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PROPAGATION OF A FOCUSED LASER BEAM IN A TURBULENT ATMOSPHERE

R. F. Lutomirski

A Report prepared for

ADVANCED RESEARCH PROJECTS AGENCY

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10. ABSTRACT

A method of calculating the performance of a laser system with beam truncated by focusing optics in a turbulent atmosphere. Previous analyses have been limited to vacuum calculations with untruncated beams, and even then have not conserved the irradiance. This approach separates the geometry of the problem (the complex aperture distribution) from the beam propagation. The turbulence parameters appear only in the mutual coherence function describing the reduction in lateral coherence with increasing distance. Based on proofs presented in RM-6055, this report shows that, while the vacuum focal point intensity for a given laser power output will increase with finer focusing, the effect of turbulence limits this increase. Turbulence can virtually eliminate the vacuum advantage of visible over infrared wavelengths in focusing the beam at practical ranges. To predict beam patterns for design purposes, we need direct mutual coherence function measurements, now notably lacking.

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PREFACE

As part of The Rand Corporation's continuing efforts for the Advanced Research Projects Agency in the study of laser propagation, a handbook is being prepared that presents data and methodology required for calculating the atmospheric degradation of laser systems in tactical missions. The theory developed in this report will be an essential input to the preparation of this laser propagation handbook.

The outputs of many tactical lasers are best approximated by a focused, truncated-gaussian disturbance across a circular aperture. A quantitative understanding of the manner in which an initially coherent beam of finite cross section propagates through a turbulent atmosphere is required for prediction of the performance of various devices employing lasers. This report calculates the mean intensity distribution for a focused gaussian disturbance in a finite aperture in both the near and far field and examines in detail the effects of truncation, wavelength, and aperture size.

These results should be of use to those interested in propagation theory and its applications to laser range finders, laser line scanners, communication systems, and various guidance and other systems employing an illuminating beam.

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SUMMARY

The first section of this report briefly outlines the development of RM-6055-ARPA, *Propagation of a Finite Optical Beam in an Inhomogeneous Medium*. A formula is derived for the mean intensity distribution from a finite beam in terms of the complex disturbance in the aperture and the mutual coherence function (MCF) for spherical wave in the medium. The formula is used to examine the effects of turbulence on the long-term average intensity produced by a focused, truncated gaussian aperture distribution. In particular, it is shown that (a) while the vacuum focal point intensity will increase as the degree of truncation decreases for a given laser output power, the effect of turbulence limits this increase, and (b) the turbulence can virtually eliminate the vacuum advantage of visible over infrared wavelengths in focusing the beam at practical ranges. Transverse beam patterns and the on-axis intensity are shown for CO_2 wavelength, and a criterion is established for the condition under which the turbulence prevents effective focusing.

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I. INTRODUCTION

The output of a laser, with a mirror system designed to keep diffraction losses small, can usually be approximated by a gaussian amplitude and uniform phase distribution[1] across a circular transmitting aperture.* Generally, in order to deliver maximum intensity to a specified receiver, the output beam will be focused onto the target, with the focusing optics truncating the gaussian. The complex field at the exit pupil of the transmitter for this configuration may be approximated by a truncated-gaussian amplitude distribution and quadratic phase distribution. In this report, we will calculate the mean intensity distribution in space, resulting from the above complex field in a transmitting aperture in the presence of a turbulent atmosphere.

Previous analyses for truncated distributions have been limited to vacuum calculations.[2,3] The atmospheric calculations have been treated[4,5,6] by introducing the fluctuation in refractive index as a small parameter in the wave equation, and expanding the fields in powers of this same parameter. This procedure results in the field being expressed as a hierarchy of Born-type integrals over the vacuum fields existing at each point in space. Because of the extreme complexity of these integrals, the atmospheric calculations have been limited to non-truncated (possibly focused) gaussians, for which a closed-form solution exists for the vacuum fields. Even in this special case, because of the approximations necessary to manipulate the expressions, it is not difficult to show that none of the expressions contained in Refs. 4, 5, and 6 conserve the long-term average irradiance over a plane normal to the direction of propagation, implying that even those solutions must be incorrect.

The present analysis is based on a proof developed in an earlier study,[7] which states that the Huygens-Fresnel principle** can be extended to a medium exhibiting a spatial variation in refractive index. From this principle, the field due to a disturbance specified over an

---

*With confocal mirrors, the lowest-order mode is exactly gaussian.

**See p. 370 of Ref. 8.

aperture can be computed by superimposing spherical wavelets radiating from all elements of the aperture. The mean intensity distribution is then found by first computing the intensity at a point from an arbitrary pair of elements in the aperture. Applying a reciprocity theorem (also proven in Ref. 7), and averaging, reveals that the above quantity is essentially the mutual coherence function (or MCF) for a spherical wave in the medium. The integration over the aperture is performed as a final step, resulting in a simple formula for the mean intensity pattern valid in both the Fresnel and Fraunhofer regions of the aperture. The development is outlined in Section II.

In contrast with previous work, the properties of the medium (e.g., the turbulence parameters) appear only in the MCF of a spherical wave. The present calculation separates the geometry of the problem (i.e., the complex aperture distribution) from the propagation problem, which is determined by the manner in which a spherical wave propagates in the medium. The MCF describes the reduction in lateral coherence between different elements of the aperture, effectively transforming it into a partially coherent radiator, with the degree of coherence decreasing as the distance from the aperture increases. In Section III, the general formula relating the MCF to the spectrum of index of refraction fluctuations for homogeneous isotropic turbulence is presented, and approximate formulas are given for the MCF at various propagation ranges for a modified Kolmogorov spectrum. A curve of normalized coherence length, ρ_o, defined as the transverse separation at which the coherence is reduced to e^{-1}, versus normalized range, is also presented.

In Section IV, the case of a focused, truncated-gaussian field distribution is examined in detail. The general features of the vacuum pattern, with particular emphasis on the effect of truncation, are discussed first. For a fixed laser output power, a curve is presented showing the vacuum variation in focal-point intensity as a function of the degree of truncation. The curve shows, for example, that when the intensity at the circumference of the transmitting aperture is reduced to e^{-3} from the intensity at the center, the intensity at the focus is reduced to $\approx$85 percent of that corresponding to a uniform distribution (this latter distribution maximizes the vacuum gain). The effect of

turbulence is shown to reduce this difference in all cases and, in fact, the distribution that maximizes the gain is no longer a uniform one. A criterion is also given for which no further increase in the on-axis intensity at a given range can be realized through focusing because of the turbulence. In a further example it is shown that for a 30-cm beam focused at 0.5 km, the effect of the turbulence is virtually to eliminate the vacuum advantage of visible over infrared wavelengths in focusing the beam.

In addition, beam patterns and the on-axis intensity are shown for CO_2 wavelength, for both a 30-cm and 1-m aperture, using turbulence parameters that are characteristic of paths of the order of a few meters above the ground.

The results are briefly summarized in Section V, where it is argued that the spherical-wave MCF is the basic quantity to be measured for computing the degradation of an intensity pattern.

II. THE HUYGENS FORMULATION

Reference 7 shows that for optical propagation in a weakly inhomogeneous, non-absorbing medium, the field at the observation point $\underline{P}$ from a complex aperture disturbance $U_A(\underline{r}_1)$ can be written as

$$U(\underline{P}) = \frac{-ik}{2\pi} \int G(\underline{P},\underline{r}_1) U_A(\underline{r}_1)\, d^2\underline{r}_1 \tag{1}$$

where $k = 2\pi/\lambda$ is the wave number, $\underline{r}_1$ is a coordinate in the plane of the aperture, and the integration is carried out over the aperture. The term $G(\underline{P},\underline{r}_1)$ is the field at point $\underline{P}$ due to the spherical wave propagating in the medium from a delta function source at $\underline{r}_1$. Equation (1) is valid for atmospheric optical propagation where the scattered field can be shown to vary slowly over a wavelength for all propagation distances of interest (e.g., $\approx 10^6$ km over horizontal paths),[7] and for sufficiently small scattering angles where the obliquity factor[8] can be taken as $-i/\lambda$.

The intensity at the point $\underline{P}$ is then given by

$$I(\underline{P}) = U(\underline{P})U^*(\underline{P}) = \left(\frac{k}{2\pi}\right)^2 \iint G(\underline{P},\underline{r}_1) G^*(\underline{P},\underline{r}_2) U_A(\underline{r}_1) U_A^*(\underline{r}_2)\, d^2\underline{r}_1\, d^2\underline{r}_2 \tag{2}$$

It is shown in Ref. 7 that in a refractive medium,* the complex field at $\underline{P}$ due to a point source at $\underline{r}_1$ is identically equal to the field at $\underline{r}_1$ due to a point source at $\underline{P}$--i.e., reciprocity exists in the form $G(\underline{P},\underline{r}_1) = G(\underline{r}_1,\underline{P})$. Hence, $G(\underline{P},\underline{r}_1)G^*(\underline{P},\underline{r}_2) = G(\underline{r}_1,\underline{P})G^*(\underline{r}_2,\underline{P})$ is the field at $\underline{r}_1$, multiplied by the complex conjugates of the field at $\underline{r}_2$, due to a spherical-wave source at $\underline{P}$. The ensemble average (denoted by angular brackets)

$$\langle G(\underline{r}_1,\underline{P})G^*(\underline{r}_2,\underline{P})\rangle \equiv \frac{e^{ik\left[|\underline{r}_1-\underline{P}|-|\underline{r}_2-\underline{P}|\right]}}{|\underline{r}_1 - \underline{P}|\ |\underline{r}_2 - \underline{P}|} M_S(\underline{r}_1,\underline{r}_2,\underline{P}) \tag{3}$$

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*A more general theorem is proven by Rayleigh (see p. 380 of Ref. 9).

is the cross correlation of the complex fields at the points $\underline{r}_1, \underline{r}_2$ due to a unit point source at $\underline{P}$. When the scattering angles are sufficiently small so that the aperture points $\underline{r}_1, \underline{r}_2$ can be considered as lying on the surface on a sphere centered at $\underline{P}$, then the function $M_S(\underline{r}_1, \underline{r}_2, \underline{P})$, defined by factoring out the vacuum fields in Eq. (3), is the mutual coherence function for a spherical wave.

If the vector $\underline{p}$ is defined as the normal from the z-axis of symmetry to the observation point $\underline{P}$, then in the small-angle approximation

$$\frac{e^{ik|\underline{r}_i - \underline{P}|}}{|\underline{r}_i - \underline{P}|} \sim \frac{e^{ik\left[z + \frac{(\underline{r}_i - \underline{p})^2}{2z}\right]}}{z} \tag{4}$$

Hence, we obtain from Eq. (2)

$$\langle I\rangle(\underline{p},z) = \left(\frac{k}{2\pi z}\right)^2 \iint \exp\left\{-\frac{ik}{2z}\left[2\underline{p}\cdot(\underline{r}_1 - \underline{r}_2) + r_1^2 - r_2^2\right]\right\}$$

$$\times M_S(\underline{r}_1, \underline{r}_2, \underline{P}) U_A(\underline{r}_1) U_A^*(\underline{r}_2)\, d^2\underline{r}_1\, d^2\underline{r}_2 \tag{5}$$

Changing variables in Eq. (5) to $\underline{\rho} = (\underline{r}_1 - \underline{r}_2)$, $\underline{r} = \frac{1}{2}(\underline{r}_1 + \underline{r}_2)$, and restricting the analysis to the case of homogeneous isotropic turbulence where $M_S(\underline{r}_1, \underline{r}_2, z) = M_S(\rho, z)$ (with $\rho = |\underline{\rho}|$), yield

$$\langle I\rangle(\underline{p},z) = \left(\frac{k}{2\pi z}\right)^2 \int d^2\underline{\rho}\, M_S(\rho,z) e^{-(ik/z)\underline{p}\cdot\underline{\rho}} \int U_A(\underline{r} + \tfrac{1}{2}\underline{\rho})$$

$$\times U_A^*(\underline{r} - \tfrac{1}{2}\underline{\rho}) e^{(ik/z)\underline{\rho}\cdot\underline{r}}\, d^2\underline{r} \tag{6}$$

Equation (6) gives the mean spatial intensity distribution in both the Fresnel and Fraunhofer regions from an arbitrary complex disturbance in a finite aperture in the presence of a homogeneous, isotropic, turbulent medium. It should be noted that integration of Eq. (6) over the $\underline{p}$ plane yields $\int \langle I\rangle\, d^2\underline{p} = \int |U_A(\underline{r})|^2\, d^2\underline{r}$. Hence, this development implies energy conservation for arbitrary aperture functions.

III. THE SPHERICAL-WAVE MCF

The long-term effect of the turbulence on the beam pattern enters through the spherical-wave MCF. Over a uniformly turbulent path, M_S is related to the spectrum of index of refraction fluctuations by the formula[10]

$$M_S(\rho,z) = \exp\left\{-\frac{2z}{z_c}\left[1 - \frac{\int_0^\infty dK K\Phi_n(K)\int_0^1 J_o(K\rho u)\,du}{\int_0^\infty \Phi_n(K)K\,dK}\right]\right\} \tag{7}$$

where

$$z_c = \left[2\pi^2 k^2 \int_0^\infty \Phi_n(K)K\,dK\right]^{-1} \tag{8}$$

is the propagation distance at which the mean spherical-wave field is reduced to e^{-1} of the vacuum field.

Lutomirski and Yura[10] have used a modified Kolmogorov spectrum for the turbulence

$$\Phi_n(K) = \frac{0.033C_n^2 e^{-(K\ell_o)^2}}{(K^2 + \mathcal{L}_o^{-2})^{11/6}} \tag{9}$$

where C_n is the index structure constant, and $\ell_o = 2\pi\mathcal{l}_o$ and $L_o = 2\pi\mathcal{L}_o$ are the inner and outer scales of turbulence, respectively.

In Fig. 1, a plot of z_c versus λ is shown for three values of C_n^2 roughly corresponding to weak (3×10^{-16} cm$^{-2/3}$), medium (3×10^{-15} cm$^{-2/3}$), and strong (3×10^{-14} cm$^{-2/3}$) turbulence. For horizontal propagation near the ground, we have used the nominal values $\mathcal{l}_o = 0.1$ cm and $\mathcal{L}_o = 100$ cm. In the usual atmospheric case, $\ell_o \ll L_o$, and the integral in Eq. (8) can be approximated to yield

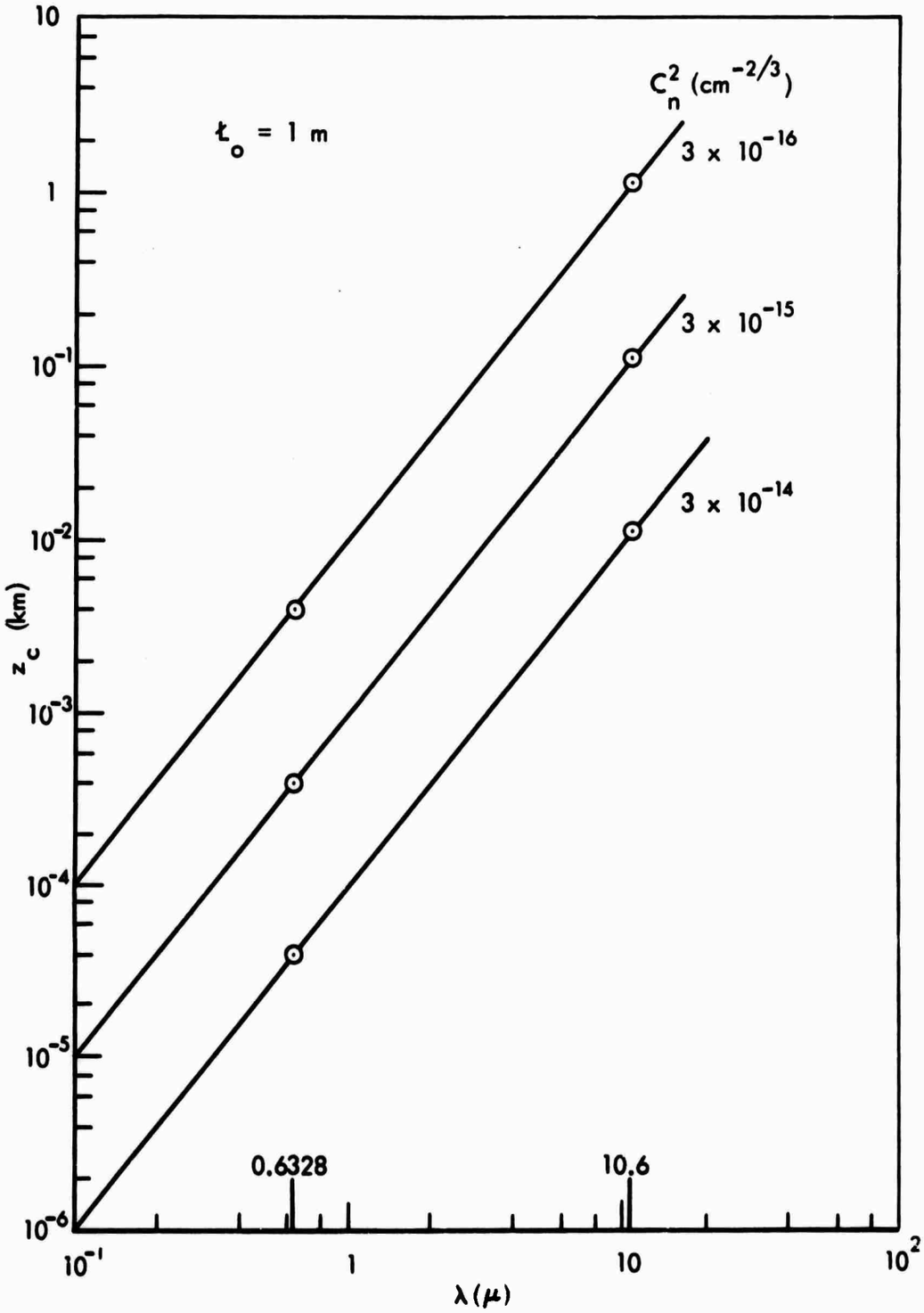

Fig. 1—Propagation distance z_c as a function of wavelength

$$z_c = (0.39k^2C_n^2\ell_o^{5/3})^{-1} \tag{10}$$

The mean field-decay length, z_c, and hence the MCF, will depend strongly on the outer scale of turbulence.

Reference 10 shows that there are three distinct propagation ranges for which approximate expressions exist for the MCF.

The first is found by observing that for any z, M_S is a monotonically decreasing function of ρ with $M_S(0,z) = 1$ and $M_S(\infty,z) = e^{-2z/z_c}$. Substituting Eq. (9) into Eq. (7), it can be seen that this asymptotic value is reached when $\rho \gg L_o$. Physically, when the separation is large compared with the distance over which the refractive index fluctuations (or temperature fluctuations) are correlated, $\sim L_o$, then

$$M_S(\rho,z) = \langle U(\underline{r}_1,z)U(\underline{r}_1 + \underline{\rho},z)\rangle \underset{\rho \gg L_o}{\rightarrow} \langle U(\underline{r}_1,z)\rangle\langle U(\underline{r}_1 + \underline{\rho},z)\rangle = e^{-2z/z_c}$$

because the light reaching the points $(\underline{r}_1,z)$, $(\underline{r}_1 + \underline{\rho},z)$ has propagated through essentially statistically independent media. Hence we define $M_S^{(1)} \equiv M_S(\underline{\rho},\ z \ll z_c) = 1$.

To describe the remaining two regions, it is convenient to first characterize the MCF at a given range by the separation ρ_o, for which $M_S(\rho_o,z) = e^{-1}$. Then the second region is found by observing that for $\ell_o \ll \rho_o \ll L_o$, the integrand in Eq. (7) can be expanded in powers of (ρ/ℓ_o) to yield[9]

$$\begin{aligned} M_S^{(2)}(\rho,z) &= \exp\left\{-0.55k^2C_n^2z\rho^{5/3}\left[1 - 0.71\left(\frac{\rho}{\ell_o}\right)^{1/3}\right]\right\} \\ &= \exp\left\{-1.4\left(\frac{z}{z_c}\right)\left(\frac{\rho}{\ell_o}\right)^{5/3}\left[1 - 0.71\left(\frac{\rho}{\ell_o}\right)^{1/3}\right]\right\} \end{aligned} \tag{11}$$

The expansion proves valid only when $M_S(\ell_o,z) \approx 1$ and $M_S(L_o,z) \ll 1$, which implies the range limitation

$$z_c << z << z_i$$

where

$$z_i \equiv (0.39k^2C_n^2\ell_o^{5/3})^{-1} = \left(\frac{L_o}{\ell_o}\right)^{5/3} z_c$$

For ranges greater than z_i, $M_S(\ell_o,z) << 1$, and all of the ρ's of interest are small compared with the inner scale. Then, for $\rho_o << \ell_o$, the Bessel function in Eq. (7) can be expanded in powers of (ρ/ℓ_o) to yield

$$M_S^{(3)}(\rho,z) = \exp\left[-k^2qz\rho^2\right]$$

$$= \exp\left[-0.80\left(\frac{z}{z_i}\right)\left(\frac{\rho}{\ell_o}\right)^2\right] \qquad z >> z_i \tag{12}$$

where

$$q = (\pi^2/3)\int_0^\infty \Phi_n(K)K^3\,dK \approx 0.56C_n^2\ell_o^{-1/3}$$

Hence, for propagation paths that are short compared with z_i, the MCF does not depend on the inner scale, and can be written as

$$M_S(\rho,z) = \exp\left\{-\frac{2z}{z_c}\left[1 - \frac{5}{3}\left(\frac{\rho}{\ell_o}\right)^{5/3}\int_0^\infty du\,uJ_o(u)\int_0^1 \frac{s^{5/3}\,ds}{\left[u^2 + \left(\frac{s\rho}{\ell_o}\right)^2\right]^{11/6}}\right]\right\}$$

$$\equiv F\left(\frac{\rho}{\ell_o},\frac{z}{z_c}\right) \qquad 0 < z << z_i \tag{13}$$

For the range $0 < z << z_i$ (for $L_o/\ell_o = 10^3$, $z_i = 10^5 z_c$), ρ_o/ℓ_o versus z/z_c, obtained by inverting the equation $F(\rho_o/\ell_o, z/z_c) = e^{-1}$, is plotted in Fig. 2. For ranges $z < \frac{1}{2}z_c$, $M_S(\infty,z) > e^{-1}$, and the coherence length as defined here is infinite.

It is apparent that for ρ_o much greater than the aperture diameter D, M_S in Eq. (6) can be replaced by unity, and the mean intensity is given by its vacuum value. When $\rho_o \lesssim D$, the turbulence reduces the average irradiance on a target.

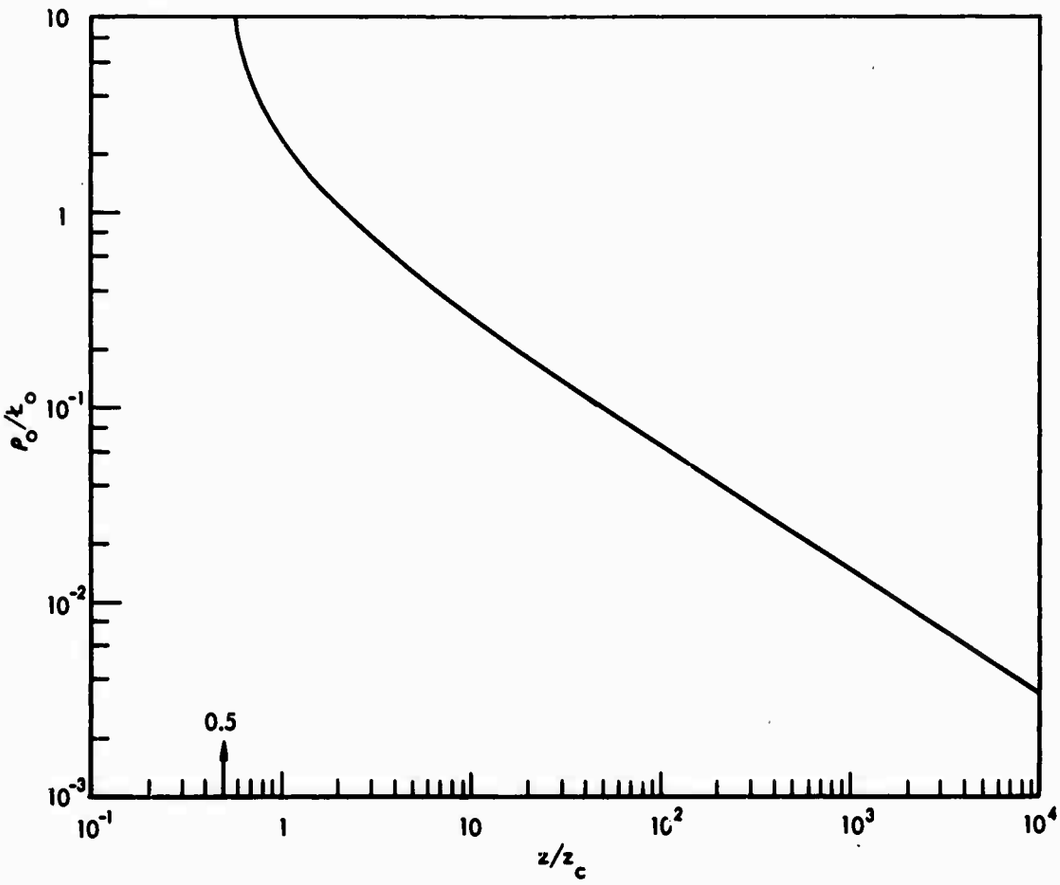

Fig. 2—The normalized coherence length, ρ_o/ℓ_o, as a function of z/z_c for a spherical wave

IV. THE FOCUSED, TRUNCATED-GAUSSIAN DISTURBANCE

THE MEAN INTENSITY

The complex field in the aperture corresponding to a gaussian amplitude distribution with standard deviation a, focused at range f, and normalized to unit amplitude at the center of the aperture is

$$U_A(|\underline{r}_i|) = e^{-\frac{1}{2}r_i^2(a^{-2}+ikf^{-1})} \qquad |\underline{r}_i| \le D/2$$
$$= 0 \qquad |\underline{r}_i| > D/2 \tag{14}$$

With U_A given by Eq. (14), the inner integral in Eq. (6) is the integration of the function

$$\exp\left[-a^{-2}(r^2 + \tfrac{1}{4}\rho^2) + ik(z^{-1} - f^{-1})\underline{\rho} \cdot \underline{r}\right]$$

over the area of overlap of two circles, each of diameter D, with centers relatively displaced at opposite ends of the vector $\underline{\rho}$. The integration is straightforward and yields for the inner integral $D^2\Gamma_{\delta,\beta}(x)$, where

$$\Gamma_{\delta,\beta}(x) = e^{-2\delta^2x^2}\int_0^{\cos^{-1}(x)} d\varphi \int_{\frac{x}{\cos\varphi}}^{1} e^{-\delta^2(\mu^2-2\mu x\cos\varphi)}$$
$$\times \cos\left[2\beta x(u\cos\varphi - x)\right]u\,du \qquad x \le 1$$
$$= 0 \qquad x > 1 \tag{15}$$

where $x = \rho/D$, $\delta = D/2a$, and $\beta = \frac{1}{4}kD^2(z^{-1} - f^{-1})$. Then, using polar coordinates for the $\underline{\rho}$ integration, performing the angular integral, and changing variables from ρ to $x = \rho/D$, yield

$$\langle I\rangle(\alpha,z) = \frac{8}{\pi}\left(\frac{kD^2}{4z}\right)^2 \int_0^1 xJ_o(2\alpha x)M_S(Dx,z)\Gamma_{\delta,\beta}(x)\,dx \qquad (16)$$

where J_o is the zero-order Bessel function, and $\alpha = kDp/2z = \frac{1}{2}kD \tan\theta$, where θ is the angle that the direction to $\underline{P}$ makes with the central direction.

THE VACUUM PATTERN AND EFFECT OF TRUNCATION

In the limit of zero turbulence, $M_S = 1$, and Eq. (16) reproduces the vacuum intensity pattern. In this limit, it follows from the definition of β that the angular pattern in the focal plane ($z = f$) is identical with the Fraunhofer pattern for the unfocused case ($f = \infty$, $z = \infty$). Further, because $\Gamma_{\delta,\beta}(x)$ is an even function of β, the angular vacuum patterns are identical in planes for which $|z^{-1} - f^{-1}| =$ constant.

For the limit $D \ll a$ ($\delta \to 0$), the problem reduces to the focusing of a plane wave diffracted by a circular aperture. No simple expression exists for the transverse vacuum intensity except in the focal plane, where the distribution is given by the Airy pattern

$$I_p(p,f) = \frac{1}{4}\left(\frac{kD^2}{4f}\right)^2\left[\frac{2J_1(\alpha)}{\alpha}\right]^2 \qquad \alpha = \frac{kDp}{2f} \qquad (17)$$

The on-axis intensity is given by

$$I_p(0,z) = \frac{1}{4}\left(\frac{kD^2}{4z}\right)^2\left(\frac{\sin\frac{\beta}{4}}{\frac{\beta}{4}}\right)^2 \qquad (18)$$

which has an absolute maximum at the focal point. The on-axis nulls are located at

$$z = \frac{f}{1 \pm \frac{16n\pi f}{kD^2}} \qquad n = 1,2,\ldots$$

When $16\pi f/kD^2 > 1$, the only on-axis nulls lie between the transmitter and the focal plane; when $16\pi f/kD^2 < 1$, there are on-axis nulls beyond the focal plane. In the limit $16\pi f/kD^2 \ll 1$, the nulls--indeed the whole pattern--are symmetric about the focal plane for ranges $|(z - f)/f| \ll 1$.

In the limit $D \gg a$ (or $\delta \to \infty$), when the aperture does not effectively truncate the gaussian, the vacuum pattern propagates as a gaussian in all transverse planes according to the equation

$$I_g(p,z) = \frac{1}{w^2} e^{-p^2/a^2w^2} \tag{19}$$

where

$$w^2 = \left(1 - \frac{z}{f}\right)^2 + \left(\frac{z}{ka^2}\right)^2$$

The maximum intensity in this case does not occur at the focal point, but rather on the axis at the shorter range $z = f/[1 + (f/ka^2)]$, where the intensity is $1 + (ka^2/f)^2$. Generally, in order to focus the laser beam effectively, the focal length should be much less than the smaller of ka^2, kD^2, in which case the intensity at the focal point is approximately the smaller of $(ka^2/f)^2$, $\frac{1}{4}(kD^2/f)^2$.

For the case of arbitrary truncation, the on-axis intensity can be shown to be

$$I(0,z) = \left(\frac{kD^2}{4z}\right)^2 \frac{\left(1 - e^{-\delta^2/2}\right)^2 + 4e^{-\delta^2/2}\sin^2\frac{\beta}{4}}{\delta^4 + \beta^2} \tag{20}$$

with no simple formula for the transverse pattern, even in the focal plane. The average aperture intensity, defined as the total laser output power, W, divided by the area of the aperture, can be computed by integrating $|U_A|^2$ over the aperture, and is

$$\bar{I} \equiv \frac{4W}{\pi D^2} = \frac{1 - e^{-\delta^2}}{\delta^2} \tag{21}$$

The effect of truncation on the intensity at the focal point for a given output power can then be computed from the gain

$$G = \frac{I(0,f)}{\bar{I}} = \left(\frac{kD^2}{4f}\right)^2 \frac{\left(1 - e^{-\delta^2/2}\right)^2}{\delta^2\left(1 - e^{-\delta^2}\right)} \tag{22}$$

It has been shown[11] that of all possible aperture distributions, the focal-point intensity for a given W is maximized by a uniform disturbance, corresponding in our case to $\delta = 0$, and yielding

$$G_u = \frac{1}{4}\left(\frac{kD^2}{4f}\right)^2$$

In Fig. 3 the normalized gain

$$G_N = \frac{G}{G_u} = \frac{4\left(1 - e^{-\delta^2/2}\right)^2}{\delta^2\left(1 - e^{-\delta^2}\right)} \tag{23}$$

is plotted versus δ^2, the power of e by which the intensity at the circumference is reduced from the intensity at the center. For example, a ratio of e^{-3} yields a vacuum reduction of ≈15 percent. The effect of turbulence will be shown to reduce this difference.

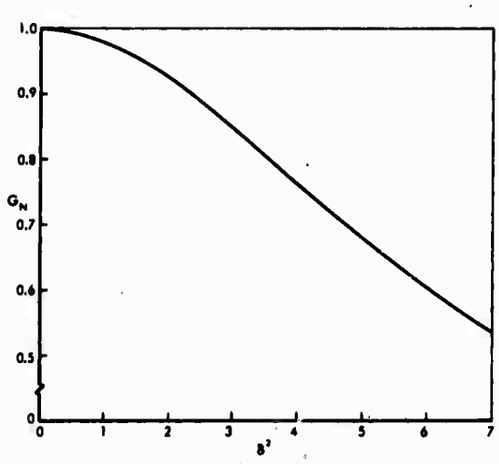

Fig. 3—The normalized gain as a function of δ^2

THE EFFECTS OF TURBULENCE

The long-term effect of the turbulence, as discussed in Section III, is to reduce the lateral coherence between the fields radiating

from different elements of the aperture for separations $\gtrsim \rho_o$. For $\rho_o < a,D$, the beam will be confined to a circularly symmetric area of dimension $\sim z/k\rho_o$ at a distance z. The ratio of the field intensity to aperture intensity is then $\sim(kD\rho_o/z)^2$, and the on-axis intensity can be increased by focusing the beam only if this quantitiy is considerably greater than unity at the range of interest. In general, this partially coherent aperture will produce an intensity pattern with less detailed structure than the vacuum distribution, and with the maxima and minima tending to be averaged out. Further, because $z_c \propto \lambda^2$, there will be less degradation of longer wavelength patterns. In Figs. 4a-4c, the focal-plane atmospheric patterns, normalized to the vacuum focal-point intensity $(1/4)(kD^2/4f)^2$, are shown for a uniform aperture disturbance with $D/\mathcal{L}_o = 0.01$, 0.1, and 1, respectively. With z_c determined from Eq. (10) (or Fig. 1), these curves can be used for arbitrary wavelengths. The dashed curve in each figure is the vacuum Airy pattern, Eq. (17), similarly normalized. For a given ratio of z/z_c, the greater the diameter $(D/\mathcal{L}_o)$, the less the effective coherence over the aperture, and hence, the greater the degradation from the vacuum pattern.

In the remaining examples, the normalized intensities $\langle I\rangle_N$, defined as the ratio of the mean intensities $\langle I\rangle$ to the mean aperture intensity $\bar{I} = 4W/\pi D^2$, are plotted. Turbulence parameters have been chosen that are characteristic of the strengths and scales found from zero to several meters above the ground: $C_n^2 = 3 \times 10^{-5}\ \mathrm{cm}^{-2/3}$, $\ell_o = 0.1$ cm, and $\mathcal{L}_o = 100$ cm.

Figures 5a and 5b display the effects of truncation on the focal-plane intensity of a 10.6-μ beam from a 1-m aperture focused at 0.5 km. The vacuum and atmospheric patterns are shown in Fig. 5a for a uniform distribution, whereas in Fig. 5b the gaussian intensity distribution is reduced to e^{-4} at the circumference from its value at the center. In vacuum the truncated distribution has a focal-point intensity of ≈25 percent less than the uniform case; however, the effect of turbulence is to reduce both intensities such that the difference is only ≈10 percent. Because the MCF reduces the effective average coherence between different radiating elements of the aperture in a manner that depends only on the distance between them, elements near the edge of the

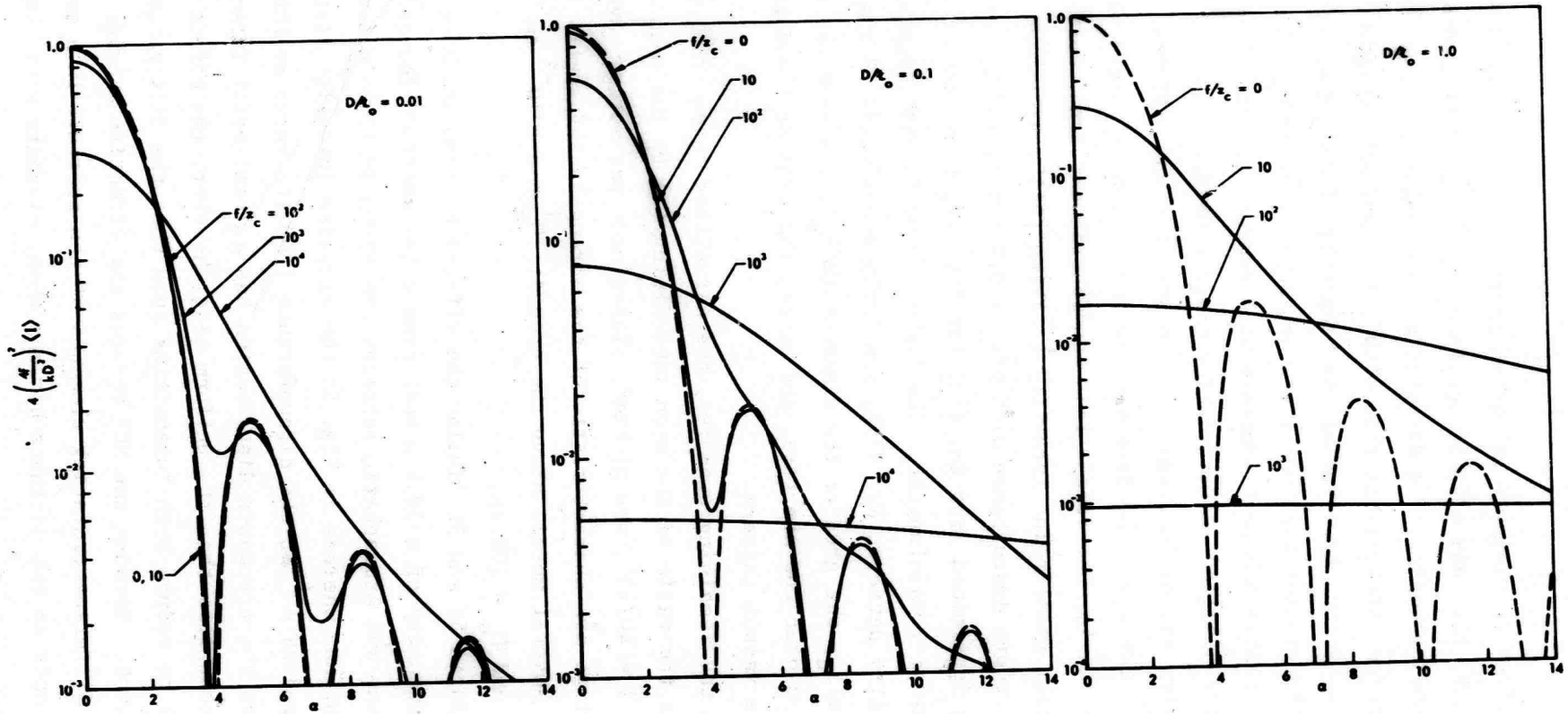

Fig. 4—Normalized focal plane intensity patterns for various values of D/ℓ_o

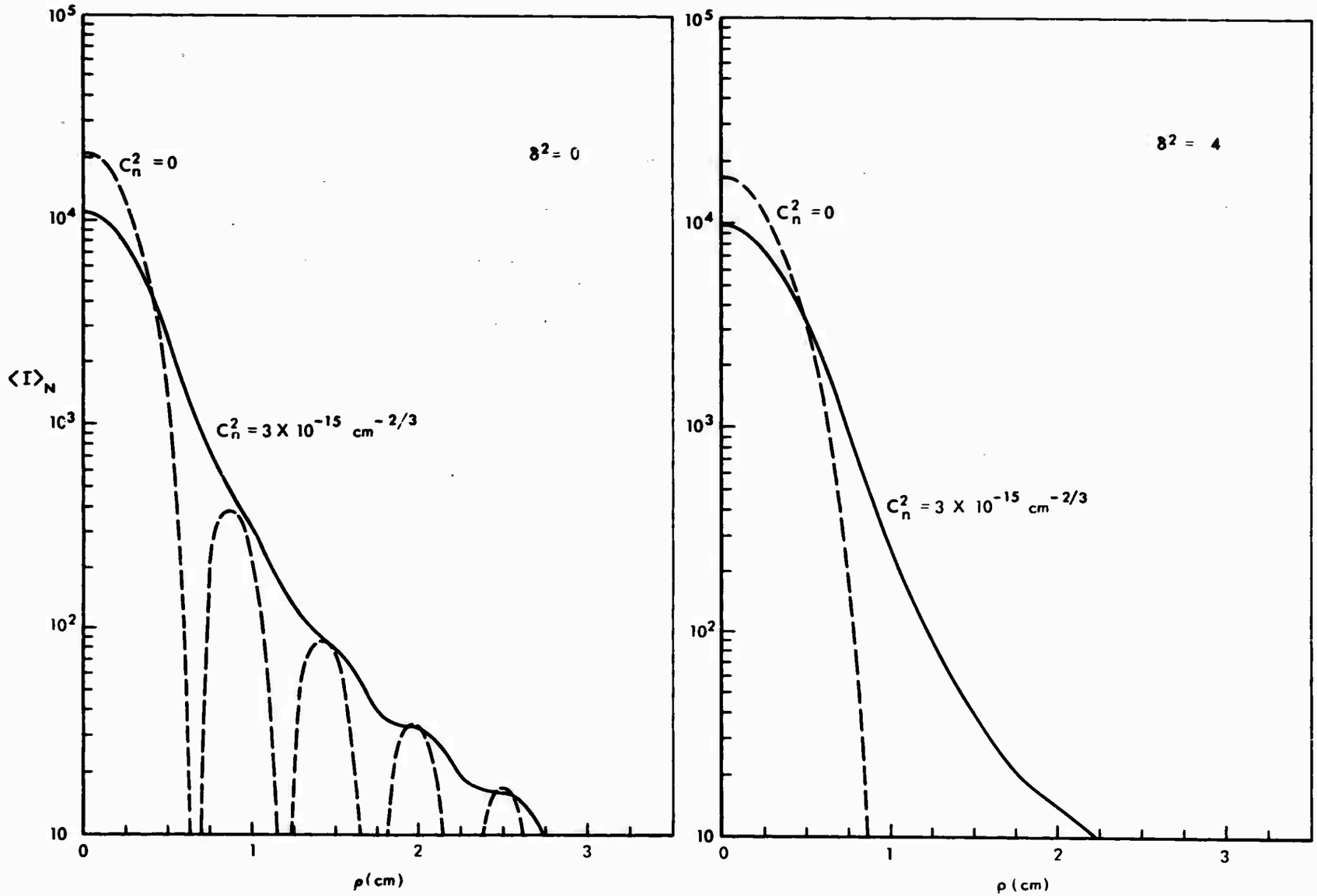

Fig. 5—Comparison of focal plane patterns at $\lambda = 10.6\ \mu$ for D = 1 m focused at 0.5 km, for (a) uniform and (b) truncated gaussian aperture distribution

aperture weigh less than those near the center in contributing to the average intensity. Hence, the relative intensity gain of a uniform distribution over a truncated-gaussian distribution will always be less than in the vacuum case. In fact, it can be shown from Eqs. (6) and (21) that in the presence of turbulence, the aperture distribution which maximizes the gain, $G = \langle I \rangle (0,f)/\overline{I}$, is, in general, not uniform, but somewhat peaked at the center of the aperture.

In Figs. 6a-6c, the focal-plane vacuum and atmospheric patterns are compared for $D = 30$ cm, $\delta = 0$, and $z = f = 0.5$ km, using the wavelengths 0.6328 μ (He-Ne), 1.06 μ (Nd), and 10.6 μ (CO_2). Although the vacuum intensities at the focal point for the shorter wavelengths are greater (the "antenna gain" is proportional to λ^{-2}), the effect of the atmosphere for the chosen geometry essentially eliminates the shorter wavelength advantage in focusing the beam. For larger apertures, the advantage of CO_2 in producing an intense focal intensity will be greater.

In Figs. 7a-7c, for $\lambda = 10.6$ μ, the vacuum and atmospheric focal-plane patterns are shown for $D = 1$ m and $\delta = 0$ (uniform distribution) at focal lengths of 0.5 km, 5 km, and 10 km. The long-term effect of the turbulence at the larger ranges is to reduce the on-axis intensity, and to spread out the beam. Hence, the positioning accuracy required to illuminate a target can be considerably less than a vacuum calculation would indicate. As discussed above, the solid curves (with turbulence present) are relatively insensitive to the degree of truncation.

Finally, in Figs. 8a and 8b, the vacuum and atmospheric on-axis intensities are compared for a 10.6-μ beam focused at 0.5 km with $\delta = 0$, and $D = 30$ cm and 1 m, respectively. The intensities are shown for a range of 50 m on either side of the focal plane. The on-axis intensity for the 30-cm-aperture case remains reasonably constant over the 100-m range, with a maximum at ≈30 m on the aperture side of the focal plane. The axial distribution for the 1-m aperture is almost symmetric with respect to the focal plane and obviously requires greater focusing accuracy to produce the available gain of $\approx 10^4$ at a specified target. At ranges of 5 and 10 km, the on-axis intensities are essentially constant within 50 m of the focal plane and are equal to the focal-point values of Figs. 7b and 7c, respectively.

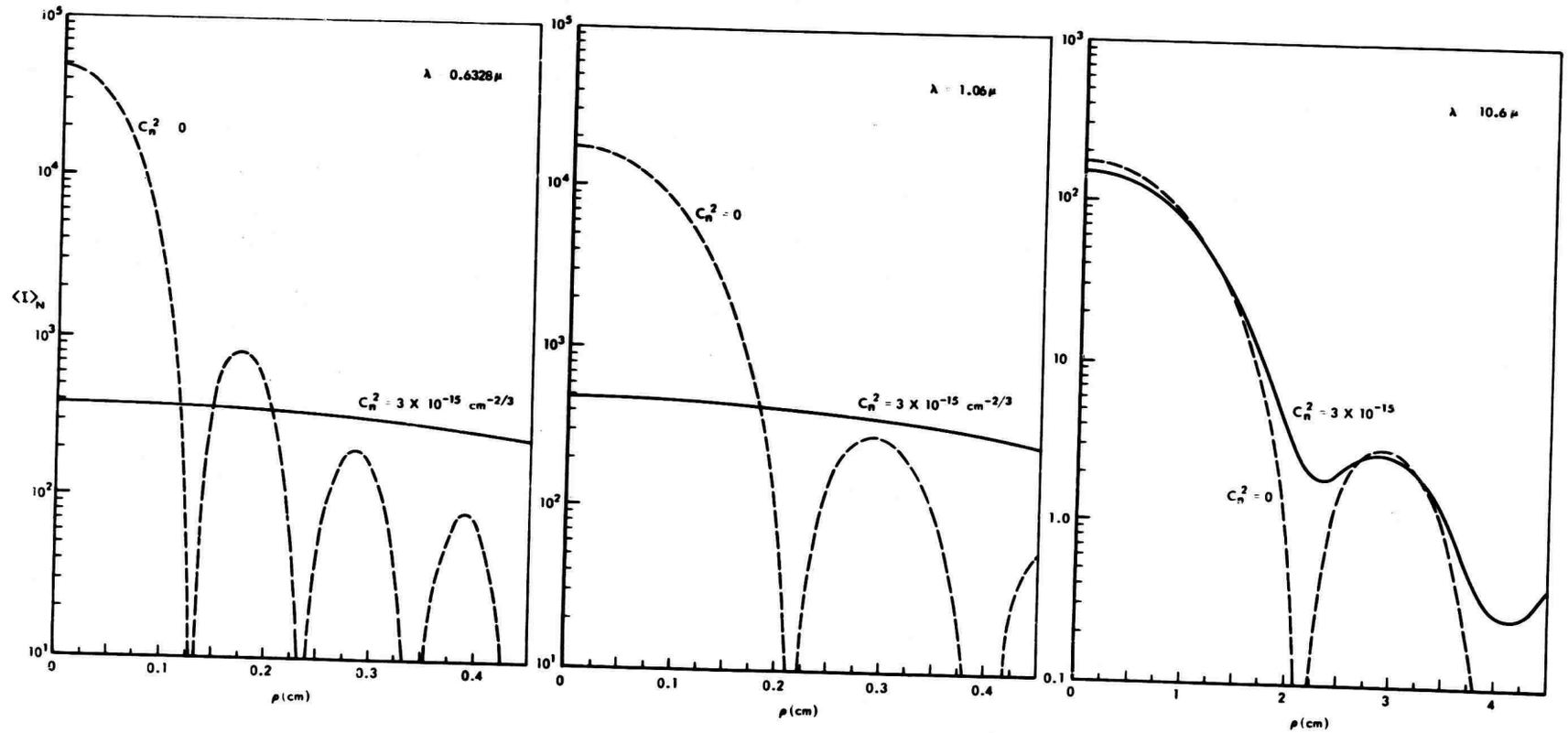

Fig. 6—Comparison of focal plane patterns, for several wavelengths, with a uniform distribution in a 30 cm aperture focused at 0.5 km

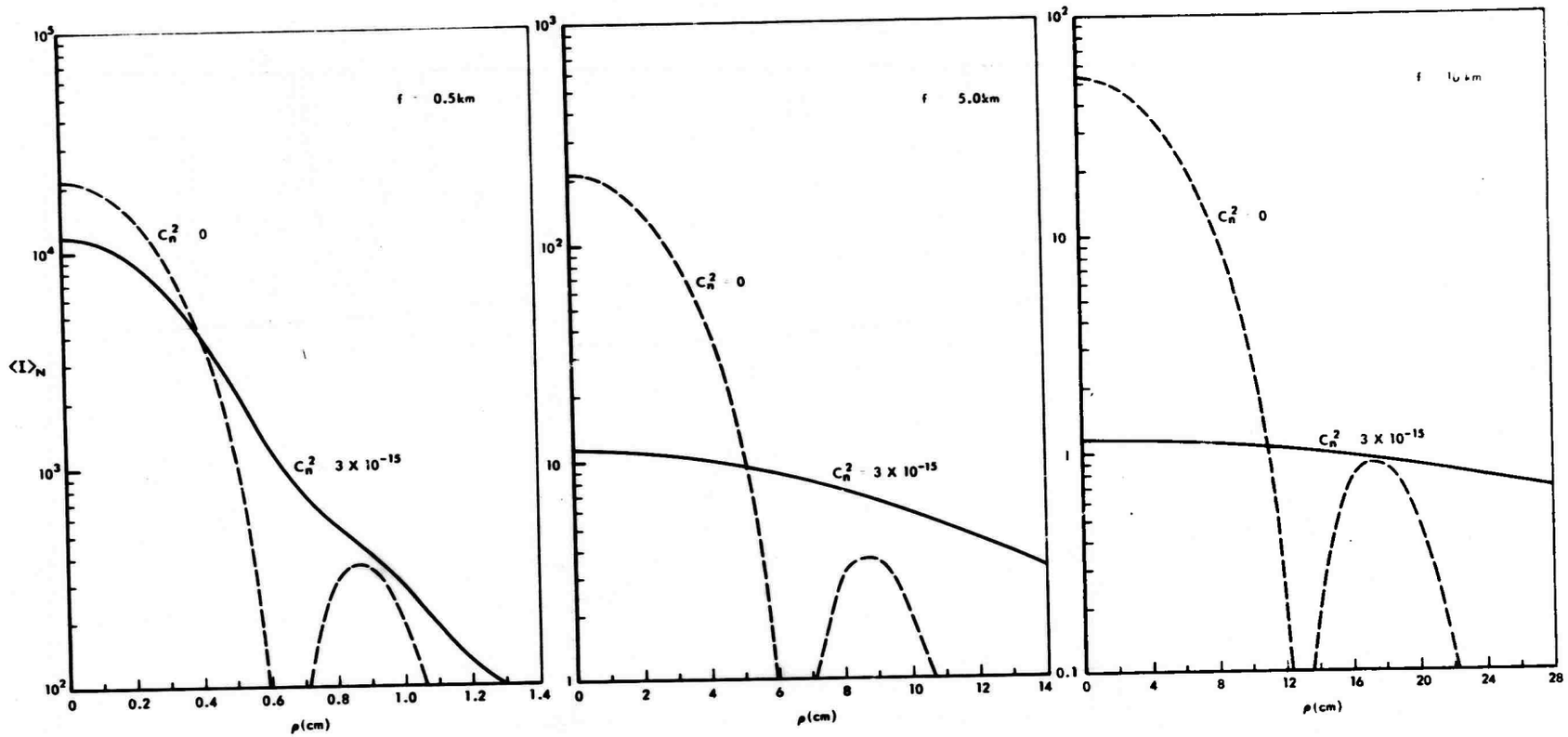

Fig. 7—Focal plane patterns at $\lambda = 10.6\,\mu$ for a uniform distribution in a 1 m aperture

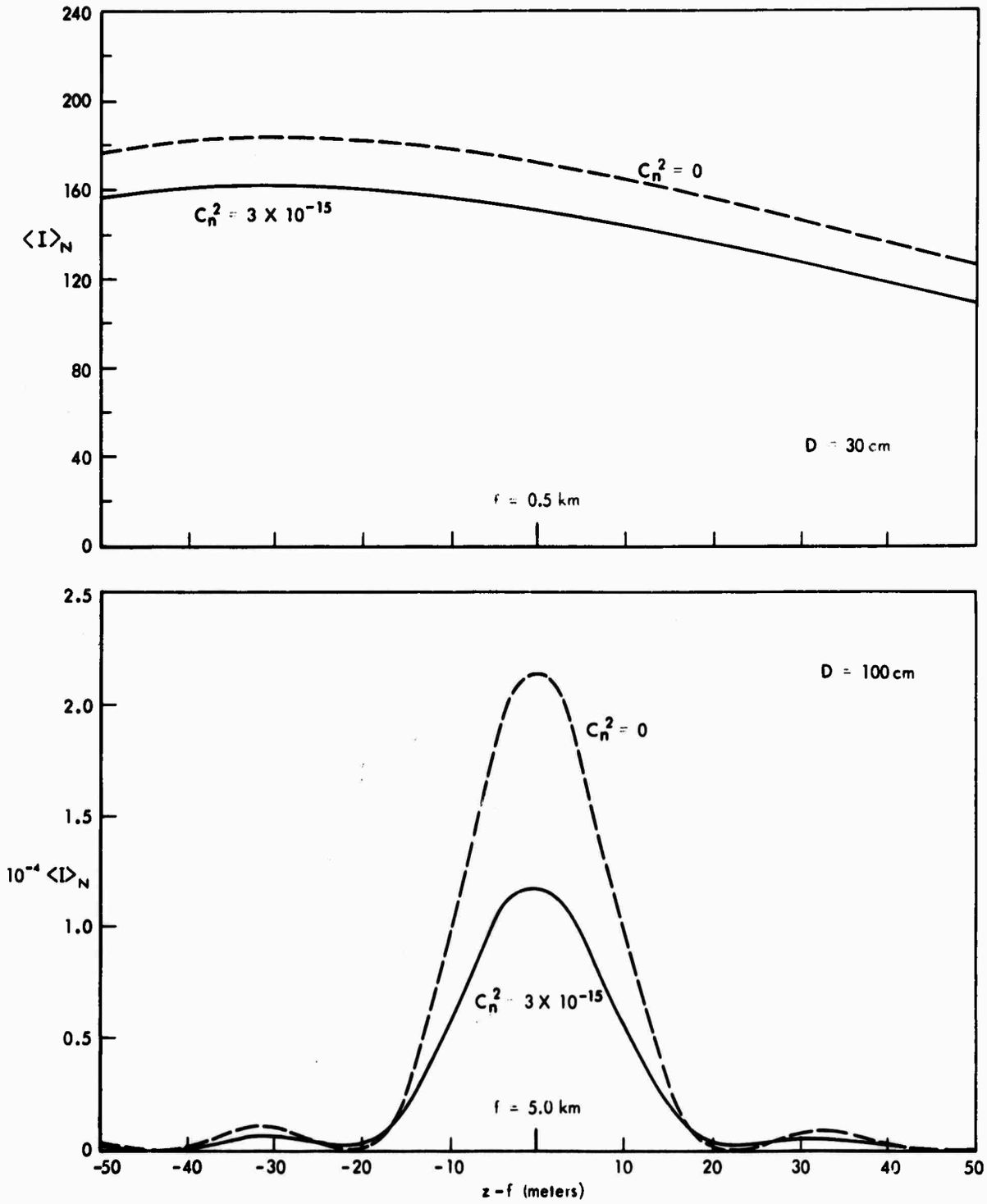

Fig. 8—Comparison of on-axis intensities at $\lambda = 10.6\ \mu$ for a uniform distribution focused at 0.5 km

V. DISCUSSION AND CONCLUSIONS

Based on the results of Sections III and IV, the following procedure can be used to estimate the focal-point intensity.

First, from a knowledge of (1) the index structure constant C_n, (2) the (outer) scale over which the temperature fluctuations are reasonably correlated, L_o and (3) the wave-number of the radiation, k, the mean-field decay length can be constructed according to Eq. (10) as

$$z_c \sim \left(0.4k^2C_n^2L_o^{5/3}\right)^{-1}$$

Then the equivalent coherent aperture size, ρ_o, at range z, can be computed from Fig. 2. If the smaller of the two quantities $(kD\rho_o/z)^2$, $(ka\rho_o/z)^2$ is considerably greater than unity, then the intensity at the range z can be increased by focusing the beam at that range. If this inequality is not satisfied, no intensity amplification can be realized by focusing the beam.

These estimates are also correct for the case of inhomogeneous turbulence when ρ_o, corresponding to the e^{-1} point of the MCF for the inhomogeneous medium, is used. The formula for the spherical-wave MCF in an inhomogeneous medium is given in Ref. 10. In either case, if a more precise estimate of the degradation due to the turbulence is required, the integral of Eq. (16) can be evaluated. This will usually be necessary when $\rho_o \sim a$ or D. When $\rho_o \ll a,D$, then the approximate formulas of Ref. 7 are applicable.

While the expressions for ρ_o depend upon particular models for the turbulence spectrum for homogeneous turbulence, and model profiles for the inhomogeneous case, the intensity distribution can always be determined from Eq. (6) if the MCF is known. Hence, in order to predict the beam pattern from an arbitrary wavefront in an aperture, it is the MCF that should be measured rather than specific beam patterns. The measurement of turbulence parameters that rely on specific models for the spectrum are also of limited utility.

This point can be clarified by first observing that Eq. (16) is the Fourier-Bessel transform of the quantity

$$\left(\frac{8}{\pi}\right)\left(\frac{kD^2}{4z}\right)^2 M_S(Dx,z)\Gamma_{\delta,\beta}(x)$$

which can be inverted to yield

$$D^2\Gamma_{\delta,\beta}\left(\frac{\rho}{D}\right) M_S(\rho,z) = 2\pi \int_0^\infty J_o\left(\frac{k\rho p}{z}\right)\langle I\rangle(p,z)p\, dp \qquad (24)$$

where $\langle I\rangle(p,z)$ is the mean intensity at a distance p from the axis at range z.

For an arbitrary disturbance in the aperture, the function $\Gamma_{\delta,\beta}(x)$ given by Eq. (15) would be replaced by the appropriate overlap integral of Eq. (6). Equation (24) thus provides a possible method for determining part of the MCF from measurements of the beam pattern.

However, because $\Gamma_{\delta,\beta}(\rho/D) = 0$ for $\rho \geq D$, inverting the intensity distribution can give no information regarding $M_S(\rho,z)$ for spatial separations larger than the diameter of the transmitting aperture. In particular, in order to determine the distance z_c from a beam pattern measurement, it would be necessary to have an aperture diameter greater than the largest scale of turbulence, L_o, which might be of the order of meters. Hence, if one can determine the spherical-wave MCF for all spatial separations at a given range (e.g., by using an interferometer), one can infer the intensity distribution from an arbitrary aperture distribution at that range, while the reverse is not true unless apertures greater than the coherence length at that range can be constructed. Even if the beam pattern were measured, $M_S(p,z)$ would first have to be constructed from Eq. (24) from the given measurement to determine the general response. Therefore, for design purposes, direct measurements of the MCF, which are notably lacking, are required.

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