20·3520000 B0
2	1·1333333 c0
3	34·0800000 C0
4	1·0000000 d0

Example 7.2.4
Circuit Figure 21

Example 7.2.3 F(s) is of the Type Q_{10},

Example 7.2.4 F(s) is of the Type Q_{10}^{-1}.

In examples 7.2.1 and 7.2.4, we split F(s) into the capacitive function s/x_t and a function $\bar{F}(s)$ of the type P_7 or P_7^{-1}. In examples 7.2.2 and 7.2.3, we split F(s) into the inductive function sv_t and a function $\bar{F}(s)$ of the type P_7 or P_7^{-1}. We apply for this purpose:

Procedure De1 to example 7.2.1 ,

Procedure De2 to example 7.2.2 ,

Procedure De3 to example 7.2.3 ,

Procedure De4 to example 7.2.4 .

Results of these procedures performed on the Programma 101 computer are shown on the preceding programmed tapes (pages 49 and 50).

8. THE DECOMPOSITIONS OF IMPEDANCE FUNCTIONS OF THE TYPES Q_{11} AND Q_{11}^{-1}

In Q-type functions the polynomials N(s) and D(s) differ by 1 in their degree. If a function F(s) = N(s)/D(s) is of the type Q_{11},

$$N(s) = \sum_{i=1}^{6} N_i s^i \tag{192}$$

and

$$D(s) = \sum_{i=0}^{5} D_i s^i, \; D_5 = 1 . \tag{193}$$

If F(s) is of the type Q_{11}^{-1},

$$N(s) = \sum_{i=0}^{5} N_i s^i \tag{194}$$

and

$$D(s) = \sum_{i=1}^{6} D_i s^i, \; D_6 = 1 . \tag{195}$$

If F(s) is of the type Q_{11},

then $F(0) = 0$ and $F(\infty) = \infty$. (196)

If F(s) is of the type Q_{11}^{-1},

then $F(0) = \infty$ and $F(\infty) = 0$. (197)

By Eqs. (196) and (197), each of the functions can be decomposed, either according to the functions behavior at $s = 0$ or at $s \to \infty$. For comparison, note that functions of the types Q_{10} and Q_{10}^{-1} can only be decomposed according to the functions behavior at $s \to \infty$, and functions of the types P_{10} and P_{10}^{-1} can only be decomposed according to the functions behavior at $s = 0$.

8.1 The Decompositions of a Function of the Type Q_{11}^{-1}

Decomposing the function according to its behavior at $s \to \infty$ yields

$$Q_{11}^{-1} = P_{10} \oplus x_d/s \quad \text{or } F(s) = F'(s) \oplus x_d/s \,, \tag{198}$$

which can also be written as

$$\frac{1}{F(s)} = \frac{1}{F'(s)} + \frac{s}{x_d} \,.$$

In Eq. (198),

$$x_d = N_5 \tag{199}$$

and F'(s) is of the type P_{10}. The coefficients of F'(s) are

$$N_i' = \frac{N_5}{N_5 - N_4} \tag{200}$$

and

$$D_i' = \frac{N_5 D_i - N_{i-1}}{N_5 - N_4} \,. \tag{201}$$

Decomposing the function according to its behavior at $s = 0$ yields

$$Q_{11}^{-1} = Q_{10}^{-1} + x_d/s \quad \text{or } F(s) = F'(s) + x_d/s \,. \tag{202}$$

In Eq. (202)

$$x_d = N_0/D_1 \tag{203}$$

and $F'(s)$ is of the type Q_{10}^{-1}. The coefficients of $F'(s)$ are

$$N_i' = N_{i+1} - x_d D_{i+2} \tag{204}$$

and

$$D_i' = D_{i+1} \, . \tag{205}$$

In the future, we shall refer to decomposition according to Eq. (198) as decomposition procedure De5, and to decomposition according to Eq. (202) as decomposition procedure De8.

8.2 The Decompositions of a Function of the Type Q11

Decomposing the function according to its behavior at $s \to \infty$ yields

$$Q_{11} = P_{10}^{-1} + v_d s \quad \text{or} \quad F(s) = F'(s) + v_d s \, . \tag{206}$$

In Eq. (206),

$$v_d = N_6 \tag{207}$$

and $F'(s)$ is of the type P_{10}^{-1}. The coefficients of $F'(s)$ are

$$N_i' = N_i - N_6 D_{i-1} \tag{208}$$

and

$$D_i' = D_i \, . \tag{209}$$

Decomposing the function according to its behavior at $s = 0$ yields

$$Q_{11} = Q_{10} \oplus v_d s \text{ or } F(s) = F'(s) \oplus v_d s \, , \tag{210}$$

which can also be written as

$$\frac{1}{F(s)} = \frac{1}{F'(s)} + \frac{1}{v_d s} \, .$$

In Eq. (210),

$$v_d = N_1/D_0 \tag{211}$$

and $F'(s)$ is of the type Q_{10}. The coefficients of $F'(s)$ are

$$N_i' = \frac{v_d}{v_d - N_6} N_{i+1} \tag{212}$$

and

$$D_i' = \frac{v_d D_{i+1} - N_{i+2}}{v_d - N_6} . \tag{213}$$

We shall refer to decomposition according to Eq. (206) as decomposition procedure De6, and to decomposition according to Eq. (210) as decomposition procedure De7.

We have presented the results of decompositions by the following circuit realizations:

Decomposition De5 realized in Figure 22
Decomposition De6 realized in Figure 23
Decomposition De7 realized in Figure 24
Decomposition De8 realized in Figure 25 .

When a function of the type Q_{11} or Q_{11}^{-1} has been decomposed, the remainding function $F'(s)$ must be decomposed as the next step in the realization of the circuit. For this we refer to Section 7. The type Q_{11} decomposition component in $F(s)$ is an inductive impedance $v_d s$; the component in $F'(s)$ then is also an inductive impedance $v_t s$. If $v_d s$ is a series element, then $v_t s$ is a shunt element and vice versa. The same holds true for the type Q_{11}^{-1} component in $F(s)$, where x_d/s is a

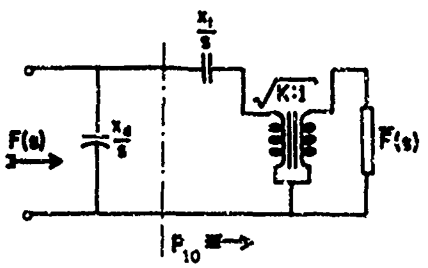


Figure 22. Transposition of Shunt Capacitance

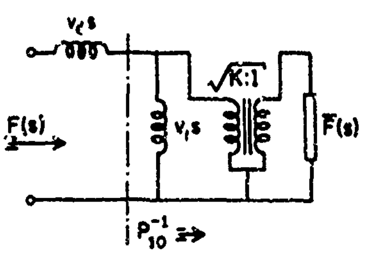


Figure 23. Transposition of Series Inductance

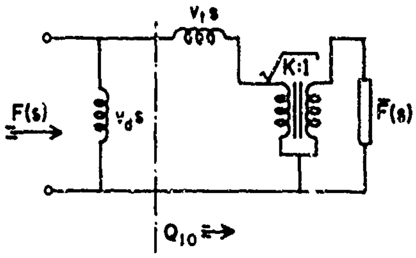


Figure 24. Transposition of Shunt Inductance

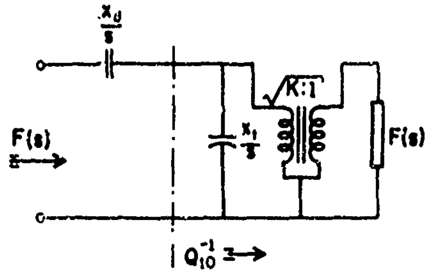


Figure 25. Transposition of Series Capacitance

capacitive impedance and x_t/s is also a capacitive impedance. If x_d/s is a shunt element, then x_t/s is a series element and vice versa.

We have devised programs for use with the Olivetti Programma 101 computer, and they are used in the following examples.

8.3 Numerical Examples

We now show four numerical examples: 8.3.1, 8.3.2, 8.3.3, and 8.3.4. In examples 8.3.1 and 8.3.4, we realize an impedance function F(s) of the type Q_{11}^{-1}, and in examples 8.3.2 and 8.3.3 we realize a function of the type Q_{11}. Each of these functions can be decomposed in two ways. The realization procedure is carried out completely. Some examples that were carried out in previous sections will tie into these four examples.

Example 8.3.1

Let F(s) = N(s)/D(s) have the coefficients:

i	N_i	D_i
0	20.2629120	0.0000000
1	1.4852059	20.1215000
2	68.0839310	1.4374599
3	2.9024125	60.4754282
4	51.6256000	2.4849205
5	1.3600000	39.3200000
6		1.0000000

The function F(s) is of the type Q_{11}^{-1}. It can be decomposed according to Eq. (198) and decomposition procedure De5 as:

Coefficients N_i (Store on Card 170 A)

$i = 0$	20•2629120	e0
1	1•4852059	E0
2	68•0839310	f0
3	2•9024125	F0
4	51•6256000	e0
5	1•3600000	E0
	0•0000000	f0
	0•0000000	F0

Coefficients D_i (Store on Card 170 B)

$i = 0$	0•0000000	e0
1	20•1216000	E0
2	1•4374599	f0
3	60•4754282	F0
4	2•4849205	e0
5	39•3200000	E0
6	1•0000000	f0
	0•0000000	F0
		V
$x_d =$	1•3600000	80

Coefficients N'_i

$i = 0$	14•8991996	e0
1	1•0920631	E0
2	50•0617127	f0
3	2•1341267	F0
		W
4	37•9599990	e0
5	0•9949999	E0
		Y
		Y

Coefficients D'_i

$i = 0$	0•0000000	e0
1	3•8400000	E0
2	0•2539681	f0
3	7•6571427	F0
		Y
4	0•2579361	e0
5	1•0000000	E0

Example 8.3.1

$$F(s) = \frac{x_d}{s} \oplus F'(s),$$

where

$F(s)$ is of the type Q_{11}^{-1}

and

$F'(s)$ is of the type P_{10}.

According to the tape record, $x_d = 1.3600000$. The coefficients of $N'(s)$ and $D'(s)$ listed on the tape are those of example 7.2.1 where we decomposed the function according to Eq. (158) and decomposition procedure De1 as:

$$F'(s) = \frac{x_t}{s} + K\bar{F}(s),$$

where

$F'(s)$ is of the type F_{10}

and

$\bar{F}(s)$ is of the type P_7^{-1}.

In example 7.2.1 we found that x_t = 3.88 and the factor K = 1. The coefficients of $\bar{F}(s)$ are those of example 4.4.1 where we proved that $\bar{F}(s)$ is a special function with Ω_0 = 0.8 and $n_1 n_2$ = -6.0.

The decomposition of F(s) in the present example is shown in Figure 26a; Figure 26b includes the decomposition of F'(s). The ideal transformer can be omitted since K = 1. Figure 26c shows the circuit with $\bar{F}(s)$ represented by the duplex Brune two-port in which, according to example 4.4.1,

$$n_1 = 2, \quad v_1 = 4.1999997, \quad k_1 = 7.0000004,$$
$$n_2 = -3, \quad x_1 = 3.3599997, \quad z' = \bar{N}_0/\bar{D}_0 = 1/36 .$$

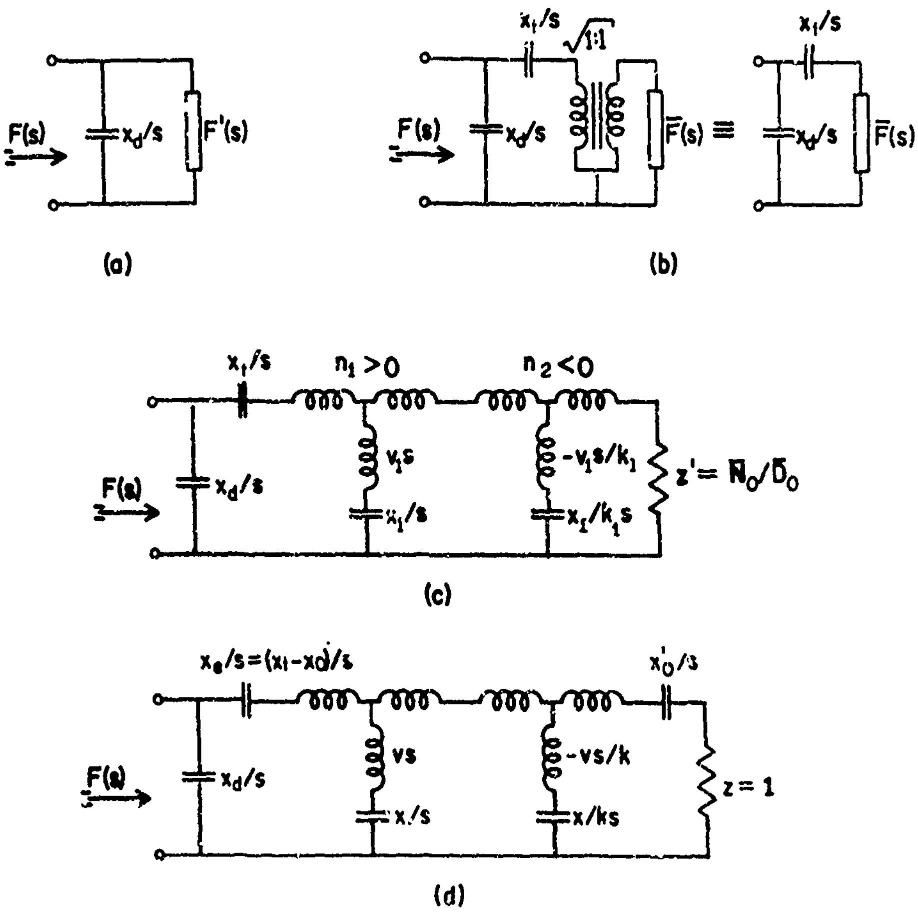


Figure 26. Circuit Expansion Example 8.3.1

Since $z' > 1$, a capacitive impedance can be transposed. This transposition was shown in example 6.5.5.1. On the tape record of this example, we find the constants

$$n = 1.2307690, \quad v = 6.3249999, \quad x_0 = 2.1000001,$$
$$k = 1.1666663, \quad x = 5.4599998, \quad x_0' = 12.6000041.$$

By the transposition, the termination of $z' = 1/36$ in Figure 26c changes to the resistive termination $z = 1$ in the circuit in Figure 26d, according to the formulas in Table 2. Taking the impedance x_0/s from the total impedance x_t/s available at the input leaves the inverse capacitance

$$x_e = x_t - x_0 = 1.78$$

at the input.

We are now able to transform the very special Brune tandem into a lattice structure by applying procedure R_2. This has already been exercised in example 5.2.1 where we obtained the constants

$$v_a = 19.0830544, \quad x_a = 2.8690976, \qquad z = 1.$$
$$v_b = 4.8653741, \quad x_b = 20.7109023,$$

The final circuit is shown in Figure 27. The turn ratio of the ideal transformer on the left side of that figure is $K = 1$. We therefore obtain for the elements of the circuit on the ride side the values

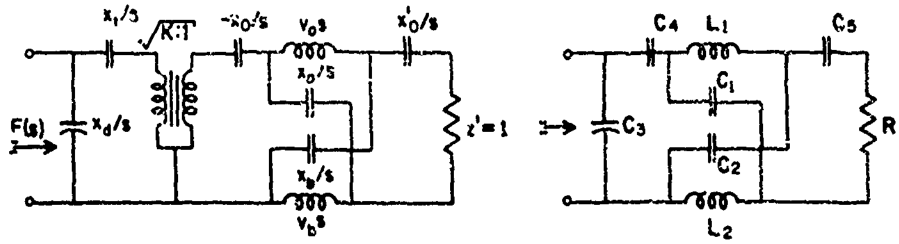


Figure 27. Final Steps in Realizing the Function in Example 8.3.1

$L_1 = v_a = 19.0830544, \quad C_1 = 1/x_a = 0.3485416, \quad R = 1.0000000.$

$L_2 = v_b = \ 4.8653741, \quad C_2 = 1/x_b = 0.0482837,$

$C_3 = 1/x_d = 0.7352941,$

$C_4 = 1/x_e = 0.5617977,$

$C_5 = 1/x_0' = 0.0793650,.$

We certainly would like to check our results. An easy way to perform such a check is to evaluate $F(1) = N(1)/D(1)$ and to analyze the final circuit in the event that $s = 1$. If both results agree, we have some assurance that they are correct. We admit that this check makes no discrimination between the evaluations of in ductances and resistances. But, as far as our experiences over many applications go, this check, which can be very easily performed, has always been sufficient.

The terminating impedance of the circuit in the right part of Figure 27 is, for $s = 1$,

$$R_t = z + 1/C_5 = 1 + x_0' = 13.6000041.$$

The driving-point impedance of the terminated lattice is

$$R_i = \frac{(1+L_1C_2)(1+L_2C_1)R_t + L_1L_2(C_1+C_2) + (L_1+L_2)}{C_1C_2(L_1+L_2) + (C_1+C_2)R_t + (1+L_1C_1)(1+L_2C_2)} \tag{214}$$

for which in our present example we obtain $R_i = 6.4562530$. The driving-point impedance of the circuit, therefore, is

$$F(1) = \frac{1 + R_iC_4}{C_3 + C_4 + R_iC_3C_4} = 1.1672702 \ .$$

Evaluating the coefficients of $F(s)$ for $s = 1$ we obtain $N(1) = 145.7200614$, $D(1) = 124.8394086$, and $F(1) = 1.1672601$, which is in agreement with the value obtained from circuit analysis.

Example 8.3.2

Assume that a driving-point function F(s) of the Type Q_{11} has the coefficients

i	N_i	D_i
0	0.0000000	38.4000000
1	35.7120000	2.6463286
2	2.4504189	96.9234280
3	88.1035880	3.7126973
4	3.3394751	44.0800000
5	37.5864000	1.0000000
6	0.8300000	

We decompose F(s) according to Eq. (206) and decomposition procedure De6 as

$$F(s) = v_d s + F'(s),$$

where

F(s) is of the type Q_{11} and F'(s) is of the type P_{10}^{-1} .

For the record of the procedure performed on the Programma 101 computer, see tape record example 8.3.2. The constant $v_d = 0.83$. The decomposed circuit is shown in Figure 28a. Next we decompose the impedance function F'(s) according to Eq. (174) and decomposition procedure De2. Performed on the Programma 101, we obtain:

Store on Card 171 A

	Coefficients N_i
i = 0	0•0000000 •◊
1	35•7120000 E◊
2	2•4504189 f◊
3	88•1035880 F◊
4	3•3394751 •◊
5	37•5864000 E◊
6	0•8300000 f◊
	0•0000000 F◊

Store on Card 171 B

	Coefficients D_i
i = 0	38•4000000 •◊
1	2•6463286 E◊
2	96•9234280 f◊
3	3•7126973 F◊
4	44•0800000 •◊
5	1•0000000 E◊
	0•0000000 f◊
	0•0000000 F◊

V
V
v_d = 0•8300000 B◊
W
W

	Coefficients N_i'
i = 0	0•0000000 •◊
1	3•8400000 E◊
2	0•2539662 f◊
3	7•6571428 F◊
	W
	W
4	0•2579364 •◊
5	1•0000000 E◊

	Coefficients D_i' Y
i = 0	38•4000000 •◊
1	2•6463286 E◊
2	96•9234280 f◊
3	3•7126973 F◊
	Y
4	44•0800000 •◊
5	1•0000000 E◊

Store on Card 161 A

	Coefficients N'_i	
i = 0	0•0000000	d◊
1	3•8400000	D◊
2	0•2539662	e◊
3	7•6571428	E◊
4	0•2579364	f◊
5	1•0000000	F◊

V

W

v_t = 0•1000000 A◊

	Coefficients $\overline{D}_i$	
i = 0	0•1066666	b◊
1	20•3520000	B◊
2	1•1333333	c◊
3	34•0800000	C◊
4	1•0000000	d◊

Store on Card 161 B

	Coefficients D'_i	
i = 0	38•4000000	d◊
1	2•6463286	D◊
2	96•9234280	e◊
3	3•7126973	E◊
4	44•0800000	f◊
5	1•0000000	F◊

W

K = 1•0000000 F◊

	Coefficients $\overline{N}_i$	
i = 0	3•8400000	b◊
1	0•2539662	B◊
2	7•6571428	c◊
3	0•2579364	C◊
4	1•0000000	d◊

Example 8.3.2

With $F'(s)$ decomposed, the circuit is shown in Figure 28b. Since $K = 1$, the ideal transformer with the turn ratio 1 can be omitted. The shunt inductance has the magnitude $v_t = 0.1$, and $\overline{F}(s)$ is a function of the type P_7 $(\overline{N}_0 > \overline{D}_0)$. The coefficients of $\overline{F}(s)$ are those of example 4.4.2. Therefore, the circuit in Figure 28c, in which $\overline{F}(s)$ is also decomposed according to Brune, has the constants

$$n_1 = 0.4999999, \quad v_1 = 0.5952378, \qquad k_1 = 1.5238088 .$$
$$n_2 = -0.3333333, \quad x_1 = 0.4761902,$$

Over this circuit we have to transpose a shunt inductance of magnitude v_0. This transposition has been exercized in example 6.5.5.2, where we found that $v_0 = 0.4761895$ and $v'_0 = 0.0793647$. The circuit is pictured in Figure 28d after the transposition. Taking the shunt inductance v_0 from the shunt inductance v_t available at the input leaves

$$1/v_e = 1/v_t - 1/v_0 = 10 - 2.1000043 = 1/0.1265823.$$

Thus the remaining inductance $v_e = 0.1265823$ is positive.

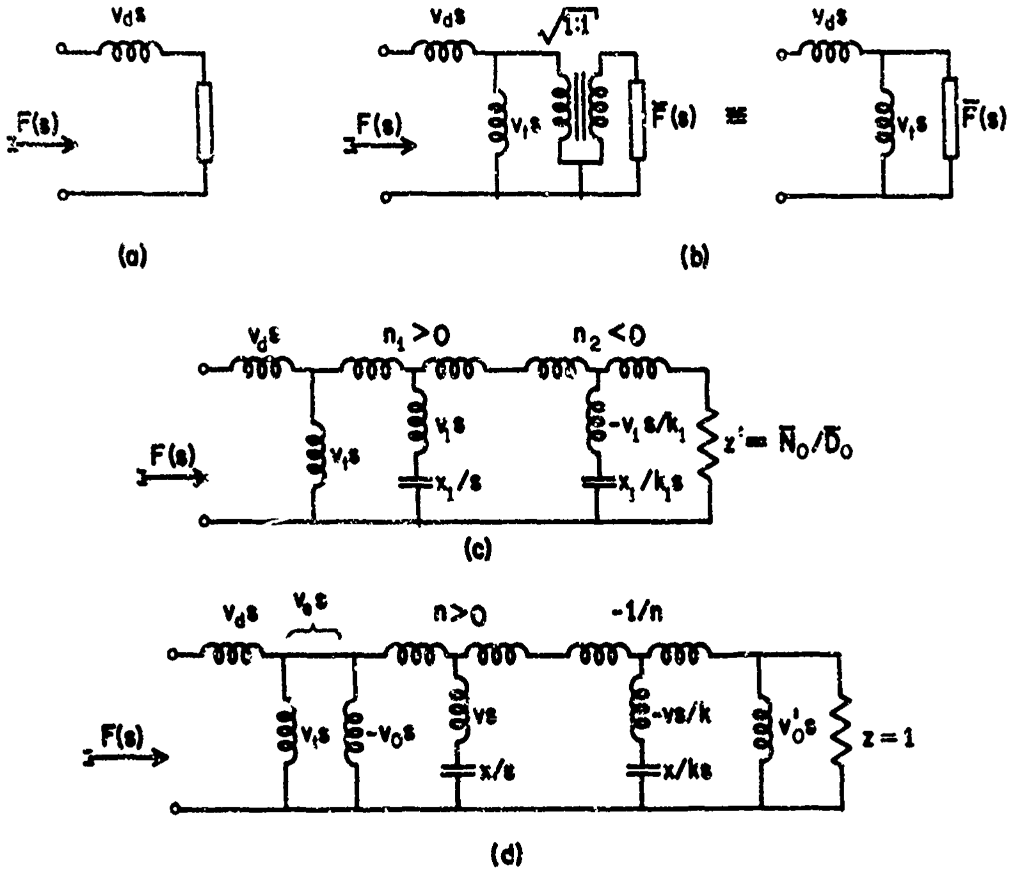


Figure 28. Circuit Expansion Example 8.3.2

In example 6.5.5.2 we also found the constants

$k = 0.8125003$, $v = 4.2328149$,

$n = 80.0955910$, $x = 3.3862516$.

The transformation of the circuit into a lattice structure was exercised in example 5.2.2 where we found the constants

$v_a = 0.0482836$, $x_a - 0.0524023$,

$v_b = 0.3485408$, $x_b = 0.2055336$.

The final circuit is pictured in Figure 29, where the transformer ratio in the left side part of the figure is 1. The elements of the circuit in the right side part of the figure are

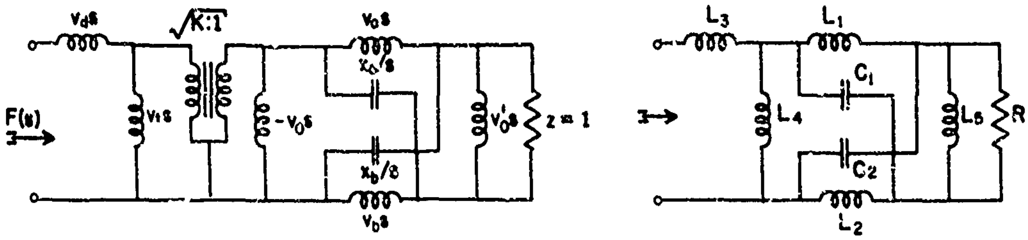


Figure 29. Final Steps in Realizing the Function in Example 8.3.2

$L_1 = v_a = 0.0482836$, $C_1 = 1/x_a = 19.0831318$, $R = 1.0000000$

$L_2 = v_b = 0.3485408$, $C_2 = 1/x_b = 4.8653845$,

$L_3 = v_d = 0.8300000$,

$L_4 = v_e = 0.1265823$,

$L_5 = v_0' = 0.0793647$.

Our results check as.

$N(1) = 168.0218820$,

$D(1) = 186.7624539$,

$F(1) = 0.8996555$.

The termination of the lattice in the right-side part of Figure 29 is

$$R_t = \frac{L_5}{1 + L_5} = 0.0735299.$$

By Eq. (214) we find $R_i = 0.1548846$. Therefore, the driving-point impedance of the circuit in Figure 29 is

$$F(1) = L_3 + L_4 R_i/(L_4 + R_i) = 0.8996550,$$

which is in agreement with evaluation F(1).

Example 8.3.3

The function F(s) of type Q_{11} that we discussed in example 8.1.2 also allows the decomposition

$$F(s) = v_d s + F'(s)$$

where

F(s) is of type Q_{11} and F'(s) is of type Q_{10}, according to Eq. (207) and decomposition De7. The decomposition performed on the Programma 101 is recorded on tape record example 8. 3. 3 The decomposed circuit in Figure 30 has a shunt inductance of magnitude $v_d = 0.93$. The further decomposition of the function F'(s) according to Eq. (166) and decomposition procedure De3 is recorded below:

Store on Card 162 A

	Coefficients N'_i	
i = 0	332·1216000	d◊
1	22·7888950	D◊
2	819·3633680	e◊
3	31·0571130	E◊
4	349·5535200	f◊
5	7·7190000	F◊

Store on Card 162 B

	Coefficients D'_i	
i = 0	0·1066666	d◊
1	20·3520000	D◊
2	1·1333330	e◊
3	34·0800000	E◊
4	1·0000000	f◊
	0·0000000	F◊

V
V
v_t = 7·7190000 B◊
W
K = 86·4900000 A◊

	Coefficients $\overline{N}_i$	
i = 0	3·8400000	b◊
1	0·2539661	B◊
3	7·6571427	c◊
2	0·2579364	C◊
4	1·0000000	d◊

W

	Coefficients $\overline{D}_i$	
i - 0	0·1066666	b◊
1	20·3520000	B◊
2	1·1333330	c◊
3	34·0800000	C◊
4	1·0000000	d◊

According to the circuit in Figure 30b, function F'(s) is decomposed into a series inductance of magnitude $v_t = 7.719$, an ideal transformer with the turn ratio K:1 where K = 86.49, and impedance function $\overline{F}(s)$, with the coefficients listed on the tape. The function is of type P_7 $(\overline{N}_0\ \overline{D}_0)$ that allows the transposition of an inductance. The Brune realization of $\overline{F}(s)$ is that of example 8. 1. 2. Therefore, the circuit in Figure 30c has the constants

$n_1 = 0.4999999$, $v_1 = 0.5952378$ $k_1 = 1.523.\ 88$,

$n_2 = -0.3333333$, $x_1 = 0.4761902$, $z' = \overline{N}_0/\overline{D}_0 = 36$.

The transposition of the series inductance shown in the circuit in Figure 30 was exercised in example 6. 5. 5. 3 where we found that

$v_0 = 0.1467136$, $n = 0.7464789$, $v = 0.5952378$,

$v'_0 = 0.8802830$, $k = 1.5238088$, $x = 0.4761902$.

Store on Card 172 A

	Coefficients N_i	
i = 0	0·0000000	e◊
1	35·7120000	E◊
2	2·4504189	f◊
3	88·1035880	F◊
4	3·3394751	e◊
5	37·5864000	E◊
6	0·8300000	f◊
	0·0000000	F◊

v_d = 0·9300000 R◊

	Coefficients N'_i	
i = 5	7·7190000	F◊
4	349·5535200	e◊
3	31·0571180	F◊
2	819·3633680	f◊
1	22·7888950	E◊
0	332·1216000	e◊

Store on Card 172 B

	Coefficients D_i	
i = 0	38·4000000	e◊
1	2·6463286	E◊
2	96·9234280	f◊
3	3·7126973	F◊
4	44·0800000	e◊
5	1·0000000	E◊
	0·0000000	f◊
	0·0000000	F◊

	Coefficients D'_i	
i = 4	1·0000000	e◊
3	34·0800000	F◊
2	1·1333330	f◊
1	20·3520000	E◊
0	0·1066660	e◊

Example 8.3.3

In order to be able to combine the negative series impedance $-v_0 s$ with the impedance $v_t s$ available at the input, we have to take inductance $-v_0$ to the left side of the ideal transformer. This means that we have to multiply with K. Thus the inductance

$$v_e = v_t - Kv_0 = 7.719 - 12.6892592 = -4.9702592$$

is left at the input. Since this inductance is negative, our attempt to obtain the anticipated realization has failed. But for tutorial reasons we will continue this example.

The transformation of the circuit into the lattice structure was exercised in example 5.2.3, with the present values of the circuit. In that example we obtained the constants

$$v_a = 1.4469391, \quad x_a = 0.2545345, \quad v'_0 = 0.8802830$$

$$v_b = 0.2004454, \quad x_b = 0.7292550, \quad z' = 36.0000000.$$

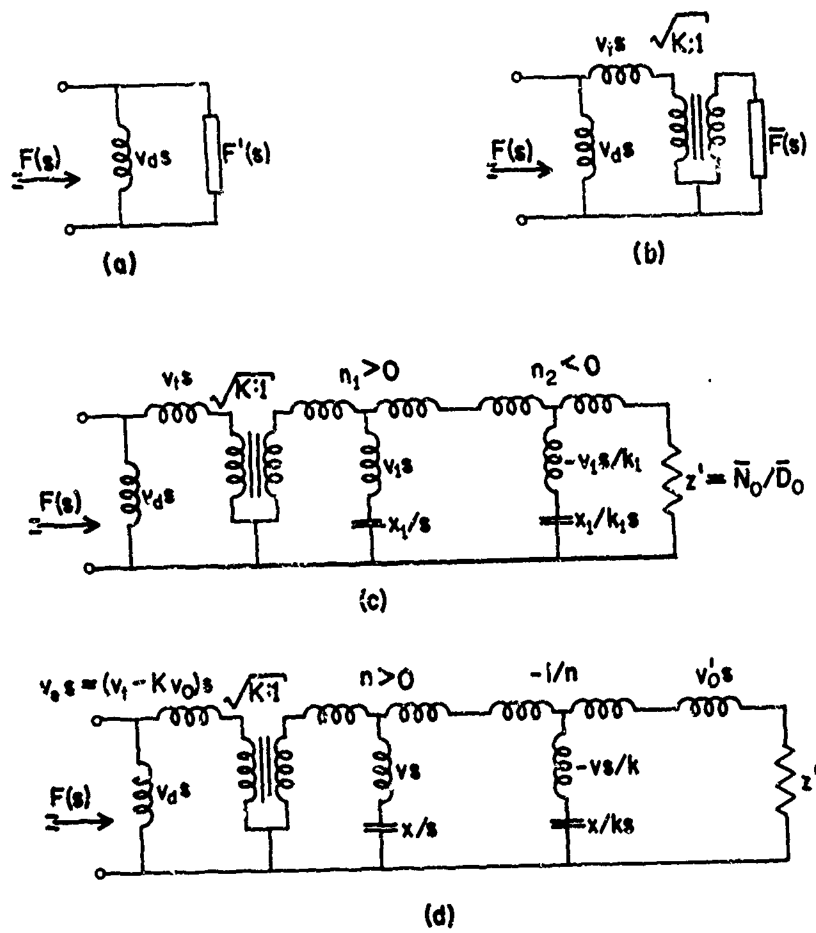


Figure 30. Circuit Expansion Example 8.3.3

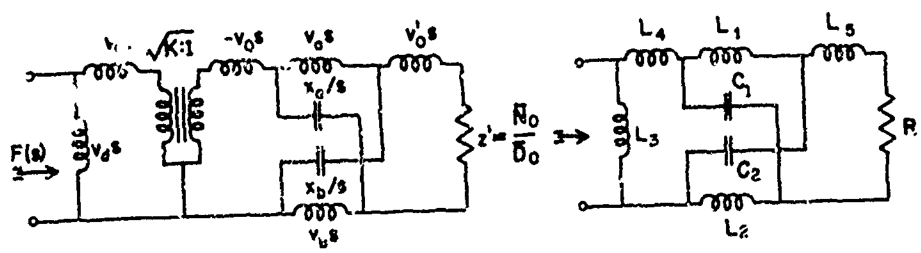


Figure 31. Final Steps in Realizing the Function in Example 8.3.3

The final circuit is shown in Figure 31. In this example the ratio of the ideal transformer is $K = 86.49$. The elements of the final circuit (right side in Figure 31) are

$L_1 = Kv_a = 125.1457627,\quad C_1 = 1/Kx_a = 0.0454242,\quad R = 36K = 3113.64$

$L_2 = Kv_b = 17.3365226,\quad C_2 = 1/Kx_b = 0.0158545,$

$L_3 = v_d = 0.93000000,$

$L_4 = v_e = -4.6892592,$

$L_5 = Kv_0' = 76.1356756,$

We check our result, and the terminating impedance of the lattice is

$$R_t = L_5 + R = 3389.7756766.$$

According to Eq. (214), we obtain

$$R_i = 32.5432262$$

as the driving-point impedance of the lattice. The driving-point impedance of the circuit in Figure 31 that was evaluated for $s = 1$ is:

$$F(1) = \frac{L_3(L_4 + R_i)}{L_3 + L_4 + R_i} = 0.8996557\ ,$$

which is in full agreement with the evaluation F(1) found in example 8.1.2.

This example has shown that a function of type Q_{11} can be decomposed eventually by decomposition procedures De6 and De7. If both procedures are successful, two equivalent realizations are obtained; otherwise, only one of them is successful, or with bad luck none of them. The same is true for a function of type Q_{11}^{-1}.

Example 8.3.4

In this example we decompose the function discussed in example 8.1.1 according to Eq. (199) and decomposition procedure De8 as

$$F(s) = \frac{x_d}{s} + F'(s),$$

where

F(s) is of the type Q_{11}^{-1} and F'(s) is of the type Q_{10}^{-1}.

Store on Card 173 A	i	Coefficients N_i	
	i = 0	20•2629120	• ◊
	1	1•4852059	E ◊
	2	68•0839310	f ◊
	3	2•9024125	F ◊
	4	51•6256000	• ◊
	5	1•3600000	E ◊
		0•0000000	f ◊
		0•0000000	F ◊

Store on Card 173 B	i	Coefficiente D_i	
	i = 0	0•0000000	• ◊
	1	20•1216000	E ◊
	2	1•4374599	f ◊
	3	60•4754282	F ◊
	4	2•4849205	• ◊
	5	39•3200000	E ◊
	6	1•0000000	f ◊
		0•0000000	F ◊

V

V

x_d = 1•0070229 8◊

W

W

i	Coefficients N_i'	
i = 4	0•3529771	• ◊
		W
		W
3	12•0294596	F ◊
2	0•4000407	f ◊
1	7•1837900	E ◊
0	0•0376509	• ◊

Y

i	Coefficients D_i'	
i = 5	1•0000000	E ◊
4	39•3200000	• ◊
		Y
3	2•4849205	F ◊
2	60•4754282	f ◊
1	1•4374599	E ◊
0	20•1216000	• ◊

Example 8.3.4

The decomposition was performed on the Programma 101, and the tape record is shown in example 8.3.4. The resulting circuit (Figure 32a) consists of a series capacitance of magnitude $1/x_d = 1/1.0070229$ and the box representing the impedance $F'(s)$. The latter function is decomposed as follows:

$$F'(s) = \frac{x_t}{s} + K\overline{F}(s),$$

where

$F'(s)$ is of the type Q_{10}^{-1} and $\overline{F}(s)$ is of the type P_7^{-1}.

The circuit is shown in Figure 32b with function $F'(s)$ decomposed. There is a shunt capacitance of magnitude $1/x_t = 2.8330449$, an ideal transformer with a turn ratio of $\sqrt{K}:1$ where $K = 0.0673620$ with a driving-point impedance $\overline{F}(s)$ on its secondary side. With the present coefficients of $\overline{F}(s)$, its Brune realization was exercised in example 4.4.1 where we found the constants

$n_1 = 2.0000001$, $v_1 = 4.1999997$, $k_1 = 7.0000004$,

$n_2 = -3.0000007$, $x_1 = 3.3599997$, $z' = 1/36$.

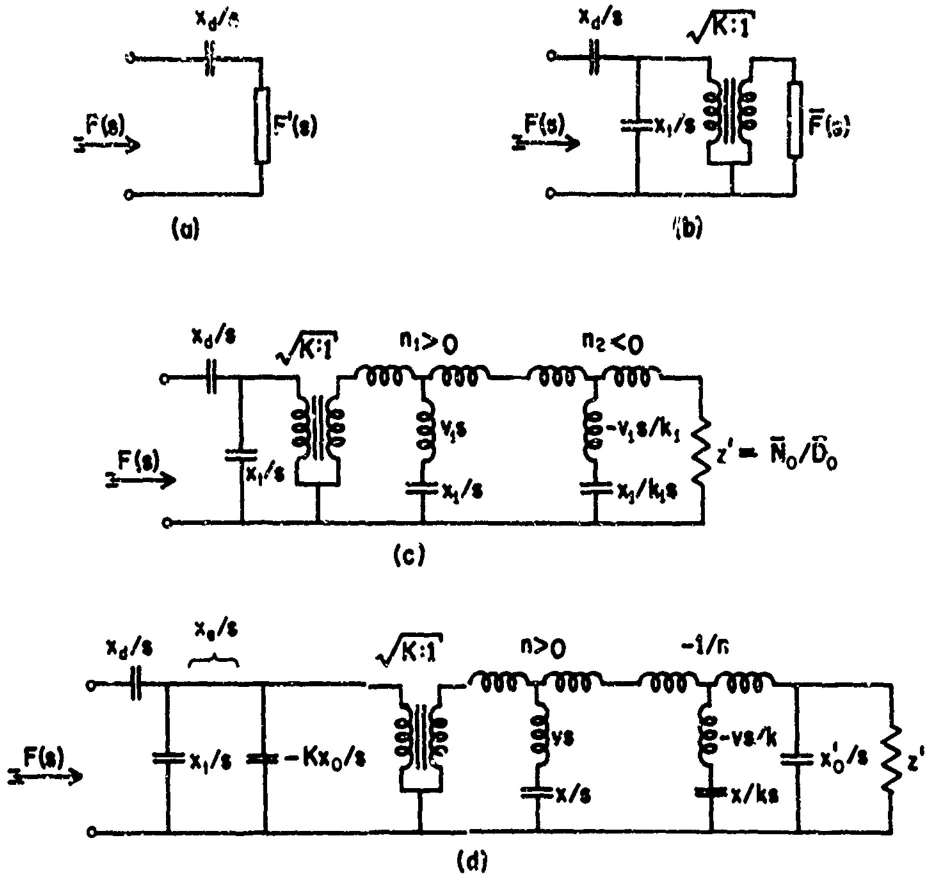


Figure 32. Circuit Expansion Example 8.3.4

The realizing circuit is shown in Figure 32c. A capacitance $1/x_0$ has to be transposed over the circuit. The transposition was exercised in example 6.5.5.4 where we found

x_0 = 6.8119995, n = 1.3396226, v = 24.3898237,

x_0' = 1.1359995, k = 31.1435267, x = 19.5118586.

As it is shown in Figure 32d, if we take the transposition capacitance to the left side of the ideal transformer, the inverse capacitance at the input will be $1/x_e = 1/x_t - 1/Kx_0 = 2.8330449 - 2.1779887 = 0.6550562$. The remaining capacitance is positive in this example and, therefore, the final realization is equivalent to that obtained in example 8.3.1. The final circuit is as shown in Figure 33. With the present coefficients, the transformation into the lattice is as exercised in example 5.2.4 where we obtained the constants

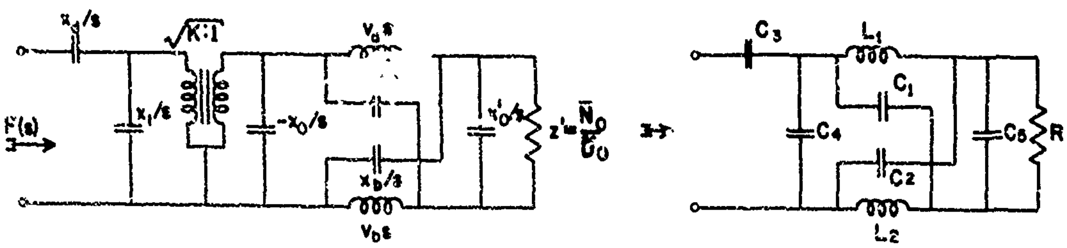


Figure 33. Final Steps in Realizing the Function in Example 8. 3. 4

$v_a = 1.3712615$, $x_a = 0.6911142$, $x_0' = 1.1359995$,
$v_b = 3.9287369$, $x_b = 4.9888857$, $z' = 0.0277777$.

According to Figure 33, the final circuit has the elements

$L_1 = Kv_a = 0.0923709$, $C_1 = 1.Kx_a = 21.4800622$, $R = Kz' = 0.0018711$.
$L_2 = Kv_b = 0.2648477$, $C_2 = 1/Kx_b = 2.9756475$,
$C_3 = 1/x_d = 0.9930260$,
$C_4 = 1/x_e = 0.6550562$,
$C_5 = 1/Kx_0' = 13.0679494$,

We check these results, and in example 8. 1. 1 find $F(1) = 1.1672601$. Analyzing the circuit in Figure 33 yields

$$R_t = \frac{R}{1 + RC_5} = 0.0018253,$$

and by Eq. (214), $R_i = 0.1790277$.

Therefore, by circuit analysis,

$$F(1) = \frac{R_i}{1 + R_i C_i} = 1.1672591\ ,$$

which is in complete agreement with the evaluation of F(s).

9. THE REALIZATION OF A DRIVING-POINT IMPEDANCE THAT DOES NOT YIELD A VERY SPECIAL FUNCTION $\bar{F}(s)$

In all examples discussed so far, the coefficients of the function F(s) to be realized were chosen so that, first, a special function was found for which the test values were $c_1 = c_2 = c_3 = 0$; then, by element transposition this function was changed to one that allowed the lattice equivalence. This situation was necessary for tutorial reason. In most practical cases, however, the function to be realized will not yield the zero identity of the test values; this may be due to truncation of the coefficients or to some other reasons. If the text values are not zero, it may be possible that by changing the coefficients slightly we will obtain a function for which the zero indentity of the test values holds. In doing this no additional elements are necessary. But we have to pay for the advantage of the economical realization by some deviation of the functions behavior. This will be the objective of this Section.

Let us discuss the technique of coefficient adjustment along with the following example. Assume we have to realize a driving-point impedance F(s) that has the coefficients

i	N_i	D_i
0	0.182	0.000
1	1.274	0.837
2	2.811	1.564
3	3.102	2.261
4	3.246	1.756
5	1.529	1.000

This function F(s) is of the type F_{10}.

We decompose F(s) according to Procedure De1 (Section 7.1.1).

Store on Card 160 A

	Coefficients N_i	
$i = 0$	0•1820000	d◊
1	1•2740000	D◊
2	2•8110000	e◊
3	3•1020000	E◊
4	3•2460000	f◊
5	1•5290000	F◊

Store on Card 160 B

	Coefficients D_i	
$i = 0$	0•0000000	d◊
1	0•8370000	D◊
2	1•5640000	e◊
3	2•2610000	E◊
4	1•7560000	f◊
5	1•0000000	F◊

V
V

x_t = 0•2174432 F◊

V

K = 1•5290000 A◊

	Coefficients $\tilde{N}_i$	
$i = 0$	0•6108037	b◊
1	1•5169136	B◊
2	1•7790515	c◊
3	1•9807434	C◊
4	1•0000000	d◊

	Coefficients $\tilde{D}_i$	
$i = 0$	0•8370000	b◊
1	1•5640000	B◊
2	2•2610000	c◊
3	1•7560000	C◊
4	1•0000000	d◊

The decomposition yields the series impedance x_t/s and a function $\widetilde{F}(s)$ that is of the type P_7^{-1} multiplied by the positive constant $K = 1.529$. The series impedance is a capacitor $1/x_t$. The decomposed function in shown in Figure 34.

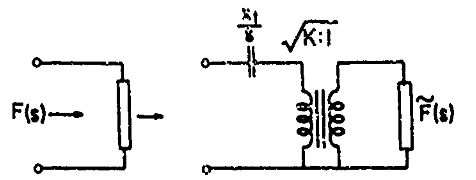


Figure 34. P_{10}-type Function F(s) and Implementation of the P_7^{-1}-type Function $\widetilde{F}(s)$

To continue our attempted realization procedure, it is necessary that F(s) be a minimum function. For this purpose we present the impedance diagram of this function in Figure 35. The figure shows that F(s) is not a minimum function. It has a distance of about $r = 0.37$ from the ordinate. We determine the exact distance by performing a regular Brune procedure on F(s) (see Hasse, 1970b), and we find that a constant $r = \text{Re } F(j\omega)_{min} = 0.3727543$ can be subtracted from F(s). This would cause the curve shown in Figure 35 to shift to the left and touch the ordinate. We expand

$$\overline{F}(s) = \frac{F(s) - r}{1 - r} = \frac{N(s) - rD(s)}{(1 - r)D(s)} .$$

Therefore,

$$F(s) = (1 - r)\overline{F}(s) .$$

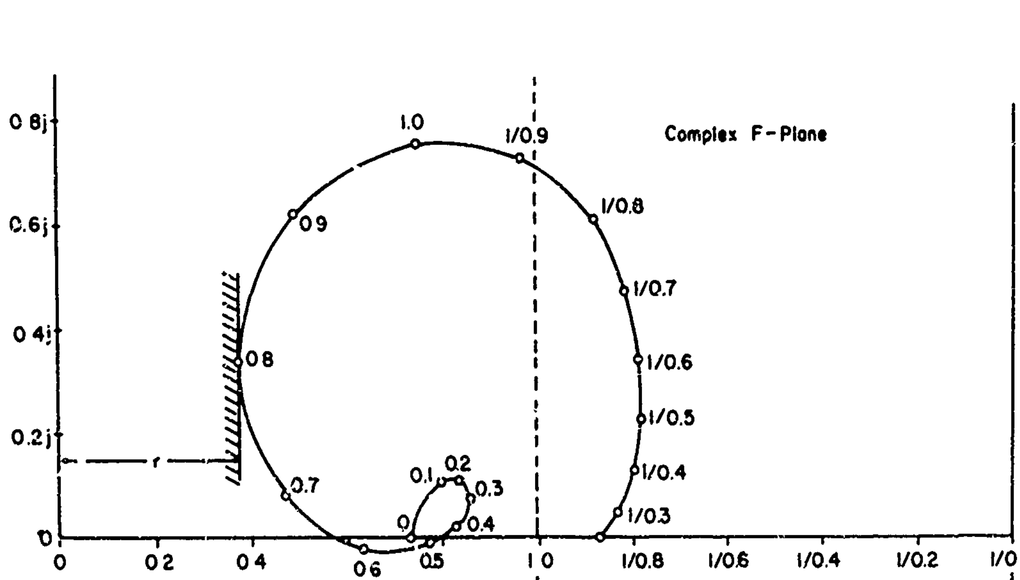


Figure 35. Driving-Point Impedance in the Complex F(s)-Plane

The coefficients of $\bar{F}(s)$ are

i	$\bar{N}_i$	$\bar{D}_i$
0	0.4763817	0.837
1	1.4889315	1.564
2	1.4926433	2.261
3	2.1143020	1.756
4	1.0000000	1.000

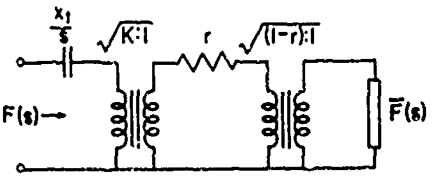


Figure 36. P_{10}-type Function F(s) with Minimum Resistance r and Normalizing Transformer Extracted

Figure 36 shows the circuit. The circuit-realizing F(s) consists of the series resistor r = 0.3727543 followed by an ideal transformer with the turn ratio $\sqrt{(1-r):1}$. This transformer represents the factor by which F(s) has to be multiplied.

Let us now test the function $\bar{F}(s)$ according to Section 4.1.

With - sign in Eq. (52)

		Coefficients $\bar{N}_i$
Store on Card 165 A	i = 0	0•4763817 d◊
	1	1•4889315 D◊
	2	1•4926433 e◊
	3	2•1143020 E◊
	4	1•0000000 f◊

		Coefficients $\bar{D}_i$
Store on Card 165 B	i = 0	0•8370000 d◊
	1	1•5640000 D◊
	2	2•2610000 e◊
	3	1•7560000 E◊
	4	1•0000000 f◊

V
V

c_1 = 0•0022474 A◊
c_2 = -2•4959874 A◊
c_3 = -0•0409290 A◊

Ω_0 = 0•7946394 A◊
n_1n_2 = -1•3255166 A◊

With + sign in Eq. (52)

		Coefficients $\bar{N}_i$
Store on Card 164 A	i = 0	0•4763817 d◊
	1	1•4889315 D◊
	2	1•4926433 e◊
	3	2•1143020 E◊
	4	1•0000000 f◊

		Coefficients $\bar{D}_i$
Store on Card 164 B	i = 0	0•8370000 d◊
	1	1•5640000 D◊
	2	2•2610000 e◊
	3	1•7560000 E◊
	4	1•0000000 f◊

V
V

c_1 = 0•0022474 A◊
c_2 = 0•0298198 A◊
c_3 = -0•0409290 A◊

Ω_0 = 0•7946394 A◊
n_1n_2 = -1•3255166 A◊

The test values are not zero. The + sign in Eq. (52) yields a smaller deviation of the magnitude of c_2 from 0. For this reason let us assume "+" in that equation.

Let us glance back to Eqs. (47), (48), and (52). Assume we would have $c_1 = c_2 = c_3 = 0$ and let us write the equations in the following form:

$$\bar{N}_1\bar{D}_1 - \bar{N}_2\bar{D}_0 = \bar{N}_0\bar{D}_2 \tag{215}$$

$$\bar{N}_1\bar{D}_3 - \bar{N}_2\bar{D}_2 + \bar{N}_3\bar{D}_1 = (\sqrt{\bar{N}_0} + \sqrt{\bar{D}_0})^2 \tag{216}$$

$$- \bar{N}_2 + \bar{N}_3\bar{D}_3 = \bar{D}_2 \,. \tag{217}$$

If we assume that in these equations the $\bar{N}_1$, $\bar{N}_2$, and $\bar{N}_3$ are the unknowns and all other coefficients are known, then the system represents a system of three linear equations that allows us to compute the unknowns. The solutions are:

$$\bar{N}_2 = \frac{\bar{N}_0\bar{D}_3(\bar{D}_2\bar{D}_3 - \bar{D}_1) + \bar{D}_1(\bar{D}_2\bar{D}_3 - \bar{D}_0\bar{D}_3 - 2\bar{D}_3\sqrt{\bar{N}_0\bar{D}_0})}{\bar{D}_3(\bar{D}_1\bar{D}_2 - \bar{D}_0\bar{D}_3) - \bar{D}_1^{\,2}} \tag{218}$$

$$\bar{N}_3 = \frac{\bar{N}_2 + \bar{D}_2}{\bar{D}_3} \,, \tag{219}$$

$$\bar{N}_1 = \frac{\bar{N}_0\bar{D}_2 + \bar{N}_2\bar{D}_0}{\bar{D}_1} \,. \tag{220}$$

We also could write the equations in the form

$$\bar{N}_1\bar{D}_1 - \bar{N}_0\bar{D}_2 = \bar{N}_2\bar{D}_0 \tag{221}$$

$$\bar{N}_3\bar{D}_1 - \bar{N}_2\bar{D}_2 + \bar{N}_1\bar{D}_3 = (\sqrt{\bar{N}_0} + \sqrt{\bar{D}_0})^2 \tag{222}$$

$$- \bar{D}_2 + \bar{N}_3\bar{D}_3 = \bar{N}_2 \tag{223}$$

where we assume that $\bar{D}_1$, $\bar{D}_2$, and $\bar{D}_3$ are the unknowns.

Eqs. (221), (222), and (223) have the solutions

$$\overline{D}_2 = \frac{\overline{N}_3\overline{D}_0(\overline{N}_2\overline{N}_3-\overline{N}_1) + \overline{N}_1(\overline{N}_1\overline{N}_2-\overline{N}_0\overline{N}_3 \overset{\downarrow}{-} 2\overline{N}_3\sqrt{\overline{N}_0\overline{D}_0})}{\overline{N}_3(\overline{N}_1\overline{N}_2 - \overline{N}_0\overline{N}_3) - \overline{N}_1^{\,2}} \tag{224}$$

$$\overline{D}_3 = \frac{D_2 + \overline{N}_2}{\overline{N}_3}, \tag{225}$$

$$D_1 = \frac{N_0D_2 + N_2D_0}{\overline{N}_1} . \tag{226}$$

If we had chosen the - sign in Eq. (52), then in Eqs. (216) and (222) the + sign marked by the arrow would have to be changed to "-", and in Eqs. (218) and (224) the - sign marked by the arrow would have to be changed to "+".

We will now gradually adjust the coefficients, but for this purpose coefficients $\overline{N}_0$ and $\overline{D}_0$ will not change. First we find for coefficients $\overline{D}_1$, $\overline{D}_2$, and D_3, by Eqs. (218), (219), and (220), coefficients $\overline{N}'_1$, $\overline{N}'_2$, and $\overline{N}'_3$. Then we average

$$N''_1 = \frac{\overline{N}_1 + \overline{N}'_1}{2}, \quad N''_2 = \frac{\overline{N}_2 + \overline{N}'_2}{2}, \quad N''_3 = \frac{\overline{N}_3 + \overline{N}'_3}{2}. \tag{227}$$

Then we find, by Eqs. (224), (225), and (226), for coefficients $\overline{N}_1$, $\overline{N}_2$, and $\overline{N}_3$, coefficients $\overline{D}'_1$, $\overline{D}'_2$, and $\overline{D}'_3$ and we average

$$\overline{D}''_1 = \frac{\overline{D}_1 + \overline{D}'_1}{2}, \quad \overline{D}''_2 = \frac{\overline{D}_2 + \overline{D}'_2}{2}, \quad \overline{D}''_3 = \frac{\overline{D}_3 + \overline{D}'_3}{2} . \tag{228}$$

Next we perform the same procedures on the coefficients with double primes and repeat until the coefficients do not change any more. In our example we arrived, after four repetitions, at the following adjusted coefficients:

i	$\overline{N}_i$	$\overline{D}_i$
0	0.4763817	0.8370000
1	1.4907697	1.5572695
2	1.4974132	2.2423101
3	2.1286690	1.7568364
4	1.0000000	1.0000000

adjusted coefficients

The test procedure T shows the following result

Store on Card 164 A		Coefficients $\bar{N}_i$
	i = 0	0•4763817 d◊
	1	1•4907697 D◊
	2	1•4974132 e◊
	3	2•1286690 E◊
	4	1•0000000 f◊

Store on Card 164 B		Coefficients $\bar{D}_i$
	i = 0	0•8370000 d◊
	1	1•5572695 D◊
	2	2•2423101 e◊
	3	1•7568364 E◊
	4	1•0000000 f◊

adjusted coefficients of $\bar{F}(s)$
Type P_7^{-1}

V
V
c_1 = -0•0000001 A◊
c_2 = 0•0000003 A◊
c_3 = -1•0000002 A◊

Ω_0 = 0•7946394 A◊
n_1n_2 = -1•3255166 Á◊

Next we realize $\bar{F}(s)$ according to realization procedure R_1 (see Section 4, example 4.4.1).

Store on Card 171 A		Coefficients $\bar{N}_i$
	i = 0	0•4763817 e◊
	1	1•4907697 E◊
	2	1•4974132 f◊
	3	2•1286690 F◊
	4	1•0000000 e◊

Store on Card 171 B		Coefficients $\bar{D}_i$
	i = 0	0•8370000 e◊
	1	1•5572695 E◊
	2	2•2423101 f◊
	3	1•7568364 F◊
	4	1•0000000 e◊

$-\Omega_0$ = -0•7946394 b!
RS
V
V
R_N = -0•0820700 e◊
S_N = -0•2007545 E◊
R_n = -0•0918656 f◊
S_n = 2•1286690 F◊
RS
V
V
R_D = -0•3133761 e◊
S_D = 0•1612181 E◊
R_d = 0•6530313 f◊
S_d = 1•7568364 F◊

Z
$+\Omega_0$ = 0•7946394 b◊
RS
V
V
R_N^* = 2•2977370 e◊
S_N^* = 3•1822939 E◊
RS
V
V
R_D^* = 3•2502797 e◊
S_D^* = 2•9533209 E◊
W
W
Y
Y
Y
W
W
n_1n_2 = -1•3255166 c!
V

n_1 = 1•1477123 D◊
n_2 = -1•1549206 30

v_1 = 4•3369353 c◊
x_1 = 3•4462996 C◊

k_1 = 10•2657930 A◊

With the constants presented in the framed field above, the circuit is now as shown in Figure 37.

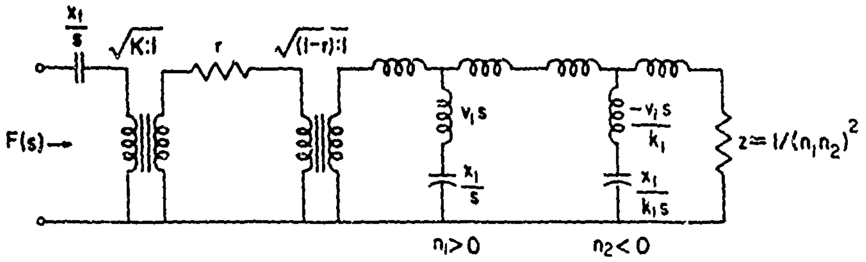


Figure 37. P_{10}-type Function Implying Brune Duplex

Since the function F(s) to be realized is of the type P_{10}, we have to transpose a series impedance x_0/s over the circuit realizing $\overline{F}(s)$ in order to obtain the very special function $\overline{F}(s)$ that has the equivalence of a lattice two-port. The following is the result of Procedure A_1 (see Section 6, example 6.5.1):

Store on Card 153 A	n_1 =	1·1477123	00
	n_2 =	-1·1540206	00
	v_1 =	4·3369353	00
	x_1 =	3·4462996	00
	k_1 =	10·2657930	00
			V
	x_0 =	0·0995780	00
	x_0' =	0·1319921	00
	n =	1·1154812	00
	k =	7·7447137	00
	v =	4·4622475	00
	x =	3·5458776	00

See the circuit realization in Figure 38

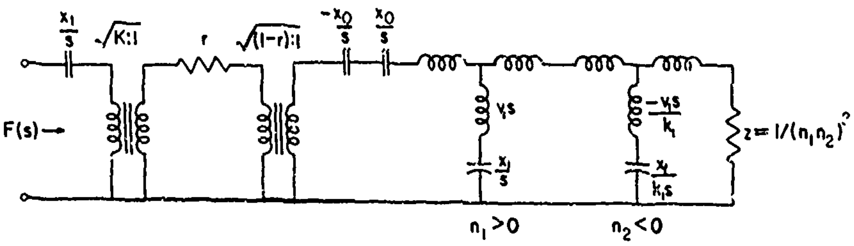


Figure 38. P_{10}-type Function with Brune Duplex Prepared for Capacitance Transposition

The elements of the equivalent lattice two-port are obtained by Procedure R_2 (see example 5.2.1).

Store on Cards 156A & B	n =	1•1151912	40
	k =	7•7447437	D0
	v =	4•4622475	e0
	x =	3•5158776	E0
			V
			V
	v_a =	1•4106371	b0
	v_b =	0•9542553	B0
	x_a =	0•5495615	c0
	x_b =	1•5466916	C0

See the circuit realization in Figure 39

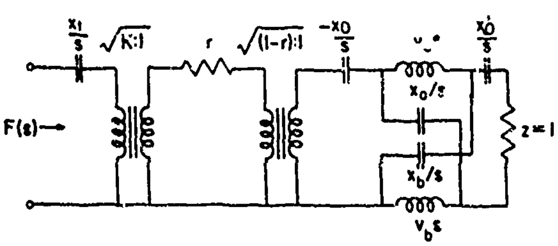


Figure 39. P_{10}-type Function with Lattice Two-port After Transposition

We now completely know the circuit shown in Figure 39. In this circuit we will now dissolve two ideal transformers. The first one has the turn ratio $\sqrt{K} = \sqrt{1.529}$, the second one has the turn ratio $\sqrt{(1-r)} = \sqrt{0.6272457}$. Dissolving the transformers means that we push them out to the right side. Therefore, we have to multiply resistance r, which is the only element that is passed by the first transformer with K. All other impedance elements are passed by both transformers and have, therefore, to be multiplied with $K(1-r) = 0.9590586$.

The impedance x_0/s that has been transposed over the circuit representing F(s) has to be induced with the "-" sign. After the transformers are dissolved, the impedance becomes $-K(1-r)x_0 = -0.9590586 \cdot 0.0995789 = -0.0955011$. Combining this capacitive impedance with impedance $x_t/s = 0.2174432/s$, the capacitive impedance

$$x_e/s = \frac{0.2174432 - 0.0955011}{s} = 0.1219421/s$$

remains at the input. Since x_e is positive, the attempt of the realization was successful from this point of view. The final circuit is shown in Figure 40 and . its elements are listed in the following table.

Resistors	Inductors
$R_1 = Kr = 0.5699413$	$L_1 = K(1-r)v_a = 1.3528836$
$R_2 = K(1-r) \cdot 1 = 0.9590586$	$L_2 = K(1-r)v_b = 0.9151867$
Capacitors	
$C_1 = 1/K(1-r)x_a = 1.8973114 = 1/0.5270616$	
$C_2 = 1/K(1-r)x_b = 0.6741416 = 1/1.4833678$	
$C_3 = 1/x_e = 8.2006132 = 1/0.1219421$	
$C_4 = 1/K(1-r)x_0' = 7.8996366 = 1/0.1265881$	

The circuit realization in Figure 40 needs two resistors, two inductors, and four capacitors, for a total of eight circuit elements.

By adjusting the coefficients, we necessarily induced an error; Figure 41 shows how this change of the coefficients affects the real and the imaginary

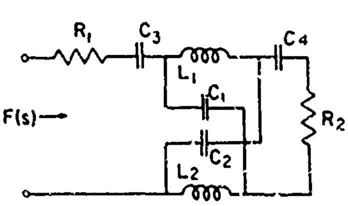


Figure 40. Final Circuit Realizing P_{10}-type Function F(s)

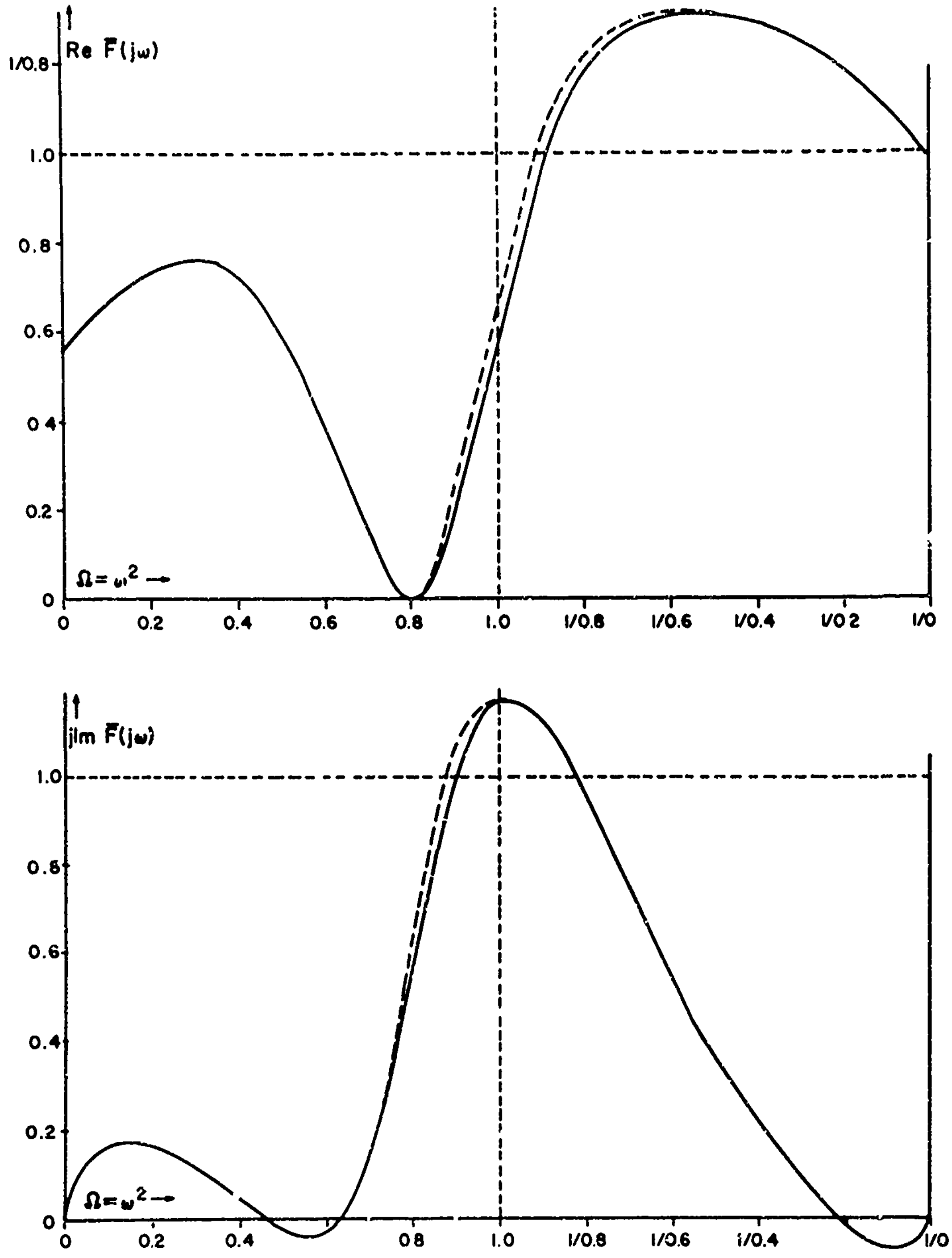


Figure 41. Coefficient Adjustment Affecting the Real and Imaginary Component of F(jω)

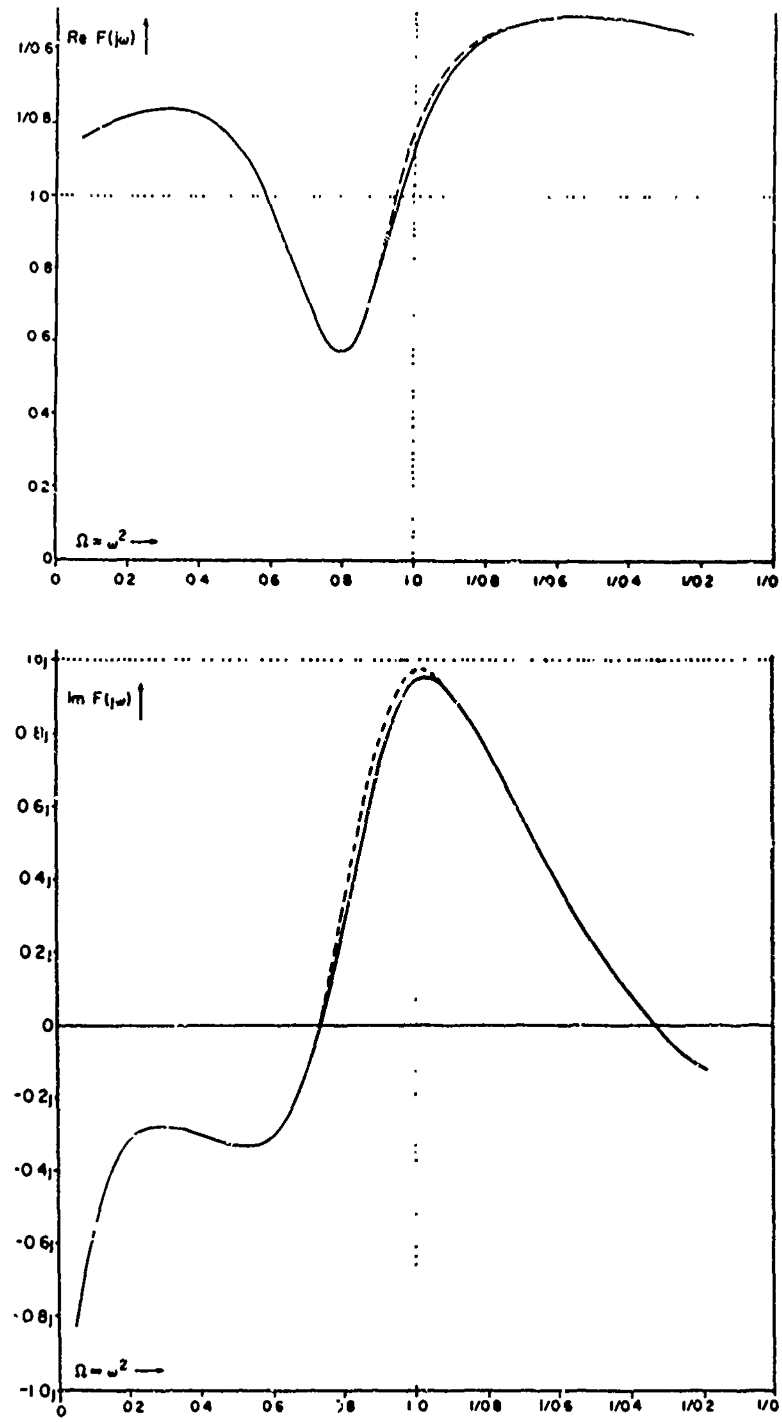


Figure 42. Coefficient Adjustment Affecting the Real and Imaginary Component of F(jω)

component of $\bar{F}(\omega)$. The solid curve refers to $\bar{F}(j\omega)$ with unchanged coefficients, and the dashed curve refers to $\bar{F}(j\omega)$ with adjusted coefficients. There is only a slight deviation around the abscissa $\Omega = \omega^2 = 1.0$.

Figure 42 shows the influence of the adjustment of the coefficients in $\bar{F}(s)$ on the real and imaginary components of the total driving-point impedance $F(j\omega)$, plotted versus the square $\Omega = \omega^2$. The same is pictured in Figure 43 in the complex F-plane. As these figures show, there is only a slight change in the functions' behavior around $\Omega = 1.0$.

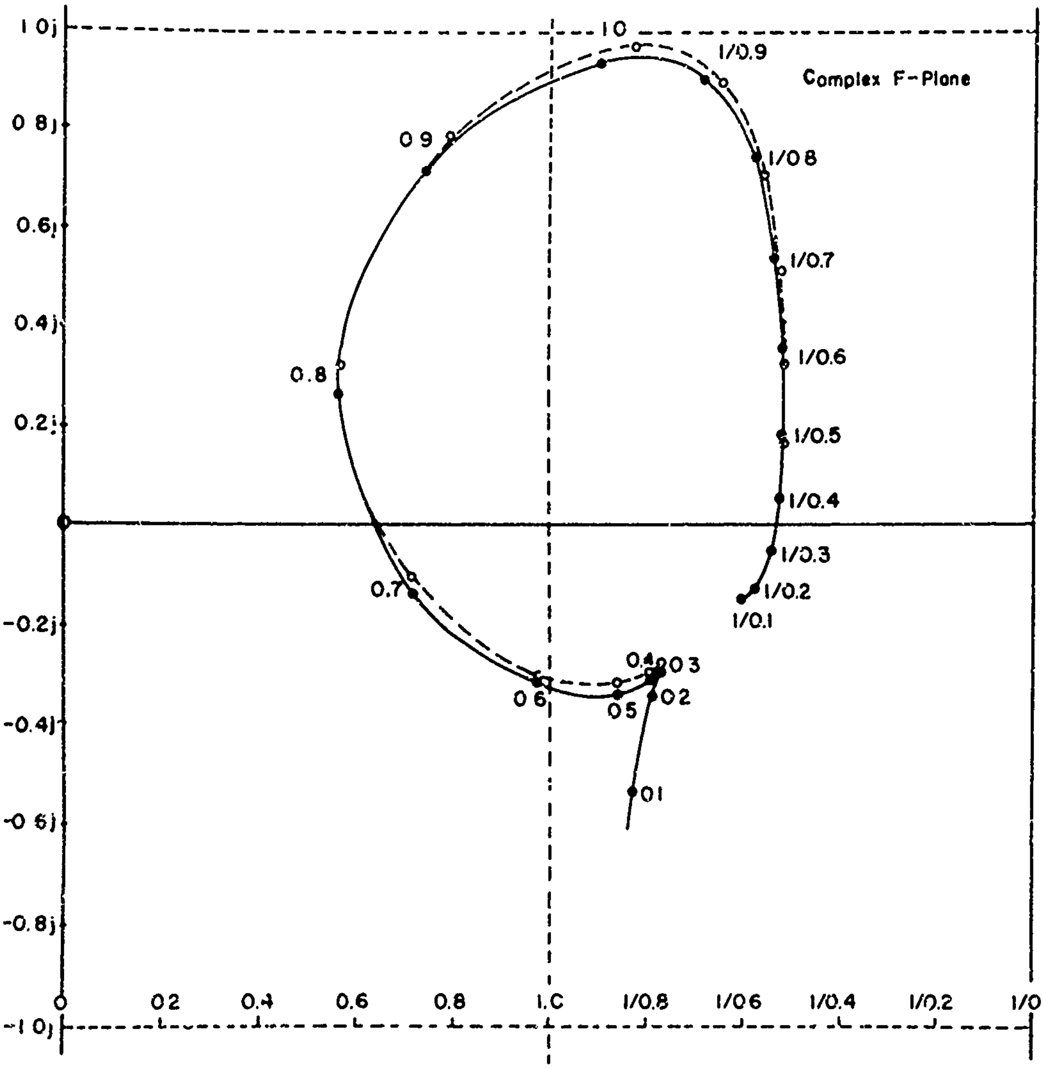


Figure 43. Coefficient Adjustment Affecting $F(j\omega)$ Represented in the Complex F(s)-Plane

10. THE CONVENTIONAL REALIZATION OF THE DRIVING-POINT IMPEDANCE DISCUSSED IN SECTION 9

In order to understand what was achieved by realizing the driving-point impedance discussed in the preceding Section where we adjusted the coefficients, let us now realize the impedance F(s) in the conventional way. For this purpose we decompose F(s) as before by decomposition procedure De1. The results of the decomposition

$$F(s) = x_t/s + K\tilde{F}(s)$$

are shown in the following table where

$x_t = 0.2174432$, and $K = 1.529$:

i	N_i	D_i	$\tilde{N}_i$	$\tilde{D}_i$
0	0.182	0.000	0.6108037	0.8370000
1	1.274	0.837	1.5169136	1.5640000
2	2.811	1.564	1.7790515	2.2610000
3	3.102	2.261	1.9807434	1.7560000
4	3.246	1.756	1.0000000	1.0000000
5	1.529	1.000		

The function F(s) has to be realized according to Brune (for instructions see Haase (1970b)). We obtain the circuit shown in Figure 44 where

resistor $r = 0.3.27543$, mutual inductance $v = 2.5861081$,
turn ratio $n = 1.1483501$, capacitor $1/x = 1/2.0695412 = 0.4831988$,
$-\Omega_0 = -0.8002532$, termination constant $z = 0.4756516$.

The coefficients of the normalized function F'(s) are

i	N_i'	D_i'
0	0.6835997	0.9108017
1	2.0416891	1.5447889
2	1.0000000	1.0000000

The driving-point function F'(s) can be realized in two ways:
(a) By the ladder realization shown in Figure 45, where

$r_0 = 0.7505472$, $\quad l_1 = 0.9686540$,
$r_1 = 0.3359745$, $\quad c_1 = 2.0440229 \approx 1/x_1$.
$r_2 = 0.2351402$,

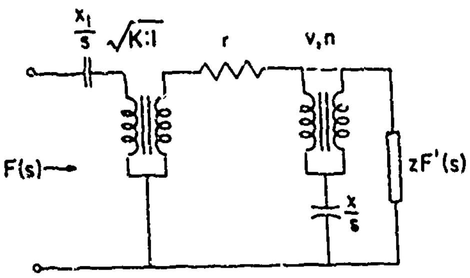


Figure 44. First Step of Realizing the P_{10}-type Function F(s) in the Conventional Brune Procedure

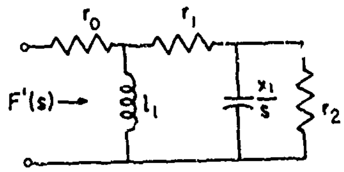


Figure 45. Ladder Realization of F'(s)

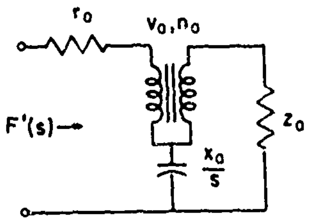


Figure 46. Brune Realization of F'(s) with Negative Mutual Inductance v_a

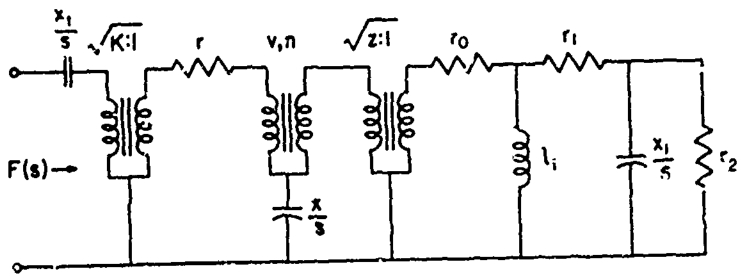


Figure 47. Tandem Circuit Implying the Circuit in Figure 45

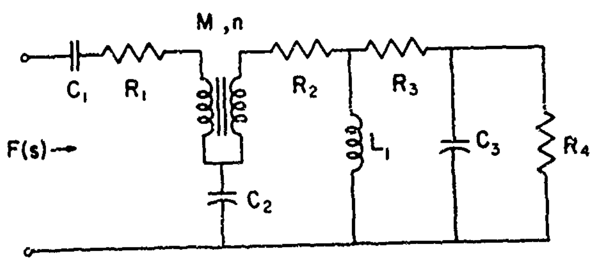


Figure 48. Final Circuit Implying One Perfectly Coupled Transformer

(b) It can be realized in Brune fashion with a perfectly coupled transformer with negative turn ratio and negative mutual inductance, as shown in Figure 46, where

$r_a = 0.0145407$, $v_a = -0.5513037$, $n_a = -1.1571209$, $x_a = 0.4339463$,
$\Omega_a = 0.7871276$.

Combining the circuits in Figures 44 and 45 we obtain the circuit in Figure 47. Dissolving the two ideal transformers with turn ratios $K{:}1 = 1.529$ and $z{:}1 = 0.4756516$ yields the circuit in Figure 48, with the circuit elements listed as follows:

$R_1 = Kr = 0.5699413$,	$M = Kv = 3.9541592$,
$R_2 = zKr_0 = 0.5458513$,	$L_1 = zKl_1 = 0.7044741$,
$R_3 = zKr_1 = 0.2443445$,	$C_1 = 1/x_t = 4.5989021$,
$R_4 = zKr_2 = 0.1710106$,	$C_2 = 1/Kx = 0.3160228$,
	$C_3 = 1/zKx_1 = 2.8105373$.

We check our results: $F(1) = 1.6370989$.

Analyzing the circuit in Figure 48, we obtain

$$F(1) = \left\{ \left(\left[(R_4 \oplus 1/C_3) + R_3 \right] \oplus L_1 \right) + R_2 + w \right\} \oplus (M + 1/C_2) + u + \\ + R_1 + 1/C_1 = 1.6370983,$$

which is correct.

Omitting the brackets, parentheses, and curled parentheses, this expression, in which

$u = Kv(n-1)$, $w = -u/n$, and $n = 1.1483501$,

can shortly be written as (see Haase (1970a)

$$F(1) \Rightarrow (R_4 \oplus 1/C_3) + R_3 \oplus L_1 + R_2 + w \oplus (M + 1/C_2) + u + R_1 + 1/C_1.$$

Besides four resistors and three capacitors, the realization shown in Figure 48 needs one inductance and one perfectly coupled transformer with the mutual inductance M and a turn ratio $n = 1.1483501$.

By combining the circuits in Figures 46 and 47, we obtain the circuit in Figure 49. Dissolving the two ideal transformers with turn ratios $\sqrt{K{:}1}$ and $\sqrt{z{:}1}$ yields the circuit in Figure 50, with the circuit elements listed as follows:

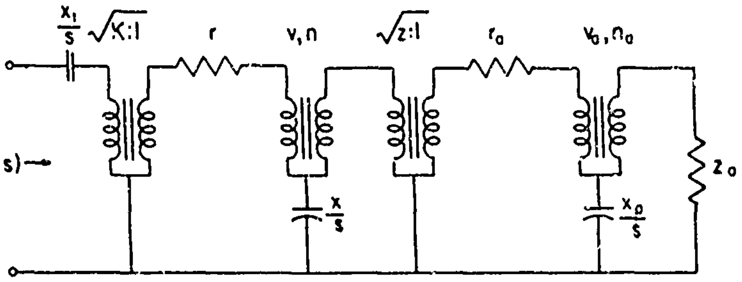


Figure 49. Tandem Circuit Implying the Circuit in Figure 46

$R_1 = Kr = C.5699413,$	$M_1 = Kv = 3.9541592,$	$n_1 = 1.1483501,$
$R_2 = zKr_a = 0.0105750,$	$M_2 = zKv_a = -0.4009473,$	$n_2 = -1.1571209,$
$R_3 = zKz_a = 0.5352757,$	$C_1 = 1/x_t = 4.5989021,$	
	$C_2 = 1/Kx = 0.3160228,$	
	$C_3 = 1/zKx_a = 3.1686019.$	

The driving-point impedance of the circuit in Figure 50 is

$$F(1) \Rightarrow (R_3 + zKw_a) \oplus (M_2 + 1/C_3) + (zKu_a + R_2 + Kw) \oplus (M_1 + 1/C_2) +$$
$$+ (Ku + R_1 + 1/C_1) = 1.6370984 \text{ (which is correct)},$$

where

$$u_a = v_a(n_2 - 1), \qquad w_a = -v_a/n_2$$
$$u = v(n_1 - 1), \qquad w = -v/n_1 .$$

The circuit realization in Figure 50 needs two perfectly coupled transformers besides three resistors and three capacitors. It would be uneconomical to use because the circuit in Figure 48 needs only one perfectly coupled transformer. Nevertheless, we designed this circuit to show a comparison with the circuit in Figure 37. This is the circuit where we have

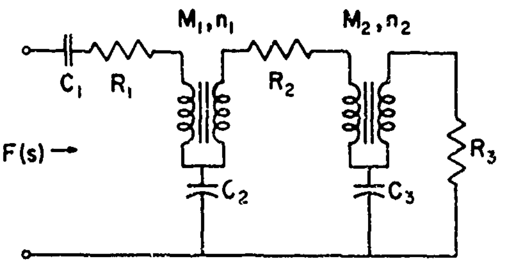


Figure 50. Final Circuit to be Compared with the Circuit in Figure 52

adjusted the coefficients of $\overline{F}(s)$. We redraw that circuit to get a better picture. The circuit in Figure 51 is the same as the one in Figure 37; its circuit elements are as follows:

$R_1 = 1/x_t = 4.5839021$,	$M_1 = K(1-r)v_1 = 4.1593753$,
$R_2 = K(1-R)/(n_1 n_2)^2 = 0.5458518$,	$M_2 = -M_1/k_1 = -0.4051684$,
	$C_1 = 1/K(1-r)x_1 = 0.3025532$,
	$C_2 = k_1 C_1 = 3.1059465$.

By adjusting the coefficients, we combined resistors R_1 and R_2 of the circuit in Figure 50 with resistor R_1 of the circuit in Figure 51. In the circuit in Figure 50 are the products $M_1C_2 = 1/0.8002532$ and $M_2C_3 = -1/0.7871274$. These two different products are combined in the circuit of Figure 51 to $M_1C_2 = -M_2C_3 = 0.7946394$.

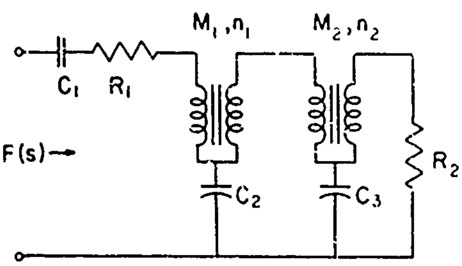


Figure 51. Final Circuit Implying Two Perfectly Coupled Transformers

After inserting impedances $(x_0 - x_0)/s$, we obtained the circuit implying the lattice section. The main advantage of the final circuit in Figure 40 is that it contains no transformers. For this reason it cannot with justice be compared to any of the circuits in Figures 48 or 50. However, it is well known that the circuit in Figure 48 can be transformed into a Bott-Duffin (1949) structure. A Bott-Duffin structure contains no transformer, but that advantage has to be paid for with a considerable number of additional circuit elements. The Bott-Duffin procedure is explained in many textbooks on passive circuit synthesis. Since its performance is rather tedious, Haase (1967) developed a shortcut procedure. Using the instructions presented in that paper, we will not transform the circuit shown in Figure 48 into the Bott-Duffin circuit pictured in Figure 52.

The bi-order function F(s) to be converted by the Bott-Duffin procedure is

$$\tilde{F}(s) = [F(s) - 1/C_1 s - R_1]/0.9590568,$$

with C_1 and R_1 listed for Figure 48. These elements reappear in the circuit in Figure 52 as C_6 and R_7 respectively. The coefficients of the F(s) point function to be converted by Bott-Duffin are

i	$\tilde{N}_i$	$\tilde{D}_i$
0	0.4763817	0.837
1	1.4889315	1.564
2	1.4926433	2.261
3	2.1143020	1.756
4	1.0000000	1.000

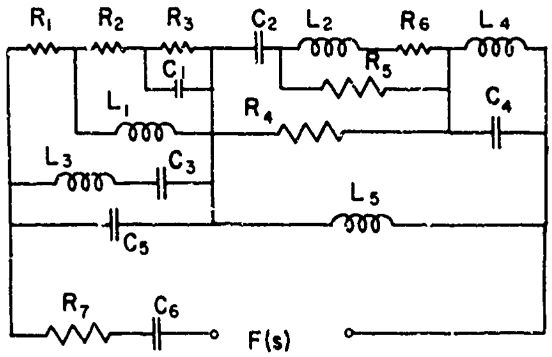


Figure 52. Conventional Bott-Duffin Circuit

The regular Brune procedure (according to Haase (1970b) yields a Brune T with the following constants:

$v = 4.1229575$, $n = 1.1483504$, $\Omega_0 = x/v = 0.8002532$, $u = v(n-1) = 0.6116424$,
$w = -u/n = -0.5326269$, $x = 3.2994099$, $z = 0.7583175$.

The terminating normalized function $F'(s)$ has the coefficients:

i	N_i'	D_i'
0	0.6241664	0.9975290
1	1.6803576	1.8932118
2	1.0000000	1.0000000

We now follow the instructions presented by Haase (1967, page 40):
We compute the coefficients of the polynomial

$$G(s) = s\tilde{D}(s) - \tilde{N}(s)/u .$$

These coefficients are listed in the table below.

i	G_i	$G(s)/(s-s_2)$	$a(s)$
0	-0.7788565	0.4994915	0.6241664
1	-1.5973170	1.3447116	1.6803576
2	-0.8763855	1.4244196	1.0000000
3	-1.1957616	1.6803576	
4	0.1210578	1.0000000	
5	1.0000000		

The polynomial G(s) has a real root $s_2 = 1.5592998$. It also contains the factor $(s^2 + \Omega_0)$. The coefficients of $G(s)/(s-s_2)$ and of $G(s)/(s-s_2)(s^2 + \Omega_0) = a(s)$ are also listed in the table above.

The constants determining the Bott-Duffin circuit are:

$$K = us_2 = 0.9537338, \text{ which yields } K^2 = 0.9036081,$$

and

$$k = s_2^2/(s_2^s + \Omega_0) = 0.7523715.$$

Next we have to compute the polynomials

$$\tilde{G}(s) = s\tilde{N}(s) - Ks_2\ddot{D}(s), \text{ and } \tilde{g}(s) = \tilde{G}(s)/(s-s_2).$$

They have the coefficients listed below:

i	$\tilde{G}(s)$	$\tilde{g}(s)$	b(s)
0	-1.2447503	0.7982756	0.9975290
1	-1.8495316	1.6980746	1.8932118
2	-1.8735302	2.2905185	1.0000000
3	-1.1188042	2.1864448	
4	0.6271451	1.0000000	
5	1.0000000		

The polynom.. ›(s) is

$$b(s) = \frac{\tilde{g}(s) - K^2 sg(s)/kv}{s^2 + \Omega_0};$$

its coefficients are listed in the table above.

The circuit with the driving-point impedance a(s)/b(s) is the same as shown in Figure 45, but its elements are

$$r_0 = 0.6257125, \quad l_1 = 0.4969794,$$
$$r_1 = 0.3742875, \quad c_1 = 14.4862453.$$
$$r_2 = 0.0605488,$$

By defining the constant H = 0.9590568, we can compute the elements of the final Bott-Duffin circuit shown in Figure 52 as follows:

$R_1 = Hr_0K^2 = 0.5458501,$	$L_1 = Hl_1K^2 = 0.4335478,$
$R_2 = Hr_1K^2 = 0.3265156,$	$L_2 = Hc_1 = 13.8931320,$
$R_3 = Hr_2K^2 = 0.0528207,$	$L_3 = Hvk = 2.9749900,$
$R_4 = H/r_0 = 1.5327435,$	$L_4 = K^2H/kx = 0.3514228,$
$R_5 = H/r_1 = 2.5623531,$	$L_5 = Hu = 0.5865998,$
$R_6 = H/r_2 = 15.8394022,$	$C_1 = c_1/HK^2 = 16.6056999,$
$R_7 = 0.5699413,$	$C_2 = l_1/H = 0.5181960,$
	$C_3 = 1/L_3\Omega_0 = 0.4200365,$
	$C_4 = 1/L_4\Omega_0 = 3.5558438,$
	$C_5 = L_5/K^2H^2 = 0.7011306,$
	$C_6 = 4.5989021.$

Circuit analysis yields $F(1) = 1.6370967$, which is in sufficient agreement with the true result of $F(1) = 1.6370989$.

We are now in the position to compare economically the result of the conventional Bott-Duffin realization with the novel circuit realization implying a lattice structure. The circuit in Figure 52 needs 18 circuit elements: 7 resistors, 5 inductors, and 6 capacitors; the circuit in Figure 40 needs 8 circuit elements: 2 resistors, 2 inductors, and 4 capacitors.

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