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ON THE ACCURACY ANALYSIS OF AIRBORNE TECHNIQUES FOR PASSIVELY LOCATING ELECTROMAGNETIC EMITTERS

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10. ABSTRACT

Discussion of methods of locating electromagnetic emitters from airborne electronic reconnaissance systems, using passive measurements on the electromagnetic waves of the emitter taken from several locations. Measurements for finding the emitter may be either the direction of arrival at 2 or more locations along the aircraft's flight path for single-aircraft DF (direction-finding) systems, or the relative time of arrival of emitter pulses at the different aircraft locations for 3-aircraft TOA (time-of-arrival) systems. Emitter location accuracy is a function of the accuracy of these indirect measurements of the emitter location and the estimation procedure used to combine the accuracies of various measurements of location and bearing. A general expression is derived for a lower bound to the covariance matrix of the unbiased estimates of emitter location coordinates (and consequently the location CEP), using the Cramer-Rao inequality. Examples illustrate the application of the Cramer-Rao lower bound to the emitter CEP for both DF and TOA systems.

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PREFACE

A primary function of electronic reconnaissance is to detect and locate enemy radars and other electromagnetic emitters such as those in communications networks. Knowledge of the location of radars in enemy air defense systems is necessary for attacking the individual radar sites and associated equipment such as surface-to-air missiles and for developing effective aircraft penetration tactics. In the battlefield area the location of enemy radars and communications emitters can be used to locate individual ground units as well as to provide important inputs to the overall intelligence picture.

As an aid in comparing the relative effectiveness of alternative airborne emitter location finding systems, a computer model was developed at The Rand Corporation to simulate the capabilities of such systems against postulated full-scale emitter environments. An important output of the model is the accuracy with which each detected emitter is located. This report provides the documentation for the methods used in the model for determining emitter location accuracy.

This report should be of interest to individuals involved in analyzing and evaluating emitter location finding systems. It presents general methods for determining location accuracy and provides a number of analytic and numerical results which facilitate an understanding of the effects of system component measurement errors and aircraft/emitter geometry on overall system location accuracy. Although most of the mathematical derivations are included for completeness and to present a unified treatment of the location accuracy problem, the reader who is not interested in the mathematical development can make practical use of the results and curves presented to make his own rapid determinations of emitter location CEP.

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SUMMARY

This report treats methods of locating electromagnetic emitters from airborne electronic reconnaissance systems, using passive measurements on the electromagnetic waves of the emitter, taken from several different locations. The measurements used for location finding may be either the direction of arrival at two or more locations along the aircraft's flight path for single-aircraft DF (direction-finding) systems, or the relative time of arrival of emitter pulses at the different aircraft locations for three-aircraft TOA (time-of-arrival) systems.

Emitter location accuracy is a function of the accuracy of these "indirect" measurements on the emitter location, the accuracy of the aircraft location measurements, the accuracy of the bearing measurement base line (in the case of DF systems), and the estimation procedure used to combine the various measurements. A general expression for a lower bound to the covariance matrix of the unbiased estimates of the emitter location coordinates (and consequently the location CEP) is derived, using the Cramér-Rao inequality. The lower bound is independent of the estimation procedure used; however, it is shown (in an appendix) that the covariance matrix of the "generalized least squares" estimate of the location coordinates approximates the lower bound. Thus, the lower bound may be used as an intrinsic measure of the location accuracy possible under the given measurement conditions.

Examples are presented illustrating the application of the Cramér-Rao lower bound to the emitter location estimate CEP for both DF and TOA systems. Approximate analytic equations for the emitter CEP are derived for both DF and TOA systems for the case where the aircraft location errors are negligible so that the emitter location errors are due solely to errors in the measurements on the emitter. For DF systems, analytic results are also derived for the case where measurements on the elevation angle from the aircraft to the emitter are taken in addition to bearing measurements. For TOA systems, additional analytic results are presented for two different aircraft location schemes in which aircraft location accuracy combines in a simple manner with time-of-arrival accuracy in determining overall emitter location accuracy.

In each instance where analytic results are obtained, data are presented in the form of graphs and simplified equations which enable the user who is not concerned with the background mathematics to quickly estimate emitter location CEPs for a number of cases of practical interest. For both DF systems and TOA systems, the general procedure using the Cramér-Rao lower bound to determine an approximate emitter location CEP from the combined accuracies in the emitter measurements and the aircraft location measurements is illustrated by sample calculations.

CONTENTS

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I. INTRODUCTION

The problem of locating an object on the surface of the earth by the use of indirect measurements has a long history. Surveyors and navigators, for example, have always been concerned with location estimation. Analytical methods for the solution of such problems can be traced back to Gauss and Laplace.(1) This report is concerned with the analysis of the accuracy with which surface-based electromagnetic emitters can be located by passive measurements on electromagnetic waves from the emitter taken from single- and multiple-aircraft electronic reconnaissance systems.

To locate an object on the surface of the earth in a given three-dimensional coordinate system requires measurements on at least three independent quantities functionally related to the three coordinate values of the object. For locating electromagnetic emitters passively, the measurements generally taken are emitter altitude (from local topographic data) and either of the following:

- o Direction of arrival of the electromagnetic waves at two or more locations of a single aircraft
- o Time of arrival of emitter pulses at three aircraft locations

These measurements are then combined with the known (or estimated) locations from which they were taken, the functional relations among the various locations and measured quantities, and assumptions about the error distributions of the measurements, to arrive at an emitter location estimate.

When the emitter measurements are taken from aircraft, the coordinates of the aircraft locations as well as the emitter location must either be known or else they must also be estimated. For each unknown aircraft location, measurements on a minimum of three additional quantities functionally related to the aircraft location are required.

The Air Force's ASQ-96 and QRC-334 systems are examples of two basically different emitter location finding systems. The ASQ-96 is a single-aircraft direction finding (DF) system which makes direction-of-arrival (directional bearing) measurements on emitter pulses from two or more locations along the aircraft's flight path. (If desired, the bearing measurement data may be combined with those of a second DF aircraft via a data link in order to obtain near-instantaneous location estimates.) The aircraft location and the bearing measurement base line are provided by a Loran-inertial navigation system. In the QRC-334 system, time-of-arrival (TOA) measurements are made on emitter pulses from three aircraft. The aircraft locations are determined from measurements of the ranges between aircraft and to two ground stations by DME (distance measuring equipment) systems, and aircraft altitudes from altimeters.

In the "just-determined" case, in which there are exactly as many functionally independent quantities measured as there are coordinates to estimate, the location estimation problem is simply one of determining the (generally) unique solution of the functional equations. However, when there are more measurements than quantities to estimate, the "overdetermined" case, the question arises as to the proper method of combining the various measurements to obtain the most accurate location estimate. Solutions for this problem depend upon the choice of criterion for the average "closeness" of an estimate as well as the probability distributions of the measurement errors and the relations among the measured quantities.

One standard measure of the "closeness" of a location estimation procedure is the location CEP (circular error probable), the circle around the true location within which 50 percent of such estimates would lie. When the emitter coordinate estimates have a multivariate normal distribution with mean values equal to the true coordinates, the location CEP can be expressed as a function of the variances and covariances of the location coordinate estimates. In Section II, a lower bound is given for the location estimate covariance matrix for unbiased location estimates. The lower bound is derived by use of the Cramér-Rao inequality under the assumption that the measurements have a multivariate normal distribution with mean values equal to the quantities measured. In Appendix A it is shown that the covariance matrix for generalized least squares location estimates approximates that given by the lower bound. In Sections III and IV, emitter location CEPs are obtained for DF systems and TOA systems using the location estimate covariance matrices obtained from the Cramér-Rao lower bounds for the system configuration examined.

II. THE ANALYTIC BACKGROUND

In the most general form considered in this report, the emitter location estimation problem is as follows. Measurements are made on an unknown emitter location (x_o, y_o, z_o) from N aircraft locations $(x_1, y_1, z_1), \cdots, (x_N, y_N, z_N)$. A total of n measurements, including the emitter measurements, are made on variables functionally related to the $p = 3(N + 1)$ location coordinates. The measurements may be either "direct," i.e., measurements on the individual location coordinates, or "indirect," i.e., related functionally to the location coordinates. The p location coordinates may be considered to be functionally independent parameters to be estimated and the n measured variables to be dependent parameters. In general, to estimate the p location coordinates requires at least p relevant measurements.

For a convenient notation, let the vector†

$$\beta = (\beta_1, \cdots, \beta_p)^*$$
$$= (x_o, y_o, z_o, \cdots, x_N, y_N, z_N)^*$$

be the parameter vector to be estimated, i.e., the location coordinates; the vector

$$m = (m_1, \cdots, m_n)^*$$

be the measurement vector on the parameter vector

$$\mu = (\mu_1, \cdots, \mu_n)^*$$

i.e., m is the measured value of μ with error $m - \mu$; and

$$f_i(\mu, \beta) = 0 \qquad i = 1, \cdots, r$$

†All symbolic vectors in this report are column vectors, so the transpose of a vector v, denoted by v^*, is a row vector.

with vector form

$$F(\mu, \beta) = (f_1(\mu, \beta), \cdots, f_r(\mu, \beta))^* = 0 \qquad (1)$$

be the functional relationships among the components of μ and β.

Although the ultimate goal is to obtain estimates of the emitter location (x_o, y_o, z_o),† it is convenient to consider the more general problem of estimating all of the components of β. If there are direct measurements on the aircraft coordinates, i.e., $\mu_i = \beta_j$ for some i and j, then the corresponding β_j may be omitted from β, if desired. When the estimates of the aircraft locations are also of interest and there are direct measurements on the aircraft locations, it may be advantageous to consider the more general problem of estimating both μ and β. This is done in the Appendices.

LEAST SQUARES AND MAXIMUM LIKELIHOOD ESTIMATES[2]

One general estimation procedure is that of "ordinary least squares," in which the estimates $\hat{\mu}$ and $\hat{\beta}$ are the values of μ and β which minimize the quadratic form

$$Q_1 = (m - \mu)^*(m - \mu)$$

subject to the constraints $F(\mu, \beta) = 0$.

When the measurements are random variables with mean vector μ and a known covariance matrix

$$\Sigma = E(m - \mu)(m - \mu)^*$$

the "generalized least squares" estimates $\hat{\mu}$, $\hat{\beta}$ are the values of μ and β which minimize the quadratic form

$$Q_2 = (m - \mu)^*\Sigma^{-1}(m - \mu) \qquad (2)$$

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†In the remainder of this report, we shall generally take z to be the altitude coordinate and assume z_o to be known with negligible error.

subject to the constraints $F(\mu, \beta) = 0$. Generalized least squares estimates will usually be better estimates than ordinary least squares estimates.

It will be assumed throughout this report that the measurement vector m is a random vector with a multivariate normal distribution with mean vector μ and known[†] covariance matrix Σ. The probability density function of m is then

$$\phi(m) = |\Sigma|^{-\frac{1}{2}}(2\pi)^{-n/2} \exp\left[-\tfrac{1}{2}(m-\mu)^{*}\Sigma^{-1}(m-\mu)\right] \qquad (3)$$

"Maximum likelihood" estimates of μ and β are the values of μ and β which maximize $\phi(m)$ subject to the constraints $F(\mu, \beta) = 0$. From the form of $\phi(m)$, it is easy to see that maximizing $\phi(m)$ is equivalent to minimizing the quadratic form in the exponent of (3),

$$Q_3 = (m - \mu)^{*}\Sigma^{-1}(m - \mu)$$

subject to the given constraints, $F(\mu, \beta) = 0$. Since Q_3 is identical to Q_2, maximum likelihood estimates are identical to generalized least squares estimates when the measurement vector has a multivariate normal distribution. This will not be true in general for other probability distributions. Maximum likelihood estimates have the desirable property that, under rather general conditions, they yield estimates which have the smallest variance when large numbers of measurements are taken.

The determination of generalized least squares estimates of μ and β requires finding the minimum of the quadratic form, Eq. (2), subject to (possibly) nonlinear constraints, $F(\mu, \beta) = 0$. In general, this requires numerical methods--one general procedure is described in Appendix A. Since the primary concern of this report is with emitter location accuracy and not estimation procedures per se, most of the

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[†] In practice, Σ is obtained from error analyses and independent testing of the location finding system and its components; or from hypothetical values when different proposed systems are being compared in analytic studies.

remainder of the discussion is concerned with the determination of location accuracy.

THE CRAMÉR-RAO INEQUALITY[2]

Maximum likelihood and ordinary and general least squares are particular procedures for obtaining estimates. Other estimation procedures exist and ad hoc estimates not based on any general procedure are used in many problems. Various criteria are available for judging the "goodness" of an estimation procedure. These usually involve some measure of average closeness of the estimate to the quantity estimated. For unbiased estimates, one commonly used measure of average closeness is the variance of the estimate. Under fairly general assumptions the Cramér-Rao inequality described below provides a useful lower bound to the covariance matrices of a large class of unbiased estimators. An unbiased estimator whose covariance matrix is close to the lower bound will then have an accuracy proximate to that of a "minimum variance unbiased estimate." It is shown in Appendix A that the covariance matrix of generalized least squares estimates is approximately equal to the Cramér-Rao lower bound.

As applied to the estimation problem of this section, the Cramér-Rao inequality takes the following form. Let the $n \times 1$ measurement vector m have a multivariate normal distribution with mean vector μ and a positive definite covariance matrix Σ. Let $F(\mu, \beta) = 0$ be an $r \times 1$ vector of constraints relating the $n \times 1$ measurement parameter vector μ to the $p \times 1$ estimation parameter vector β. Assume that $F(\mu, \beta)$ has continuous partial derivatives. Suppose $p \leq r \leq n$, and define

$$F_\mu = \frac{\partial F}{\partial \mu} = \left(\frac{\partial f_i}{\partial \mu_j}\right) \qquad i = 1, \cdots, r \qquad j = 1, \cdots, n \tag{4}$$

$$F_\beta = \frac{\partial F}{\partial \beta} = \left(\frac{\partial f_i}{\partial \beta_k}\right) \qquad i = 1, \cdots, r \qquad k = 1, \cdots, p \tag{5}$$

where F_μ and F_β are evaluated at the true values of μ and β.

Let $\Sigma_{\hat\beta}$ be the covariance matrix of an arbitrary unbiased estimate of β. Then, assuming that the regularity conditions given in Appendix B hold and that the rank of F_μ is r and the rank of F_β is p, we have from Appendix B,

$$\Sigma_{\hat\beta} \geq \Sigma_\beta \equiv (F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1} F_\beta)^{-1} \tag{6}$$

Equation (6) is the Cramér-Rao inequality[†] for the covariance matrix of unbiased estimates[††] of β, and Σ_β is the Cramér-Rao lower bound.

The inequality $\Sigma_{\hat\beta} \geq \Sigma_\beta$ means that the matrix $\Sigma_{\hat\beta} - \Sigma_\beta$ is non-negative definite. This implies, for example, that the variance of any unbiased estimate of an individual component parameter of β is larger than the corresponding diagonal element of Σ_β. Also, if the component estimates are approximately normally distributed, the smallest volume with a specified probability content will be larger for a covariance matrix from $\Sigma_{\hat\beta}$ than from Σ_β.

Equation (6) can be written in alternative computational forms in special cases. From Eq. (6),

$$\begin{aligned} \Sigma_\beta &= (F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1} F_\beta)^{-1} \\ &= ((AF_\beta)^*(AF_\mu \Sigma (AF_\mu)^*)^{-1} AF_\beta)^{-1} \end{aligned} \tag{7}$$

where A is any nonsingular r x r matrix. Thus, for example, in the "just-determined" case where r = p we may set $A = F_\beta^{-1}$ so that

$$\Sigma_\beta = F_\beta^{-1} F_\mu \Sigma F_\mu^* (F_\beta^{-1})^* \tag{8}$$

[†] Appendix B contains a proof of a more general version of the Cramér-Rao equality in which unbiased estimates of both μ and β are considered.

[††] It is shown in Appendix A that the covariance matrix of generalized least squares estimates is approximately the Cramér-Rao lower bound.

Equation (8) is equivalent to the classic "propagation of error" variance relationship given in many texts on applied analysis. If $r = n$ and F_μ is nonsingular, we may set $A = F_\mu^{-1}$ in Eq. (7) to obtain

$$\Sigma_\beta = ((F_\mu^{-1}F_\beta)^* \Sigma^{-1} F_\mu^{-1} F_\beta)^{-1} \tag{9}$$

Equations (7), (8), and (9) may be evaluated directly in terms of F_μ and F_β or in the following manner. Taking the total differential of $F(\mu, \beta) = 0$, we obtain the set of simultaneous equations

$$F_\mu d\mu + F_\beta d\beta = 0 \tag{10}$$

Any permissible linear operations on Eq. (10) to simplify it are equivalent to multiplying by a nonsingular matrix A so that

$$AF_\mu d\mu + AF_\beta d\beta = 0 \tag{11}$$

Comparing the coefficients in Eq. (11) to the expressions in Eq. (7), we see that if Eq. (11) is written in an equivalent form

$$Bd\mu + Cd\beta = 0$$

then

$$\Sigma_\beta = (C^* (B\Sigma B^*)^{-1} C)^{-1} \tag{12}$$

For example, if F_β is nonsingular, and Eq. (10) is solved for $d\beta$ so that

$$d\beta = Bd\mu$$

then

$$\Sigma_\beta = B\Sigma B^* \tag{13}$$

Suppose that $F(\mu, \beta)$ and Σ can be written (by reordering the parameters if necessary) as

$$F(\mu, \beta) = \begin{bmatrix} F_1(\mu_1, 0, \beta_1, \beta_2) \\ F_2(0, \mu_2, 0, \beta_2) \end{bmatrix} \tag{14}$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{15}$$

(Many of the problems in Section IV satisfy Eqs. (14) and (15) with β_1 the emitter coordinates and β_2 the aircraft coordinates.) Then Eq. (6) reduces to

$$\Sigma_\beta^{-1} = \begin{bmatrix} F_{1\beta_1} & F_{1\beta_2} \\ F_{2\beta_2} & 0 \end{bmatrix}^* \begin{bmatrix} F_{1\mu_1}\Sigma_1 F^*_{1\mu_1} & 0 \\ 0 & F_{2\mu_2}\Sigma_2 F^*_{2\mu_2} \end{bmatrix}^{-1} \begin{bmatrix} F_{1\beta_1} & F_{1\beta_2} \\ 0 & F_{2\beta_2} \end{bmatrix}$$

$$\Sigma_\beta = \begin{bmatrix} F^*_{1\beta_1} A_1^{-1} F_{1\beta_1} & F^*_{1\beta_1} A_1^{-1} F_{1\beta_2} \\ F^*_{1\beta_2} A_1^{-1} F_{1\beta_1} & F^*_{1\beta_2} A_1^{-1} F_{1\beta_2} + F^*_{2\beta_2} A_2^{-1} F_{2\beta_2} \end{bmatrix}^{-1} \tag{16}$$

where $A_1 = F_{1\mu_1}\Sigma_1 F^*_{1\mu_1}$ and $A_2 = F_{2\mu_2}\Sigma_2 F^*_{2\mu_2}$. Inverting Eq. (16) by the rule for partitioned matrices[2] yields (if $F_{1\beta_1}$ is nonsingular)

$$\Sigma_{\beta_1} = F^{-1}_{1\beta_1}[A_1 + F_{1\beta_2}(F^*_{2\beta_2} A_2^{-1} F_{2\beta_2})^{-1} F^*_{1\beta_2}] F^{*-1}_{1\beta_1}$$

$$\Sigma_{\beta_2} = (F^*_{2\beta_2} A_2^{-1} F_{2\beta_2})^{-1}$$

so that

$$\Sigma_{\beta_1} = F^{-1}_{1\beta_1} F_{1\mu_1} \Sigma_1 F^{*}_{1\mu_1} F^{*-1}_{1\beta_1} + F^{-1}_{1\beta_1} F_{1\beta_2} \Sigma_{\beta_2} F_{1\beta_2} F^{*-1}_{1\beta_1} \tag{17}$$

$$\Sigma_{\beta_2} = (F^{*}_{2\beta_2} (F_{2\mu_2} \Sigma_2 F^{*}_{2\mu_2})^{-1} F^{*}_{2\beta_2})^{-1} \tag{18}$$

Equations (17) and (18) are thus equivalent to the intuitively reasonable result that we may estimate β_2 from the measurements on μ_2 alone and then estimate β_1 from the measurements on μ_1 and the estimate of β_2. However, this result is not true in general. For example, if β_1 is overdetermined (in which case $F_{1\beta_1}$ is not a square matrix), then the measurements on μ_1 can contribute to improved estimates of β_2 and a smaller covariance matrix than that of Eq. (18).

When $n \geq r > p$ (the so-called overdetermined case), it is sometimes useful to introduce $r - p$ extra parameters into β (by defining $\beta_{p+1}, \cdots, \beta_r$ equal to a subset of $\mu_1, \cdots, \mu_n$, for example) so that the new F_β is nonsingular. This is particularly convenient if the resultant matrix for $F_\beta^{-1} F_\mu$ can be determined easily so that Eq. (8), which involves no further matrix inversions, can be used. The new parameter vector is $\beta = (\beta_1, \cdots, \beta_p, \beta_{p+1}, \cdots, \beta_r)^*$ and the Cramér-Rao lower bound for the covariance matrix for unbiased estimates of $(\beta_1, \cdots, \beta_p)^*$ is the upper left-hand corner of the new Σ_β.

ADDITIVITY OF INFORMATION MATRICES FOR INDEPENDENT MEASUREMENTS

The matrix

$$\Phi = F^{*}_{\beta} (F_\mu \Sigma F^{*}_{\mu})^{-1} F_\beta \tag{19}$$

in Eq. (6) is the "information" matrix for the measurement vector m relative to the independent parameter vector β.

Suppose that m consists of two stochastically independent component vectors m_1 and m_2 with mean vectors μ_1, μ_2, covariance matrices

Σ_1, Σ_2, and independent constraint relations $F_1(\mu_1, \beta) = 0$, $F_2(\mu_2, \beta) = 0$. Then, for the combined measurement vector m,

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$F_\beta^* = (F_{1\beta}^*, F_{2\beta}^*)$$

$$F_\mu = \begin{bmatrix} F_{1\mu_1} & 0 \\ 0 & F_{2\mu_2} \end{bmatrix}$$

Substituting in Eq. (19) and expanding,

$$\Phi = F_{1\beta}^*(F_{1\mu_1}\Sigma_1 F_{1\mu_1}^*)^{-1}F_{1\beta} + F_{2\beta}^*(F_{2\mu_2}\Sigma_2 F_{2\mu_2}^*)^{-1}F_{2\beta}$$

$$= \Phi_1 + \Phi_2 \tag{20}$$

so that for independent measurements and constraints, information matrices are additive. Furthermore,

$$\Sigma_\beta = \Phi^{-1} = (\Phi_1 + \Phi_2)^{-1} \tag{21}$$

$$= \Sigma_{1\beta} - \Sigma_{1\beta}(\Sigma_{1\beta} + \Sigma_{2\beta})^{-1}\Sigma_{1\beta} \tag{22}$$

from Ref. 2, p. 29, where $\Sigma_{1\beta}$ and $\Sigma_{2\beta}$ are the matrices in the Cramér-Rao inequality estimates of β corresponding to the measurement vectors m_1 and m_2, respectively.

In the computer simulation described in the Preface, Eq. (21) provides an efficient means for accumulating location accuracy data on individual emitters when a number of independent measurements are taken during the course of the simulation. Only the distinct elements of the information matrix for the emitter location accuracy need be saved in the computer for each emitter. As additional measurements are made, e.g., bearing measurements from a single aircraft, the information matrix is updated by means of Eq. (20). At the end of the simulation run, the final information matrix for each emitter is inverted to obtain Σ_β, and then Eq. (24) of the next section is used to obtain the location CEP.

THE EMITTER LOCATION ERROR CEP

If the measurements have a multivariate normal distribution, with mean equal to the measured quantity, the generalized least squares estimate of β will be approximately normally distributed with mean equal to the true value of β and covariance matrix given by Eq. (6) under rather general assumptions (see Appendix A). We shall assume that the location coordinate estimates are approximately normally distributed.

One measure of location accuracy is the size of the minimum area having a specified probability of containing the estimate. From normal distribution theory, this is the area in an elliptical region around the true location with area

$$-2\pi\sigma_{x_o}\sigma_{y_o}(1 - \rho^2)^{\frac{1}{2}} \ln (1 - P) \tag{23}$$

where $\sigma^2_{x_o}$ and $\sigma^2_{y_o}$ are the variances of the coordinate estimates $\hat{x}_o$, $\hat{y}_o$; ρ is the correlation between the estimates; and P is the specified probability. This measure of accuracy has disadvantages. If the variances are quite dissimilar in magnitude or if the estimates are highly correlated, the ellipse is quite elongated so that while the area may be small, the error in one direction may be quite large.

As a seemingly fairer single measure of accuracy, the CEP of the location estimate is commonly used. The CEP can be found from standard tables, as a function of σ_{x_o}, σ_{y_o}, and ρ.[3] When $\rho = 0$, so that the coordinate estimates are uncorrelated, an empirical approximation with a maximum error of 1 percent is

$$\mathrm{CEP} \approx 0.59(\sigma_S + \sigma_L) \qquad \sigma_S/\sigma_L \geq 0.5 \tag{24a}$$

$$\approx [0.67 + 0.8(\sigma_S/\sigma_L)^2]\sigma_L \qquad \sigma_S/\sigma_L < 0.5 \tag{24b}$$

where σ_S is the smaller of σ_{x_o} and σ_{y_o}, and σ_L is the larger.

When $\rho \neq 0$, the coordinate system may be rotated to obtain new location coordinate estimates for which the correlation is zero. The angle of rotation is

$$\alpha = \tfrac{1}{2}\tan^{-1}\left[2\rho\sigma_{x_o}\sigma_{y_o}\bigg/\left(\sigma_{x_o}^2 - \sigma_{y_o}^2\right)\right] \tag{25}$$

and the resultant values of σ_S^2 and σ_L^2 are

$$\sigma_S^2 = \tfrac{1}{2}\left\{\sigma_{x_o}^2 + \sigma_{y_o}^2 - \left[\left(\sigma_{x_o} - \sigma_{y_o}\right)^2 + 4\left(\rho\sigma_{x_o}\sigma_{y_o}\right)^2\right]^{\frac{1}{2}}\right\} \tag{26}$$

$$\sigma_L^2 = \tfrac{1}{2}\left\{\sigma_{x_o}^2 + \sigma_{y_o}^2 + \left[\left(\sigma_{x_o} - \sigma_{y_o}\right)^2 + 4\left(\rho\sigma_{x_o}\sigma_{y_o}\right)^2\right]^{\frac{1}{2}}\right\} \tag{27}$$

A simple approximation to the CEP with a maximum error of 10 percent and which does not require evaluating Eqs. (25) through (27) is obtained as follows: From Eqs. (26) and (27),

$$\sigma_S^2 + \sigma_L^2 = \sigma_{x_o}^2 + \sigma_{y_o}^2$$

and, from Eqs. (24a) and (24b),

$$\mathrm{CEP} \approx 0.83\left(\sigma_S^2 + \sigma_L^2\right)^{\frac{1}{2}} \qquad \sigma_L \approx \sigma_S \tag{28}$$

$$\approx 0.67\left(\sigma_S^2 + \sigma_L^2\right)^{\frac{1}{2}} \qquad \sigma_L \gg \sigma_S \tag{29}$$

The arithmetic average of Eqs. (28) and (29),

$$\mathrm{CEP} \approx 0.75\left(\sigma_S^2 + \sigma_L^2\right)^{\frac{1}{2}}$$

$$\approx 0.75\left(\sigma_{x_o}^2 + \sigma_{y_o}^2\right)^{\frac{1}{2}} \tag{30}$$

approximates the true CEP to within 10 percent.

III. LOCATION BY DIRECTION FINDING (DF)

Direction finding techniques for passive emitter location utilize directional bearing measurements to the emitter from two or more locations along the reconnaissance aircraft's flight path. The emitter location estimate is then taken to be some measure of the center of the intersection points of the bearing lines (actually curves) formed by the intersection of the bearing measurement planes and the earth's surface.

A drawback to location by direction finding from a single aircraft is that for accurate locations the emitter must be turned on sufficiently long for two or more bearing measurements to be made some distance apart. One way to overcome this drawback is to take elevation angle measurements as well as bearing measurements. However, as will be shown, location estimates from a combination of a single bearing measurement plus an elevation angle measurement are as accurate as two well-spaced bearing measurements only when the ground range to the emitter is of the same order as the aircraft altitude, and the relative accuracy becomes progressively poorer as the ground range increases.

It will be shown that when an emitter remains on sufficiently long for multiple bearing measurements to be made, the CEP decreases as the inverse square root of the number of independent measurements taken. However, if the measurement errors are correlated, which is likely when the measurements are taken at short intervals, there is a limit to the "equivalent" number of independent measurements that can be obtained (see p. 32) and hence a lower limit to the CEP achievable by taking multiple bearing measurements.

A BRIEF DISCUSSION OF THE MEASUREMENT ERRORS

A comprehensive error analysis for a proposed DF system would attempt to account for all sources of error which contribute to the final emitter location estimate error. The sources of error depend upon the detailed design characteristics of the system. A discussion of these sources is beyond the scope of this report.

The errors may be broadly categorized as either aircraft navigation errors or bearing measurement errors. The aircraft navigation errors result in errors in the estimated aircraft location and the estimated direction of the base line from which the direction bearings to the emitter are measured. The bearing measurement errors are combinations of errors in the sensor which measures the angle of arrival of the electromagnetic wave and distortions in the electromagnetic wave due to interactions with the aircraft structure, wave propagation anomalies, etc. The latter errors may be functions of frequency, direction of arrival, elevation angle, and aircraft attitude.

One simplistic categorization of the errors is given below. Fixed errors are those which may be considered to be constant throughout a flight. Random errors may be correlated from bearing measurement to bearing measurement, but have mean zero.

I. Navigation Errors
   a. Fixed navigation position error
   b. Fixed navigation heading error
   c. Random navigation position error
   d. Random navigation heading error

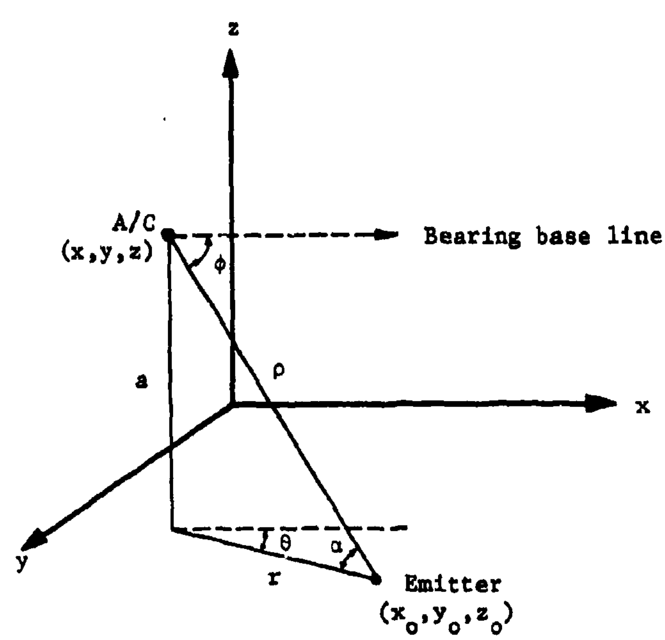
II. Bearing Measurement Errors
   a. Fixed bearing error
   b. Random bearing error

If the fixed errors can be assumed to be random errors which are fixed for a given flight, then these errors may be treated as additive to the random errors at any given bearing measurement and are common to all. The net effect in this case is that a given error type, bearing measurement errors, for example, can be treated as random correlated errors.

Certain of the errors may be combined for error analysis purposes. The fixed navigation position errors, Ia, will simply add to the emitter location errors if they are constant in each coordinate. The fixed navigation heading error, Ib, and the fixed bearing error, IIa, may be added together as a single fixed bias. The random navigation heading error, Ic, and the random bearing error, IIb, may be added together as a single random bearing error.

THE FUNCTIONAL EQUATIONS RELATING THE PARAMETERS

The coordinate system used for DF systems is generally some form of earth surface coordinates and altitude. For accurate location estimation procedures, correction for earth curvature must be made when the surface coordinate system is approximated locally by a rectangular coordinate system. However, since the earth central angle between aircraft and emitter is quite small, the corrections are minor and may be considered equivalent to removing biases in the estimates. Thus, for location error analysis purposes it is adequate to assume a local rectangular coordinate system tangent to the earth's surface in the vicinity of the aircraft and the emitter. Let x, y be rectangular coordinates in the tangent plane and z be the altitude above that plane. Denote the aircraft location by x, y, z and the emitter location by x_o, y_o, z_o as indicated in the following sketch.



Let θ be the directional bearing to the emitter from the aircraft, as projected onto the x-y plane. Since the aircraft and emitter are not, in general, coaltitude, the actual measured bearing angle is in a plane determined by the locations of the aircraft and the emitter and the axis of the DF antenna system which we take to be the bearing base line from which the directional bearings are measured (cf. the preceding sketch). With no loss in generality, assume (temporarily) that the bearing base line is parallel to the x-y plane.

Let ϕ be the actual measured bearing angle. Then

$$\cos\phi = \frac{V \cdot R}{|V|\,|R|} \tag{31a}$$

where V is a vector along the bearing base line, R is the vector from the aircraft to the emitter, and V•R is the vector dot product.

Let i, j, and k be unit vectors in the directions of the orthogonal coordinate axes. Then

$$R = (x_o - x)i + (y_o - y)j + (z_o - z)k$$

and

$$\cos\theta = \frac{V \cdot (R - (z_o - z)k)}{|V|\,|R - (z_o - z)k|} = \frac{V \cdot R}{|V|\,|R - (z_o - z)k|} \tag{31b}$$

since V is perpendicular to k. Thus, from Eqs. (31a) and (31b),

$$\frac{\cos\phi}{\cos\theta} = \frac{|R - (z_o - z)k|}{|R|} = \frac{\sqrt{\rho^2 - a^2}}{\rho} = \cos\alpha \tag{32}$$

where $\rho = |R|$, $a = z_o - z$, and α is the elevation angle from the emitter to the aircraft. Equation (32) is valid for any bearing base line in the constant altitude plane.

Now assume that the bearing base line is parallel to the x-axis. We have

$$\tan\theta = \frac{y_o - y}{x_o - x} \tag{33}$$

$$\cos\phi = \cos\theta\cos\alpha \tag{34}$$

$$\cos^2\alpha = \frac{\rho^2 - a^2}{\rho^2} = \frac{(x_o - x)^2 + (y_o - y)^2}{(x_o - x)^2 + (y_o - y)^2 + (z_o - z)^2} \tag{35}$$

Equations (33), (34), and (35) are the primary equations relating the bearing angle ϕ and the elevation angle α to the aircraft and emitter locations.

The contribution of altitude errors to the total emitter location error is generally quite small. This can be indicated as follows:

From Eqs. (34) and (35),

$$\cos\theta = \cos\phi\left(1 - \frac{a^2}{\rho^2}\right)^{-1/2}$$

$$-\sin\theta d\theta = \cos\phi\left(1 - \frac{a^2}{\rho^2}\right)^{-3/2}\frac{a}{\rho^2}\,da$$

$$= \cos\theta\; a(\rho^2 - a^2)^{-1}\,da$$

$$d\theta = -\cot\theta\; ar^{-2}\,da \tag{36}$$

where $r^2 = (\rho^2 - a^2) = (x_o - x)^2 + (y - y_o)^2$. From Eq. (33),

$$\sec^2\theta d\theta = \frac{y_o - y}{(x_o - x)^2}\,dx_o - \frac{1}{x_o - x}\,dy_o$$

$$d\theta = \frac{y_o - y}{r^2}\,dx_o - \frac{x_o - x}{r^2}\,dy_o$$

$$= \frac{1}{r}(\sin\theta\,dx_o - \cos\theta\,dy_o) \tag{37}$$

Substituting Eq. (37) in Eq. (36) and simplifying,

$$\frac{a}{r}\,da = \sin\theta\,dy_o - \tan\theta\,\sin\theta\,dx_o \tag{38}$$

The bearing angle θ is generally limited to an included angle of 90 deg or less centered on the perpendicular to the aircraft flight path. Taking $\theta = 45$ deg as a typical worst-case condition and setting $dx_o = 0$ and $dy_o = 0$ in turn, we obtain, from Eq. (38), the first-order worst-case effect of altitude errors,

$$|dy_o| = |dx_o| = \sqrt{2}\,\left|\frac{a}{r}\,da\right| \tag{39}$$

Thus, the first-order effect of altitude error is *directly* proportional to a, the difference in aircraft and emitter altitudes, *inversely* proportional to r, the ground range between aircraft and emitter, and *directly* proportional to da, the altitude difference measurement error. Since altitude difference errors will generally be hundreds of feet or less, the contribution of altitude errors to the total error will be relatively small whenever the aircraft-to-emitter ground range is much greater than the altitude of the aircraft.

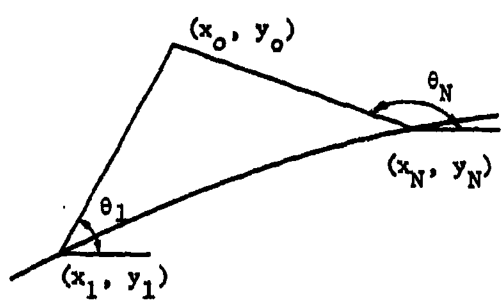
Since the primary use of aircraft-to-emitter altitude difference is in finding the projected angle θ in the x-y plane from the measured bearing angle ϕ, and since altitude measurement errors are generally of secondary importance, the altitude coordinates will be suppressed in the following development.

We shall next derive analytic results for the emitter location CEP in the special case where errors in the bearing measurements from the aircraft to the emitter dominate the aircraft location errors. Then the general procedure to be used when both bearing measurement accuracy and aircraft location accuracy contribute substantially to the emitter location CEP will be illustrated by some sample calculations.

EMITTER LOCATION CEP FROM BEARING MEASUREMENT ERRORS ONLY

The emitter location CEP will depend, in general, upon the aircraft location-error and bearing-measurement-error covariance matrices and the relative locations of the emitter and the aircraft when the bearing measurements are made. In this subsection analytic results will be obtained for the contribution of independent bearing measurement errors to the emitter location CEP, assuming that the aircraft locations are known with negligible error.

Let N independent bearing measurements $\theta_1^m, \cdots, \theta_N^m$ be taken on the true bearing $\theta_1, \cdots, \theta_N$ at locations $(x_1, y_1), \cdots, (x_N, y_N)$ along the aircraft flight path. Let σ_θ be the common bearing measurement standard deviation. Let the bearing measurements be taken relative to a bearing line parallel to the x-axis so that we have the situation depicted below:



From Eq. (33), the functional relations between the true bearings and the emitter location are

$$\theta_i = \tan^{-1}(y_o - y_i)/(x_o - x_i) \qquad i = 1, \cdots, N \tag{40}$$

so that, taking the total differential and simplifying,

$$d\theta_i = \frac{y_o - y_i}{r_i^2} dx_o - \frac{x_o - x_i}{r_i^2} dy_o \tag{41}$$

$$= \frac{1}{r_i}(\sin \theta_i dx_o - \cos \theta_i dy_o) \tag{42}$$

where $r_i^2 = (x_o - x_i)^2 + (y_o - y_i)^2$.

Identifying $\theta = (\theta_1, \cdots, \theta_N)^*$ with μ and $(x_o, y_o)^*$ with β in the development of Section II, from Eq. (41), F_μ is the identity matrix and

$$F_\beta = \begin{bmatrix} -(y_o - y_1)/r_1^2 & (x_o - x_1)/r_1^2 \\ \vdots & \vdots \\ -(y_o - y_N)/r_N^2 & (x_o - x_N)/r_N^2 \end{bmatrix} \tag{43}$$

so that, from Eq. (9), the Cramér-Rao lower bound for the covariance matrix of any unbiased estimate of $\beta = (x_o, y_o)^*$ is

$$\begin{aligned} \Sigma_\beta &= (F_\beta^* \Sigma^{-1} F_\beta)^{-1} \\ &= (F_\beta^* F_\beta)^{-1} \sigma_\theta^2 \\ &= \begin{bmatrix} \sum_i (y_o - y_i)^2/r_i^4 & -\sum_i (y_o - y_i)(x_o - x_i)/r_i^4 \\ -\sum_i (y_o - y_i)(x_o - x_i)/r_i^4 & \sum_i (x_o - x_i)^2/r_i^4 \end{bmatrix}^{-1} \sigma_\theta^2 \end{aligned} \tag{44}$$

Two Bearing Measurements

Choose the coordinate system so that the x-axis is parallel to the line segment joining the two aircraft positions and let

$$h = y_o - y_1 = y_o - y_2$$

Then, Eq. (44) can be written as

$$\Sigma_\beta = \begin{bmatrix} \sum_i \sin^4\theta_i & -\sum_i \sin^3\theta_i \cos\theta_i \\ -\sum_i \sin^3\theta_i \cos\theta_i & \sum_i \sin^2\theta_i \cos^2\theta_i \end{bmatrix}^{-1} h^2\sigma_\theta^2 \tag{45}$$

Since

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix}^{-1} = (ab - c^2)^{-1} \begin{bmatrix} b & -c \\ -c & a \end{bmatrix} \tag{46}$$

the variances and covariance in Σ_β are

$$\sigma_{x_o}^2 = h^2\sigma_\theta^2 D^{-1}(\sin^2\theta_1 \cos^2\theta_1 + \sin^2\theta_2 \cos^2\theta_2) \tag{47}$$

$$\sigma_{y_o}^2 = h^2\sigma_\theta^2 D^{-1}(\sin^4\theta_1 + \sin^4\theta_2) \tag{48}$$

$$\sigma_{x_o y_o} = h^2\sigma_\theta^2 D^{-1}(\sin^3\theta_1 \cos\theta_1 + \sin^3\theta_2 \cos\theta_2) \tag{49}$$

where D, after simplifying, is

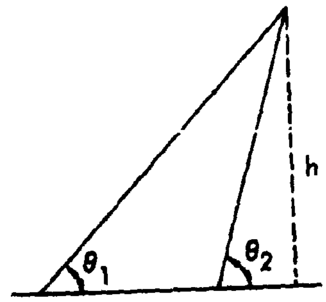
$$D = \sin^2\theta_1 \sin^2\theta_2 \sin^2(\theta_2 - \theta_1) \tag{50}$$

Substituting Eqs. (47) through (49) into Eqs. (26a) and (26b), and simplifying,

$$\sigma_L^2 = h^2\sigma_\theta^2 D^{-1}[\sin^2\theta_1 + \sin^2\theta_2 + (\sin^4\theta_1 + \sin^4\theta_2 + 2\sin^2\theta_1 \sin^2\theta_2 \cos 2(\theta_2 - \theta_1))^{\frac{1}{2}}] \tag{51}$$

$$\sigma_S^2 = h^2\sigma_\theta^2 D^{-1}[\sin^2\theta_1 + \sin^2\theta_2 - (\sin^4\theta_1 + \sin^4\theta_2 + 2\sin^2\theta_1 \sin^2\theta_2 \cos 2(\theta_2 - \theta_1))^{\frac{1}{2}}] \tag{52}$$

Figure 1 contains CEP isocontours as a function of $\theta = \theta_1$ and $\Delta\theta = \theta_2 - \theta_1$ as obtained using σ_S and σ_L from Eqs. (51) and (52) in Eqs. (24a) and (24b).



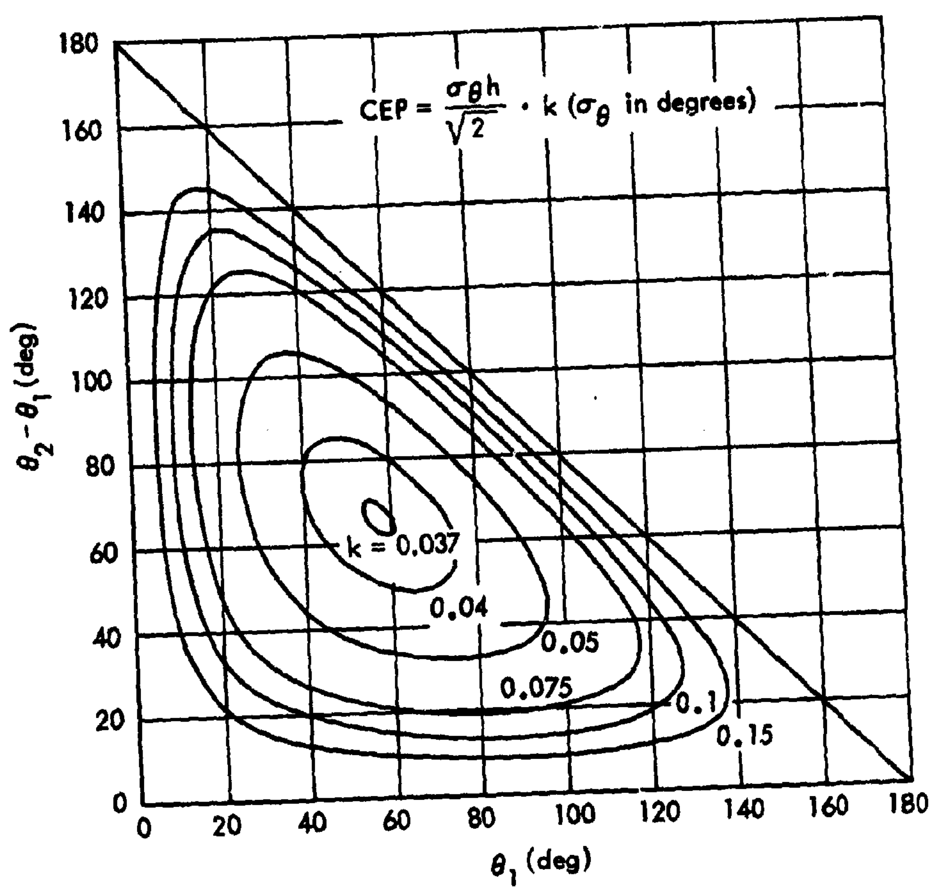


Fig. 1—CEP isocontours for two bearing measurements

From Eq. (30), the CEP may be approximated as

$$CEP = 0.75 \sqrt{\sigma^2_{x_o} + \sigma^2_{y_o}} \qquad (53)$$

where from Eqs. (47) and (48),

$$\sigma^2_{x_o} + \sigma^2_{y_o} = h^2\sigma^2_\theta(\sin^2\theta_1 + \sin^2\theta_2)/(\sin^2\theta_1 \sin^2\theta_2 \sin^2(\theta_2 - \theta_1))$$

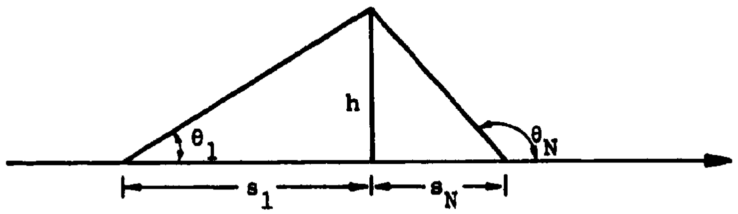
The minimum value of the approximate CEP, Eq. (53), is at $\theta_1 = 60$ deg, $\theta_2 = 120$ deg, where

$$\min CEP = \sqrt{2}\, h\sigma_\theta \qquad (\sigma_\theta \text{ in radians}) \qquad (54)$$

$$= 0.025\, h\sigma_\theta \qquad (\sigma_\theta \text{ in degrees}) \qquad (55)$$

Multiple Bearing Measurements

When the aircraft is flying a straight path and a number of bearing measurements are taken at nearly equally spaced intervals along the flight path, a simple approximation to the location CEP can be derived. For convenience in the derivation, assume a coordinate system as indicated below:



where the aircraft is flying in the direction indicated, h is the perpendicular distance from the emitter to the flight path, s is negative to the left, and $\theta_1, \theta_2, \cdots, \theta_N$ are the true directions from the aircraft flight path to the emitter at locations $s_1, s_2, \cdots, s_N$ along the flight path.

Identifying s_i with $x_i - x_o$ and h with $y_i - y_o$ in Eq. (44), and defining $L = s_N - s_1$, $\Delta\theta = \theta_N - \theta_1$, and $\{I_{ij}\}$ as the elements in the "information" matrix on the right-hand side of Eq. (44), we have

$$\sigma_\theta^2 I_{11} = \Sigma\, h^2/(h^2 + s_i^2)^2$$

$$\simeq \frac{N}{L} \int_{s_1}^{s_N} h^2/(h^2 + s^2)^2\, ds$$

$$\simeq \frac{N}{2Lh} [sh/(h^2 + s^2) + \tan^{-1} \frac{s}{h}]_{s_1}^{s_N}$$

$$\simeq \frac{N}{2Lh} [\tfrac{1}{2} \sin 2\theta + \theta - \frac{\pi}{2}]_{\theta_1}^{\theta_N}$$

$$\simeq \frac{N}{2Lh} (\tfrac{1}{2} \sin 2\theta_N - \tfrac{1}{2} \sin 2\theta_1 + \Delta\theta)$$

$$\simeq \frac{N}{2Lh} (\sin \Delta\theta \cos (\theta_1 + \theta_N) + \Delta\theta) \qquad (56)$$

Similarly,

$$\sigma_\theta^2 I_{22} = \Sigma\, s_i^2/(h^2 + s_i^2)^2$$

$$\simeq \frac{N}{L} \int_{s_1}^{s_N} s^2/(h^2 + s^2)^2\, ds$$

$$\simeq \frac{N}{2Lh} (-\sin \Delta\theta \cos (\theta_1 + \theta_N) + \Delta\theta) \qquad (57)$$

$$\sigma_\theta^2 \, I_{12} = -\Sigma \, hs_i/(h^2 + s_i^2)^2$$

$$\simeq -\frac{N}{L} \int_{s_1}^{s_N} hs/(h^2 + s^2)^2 \, ds$$

$$\simeq \frac{N}{2Lh} \sin \Delta\theta \sin (\theta_1 + \theta_N) \tag{58}$$

From Eqs. (56) through (58) and Eq. (46) it follows that

$$\sigma_{x_o}^2 = \frac{2Lh}{N} \frac{\Delta\theta - \sin \Delta\theta \cos (\theta_1 + \theta_N)}{(\Delta\theta)^2 - \sin^2 \Delta\theta} \sigma_\theta^2 \tag{59}$$

$$\sigma_{y_o}^2 = \frac{2Lh}{N} \frac{\Delta\theta + \sin \Delta\theta \cos (\theta_1 + \theta_N)}{(\Delta\theta)^2 - \sin^2 \Delta\theta} \sigma_\theta^2 \tag{60}$$

$$\sigma_{x_o y_o} = -\frac{2Lh}{N} \frac{\sin \Delta\theta \sin (\theta_1 + \theta_N)}{(\Delta\theta)^2 - \sin^2 \Delta\theta} \sigma_\theta^2 \tag{61}$$

The rotation to a coordinate system in which the estimates of the emitter coordinates are independent is, from Eq. (25), through the angle

$$\alpha = \tfrac{1}{2} \tan^{-1} \frac{2\sigma_{x_o y_o}}{\sigma_{x_o}^2 - \sigma_{y_o}^2}$$

$$= \tfrac{1}{2} \tan^{-1} \frac{2 \sin \Delta\theta \sin (\theta_1 + \theta_N)}{2 \sin \Delta\theta \cos (\theta_1 + \theta_N)}$$

$$= \tfrac{1}{2} (\theta_1 + \theta_N)$$

which shows that the direction of largest variation in the joint distribution of estimated emitter coordinates is in the direction $(\theta_1 + \theta_N)/2$ relative to the flight path.

Substituting Eqs. (59) through (61) into Eq. (27) yields

$$\sigma_L^2 \simeq \frac{2Lh}{N} \frac{2\,\Delta\theta + [4\sin^2\Delta\theta\cos^2(\theta_1+\theta_N) + 4\sin^2\Delta\theta\sin^2(\theta_1+\theta_N)]^{\frac{1}{2}}}{2(\Delta\theta + \sin\Delta\theta)(\Delta\theta - \sin\Delta\theta)}\,\sigma_\theta^2$$

$$\simeq \frac{2Lh}{N} \frac{1}{\Delta\theta - \sin\Delta\theta}\,\sigma_\theta^2 \tag{62}$$

Similarly, from Eq. (26),

$$\sigma_S^2 \simeq \frac{2Lh}{N} \frac{1}{\Delta\theta + \sin\Delta\theta}\,\sigma_\theta^2 \tag{63}$$

Since $L \simeq s_N - s_1 = h\cot\theta_N - h\cot\theta_1$, we have

$$\sigma_L \simeq \frac{h}{\sqrt{N}}\left(\frac{2(\cot\theta_N - \cot\theta_1)}{\Delta\theta - \sin\Delta\theta}\right)^{\frac{1}{2}}\sigma_\theta \tag{64}$$

$$\sigma_S \simeq \frac{h}{\sqrt{N}}\left(\frac{2(\cot\theta_N - \cot\theta_1)}{\Delta\theta + \sin\Delta\theta}\right)^{\frac{1}{2}}\sigma_\theta \tag{65}$$

$$\frac{\sigma_L}{\sigma_S} \simeq \left(\frac{\Delta\theta + \sin\Delta\theta}{\Delta\theta - \sin\Delta\theta}\right)^{\frac{1}{2}} \tag{66}$$

Figure 2 shows how the ratio σ_L/σ_S in Eq. (66) varies as a function of the emitter included angle $\Delta\theta = \theta_N - \theta_1$.

Figures 3a and 3b contain values of $k \equiv k(\theta_1, \Delta\theta)$ obtained using Eqs. (64) and (65) in the CEP approximation, Eqs. (24a) and (24b), so that

$$\mathrm{CEP} = k\,\frac{h\sigma_\theta}{\sqrt{N}}$$

where σ_θ is measured in degrees. Figure 3 is accurate enough for most purposes when N is greater than about 5 and the bearing measurements are more or less uniformly distributed along the flight path.

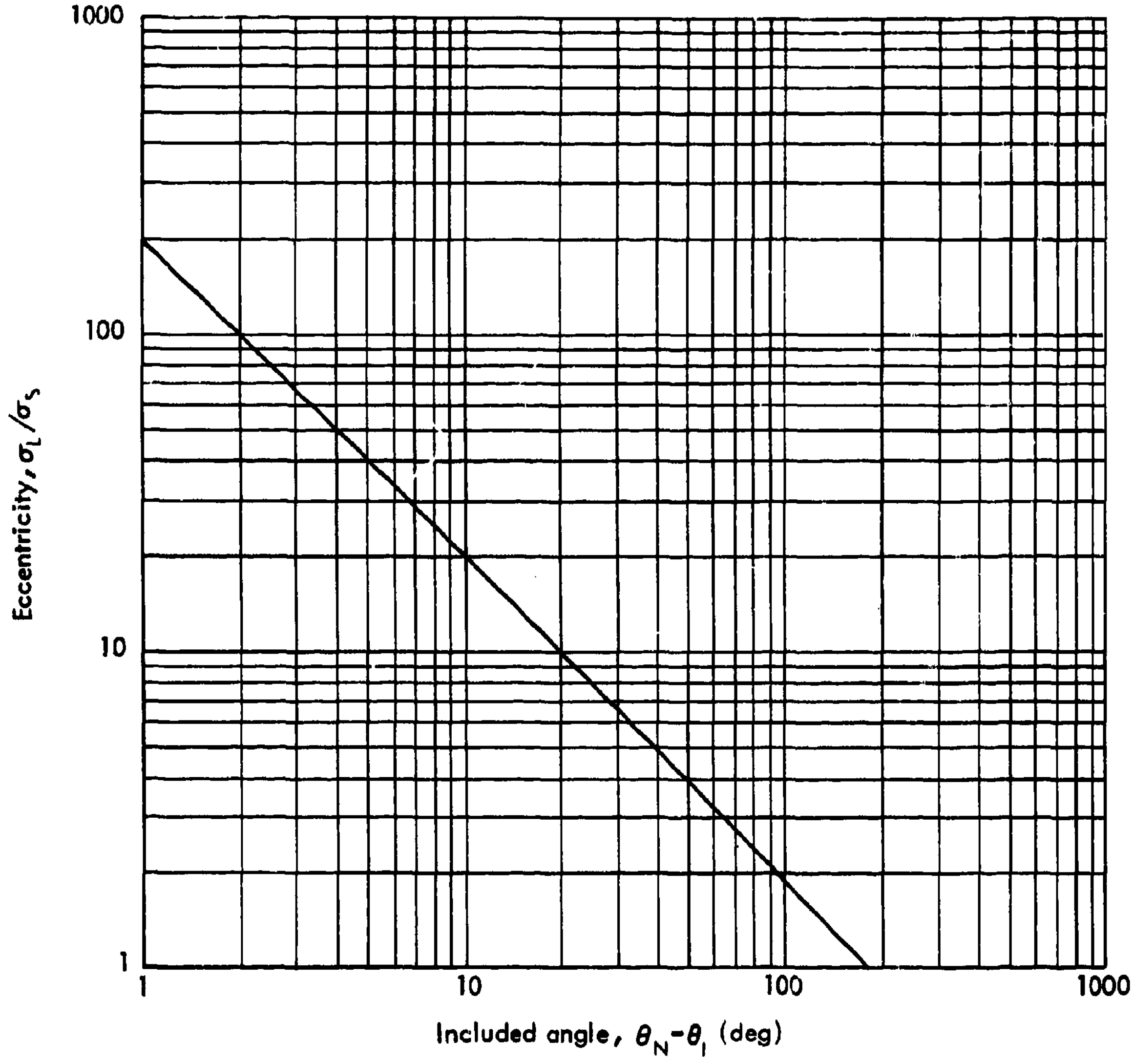
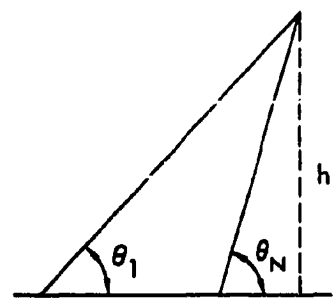


Fig.2—Eccentricity of location error distribution as a function of emitter included angle



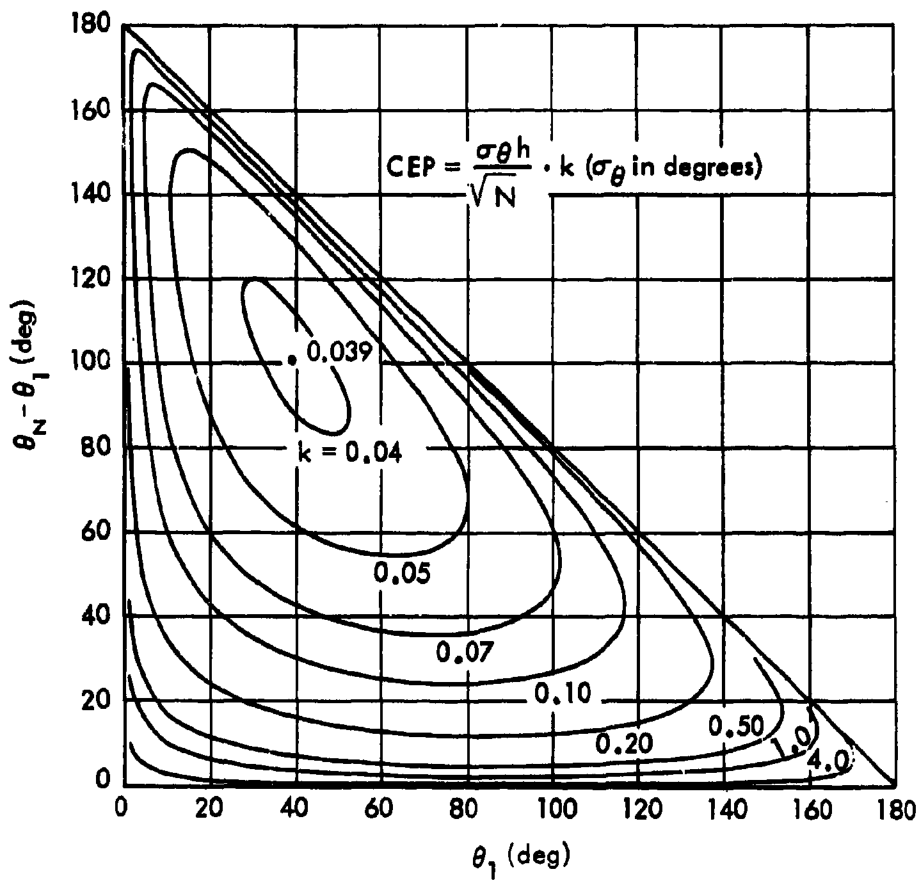


Fig.3a—CEP isocontours for multiple bearing measurements

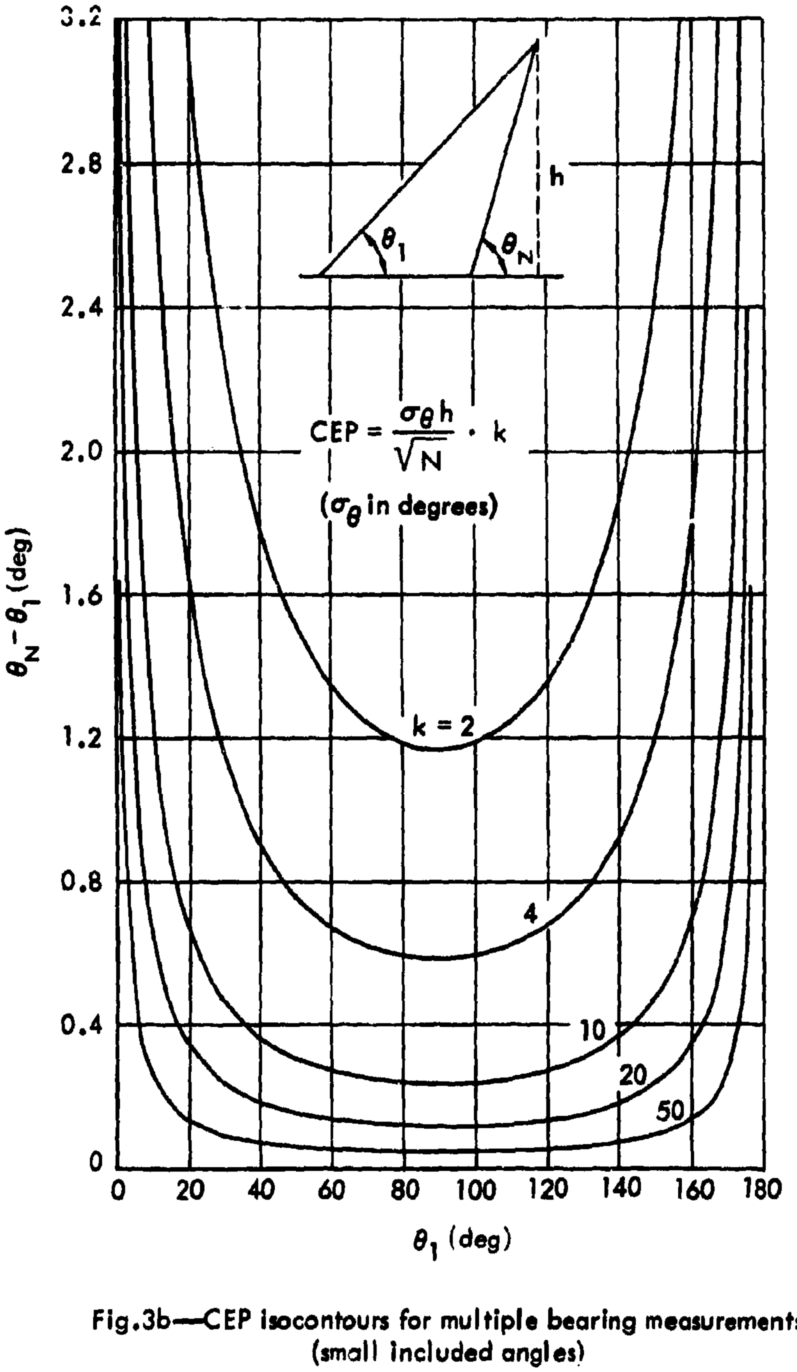


Fig.3b—CEP isocontours for multiple bearing measurements (small included angles)

MINIMUM CEP FOR BEARING MEASUREMENTS TAKEN AT A CONSTANT RATE

When bearing measurements are taken at equally spaced points along the flight path, the CEP decreases as each new measurement is taken. However, the minimum value of the CEP is not zero. In practical terms, the lower limit is reached when the emitter passes beyond the line of sight from the airborne platform. A general lower bound can be obtained as follows. Since the CEP decreases as $\Delta\theta$ increases, and σ_S approaches σ_L as $\Delta\theta$ approaches its maximum of π radians, we may simply evaluate Eq. (24a) at $\Delta\theta = \pi$. We thus obtain

$$\mathrm{CEP} \geq 0.59 \sqrt{2Lh/N}\; 2\pi^{-\frac{1}{2}}\, \sigma_\theta \qquad (\sigma_\theta \text{ in radians})$$

$$\geq 0.016 \sqrt{h/N'}\; \sigma_\theta \qquad (\sigma_\theta \text{ in degrees})$$

where N' is the bearing measurement density in measurements per unit distance (in the same units as h).

CORRELATED BEARING MEASUREMENTS

When bearing measurements are taken on an emitter with only small changes in the bearing angle between consecutive measurements, the bearing measurement errors may be highly correlated. The simplest type of correlation structure to treat is that arising when the bearing measurement errors form a stationary Markoff process. The increase in the location estimate CEP due to correlation in the bearing measurement errors will be illustrated for this case.

Let the correlation between successive measurements be equal to ρ. From the assumption that the bearing measurement errors form a stationary Markoff process,

$$\Sigma = \begin{bmatrix} 1 & \rho & \cdots & \rho^n \\ \rho & 1 & & \vdots \\ \vdots & & \ddots & \rho \\ \rho^n & \cdots & \rho & 1 \end{bmatrix} \sigma_\theta^2 \quad \text{and} \quad \Sigma^{-1} = \frac{1}{\sigma_\theta^{-2}(1-\rho^2)} \begin{bmatrix} 1 & -\rho & & & 0 \\ -\rho & 1+\rho^2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1+\rho^2 & -\rho \\ 0 & & & -\rho & 1 \end{bmatrix} \tag{67}$$

Assuming a linear flight path, we have, from Eqs. (43) and (44),

$$F_\beta = \begin{bmatrix} -h/r_1^2 & s_1/r_1^2 \\ \vdots & \vdots \\ -h/r_n^2 & s_n/r_n^2 \end{bmatrix} \tag{68}$$

$$\Sigma_\beta = (F_\beta^* \Sigma^{-1} F_\beta)^{-1} \tag{69}$$

where $r_i^2 = h^2 + s_i^2$, and h and s_i are defined as in the sketch on p. 26. Substituting Eqs. (67) and (68) into Eq. (69), performing the indicated matrix multiplications, and simplifying, we find that for large N,

$$\Sigma_\beta \approx (F_\beta^* F_\beta)^{-1}\, \sigma_\theta^2\, (1 + \rho)/(1 - \rho) \tag{70}$$

Comparing Eq. (70) with Eq. (44), we see that Figs. 2 and 3 may be used with σ_θ replaced by $\sigma_\theta\sqrt{(1+\rho)/(1-\rho)}$ when the bearing measurement errors form a stationary Markoff process with correlation between successive measurements equal to ρ.

BEARING MEASUREMENTS PLUS ELEVATION ANGLE MEASUREMENTS

If an elevation angle measurement is made along with each bearing measurement, an emitter location estimate can be made from measurements taken from a single aircraft position. It is shown in the following that when the elevation angle measurement accuracy is of the same order as the bearing measurement accuracy, the combined bearing measurement plus elevation angle measurement is roughly equivalent to two optimally spaced bearing measurements when the aircraft-to-emitter ground range is equal to the aircraft-to-emitter altitude difference. When the ground range to the emitter is much larger than the aircraft altitude, elevation angle measurements must be much more accurate than bearing measurements to provide equivalent emitter location accuracy. Thus, combined bearing and angle measurements are most useful against emitters that are close to the aircraft and when measurements can be made at only a few positions along the flight path (e.g., measurements on emitters with very low power or intermittent operation).

Let a be the aircraft-to-emitter altitude difference, r be the aircraft-to-emitter ground range, and α be the elevation angle from the emitter to the aircraft. Then

$$\tan \alpha = \frac{a}{r}$$

$$\sec^2 \alpha d\alpha = \frac{a}{r}\left(\frac{x - x_o}{r^2}\, dx_o + \frac{y - y_o}{r^2}\, dy_o\right)$$

since $r^2 = (x - x_o)^2 + (y - y_o)^2$. Thus

$$d\alpha = \sin \alpha \cos \alpha \left(\frac{x - x_o}{r^2}\, dx_o + \frac{y - y_o}{r^2}\, dy_o\right) \tag{71}$$

From Eq. (41),

$$d\theta = -\frac{y - y_o}{r^2}\, dx_o + \frac{x - x_o}{r^2}\, dy_o \tag{72}$$

Identifying $(\theta, \alpha)^*$ with μ and $(x_o, y_o)^*$ with β in Section II, we have, from Eqs. (71) and (72),

$$F_\mu^{-1} F_\beta = \frac{1}{r^2}\begin{bmatrix} -(y - y_o) & x - x_o \\ (x - x_o) \sin \alpha \cos \alpha & (y - y_o) \sin \alpha \cos \alpha \end{bmatrix} \tag{73}$$

so that for independent angle measurements, from Eqs. (9) and (73),

$$\begin{aligned} \Sigma_\beta^{-1} &= (F_\mu^{-1} F_\beta)^* \begin{bmatrix} \sigma_\theta^{-2} & 0 \\ 0 & \sigma_\alpha^{-2} \end{bmatrix} F_\mu^{-1} F_\beta \\ &= \begin{bmatrix} (y - y_o)^2 + c(x - x_o)^2 & (c - 1)(x - x_o)(y - y_o) \\ (c - 1)(x - x_o)(y - y_o) & (x - x_o)^2 + c(y - y_o)^2 \end{bmatrix} r^{-4} \sigma_\theta^{-2} \end{aligned} \tag{74}$$

where $c = \sin^2 \alpha \cos^2 \alpha \, \sigma_\theta^2/\sigma_\alpha^2$.

Now choose the coordinate system so that $x = x_o$ and, therefore, $r^2 = (y - y_o)^2$. Then Eq. (74) can be written as

$$\Sigma_\beta^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} (r\sigma_\theta)^{-2}$$

Thus,

$$\sigma_{x_o} = r\sigma_\theta \tag{75}$$

$$\begin{aligned} \sigma_{y_o} &= r\sigma_\theta/\sqrt{c} \\ &= r\sigma_\alpha/\sin \alpha \cos \alpha \\ &= \frac{r^2 + a^2}{a} \sigma_\alpha \end{aligned} \tag{76}$$

where σ_{x_o} and σ_{y_o} are equivalent to the crossrange error and range error standard deviations, respectively.

From Eq. (30), the approximate emitter location CEP from the bearing measurement and elevation measurement combination is then

$$\begin{aligned} \text{CEP} &= 0.75(\sigma_{x_o}^2 + \sigma_{y_o}^2)^{\frac{1}{2}} \\ &= 0.75 \, r\sigma_\theta \left[1 + \left(\frac{r}{a} + \frac{a}{r} \right)^2 \frac{\sigma_\alpha^2}{\sigma_\theta^2} \right]^{\frac{1}{2}} \end{aligned} \tag{77}$$

When $\sigma_\alpha = \sigma_\theta = \sigma$, Eq. (77) can be approximated as

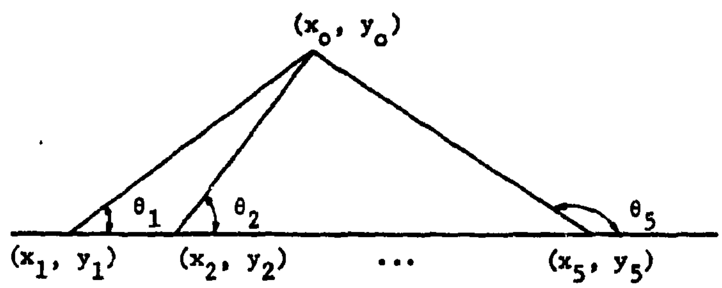
$$\text{CEP} \simeq 0.75 \frac{r^2}{a} \sigma \qquad r >> a \tag{78}$$

$$\simeq 0.75 \sqrt{5} \, r\sigma \qquad r \simeq a \tag{79}$$

$$\simeq 0.75 \, a\sigma \qquad a >> r \tag{80}$$

EMITTER LOCATION CEP CALCULATIONS FOR THE GENERAL CASE

Consider the following sketch depicting an emitter location (x_o, y_o) and five locations $(x_1, y_1), \cdots, (x_5, y_5)$ along the aircraft flight path:



Assuming the bearing measurement base line is along the x-axis, we have for an aircraft location (x, y)

$$\tan \theta = (y_o - y)/(x_o - x)$$

$$\sec^2 \theta d\theta = (x_o - x)^{-1}(dy_o - dy) - (x_o - x)^{-2}(y_o - y)(dx_o - dx)$$

$$r^2 d\theta = (x_o - x)(dy_o - dy) - (y_o - y)(dx_o - dx)$$

$$r^2 d\theta + (x_o - x)dy - (y_o - y)dx = (x_o - x)dy_o - (y_o - y)dx_o \tag{81}$$

where $r^2 = (x_o - x)^2 + (y_o - y)^2$. Taking $x_o = 0$ and setting $h = y_o - y_i$ (assuming a linear flight path), Eq. (81) may be written for each aircraft location as

$$hdx_o + x_i dy_o = -r_i^2 d\theta_i + hdx_i + x_i dy_i \qquad i = 1, \cdots, 5 \tag{82}$$

Suppose that the aircraft location is estimated by a navigation system with independent errors in each coordinate with mean zero and common variance σ_L^2 and that the bearing measurement plus navigation heading errors have mean zero and common variance σ_θ^2. Setting $\beta^* = (x_o, y_o)$ and $\mu^* = (\theta_1, x_1, y_1, \cdots, \theta_5, x_5, y_5)$, the Cramér-Rao lower bound for the covariance matrix of unbiased estimates of β is, from E[illegible]),

$$\Sigma_\beta = (F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1} F_\beta)^{-1} \tag{83}$$

where, from Eq. (82), the i^{th} component of $dF(\mu, \beta)$ is

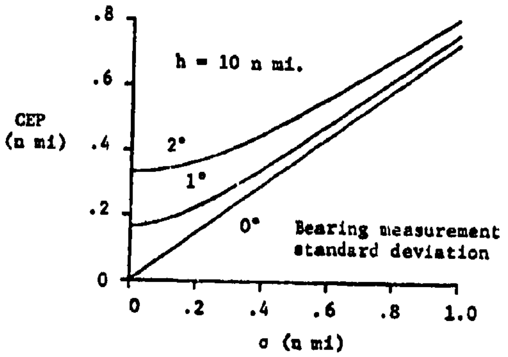
$$-r_i^2 d\theta_i + hdx_i + x_i dy_i - hdx_o - x_i dy_o \tag{84}$$

so that

$$F_\mu = \begin{bmatrix} -r_1^2 & h & x_1 & & & & & & & \\ & & & -r_2^2 & h & x_2 & & & \bigcirc & \\ & \bigcirc & & & & & \ddots & & & \\ & & & & & & & -r_5^2 & h & x_5 \end{bmatrix}$$

$$-F_\beta = \begin{bmatrix} h & x_1 \\ & \vdots \\ h & x_5 \end{bmatrix}$$

Using equally spaced aircraft locations from $\theta_1 = 45$ deg to $\theta_5 = 135$ deg, a computer program to evaluate a general version of Eq. (83) produced the illustrative results in the sketch at the left for the emitter location error CEP. The sketch illustrates the relative contribution to the emitter location CEP from bearing measurement errors and aircraft location errors.



Aircraft location standard deviation σ (n mi)

IV. LOCATION BY TIME OF ARRIVAL (TOA)

Time-of-arrival techniques for estimating emitter location use the arrival time of an emitter pulse at two different aircraft as a primary measurement. The difference in arrival time multiplied by the pulse propagation velocity measures the difference in range between the emitter and the two aircraft. Curves of constant range differences are approximately hyperbolae on the earth's surface, and the emitter location is estimated as the intersection of two such curves. Consequently, three or more aircraft must be used for an instantaneous location "fix" on an emitter. Only two aircraft are required if two or more range difference measurements can be made on the same emitter over some elapsed time interval.

Two hyperbolae can result in as many as four points of intersection. When there are multiple intersection points, additional information is required to eliminate the "ghost" intersections from the one representing the true emitter location. The additional information may be provided by TOA measurements from new aircraft locations, a crude direction-of-arrival measurement, or knowledge of the general location of the emitter.

THE TOA HYPERBOLA

For illustrative purposes, consider the two-dimensional flat-earth approximation with the aircraft and emitter taken to be in the same plane. Assume a coordinate system in which the two aircraft are one unit apart on the x-axis, with the origin at the midpoint. Then the range difference to a target at (x, y) is

$$d = [(x + \tfrac{1}{2})^2 + y^2]^{\frac{1}{2}} - [(x - \tfrac{1}{2})^2 + y^2]^{\frac{1}{2}} \qquad (85)$$

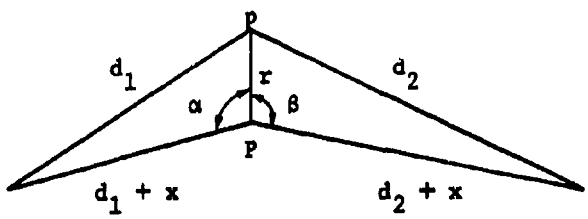
For fixed d, Eq. (85) reduces to the branch of the hyperbola,

$$\frac{x^2}{d^2} - \frac{y^2}{1 - d^2} = \frac{1}{4} \qquad d^2 \leq 1 \qquad (86)$$

which passes through the point $y = 0$, $x = d/2$ and whose values are asymptotically equal to the lines through the origin with slopes

$$m = \pm (1/d^2 - 1)^{\frac{1}{2}}$$

The tangent to the hyperbola bisects the angle formed by the lines from the emitter to the two aircraft locations. This can be shown as follows:



Let P and p be two points on a given hyperbola. Then by the law of cosines

$$d_1^2 = r^2 + (d_1 + x)^2 - 2r(d_1 + x) \cos \alpha$$

so that

$$\cos \alpha = (d_1^2 - r^2 - (d_1 + x)^2)/2r(d_1 + x)$$

Thus,

$$\lim_{P \to p} \cos \alpha = - \lim_{P \to p} \frac{x}{r}$$

Similarly,

$$\lim_{P \to p} \cos \beta = - \lim_{P \to p} \frac{x}{r}$$

so that

$$\lim_{P \to p} \alpha = \lim_{P \to p} \beta$$

showing that the tangent to the hyperbola bisects the angle formed by the lines from the emitter to the two aircraft.

ERROR ANALYSIS EQUATIONS FOR THREE-AIRCRAFT TOA

For error analysis purposes and distances up to a few hundred miles, it is generally adequate to assume a flat-earth coordinate system consisting of a rectangular coordinate system in a plane tangent to the earth at a point in the vicinity of the emitter, with altitude above the earth's surface as the third rectangular coordinate. The errors in using such a coordinate system may be considered as biases to be corrected by the location estimation procedure.

Let the aircraft coordinates be (x_i, y_i, z_i) $i = 1, 2, 3$ and the emitter coordinates be (x_o, y_o, z_o). Let r_{ij} be the distance from location i to location j, and

$$\delta_{ij} = r_{io} - r_{jo} \tag{87}$$

be the difference between the distances from the i^{th} and j^{th} aircraft locations to the emitter. Since $\delta_{ij} = \delta_{ik} + \delta_{kj}$, $k \neq i, j$, any two of δ_{12}, δ_{13}, or δ_{31} may be taken to be the TOA range difference parameters (TOA time difference multiplied by the speed of propagation, about 1 ft/nanosecond.)

The equations relating the emitter and aircraft locations are

$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \tag{88}$$

Taking differentials of both sides of Eq. (88) gives

$$dr_{ij} = c_{ijx}(dx_i - dx_j) + c_{ijy}(dy_i - dy_j) + c_{ijz}(dz_i - dz_j) \tag{89}$$

where

$$c_{ijx} = (x_i - x_j)/r_{ij} \tag{90}$$

$$c_{ijy} = (y_i - y_j)/r_{ij} \tag{91}$$

$$c_{ijz} = (z_i - z_j)/r_{ij} \tag{92}$$

are the direction cosines of the line joining location i to location j.

From Eqs. (87) and (89),

$$\begin{aligned} d\delta_{1i} &= dr_{1o} - dr_{io} \\ &= (c_{iox} - c_{1ox})dx_o + c_{1ox}dx_1 - c_{iox}dx_i \\ &\quad + (c_{ioy} - c_{1oy})dy_o + c_{1oy}dy_1 - c_{ioy}dy_i \\ &\quad + (c_{ioz} - c_{1oz})dz_o + c_{1oz}dz_1 - c_{ioz}dz_i \qquad i = 2,3 \end{aligned} \tag{93}$$

Equations (89) and (93) provide the basic relationships for the analysis of TOA emitter location accuracy. Equation (93) relates errors in the range difference measurements to the errors in the location coordinate vector $\beta^* = (x_o, y_o, z_o, \ldots, x_3, y_3, z_3)$, whereas Eq. (89) relates errors in range measurements between aircraft to errors in β^*. Similar equations hold if ground-station-to-aircraft range measurements are included.

In the following, we shall first obtain analytical results for the emitter location CEP for cases in which certain of the measurement error accuracies dominate the remainder. Then, sample calculations will be presented illustrating the general procedure for obtaining the emitter location CEP when both the aircraft location accuracies and the aircraft-to-emitter TOA measurement accuracies are important.

EMITTER-TO-AIRCRAFT RANGE DIFFERENCE ERRORS ONLY

Suppose the three aircraft locations and the emitter altitude are estimated with negligible error.

Let

$$\delta_1 = \delta_{12} = r_{1o} - r_{2o} \tag{94}$$

$$\delta_2 = \delta_{13} = r_{1o} - r_{3o} \tag{95}$$

be measured with measurement error variances σ_1^2, σ_2^2 and correlation coefficient ρ.

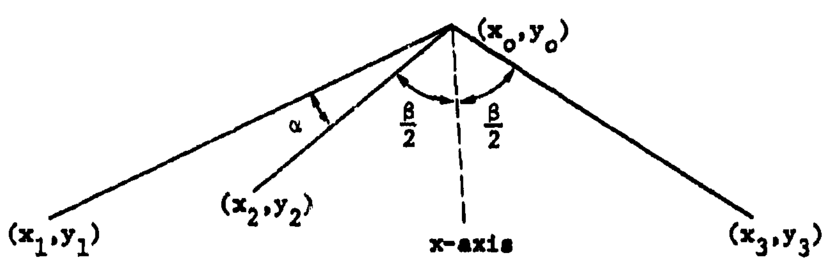
Assume also that the aircraft and the emitter are in the same geometric plane (or that the altitude differences are inconsequential). Then Eq. (93) can be written as

$$d\delta_1 = (\cos\gamma_2 - \cos\gamma_1)dx_o + (\sin\gamma_2 - \sin\gamma_1)dy_o \tag{96}$$

$$d\delta_2 = (\cos\gamma_3 - \cos\gamma_1)dx_o + (\sin\gamma_3 - \sin\gamma_1)dy_o \tag{97}$$

where γ_i is the angle between the line joining the i^{th} aircraft and the emitter and the x-axis.

Now choose the x-axis to bisect the angles between the lines joining two of the aircraft and the emitter as indicated in the sketch below:



Then, from Eqs. (96) and (97),

$$d\delta_1 = [\cos(\alpha + \beta/2) - \cos\beta/2]dx_o + [\sin(\alpha + \beta/2) - \sin\beta/2]dy_o$$

$$= -2 \sin\frac{\alpha+\beta}{2} \sin\frac{\alpha}{2}\, dx_o + 2 \sin\frac{\alpha}{2} \cos\frac{\alpha+\beta}{2}\, dy_o \qquad (98a)$$

$$d\delta_2 = [\cos(-\beta/2) - \cos\beta/2]dx_o + [\sin(-\beta/2) - \sin(\beta/2)]dy_o$$

$$= -2 \sin\frac{\beta}{2}\, dy_o \qquad (98b)$$

Solving Eqs. (98a) and (98b) for dx_o and dy_o,

$$dx_o = -\tfrac{1}{2} \csc\frac{\alpha+\beta}{2} \left(\csc\frac{\alpha}{2}\, d\delta_1 + \csc\frac{\beta}{2} \cos\frac{\alpha+\beta}{2}\, d\delta_2\right) \qquad (99a)$$

$$dy_o = -\tfrac{1}{2} \csc\frac{\beta}{2}\, d\delta_2 \qquad (99b)$$

From Eqs. (99a) and (99b), after some reduction,

$$\sigma_{x_o}^2 + \sigma_{y_o}^2 = \tfrac{1}{4} \csc^2\frac{\alpha+\beta}{2} \left(\sigma_1^2 \csc^2\frac{\alpha}{2} + \sigma_2^2 \csc^2\frac{\beta}{2} + 2\rho\sigma_1\sigma_2 \csc\frac{\alpha}{2} \csc\frac{\beta}{2} \cos\frac{\alpha+\beta}{2}\right)$$

$$= D^{-1}\left(\sigma_1^2 \sin^2\frac{\beta}{2} + \sigma_2^2 \sin^2\frac{\alpha}{2} + 2\rho\sigma_1\sigma_2 \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \cos\frac{\alpha+\beta}{2}\right) \qquad (100)$$

where

$$D = 4 \sin^2\frac{\alpha}{2} \sin^2\frac{\beta}{2} \sin^2\frac{\alpha+\beta}{2}$$

$$= \tfrac{1}{4}[\sin\alpha + \sin\beta - \sin(\alpha+\beta)]^2 \qquad (101)$$

Substituting half-angle formulas in Eq. (100) and simplifying yields

$$\sigma_{x_o}^2 + \sigma_{y_o}^2 = 2\,\frac{(1-\cos\alpha)\sigma_1^2 + (1-\cos\beta)\sigma_2^2 + \rho\sigma_1\sigma_2(\cos\alpha + \cos\beta - 1 - \cos(\alpha+\beta))}{(\sin\alpha + \sin\beta - \sin(\alpha+\beta))^2} \tag{102}$$

From Eq. (96),

$$\delta_1 = r_{1o} - r_{2o} = t_{1o}v_{1o} - t_{2o}v_{2o}$$

$$\delta_2 = r_{1o} - r_{2o} = t_{1o}v_{1o} - t_{2o}v_{2o}$$

where t_{io} is the propagation time and v_{io} the average propagation velocity (about 1 ft/nanosecond) from the emitter to the i^{th} aircraft. Assuming that the variances and covariances of the errors in the measurements of $t_{io}v_{io}$ are equal with common variance σ_d^2 and common correlation ρ_d (which is reasonable if either the velocity estimate errors are negligible compared to the time-of-arrival errors or if the emitter distances from each aircraft are comparable), the covariance matrix of the measurement errors of δ_1 and δ_2 is

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \sigma_{TOA}^2 \tag{103}$$

where $\sigma_{TOA}^2 = (1 - \rho_d)\sigma_d^2$ is the "effective" range difference error variance.

Substituting Eq. (103) in Eq. (102) gives

$$\begin{aligned} \sigma_{x_o}^2 + \sigma_{y_o}^2 &= \frac{3 - \cos\alpha - \cos\beta - \cos(\alpha+\beta)}{(\sin\alpha + \sin\beta + \sin(\alpha+\beta))^2}\; 2\sigma_{TOA}^2 \\ &= \frac{3 - \cos\eta_1 - \cos\eta_2 - \cos\eta_3}{(\sin\eta_1 + \sin\eta_2 + \sin\eta_3)^2}\; 2\sigma_{TOA}^2 \end{aligned} \tag{104}$$

where $\eta_1 \geq 0$, $\eta_1 + \eta_2 + \eta_3 = 360$ deg, and η_1, η_2, η_3 are the angles between the lines from the emitter to the aircraft, measured in a clockwise direction.

Equation (104) is a minimum when $\eta_1 = \eta_2 = \eta_3 = 120$ deg, which implies that the emitter location errors tend to be smallest near the center of the aircraft triangle.

It may be noted that the emitter location CEP is proportional to σ_{TOA} and independent of scale, i.e., Eq. (104) is a function only of the angles between the aircraft and the emitter.

Figure 4 contains CEP isocontours as a function of α and β ($\alpha \leq \beta$), the two smallest angles from η_1, η_2, and η_3. This figure is thus valid for all aircraft triangles. Figures 5, 6, and 7 contain CEP isocontours for three isosceles aircraft triangles, as obtained from a more accurate CEP calculation than the approximation of Section II. The figures are not valid for emitter locations near the extensions of the aircraft triangle base legs, where certain assumptions made in the analysis are not met. In fact, Eq. (104) implies an infinite CEP along the base leg extensions--which is not true for the location estimate that is the intersection of the range difference hyperbolae.

From the figures, at a given distance from the centroid of the isosceles triangles, the CEP is a minimum on the perpendicular bisector of the unit base leg. Along the perpendicular bisector, $\eta_1 = \eta_2 = \eta$ and $\eta_3 = 2\pi - \eta$, so that Eq. (104) becomes

$$\sigma_{x_o}^2 + \sigma_{y_o}^2 = \frac{3 - 2\cos\eta - \cos 2\eta}{(2\sin\eta - \sin 2\eta)^2}\, 2\sigma_{TOA}^2$$

$$\approx \left[(2\eta^2 + (2\eta)^2)\Big/\left(\frac{\eta^3}{3} - \frac{(2\eta)^3}{6}\right)^2\right]\sigma_{TOA}^2 \qquad \eta << 1$$

$$\approx (6/\eta^4)\, \sigma_{TOA}^2$$

$$\approx 96\, R^4 \sigma_{TOA}^2 \qquad R >> 1$$

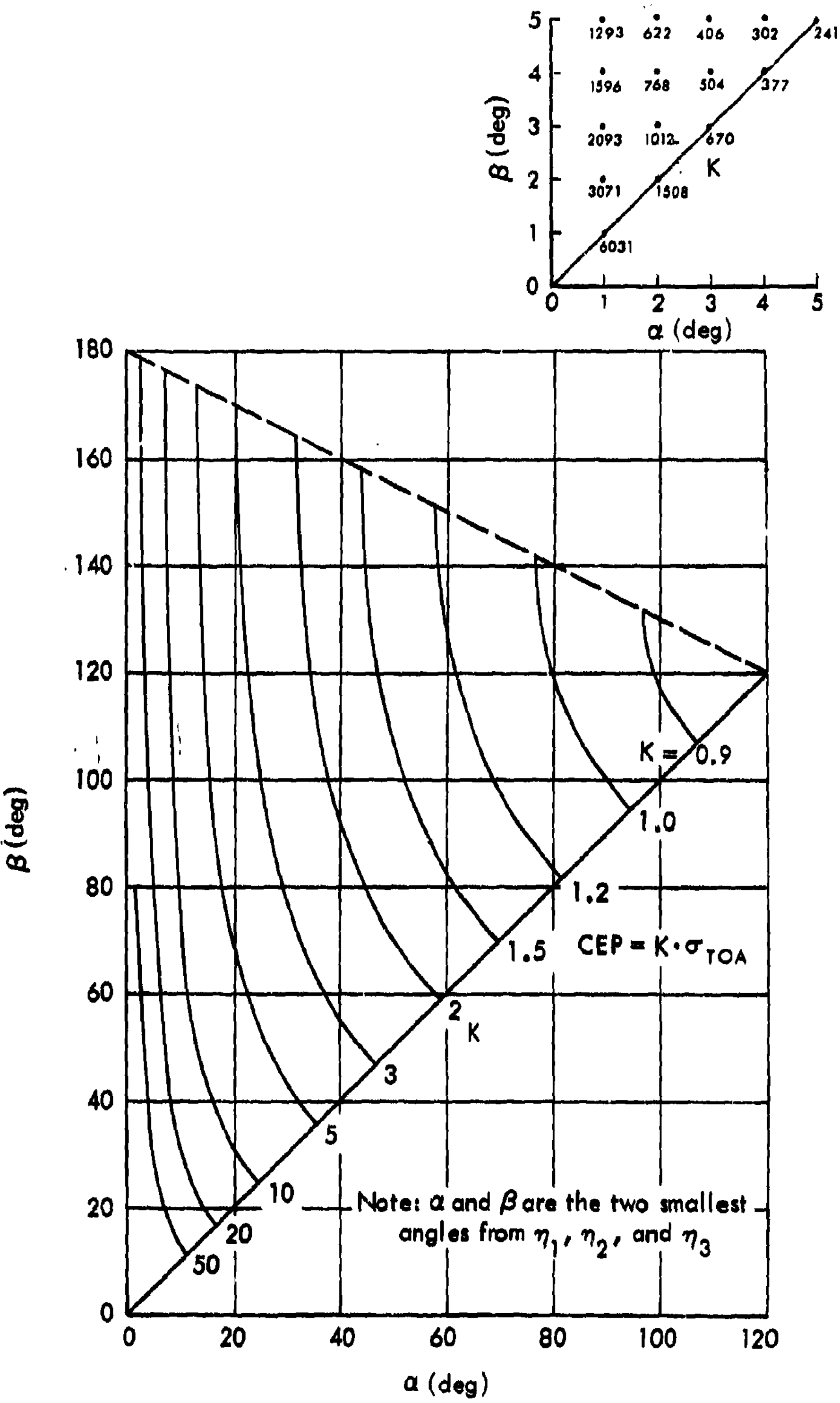


Fig. 4—CEP isocontours for arbitrary TOA aircraft triangles

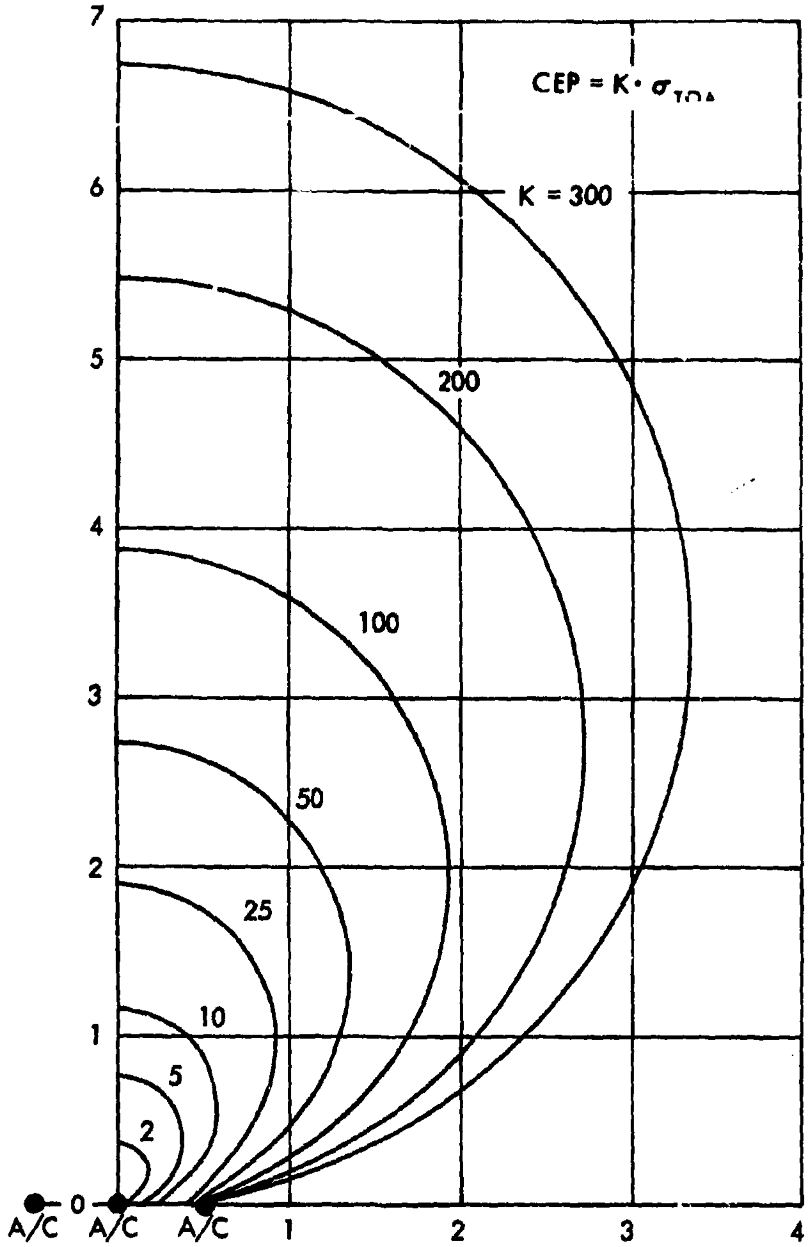


Fig.5—CEP isocontours for 180-deg isosceles (line) triangle (TOA)

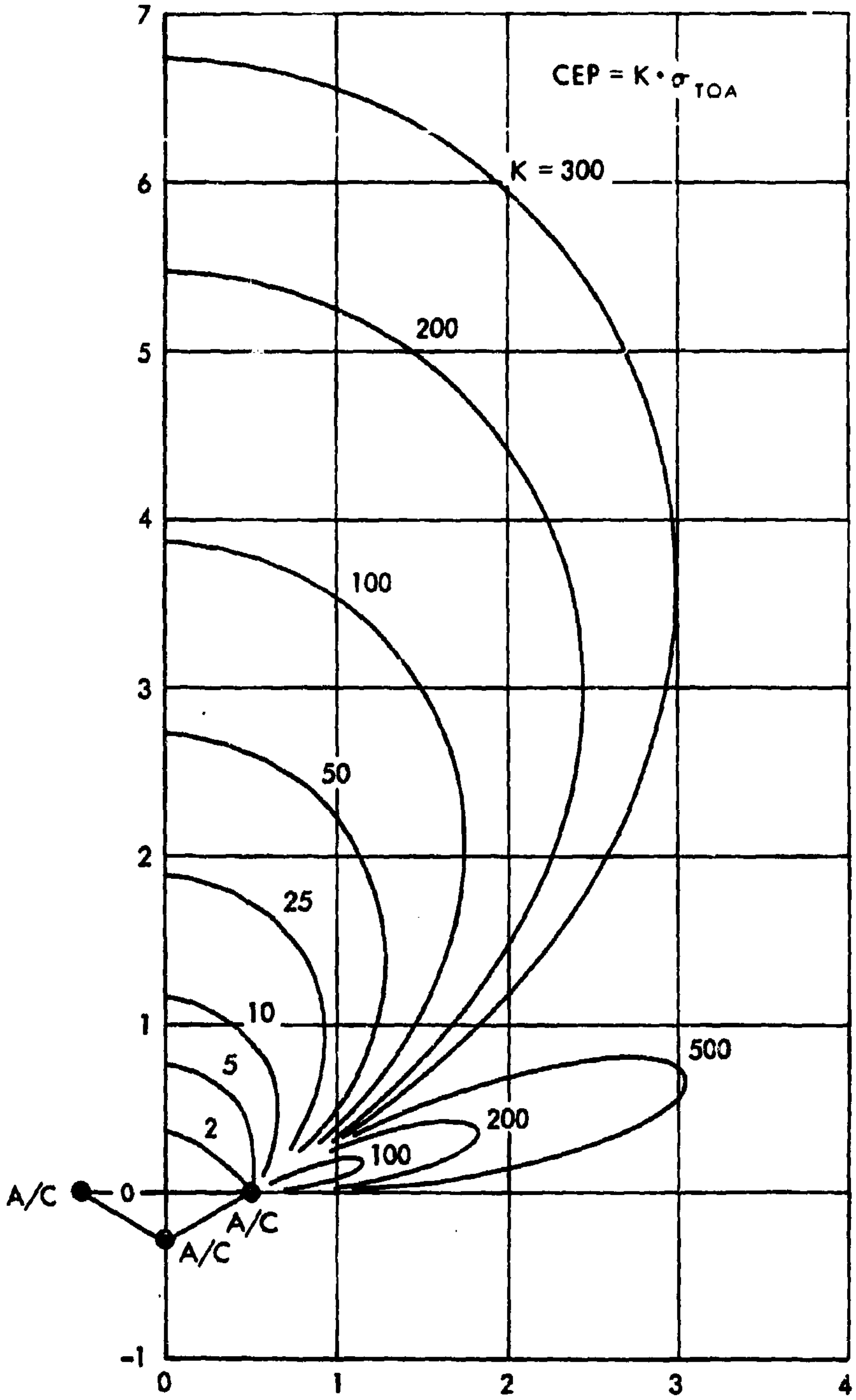


Fig.6—CEP isocontours for 120-deg isosceles triangle (TOA)

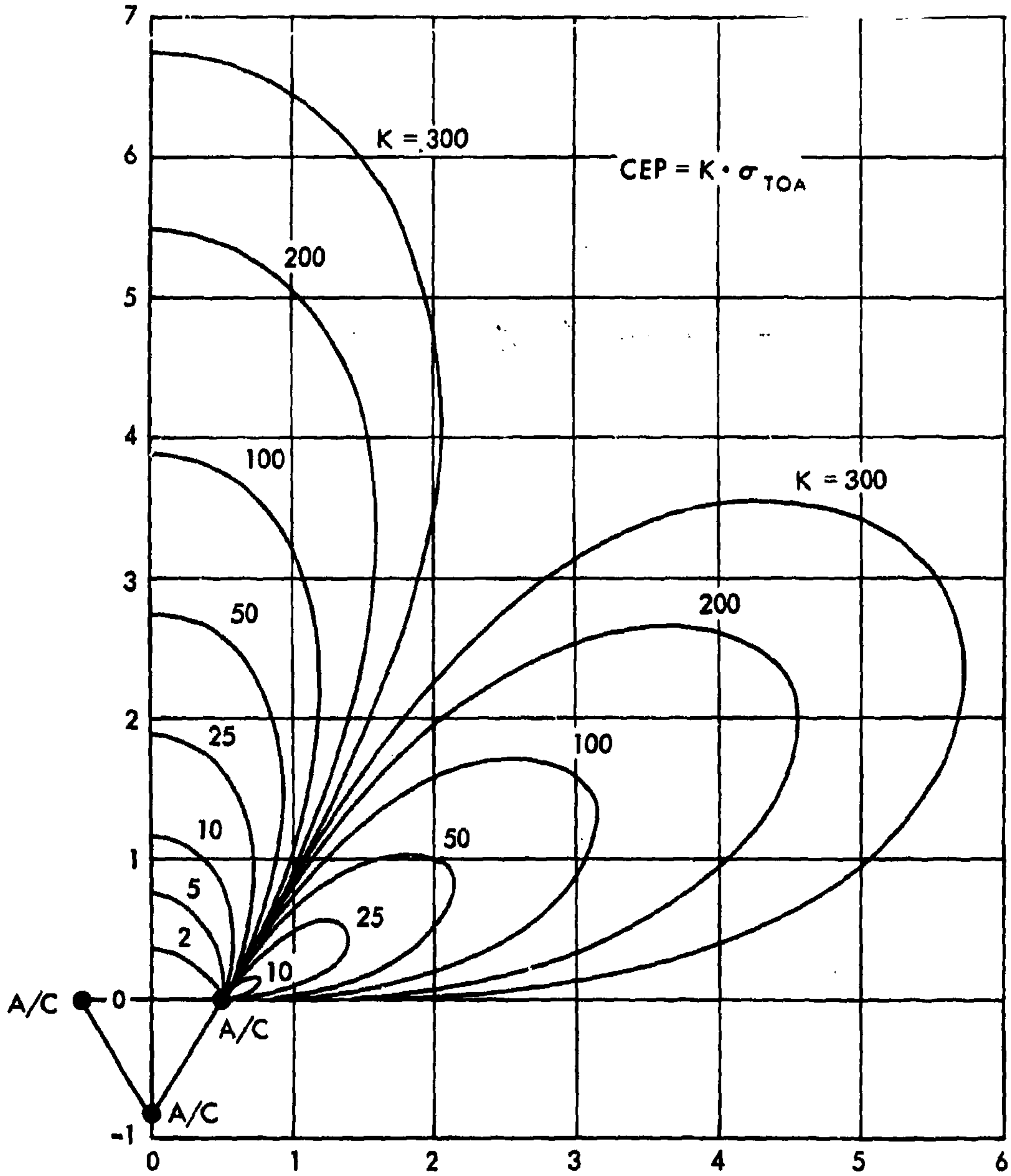


Fig.7—CEP isocontours for 60-deg isosceles (equilateral) triangle (TOA)

where R is the distance from the emitter to the unit base leg. Thus,

$$CEP \approx 0.75 \sqrt{96}\, R^2 \sigma_{TOA}$$

$$\approx 7\, R^2 \sigma_{TOA} \qquad R > 1$$

A similar reduction for $\sigma^2_{x_o}$ and $\sigma^2_{y_o}$ for locations along the perpendicular bisector yields

$$\sigma_{x_o} \approx 4\sqrt{6}\, R^2 \sigma_{TOA} \qquad R >> 1$$

$$\sigma_{y_o} \approx \sqrt{2}\, R\sigma_{TOA} \qquad R >> 1$$

and the correlation term, by symmetry, is zero. Considering the y-axis to be "downrange" and the x-axis to be "crossrange,"

$$\sigma_{DR}/\sigma_{CR} = 4\sqrt{3}\, R$$

$$\approx 7R \qquad R >> 1$$

so that the downrange errors are much larger than the crossrange. This is also true for locations off the perpendicular bisector.

AIRCRAFT LOCATION ERRORS ONLY--I

Suppose that the aircraft locations are estimated from independent navigation systems and that the TOA errors are negligible relative to the aircraft location errors. Suppose that the aircraft and emitter altitude estimate errors are also negligible. Assume that the aircraft and emitter are in the same plane (or that the altitude differences are inconsequential). Let the aircraft locations be (x_i, y_k) $i = 1, 2, 3$ and the emitter location be (x_o, y_o). Suppose also that the aircraft navigation system position errors are independent in each coordinate with common variance σ^2.

Since δ_{12} and δ_{13} are assumed known, from Eq. (93) with $d\delta_{12} = d\delta_{13} = 0$, we obtain

$$(\cos\gamma_2 - \cos\gamma_1)dx_o + (\sin\gamma_2 - \sin\gamma_1)dy_o$$
$$= \cos\gamma_2 dx_2 - \cos\gamma_1 dx_1 + \sin\gamma_2 dy_2 - \sin\gamma_1 dy_1 \quad (105a)$$

$$(\cos\gamma_3 - \cos\gamma_1)dx_o + (\sin\gamma_2 - \sin\gamma_1)dy_o$$
$$= \cos\gamma_3 dx_3 - \cos\gamma_1 dx_1 + \sin\gamma_3 dy_3 - \sin\gamma_1 dy_1 \quad (105b)$$

Taking the measurement parameter vector as $\mu^* = (x_1, y_1, x_2, y_2, x_3, y_3)$, from Eq. (105),

$$F_\mu = \begin{bmatrix} -\cos\gamma_1 & -\sin\gamma_1 & \cos\gamma_2 & \sin\gamma_2 & 0 & 0 \\ -\cos\gamma_1 & -\sin\gamma_1 & 0 & 0 & \cos\gamma_3 & \sin\gamma_3 \end{bmatrix}$$

so that, in Eq. (6),

$$F_\mu \Sigma F_\mu^* = F_\mu F_\mu^* \sigma^2$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \sigma^2 \quad (106)$$

Comparing Eq. (106) with Eq. (103), and the left-hand side of Eq. (105) with the right-hand side of Eqs. (96) and (97), and using Eq. (104), it follows that

$$\sigma_{x_o}^2 + \sigma_{y_o}^2 = \frac{3 - \cos\eta_1 - \cos\eta_2 - \cos\eta_3}{(\sin\eta_1 + \sin\eta_2 + \sin\eta_3)^2} 2\sigma^2 \quad (107)$$

Thus, Figs. 4 through 7 can be used for the CEP from only the aircraft navigation errors by replacing σ_{TOA} by σ.

Also, assuming that the aircraft navigation system errors are independent of the emitter-to aircraft range difference errors, from Eq. (104),

$$\sigma_{x_o}^2 + \sigma_{y_o}^2 = \frac{3 - \cos \eta_1 - \cos \eta_2 - \cos \eta_3}{(\sin \eta_1 + \sin \eta_2 + \sin \eta_3)^2} 2(\sigma^2 + \sigma_{TOA}^2) \qquad (108)$$

for the combined effects of the measurement errors. Thus, Figs. 4 through 7 may also be used for the combined errors by replacing σ_{TOA} by $(\sigma^2 + \sigma_{TOA}^2)^{\frac{1}{2}}$.

AIRCRAFT LOCATION ERRORS ONLY--II

As another example in which analytic results can be obtained, consider a TOA system in which the aircraft locations are estimated from DME measurements between aircraft and aircraft position measurements from independent navigation systems.

Suppose that the TOA measurement errors, the aircraft and emitter altitude measurements errors, and the DME errors are negligible relative to the aircraft position errors. Assume also that the aircraft and emitter are in approximately the same plane and denote the aircraft positions by (x_i, y_i) $i = 1, 2, 3$ and the emitter position by (x_o, y_o). Suppose also that the aircraft position errors are independent in each coordinate and from aircraft to aircraft, and let the errors have common variance σ^2.

For convenience, let the coordinate system have its origin at the centroid of the aircraft triangle so that $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$. Let ϕ_i be the angle between the x-axis and the line joining (x_i, y_i) and (x_o, y_o) and ϕ the angle between the x-axis and the line from the origin to (x_o, y_o). Then

$$x_i - x_o = r_{io} \cos (\phi + \Delta\phi_i) \qquad i = 1, 2, 3 \qquad (109a)$$

$$y_i - y_o = r_{io} \sin (\phi + \Delta\phi_i) \qquad i = 1, 2, 3 \qquad (109b)$$

where $\Delta\phi_i = \phi - \phi_i$ is known.

From Eq. (109),

$$dx_i = dx_o - r_{io} \sin (\phi - \Delta\phi_i)\, d\phi$$

$$= dx_o - (y_i - y_o)\, d\phi \qquad i = 1, 2, 3 \qquad (110a)$$

$$dy_i = dy_o + r_{io} \cos (\phi - \Delta\phi_i)\, d\phi$$

$$= dy_o + (x_i - x_o)\, d\phi \qquad i = 1, 2, 3 \qquad (110b)$$

Setting the measurement parameter vector $\mu^* = (x_1, x_2, x_3, y_1, y_2, y_3)$ and the estimation parameter vector $\beta^* = (x_o, y_o, \phi)$, from Eqs. (110) and (12),

$$\Sigma_\beta = (C^*C)^{-1}\, \sigma^2 \qquad (111)$$

where

$$C = \begin{bmatrix} 1 & 0 & y_o-y_1 \\ 1 & 0 & y_o-y_2 \\ 1 & 0 & y_o-y_3 \\ 0 & 1 & x_1-x_o \\ 0 & 1 & x_2-x_o \\ 0 & 1 & x_3-x_o \end{bmatrix}$$

Thus,

$$C^*C = \begin{bmatrix} 3 & 0 & a \\ 0 & 3 & b \\ a & b & c \end{bmatrix}$$

where

$$a = \sum_{i=1}^{3} (y_o - y_i) = 3y_o$$

$$b = \sum_{i=1}^{3} (x_i - x_o) = 3x_o$$

$$c = \sum_{i=1}^{3} [(x_i - x_o)^2 + (y_i - y_o)^2]$$

$$= \sum_{i=1}^{3} (x_i^2 + y_i^2) + 3(x_o^2 + y_o^2)$$

$$= 3(r^2 + R^2)$$

r^2 is the *average* squared range between the aircraft locations and the centroid, and R^2 is the squared range from the emitter to the centroid.

Now choose the x-axis to pass through the emitter location so that $y_o = 0$. The error in the x-direction may then be considered to be the range error and the error in the y direction the crossrange error. Then

$$C^*C = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & R \\ 0 & R & R^2+r^2 \end{bmatrix}$$

and

$$(C^*C)^{-1} = \frac{1}{3r^2}\begin{bmatrix} r^2 & 0 & 0 \\ 0 & R^2+r^2 & -R \\ 0 & -R & 1 \end{bmatrix} \qquad (112)$$

Substituting Eq. (112) in Eq. (111),

$$\sigma_{x_o}^2 = \sigma^2/3 \tag{113a}$$

$$\sigma_{y_o}^2 = (1 + R^2/r^2)\sigma^2/3 \tag{113b}$$

so that

$$\begin{aligned} CEP &\approx 0.75\ (\sigma_{x_o}^2 + \sigma_{y_o}^2)^{\frac{1}{2}} \\ &\approx 0.43\ (2 + R^2/r^2)^{\frac{1}{2}}\ \sigma \end{aligned} \tag{114}$$

When the aircraft triangle is equilateral with side length ℓ, the distance from the aircraft to the centroid is $\ell/\sqrt{3}$, so that $r^2 = \ell^2/3$ and Eq. (114) becomes

$$\begin{aligned} CEP &\approx 0.43\ (2 + 3(R/\ell)^2)^{\frac{1}{2}}\ \sigma \\ &\approx 0.75\ (R/\ell)\ \sigma \qquad\qquad R \gg \ell \end{aligned}$$

EMITTER LOCATION CEP CALCULATION FOR THE GENERAL CASE

In the preceding, we have derived analytic results for the emitter location estimate CEP for the TOA systems for cases in which certain of the measurement errors dominated the remainder. The following example illustrates the general procedure for determining the emitter location CEP which combines the contributions from all the measurement errors. Suppose that the aircraft locations are determined by combining aircraft altimeter measurements with DME range measurements between aircraft and between the aircraft and two ground stations (as in one mode of the Air Force's QRC-334 system). Suppose also that the emitter altitude is estimated separately by an independent measurement with known standard deviation. Denote the (known) ground station locations by

(x_4, y_4, z_4) and (x_5, y_5, z_5). The parameter vector to be estimated can then be taken as

$$\beta^* = (x_0, y_0, z_0, x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3)$$

and the measurement parameter vector as

$$\mu^* = (\delta_{12}, \delta_{13}, r_{14}, r_{15}, r_{24}, r_{25}, r_{34}, r_{35}, r_{12}, r_{13}, r_{23}, z_0, z_1, z_2, z_3)$$

From Eq. (6), the Cramér-Rao lower bound is

$$\Sigma_\beta = F_\beta^*(F_\mu \Sigma^{-1} F_\mu)^{-1} F_\beta)^{-1} \qquad (115)$$

where, from Eqs. (89) and (93), $F_\mu = I$, and F_β is the matrix on p. 58.

Suppose that the standard deviation of the DME range measurements is 10 ft, that of the TOA range differences is 20 ft with a correlation of 1/2 (i.e., assuming independent time-of-arrival measurement errors at the three aircraft), that of the aircraft altitudes is 40 ft, and that of the emitter altitude, 100 ft. Assuming that all measurements except TOA range differences are uncorrelated, the non-zero elements of the measurement covariance matrix Σ are $\Sigma_{11} = \Sigma_{22} = 400$, $\Sigma_{12} = \Sigma_{21} = 200$, $\Sigma_{33} = \ldots = \Sigma_{11,11} = 100$, $\Sigma_{12,12} = 10{,}000$, $\Sigma_{13,13} = \Sigma_{14,14} = \Sigma_{15,15} = 1600$.

The following figure illustrates the output of a computer program which calculates Σ_β from Eq. (115) and then the approximate CEP of the emitter location estimate from Eqs. (26), (27), and (24):

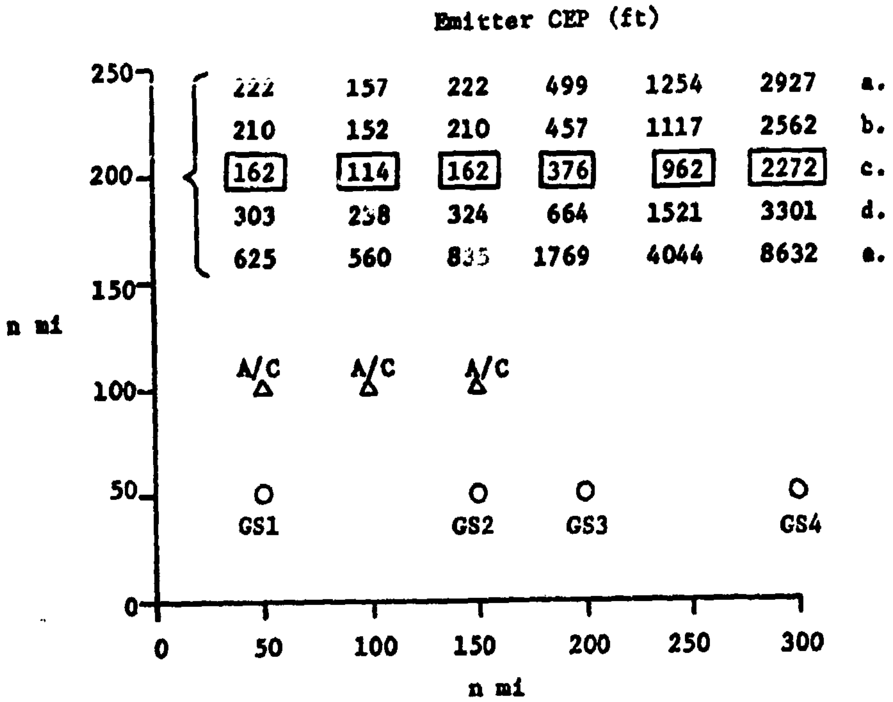
Numbers in a given column are approximate emitter location CEPs for emitters at the points indicated by the blocked number in the column when:

Ground stations GS1 and GS2 only are used and

a. Aircraft-to-aircraft distance measurements are omitted
b. All measurements are included
c. There are TOA measurement errors only--known aircraft locations

THE MATRIX $-F_\beta$ FOR THE GENERAL-CASE EXAMPLE

$$
\begin{bmatrix}
c_{2ox} - c_{1ox} & c_{2oy} - c_{1oy} & c_{2oz} - c_{1oz} & c_{1ox} & c_{1oy} & c_{1oz} & -c_{2ox} & -c_{2oy} & -c_{2oz} & 0 & 0 & 0 \\
c_{3ox} - c_{1ox} & c_{3oy} - c_{1oy} & c_{3oz} - c_{1oz} & c_{1ox} & c_{1oy} & c_{1oz} & 0 & 0 & 0 & -c_{3ox} & -c_{3oy} & -c_{3oz} \\
0 & 0 & 0 & c_{14x} & c_{14y} & c_{14z} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{15x} & c_{15y} & c_{15z} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{24x} & c_{24y} & c_{24z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{25x} & c_{25y} & c_{25z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{34x} & c_{34y} & c_{34z} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{35x} & c_{35y} & c_{35z} \\
0 & 0 & 0 & c_{12x} & c_{12y} & c_{12z} & -c_{12x} & -c_{12y} & -c_{12z} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{13x} & c_{13y} & c_{13z} & 0 & 0 & 0 & -c_{13x} & -c_{13y} & -c_{13z} \\
0 & 0 & 0 & 0 & 0 & 0 & c_{23x} & c_{23y} & c_{23z} & -c_{23x} & -c_{23y} & -c_{23z} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$



Ground stations GS3 and GS4 only are used and

c. There are TOA measurement errors only--known aircraft locations

d. All measurements are included

e. Aircraft-to-aircraft distance measurements are omitted

As indicated in the figure above, the CEP depends on the relative locations of the ground stations and the aircraft, so different results obtain for each configuration of ground stations and aircraft triangle considered.

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Appendix A

GENERALIZED LEAST SQUARES ESTIMATION

As described in Section II, the general estimation problem in location finding involves a set of measurements $m = (m_1, \cdots, m_n)$ on the observed parameter vector $\mu = (\mu_1, \cdots, \mu_n)^*$ and a set of functional relationships between μ and the (possibly) unobserved parameter vector $\beta = (\beta_1, \cdots, \beta_p)^*$,

$$F(\mu, \beta) = (f_1(\mu, \beta), \cdots, f_r(\mu, \beta))^* = 0 \tag{116}$$

$F(\mu, \beta)$ may include relationships on μ or β alone, i.e., certain of the $f_i(\mu, \beta)$ may be independent of μ or of β. The general estimation problem is to estimate μ and β subject to the constraints, Eq. (116).

Generalized least squares estimates of μ and β are the values $\hat{\mu}$ and $\hat{\beta}$ which minimize the quadratic form

$$Q = (m - \mu)^* \Sigma^{-1} (m - \mu) \tag{117}$$

subject to the constraints $F(\mu, \beta) = 0$. In general, when $F(\mu, \beta)$ is nonlinear, the determination of $\hat{\mu}$ and $\hat{\beta}$ require numerical methods. One widely used iterative procedure is an adaptation of Newton's method, sometimes called the method of linearization.

Let μ^o, β^o be initial guesses of $\hat{\mu}$ and $\hat{\beta}$. Approximating the i^{th} component of $F(\mu, \beta)$ by the first-order terms in the Taylor series expansion around μ^o, β^o,

$$\begin{aligned} f_i(\mu, \beta) \approx f_i(\mu^o, \beta^o) &+ \sum_{j=1}^{n} \left.\frac{\partial f_i}{\partial \mu_j}\right|_{\mu = \mu^o,\ \beta = \beta^o} (\mu_j - \mu_j^o) \\ &+ \sum_{k=1}^{p} \left.\frac{\partial f_i}{\partial \beta_k}\right|_{\mu = \mu^o,\ \beta = \beta^o} (\beta_j - \beta_j^o) \end{aligned} \tag{118}$$

Defining[†] F_μ as the r x n matrix

$$F_\mu = \left(\frac{\partial f_i}{\partial \mu_j}\bigg|_{\mu = \mu^o,\ \beta = \beta^o}\right)$$

and F_β as the r x p matrix

$$F_\beta = \left(\frac{\partial f_i}{\partial \beta_k}\bigg|_{\mu = \mu^o,\ \beta = \beta^o}\right)$$

Eq. (118) becomes, in matrix form,

$$F(\mu, \beta) \simeq F(\mu^o, \beta^o) + F_\mu(\mu - \mu^o) + F_\beta(\beta - \beta^o) \tag{119}$$

New approximations to $\hat{\mu}$ and $\hat{\beta}$ are obtained by first imposing the constraint equation $F(\mu, \beta) = 0$ on Eq. (119) to obtain

$$0 = F(\mu^o, \beta^o) + F_\mu(\mu - \mu^o) + F_\beta(\beta - \beta^o) \tag{120}$$

and then minimizing the quadratic form, Eq. (117), subject to the *linear* constraints of Eq. (120). The latter problem is a generalization of the usual linear least squares estimation problem. A quite general solution is given in Ref. 4. When F_μ and F_β have full rank, the new approximations to $\hat{\mu}$ and $\hat{\beta}$ are

$$\mu^1 = m - F_\mu^*(F_\mu \Sigma F_\mu^*)^{-1}[I - F_\beta(F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta)^{-1}F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}](F_\mu m - v_o) \tag{121a}$$

[†] A more standard notation for F_μ and F_β would be $F_\mu(\mu^o, \beta^o)$ and $F_\beta(\mu^o, \beta^o)$, indicating the values of the arguments at which the functions are evaluated. However, this is cumbersome. The values of μ and β at which F_μ and F_β are evaluated should be clear from the context.

$$\beta^1 = -(F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta)^{-1}F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}(F_\mu m - v_o) \tag{121b}$$

where $v_o = F_\mu \mu^o + F_\beta \beta^o - F(\mu^o, \beta^o)$.

Approximate covariance matrices and cross-covariance matrices of μ^1 and β^1 are

$$\Sigma_{\mu^1} = \Sigma - \Sigma F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}[I - F_\beta(F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta)^{-1}F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}]F_\mu \Sigma \tag{122a}$$

$$\Sigma_{\beta^1} = (F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta)^{-1} \tag{122b}$$

$$\Sigma_{\mu^1\beta^1} = \Sigma F_\mu^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta(F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta)^{-1} \tag{122c}$$

If the initial guesses μ^o and β^o are close to $\hat{\mu}$ and $\hat{\beta}$, minimizing the quadratic form of Eq. (117) subject to the linear constraints of Eq. (120) will be approximately equivalent to minimizing the quadratic form subject to $F(\mu, \beta) = 0$. The result is a new approximation μ^1, β^1 to $\hat{\mu}$, $\hat{\beta}$. The entire procedure may be iterated until convergence of the sequence (μ^i, β^i) is indicated. Conditions under which the sequence converges to $\hat{\mu}$, $\hat{\beta}$ are contained in Ref. 5, for example.

Comparing Eq. (122) with Eq. (125) of Appendix B, we see that the only difference is the value of (μ, β) at which F_μ and F_β are evaluated. When the covariance matrices of μ and β are small so that $\hat{\mu}$ and $\hat{\beta}$ are close to μ and β with high probability, the approximate covariance and cross-covariance matrices of Eq. (122) will be close to those of the Cramér-Rao lower bound for unbiased estimates of μ and β.

Appendix B

THE CRAMÉR-RAO INEQUALITY

Let $p(X, \theta)$ be the probability density function of a vector random variable X with an unknown parameter vector $\theta = (\theta_1, \ldots, \theta_k)^*$. Define

$$p_{ij} = -\frac{\partial^2 \ln p}{\partial\theta_i \, \partial\theta_j}$$

$$\phi_{ij} = E(p_{ij})$$

The matrix

$$\Phi = (\phi_{ij}) \qquad i, j = 1, \ldots, k$$

is called the information matrix for $p(X, \theta)$.

Theorem 1. *The Cramér-Rao inequality for unbiased estimators.*† *Let* $s_1(X), s_2(X), \ldots, s_t(X)$ *be* t *statistics such that*

$$\left.\begin{aligned} E[s_i(X)] &= g_i(\theta_1, \ldots, \theta_k) \\ E[(s_i - g_i)(s_j - g_j)] &= V_{ij} \end{aligned}\right\} \quad i, j = 1, \ldots, t$$

Suppose $\partial g_i/\partial\theta_j$ *exists and define the matrices*

$$V = (V_{ij}) \qquad i,j = 1, \ldots, t$$

$$\Delta = (\partial g_i/\partial\theta_j) \qquad i = 1, \ldots, t \qquad j = 1, \ldots, k$$

---

†Reference 2, p. 265.

If differentiation may be passed under the integration sign so that

$$\frac{\partial g_i}{\partial \theta_j} = \frac{\partial}{\partial \theta_j}\int g_i(X)p(X, \theta)dX$$

$$= \int g_i(X)\,\frac{\partial p(X, \theta)}{\partial \theta_j}\,dX$$

then

$$V - \Delta\Phi^{-}\Delta^{*} \geq 0 \tag{123}$$

i.e., the matrix $V - \Delta\Phi^{-}\Delta^{*}$ *is nonnegative definite.*†

In Section II, the logarithm of the multivariate normal density function for the measurement vector m is (aside from a term independent of μ) proportional to

$$-\tfrac{1}{2}\,(m - \mu)^{*}\Sigma^{-1}(m - \mu) \tag{124}$$

where the n x 1 measurement parameter vector μ is related to the p x 1 independent parameter vector β by the r x 1 vector of constraints

$$F(\mu, \beta) = 0$$

Let μ^{o}, β^{o} satisfy $F(\mu^{o}, \beta^{o}) = 0$ and define

$$F_{\mu} = \left.\frac{\partial F}{\partial \mu}\right|_{\mu = \mu^{o},\ \beta = \beta^{o}} \qquad (r \times n)$$

$$F_{\beta} = \left.\frac{\partial F}{\partial \beta}\right|_{\mu = \mu^{o},\ \beta = \beta^{o}} \qquad (r \times p)$$

†The matrix Φ^{-} is any generalized inverse of Φ, i.e., any matrix satisfying $\Phi\Phi^{-}\Phi = \Phi$. If Φ is positive definite, $\Phi^{-} = \Phi^{-1}$.

Assuming that F has continuous partial derivatives at (μ^o, β^o), F_μ has full rank r, F_β has full rank p, $n \geq r \geq p$, and that the requirements above for the Cramér-Rao inequality hold, we obtain the following result.

<u>Theorem 2</u>. *With the above assumptions, the dispersion matrix* $\Sigma_{\hat{\mu},\hat{\beta}}$ *of an unbiased estimate* $\hat{\mu}$, $\hat{\beta}$ *of* μ^o, β^o *satisfies the matrix inequality* ($A \geq B$ *means that* $A - B$ *is positive definite*),

$$\Sigma_{\hat{\mu},\hat{\beta}} = \begin{bmatrix} \Sigma_{\hat{\mu}} & \Sigma_{\hat{\mu}\hat{\beta}} \\ \Sigma_{\hat{\beta}\hat{\mu}} & \Sigma_{\hat{\beta}} \end{bmatrix} \geq \begin{bmatrix} \Sigma_{\mu} & \Sigma_{\mu\beta} \\ \Sigma_{\beta\mu} & \Sigma_{\beta} \end{bmatrix} = \Sigma_{\mu,\beta} \tag{125a}$$

where

$$\Sigma_\mu = \Sigma - \Sigma F_\mu^*(F_\mu \Sigma F_\mu^*)^{-1}[I - F_\beta(F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta)^{-1}F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}]F_\mu \Sigma \tag{125b}$$

$$\Sigma_\beta = (F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta)^{-1} \tag{125c}$$

$$\Sigma_{\mu\beta} = -\Sigma F_\mu^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta(F_\beta^*(F_\mu \Sigma F_\mu^*)^{-1}F_\beta)^{-1} \tag{125d}$$

$$\Sigma_{\beta\mu} = \Sigma_{\mu\beta}^* \tag{125e}$$

<u>Proof</u>.† First, suppose that Σ is the identity. The total differential of $F(\mu, \beta) = 0$ at μ^o, β^o is

$$F_\mu d\mu + F_\beta d\beta = 0 \tag{126}$$

Let the columns of the $n \times n - r$ matrix U and the columns of the $n \times r$ matrix W be orthonormal, and let the columns of U span the null

†The proof given here uses properties of the pseudo-inverse of a matrix. The pseudo-inverse of a matrix A is the unique matrix A^+ satisfying (a) $AA^+A = A$, (b) $A^+AA^+ = A^+$, (c) $(AA^+)^* = AA^+$, (d) $(A^+A)^* = A^+A$. The properties of the pseudo-inverse used here are contained in Ref. 4, for example.

space of F_μ and the columns of W span the space orthogonal to the null space of F_μ.

Define δ and γ by the invertible transformation

$$\mu = W\delta + U\gamma + \mu^o \tag{127}$$

and $H(\delta, \theta)$ by

$$H(\delta, \theta) = F(W\delta + U\gamma + \mu^o, \beta) \tag{128}$$

where $\theta = (\beta^*, -\gamma^*)^*$. Then $H(\delta, \theta)$ has continuous partial derivatives, $H(0, \theta_o) = 0$, where $\theta_o^* = (\beta^{o*}, 0)^*$, and $\left.\frac{\partial H^*}{\partial \delta}\right|^*_{\delta = 0,\ \theta = \theta_o} = F_\mu W$ is nonsingular. By the implicit function theorem,[†] $H(\delta, \theta) = 0$ has a unique solution $\delta = h(\theta)$, i.e., $H(h(\theta), \theta) = 0$, in a neighborhood of θ_o. Thus θ is a vector parameter indexing the normal probability density $n(m, \mu)$ in that neighborhood.

From Eq. (127),

$$d\mu = Wd\delta + Ud\gamma \tag{129}$$

Substituting Eq. (129) into Eq. (126), we obtain

$$F_\mu(Wd\delta + Ud\gamma) + F_\beta d\beta = 0$$

$$F_\mu Wd\delta + 0 + F_\beta d\beta = 0$$

so that, since $F_\mu W$ is nonsingular,

$$d\delta = -(F_\mu W)^{-1} F_\beta d\beta \tag{130}$$

[†]Reference 6, p. 138.

Substituting Eq. (130) in Eq. (129),

$$d\mu = -W(F_\mu W)^{-1}F_\beta d\beta + Ud\gamma$$

$$= -WW^+F_\mu^+F_\beta d\beta + Ud\gamma$$

$$= -F_\mu^+F_\beta d\beta + Ud\gamma \tag{131}$$

or, in matrix form,

$$d\mu = -(F_\mu^+F_\beta, U)d\theta \tag{132}$$

The information matrix ϕ is

$$\phi = \left(E\left[-\frac{\partial \ell n\ p}{\partial\theta_i\partial\theta_j}\right]\right)$$

$$= E\left(-\frac{\partial}{\partial\theta}\left[\frac{\partial \ell n\ p}{\partial\theta}\right]^*\right) \tag{133}$$

where $\frac{\partial}{\partial\theta} = \left(\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_n}\right)^*$. We have

$$-\frac{\partial \ell n\ p}{\partial\theta} = \frac{1}{2}\frac{\partial}{\partial\theta}(m-\mu)^*(m-\mu)$$

$$= \frac{1}{2}\frac{\partial\mu^*}{\partial\theta}\frac{\partial}{\partial\mu}(m-\mu)^*(m-\mu)$$

$$= (F_\mu^+F_\beta, U)^*(m-\mu) \tag{134}$$

from Eq. (132). Thus,

$$-\frac{\partial}{\partial\theta}\left[\frac{\partial\ \ln p}{\partial\theta}\right]^{*} = \frac{\partial\mu^{*}}{\partial\theta}\ \frac{\partial}{\partial\mu}(m-\mu)^{*}(F_{\mu}^{+}F_{\beta},\ U)$$

$$= (F_{\mu}^{+}F_{\beta},\ U)^{*}(F_{\mu}^{+}F_{\beta},\ U)$$

$$= \begin{bmatrix} (F_{\mu}^{+}F_{\beta})^{*} \\ U^{*} \end{bmatrix}(F_{\mu}^{+}F_{\beta},\ U)$$

$$= \begin{bmatrix} (F_{\mu}^{+}F_{\beta})^{*}F_{\mu}^{+}F_{\beta} & (F_{\mu}^{+}F_{\beta})^{*}U \\ U^{*}F_{\mu}^{+}F_{\beta} & U^{*}U \end{bmatrix}$$

$$= \begin{bmatrix} (F_{\mu}^{+}F_{\beta})^{*}F_{\mu}^{+}F_{\beta} & 0 \\ 0 & I \end{bmatrix} \tag{135}$$

since $U^{*}U = I$ and

$$U^{*}F_{\mu}^{+} = U^{*}F_{\mu}^{*}(F_{\mu}F_{\mu}^{*})^{+}$$

$$= [(F_{\mu}F_{\mu}^{*})^{+}F_{\mu}U]^{*}$$

$$= 0$$

Since the right-hand side of Eq. (135) is a constant, it is equal to its expected value and therefore to Φ. It is easy to verify that

$$\Phi^{+} = \begin{bmatrix} [(F_{\mu}^{+}F_{\beta})^{*}(F_{\mu}^{+}F_{\beta})]^{+} & 0 \\ 0 & I \end{bmatrix} \tag{136a}$$

To apply the Cramér-Rao inequality to estimates of μ and β, set

$$G(\theta) = \begin{bmatrix} \mu \\ \beta \end{bmatrix}$$

Then, in Theorem 1,

$$\Delta = \frac{\partial G^*}{\partial \theta}\bigg|^*_{\mu = \mu_o,\ \beta = \beta_o}$$

$$= \begin{bmatrix} -F_\mu^+ F_\beta & -U \\ I & 0 \end{bmatrix} \tag{136b}$$

from Eq. (132). From Eqs. (136a) and (136b),

$$\Delta\Phi^+\Delta^* = \begin{bmatrix} -F_\mu^+F_\beta & -U \\ I & 0 \end{bmatrix} \begin{bmatrix} \Sigma_\beta & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -(F_\mu^+F_\beta)^* & I \\ -U^* & 0 \end{bmatrix} \tag{137}$$

where

$$\Sigma_\beta = [(F_\mu^+F_\beta)^* F_\mu^+ F_\beta]^+$$

$$= (F_\beta^*(F_\mu F_\mu^*)^+ F_\beta)^+$$

Performing the indicated matrix multiplications in Eq. (137), we have

$$\Delta\Phi^+\Delta^* = \begin{bmatrix} F_\mu^+F_\beta\Sigma_\beta(F_\mu^+F_\beta)^* + UU^* & -F_\mu^+F_\beta\Sigma_\beta \\ -\Sigma_\beta(F_\mu^+F_\beta)^* & \Sigma_\beta \end{bmatrix} \tag{138}$$

where, from the definition of U, $UU^* = I - F_\mu^+F_\mu$ (see Ref. 4).

Substituting

$$F_\mu^+ = F_\mu^*(F_\mu F_\mu^*)^{-1} \tag{139}$$

in Eq. (138), we obtain the result stated in the theorem when Σ is the identity.

When Σ is an arbitrary positive definite matrix, there exists a nonsingular matrix A such that

$$\Sigma = AA^* \tag{140}$$

Equation (124) can then be written as

$$-\tfrac{1}{2}(A^{-1}m - A^{-1}\mu)^*(A^{-1}m - A^{-1}\mu) \tag{141}$$

Setting $\mu = A\eta$, Eq. (126) becomes $F_\mu A d\eta + F_\beta d\beta = 0$. The previous development for Σ as the identity can then be followed through Eq. (135a) with η replacing μ and $F_\mu A$ replacing F_μ. To apply the Cramér-Rao inequality to estimates of μ and β, set

$$G(\theta) = \begin{bmatrix}\mu \\ \beta\end{bmatrix} = \begin{bmatrix}A\eta \\ \beta\end{bmatrix}$$

Then,

$$\Delta = \begin{bmatrix} -A(F_\mu A)^+F_\beta & -AU \\ I & 0 \end{bmatrix}$$

so that

$$\Delta\Phi^+\Delta^* = \begin{bmatrix} A(F_\mu A)^+F_\beta\Sigma_\beta(A(F_\mu A)^+F_\beta)^* + AUU^*A^* & -A(F_\mu A)^+F_\beta\Sigma_\beta \\ -\Sigma_\beta(A(F_\mu A)^+F_\beta)^* & \Sigma_\beta \end{bmatrix} \tag{142}$$

where,

$$\Sigma_\beta = ((F_\mu A)^+ F_\beta)^* (F_\mu A)^+ F_\beta)^+$$

$$= (F_\beta^* (F_\mu A)^{+*} (F_\mu A)^+ F_\beta)^+$$

$$= (F_\beta^* (F_\mu AA^* F_\mu^*)^+ F_\beta)^+$$

$$= (F_\beta^* (F_\mu \Sigma F_\mu^*)^{-1} F_\beta)^{-1} \qquad (143)$$

$$AUU^* A^* = A(I - (F_\mu A)^+ (F_\mu A))A^*$$

$$= A(I - (F_\mu A)^* (F_\mu A (F_\mu A)^*)^+ F_\mu A)A^*$$

$$= A(I - A^* F_\mu^* (F_\mu \Sigma F_\mu^*)^+ F_\mu A)A^*$$

$$= \Sigma - \Sigma F^* (F_\mu \Sigma F_\mu^*)^{-1} F_\mu \Sigma \qquad (144)$$

and

$$A(F_\mu A)^+ = A(F_\mu A)^* (F_\mu A (F_\mu A)^*)^+$$

$$= \Sigma F_\mu^* (F_\mu \Sigma F_\mu^*)^{-1} \qquad (145)$$

Substituting Eqs. (143), (144), and (145) into Eq. (142) and applying the Cramér-Rao inequality gives the result stated in the Theorem.

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