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ON THE PHASE STRUCTURE AND MUTUAL COHERENCE FUNCTION OF AN OPTICAL WAVE IN A TURBULENT ATMOSPHERE

R. F. Lutomirski and H. T. Yura

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10. ABSTRACT

The present work shows that the most commonly used expression for the mutual coherence function (MCF) for an optical wave propagating in a turbulent atmosphere is, in general, incorrect. This expression is based on an unphysical extrapolation of the Kolmogorov spectrum. Along an atmospheric path, with specified turbulence parameters, the new MCF is shown to imply greater resolution, less beam spreading, and greater heterodyne signal-to-noise ratios than indicated by previous calculations. By comparing these results with those previously obtained for heterodyne detection, the percentage errors in the previous calculations are shown to increase with decreasing propagation paths. In particular, where it was formerly thought that the atmosphere limited the effective coherent detection size in heterodyne detection at all ranges, the present calculation reveals that over sufficiently short paths, there is no size limit imposed by the atmosphere.

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PREFACE

This Memorandum, prepared for the Advanced Research Projects Agency, is part of a study of those phenomena which affect the performance of optical or infrared reconnaissance and guidance equipment. The objective of these studies is to provide sufficient understanding for the system analyst to compute performance estimates under various operational conditions.

In light of some recent experimental measurements of the lateral phase coherence, the results of RM-6266-ARPA have been extended to include a theoretical treatment of the phase structure function. The agreement between theory and experiment is found to be very good and supports our theory of the mutual coherence function (MCF).

A quantitative understanding of the effect of atmospheric turbulence in reducing the lateral coherence of an initially coherent wavefront is required for the prediction of the performance of various devices employing lasers for target acquisition or guidance in tactical missions. The MCF is of fundamental importance in these applications because it determines (a) the limiting resolution obtainable along an atmospheric path, (b) the amount by which a finite beam spreads, and (c) the atmospherically limited signal-to-noise ratio using heterodyne detection.

SUMMARY

The most commonly used expression for the wave structure and mutual coherence function (MCF) for an optical wave propagating in a turbulent atmosphere, which is based on an unphysical extrapolation of the Kolmogorov spectrum, is shown, in general, to be incorrect. Along an atmospheric path, with specified turbulence parameters, the new MCF is shown to imply greater resolution, less beam spreading, and greater heterodyne signal-to-noise ratios than indicated by previous calculations. These comparisons are made, and it is shown that the percentage errors in the previous calculations increase with decreasing propagation paths. In particular, where it was previously thought that the atmosphere limited the effective coherent detection size in heterodyne detection at all ranges, the present calculation reveals that over sufficiently short paths, there is no size limit imposed by the atmosphere.

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I. INTRODUCTION

The mutual coherence function (MCF), defined as the cross-correlation function of the complex fields in a direction transverse to the direction of propagation, is the quantity that describes the loss of coherence of an initially coherent wave propagating in a turbulent medium. As a result, the mutual coherence function is important for a number of practical applications. It determines the signal-to-noise ratio of an optical heterodyne detector, the limiting resolution obtainable along an atmospheric path, and the mean intensity distribution from an initially coherent wave emanating from a finite aperture. For a medium characterized by an index variation that is a gaussian random variable with zero mean, the mutual coherence function is given by $\exp(-\frac{1}{2}D)$, where D is the wave structure function.[1,2]

This study first demonstrates that the most commonly used expression[1-3] for the wave structure function (given in Eq. (5) on p. 4) is, in general, of limited validity. This expression was derived on the assumption that the Kolmogorov spectrum[1] of index of refraction fluctuations can be extrapolated to arbitrarily small wave numbers, K, for the purpose of computing the structure function. The apparent justification for using the Kolmogorov power law for $K \lesssim L_o^{-1}$ (where L_o is the outer scale of turbulence) is that, although the extrapolated spectrum diverges, the integrals necessary to compute the structure function from the spectrum remain convergent. However, the sensitivity of the result to the divergent spectrum proves to be considerable. Although Tatarski (Ref. 1, pp. 33-34) has cautioned his readers against an unlimited extrapolation of the Kolmogorov spectrum, Eq. (5) has been based on just such an extrapolation.

In Section II arguments are given against the use of an unbounded spectrum (e.g., the extrapolated Kolmogorov spectrum), and the insensitivity of the wave structure function to *any bounded* spectrum is demonstrated. Based on this insensitivity, a modified von Karman spectrum (which levels off for $K \lesssim L_o^{-1}$) is used to examine the effect of the outer scale on the structure function. The present analysis predicts a transverse (ρ) dependence of the plane wave structure function given

by $\rho^{5/3}[1 - 0.8(\rho/Ł_o)^{1/3}]$ (where $Ł_o = L_o/2\pi$) for ρ in the Kolmogorov inertial subrange, $\ell_o \ll \rho \ll L_o$. Varying the form of the spectrum for $K < Ł_o^{-1}$ produces small variations in the coefficient of the $(\rho/Ł_o)^{1/3}$ term. In addition, for $\rho \sim 0.1\ Ł_o$, our structure function has a bilogarithmic slope closer to 3/2 than 5/3 and levels off for $\rho \gtrsim Ł_o$.

Bouricius and Clifford[4] recently measured the phase structure function at 0.6328 μ over a 50-m path 1.6 m above the ground for transverse separations varying from 1 cm to 2 m.* From temperature measurements, they also deduced the index structure constant and the outer scale (the latter agreeing reasonably well with the height above the ground). Their results are in agreement with the 3/2 dependence discussed above and display a leveling off for separations of the order $\rho \sim L_o$. Buser[5] has also made measurements of the phase structure function at 0.6328 μ over a 50-m path 1.75 m above the ground for ρ = 22 cm; he obtains a slope always less than the previously predicted $\rho^{5/3}$.

In Section III it is shown that there exist three distinct propagation distance regimes for which approximate expressions for the mutual coherence function can be found. These formulae with the respective ranges of validity are presented in the Table for the modified von Karman spectrum of Section II.

In Section IV we calculate the implications of our expression for the MCF with regard to resolution and beam spreading. In particular, we show the coherence length (defined as the transverse separation at which the MCF is equal to e^{-1}) can be considerably greater than the previously accepted value. Comparison with this value predicts an increase in the implied resolution and a decrease in the implied beam spreading.

Finally, in Section V we examine the implications with regard to coherent optical detection. Comparing our results for heterodyne detection with those of Fried,[3] we predict a greater long-term average signal-to-noise ratio. In particular, where Fried finds a maximum useful receiver diameter for all ranges, we find that, for distances small compared with the mean field decay length z_c (defined in Section II), the signal-to-noise ratio increases indefinitely with receiver size.

---

*For short propagation paths, the wave and phase structure function are identical (Ref. 1).

II. THE SPECTRUM AND WAVE STRUCTURE FUNCTION

The analysis is based on the expression for the wave structure function for the case of a plane wave incident upon a homogeneous, isotropic turbulent medium:[1]

$$D(\rho,z) = 8\pi^2 k^2 z \int_0^\infty [1 - J_o(k\rho)]\Phi_n(K)\ KdK \tag{1}$$

where k is the optical wave number, ρ is the transverse separation at propagation distance z, and $\Phi_n(K)$ is the three-dimensional spectral density of the index of refraction fluctuation. Equation (1) may be written in the form*

$$D(\rho,z) = \frac{4z}{z_c}\left[1 - \frac{\int_0^\infty J_o(K\rho)\Phi_n(K)\ KdK}{\int_0^\infty \Phi_n(K)\ KdK}\right] \tag{2}$$

where

$$z_c = \left[2\pi^2 k^2 \int_0^\infty \Phi_n(K)\ KdK\right]^{-1} \tag{3}$$

can be shown[1] to be the propagation distance in which the mean field of a plane (or spherical) wave, $\langle U\rangle$, decays to e^{-1} from its value at the source (the angular brackets denote an ensemble average).

The spectral density most commonly used to represent atmospheric index of refraction fluctuations is that due to Kolmogorov[1]

$$\Phi_n(K) = 0.033 C_n^2 K^{-11/3} \tag{4}$$

---

*In deriving Eq. (1) we have assumed that Φ_n is not a function of propagation distance. The modification of Eq. (1) to include an explicit dependence on range is that $z\Phi_n(K) \rightarrow \int_0^z \Phi_n(K,z')\,dz'$. For spherical wave propagation let $z[1 - J_o(K\rho)] \rightarrow \int_0^z dz'\,[1 - J_o(K\rho z'/z)]$.

valid within the inertial subrange $\text{Ł}_o^{-1} << K << \ell_o^{-1}$, where $\ell_o = 2\pi\ell_o$ and $L_o = 2\pi\text{Ł}_o$ are the inner and outer scales of turbulence, respectively, and C_n is the index structure constant.

In order to compute D from Eq. (1), it is necessary to make certain reasonable assumptions regarding the spectrum outside of the inertial subrange. Tatarski[1] and others,[2,3] using the spectrum of Eq. (4) for all K, have computed the wave structure function given by

$$D_o(\rho,z) = 2.91k^2C_n^2z\rho^{5/3}, \qquad \ell_o << \rho << L_o \tag{5}$$

which is used for all z, the apparent justification being that the integral in Eq. (1) converges. However, the sensitivity of D to the form of the spectrum for $K < \text{Ł}_o^{-1}$ is considerable. This can be seen by carrying out the integration in Eq. (1) for $\text{Ł}_o^{-1} < K < \infty$, which gives

$$D_1(\rho,z) = 2.91k^2C_n^2\rho^{5/3}\left[1 - 0.67\left(\frac{\rho}{\text{Ł}_o}\right)^{1/3} + 0\left(\frac{\rho}{\text{Ł}_o}\right)^2\right],$$

$$\text{for } \ell_o << \rho << L_o \tag{6}$$

The percentage difference is $[(D_1 - D_o)/D_1] \times 100 = 67(\rho/\text{Ł}_o)^{1/3}[1 - 0.67(\rho/\text{Ł}_o)^{1/3}]^{-1}$ and equals $\approx$ 45 percent for $\rho/\text{Ł}_o = 0.1$. This example reveals the sensitivity of the structure function, in the inertial subrange, to the physically unreasonable extrapolation of Eq. (4) for $K < \text{Ł}_o^{-1}$. Although the integral in Eq. (1) converges, an extrapolation would lead to divergent integrals for both the energy per unit volume of the fluctuations and the distance over which the mean field decays (i.e., z_c). On physical grounds, the spectrum must begin leveling off for K corresponding to scales large compared with the separations over which the temperature fluctuations exhibit appreciable correlation, with a finite upper bound as $K \to 0$. Further, careful examination of Eqs. (1) or (2) reveals the insensitivity of the integrals to the spectral density for any spectrum which remains bounded for $K < \text{Ł}_o^{-1}$, and hence, for the purpose of computing the wave structue function, we suggest the use

of any bounded spectrum in this range. The spectrum falls off very rapidly for $K \gtrsim \ell_o^{-1}$ due to viscous damping, and it is customary[6] to use a gaussian decay in this region.

An example of a spectrum that is convenient for computational purposes is the modified von Karman spectrum

$$\Phi_n(K) = \frac{0.033C_n^2 e^{-(K\ell_o)^2}}{(K^2 + L_o^{-2})^{11/6}} \tag{7}$$

which implies a flat spectrum for $K \lesssim L_o^{-1}$.* For example, substituting Eq. (6) into Eq. (3) yields, for $\ell_o \ll L_o$,

$$z_c \cong \left(0.39k^2 C_n^2 L_o^{5/3}\right)^{-1} \tag{8a}$$

and a phase structure given by

$$D(\rho,z) = 2.91k^2 C_n^2 \rho^{5/3}\left(1 - 0.80\left(\frac{\rho}{L_o}\right)^{1/3}\right), \qquad \ell_o \ll \rho \ll L_o \tag{8b}$$

which may be compared with Eq. (6). Figure 1 is a graph of the mean field decay length z_c plotted versus wavelength λ, for $L_o = 100$ cm and typical values of C_n^2.

---

*The quantity $2\pi/L_o$, rather than $1/L_o$, is introduced in Eq. (7) because we are comparing a wave number with a length. In any case, this parameter is to be regarded as the reciprocal of the wave number where the spectrum, in the low frequency regime, begins to deviate from a $K^{-11/3}$ dependence. As has been noted, the resulting functional dependence on ρ is insensitive to the details of the low frequency behavior of the spectrum, but depends only on the value of K where the spectrum begins to level off.

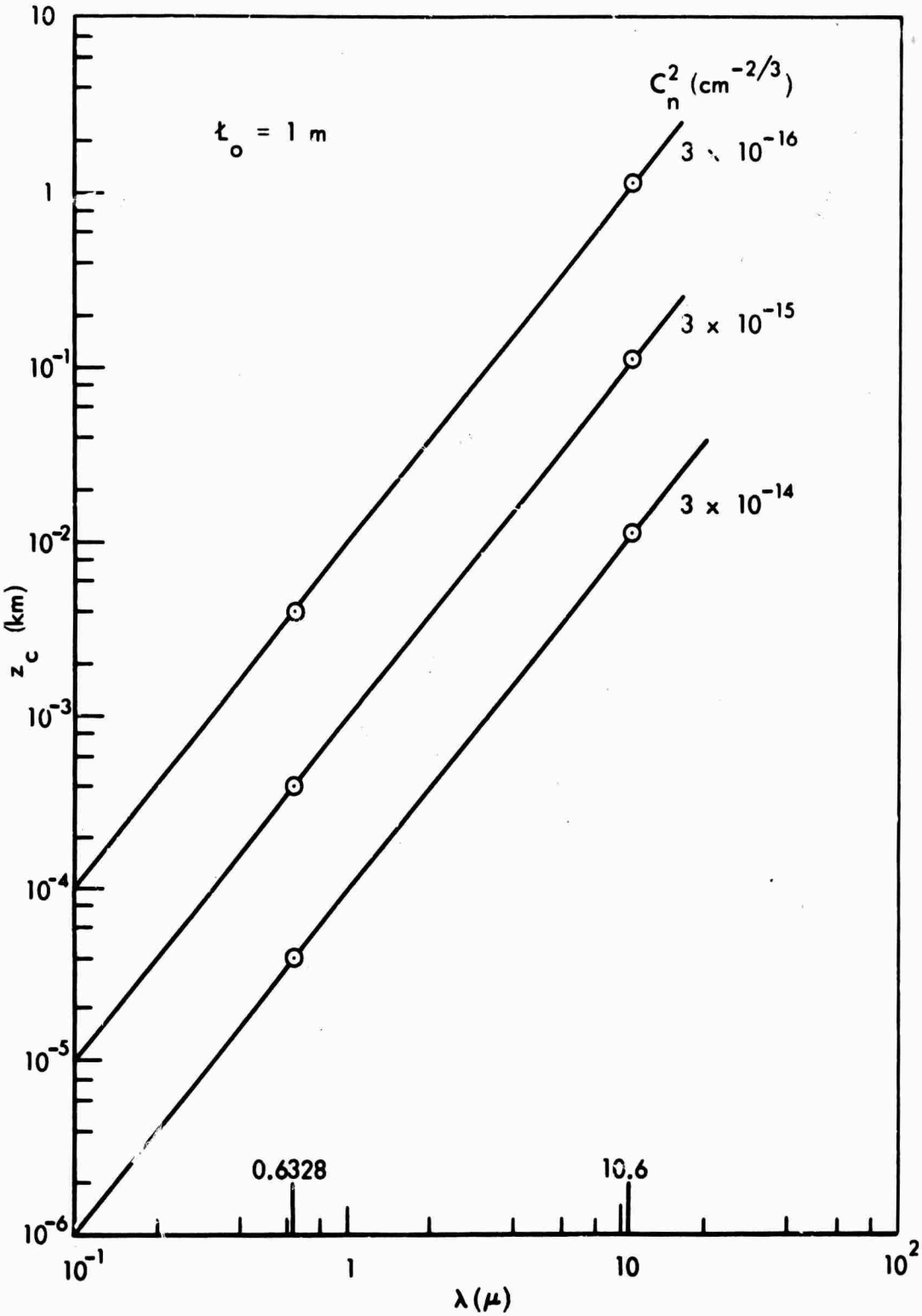

Fig. 1—Propagation distance z_c as a function of wavelength

Substituting Eq. (7) into Eq. (2) and assuming $\rho >> \ell_o$ yields

$$\Gamma_p(\rho,z) = \frac{4z}{z_c}\left\{1 - \frac{5}{3}\left(\frac{\rho}{\text{Ł}_o}\right)^{5/3}\int_0^\infty \frac{J_o(u)u\,du}{\left[u^2 + (\rho/\text{Ł}_o)^2\right]^{11/6}}\right\} \qquad (\rho >> \ell_o) \qquad (9a)$$

The modification of Eq. (9a) for spherical waves is given by

$$D_s(\rho,z) = \frac{4z}{z_{c'}}\left\{1 - \frac{5}{3}\left(\frac{\rho}{\text{Ł}_o}\right)^{5/3}\int_0^\infty du\; uJ_o(u)\int_0^1 \frac{s^{5/3}\,ds}{\left[u^2 + (s\rho/\text{Ł}_o)^2\right]^{11/6}}\right\}$$

$$(\rho >> \ell_o) \qquad (9b)$$

In Fig. 2, the quantity $(z_c/4z)D$, which is independent of λ, z, and C_n^2 (and ℓ_o for $\rho >> \ell_o$) is plotted as a function of $\rho/\text{Ł}_o$ for both plane and spherical waves. Note that

a. The theory predicts a power law dependence closer to 3/2 than 5/3 for separations of $\sim 0.1\ \text{Ł}_o$, and

b. The structure function saturates to a value of $4z/z_c$ for $\rho >> \text{Ł}_o$.

The power law dependence described in (a) has recently been observed by Bouricius and Clifford.[4]

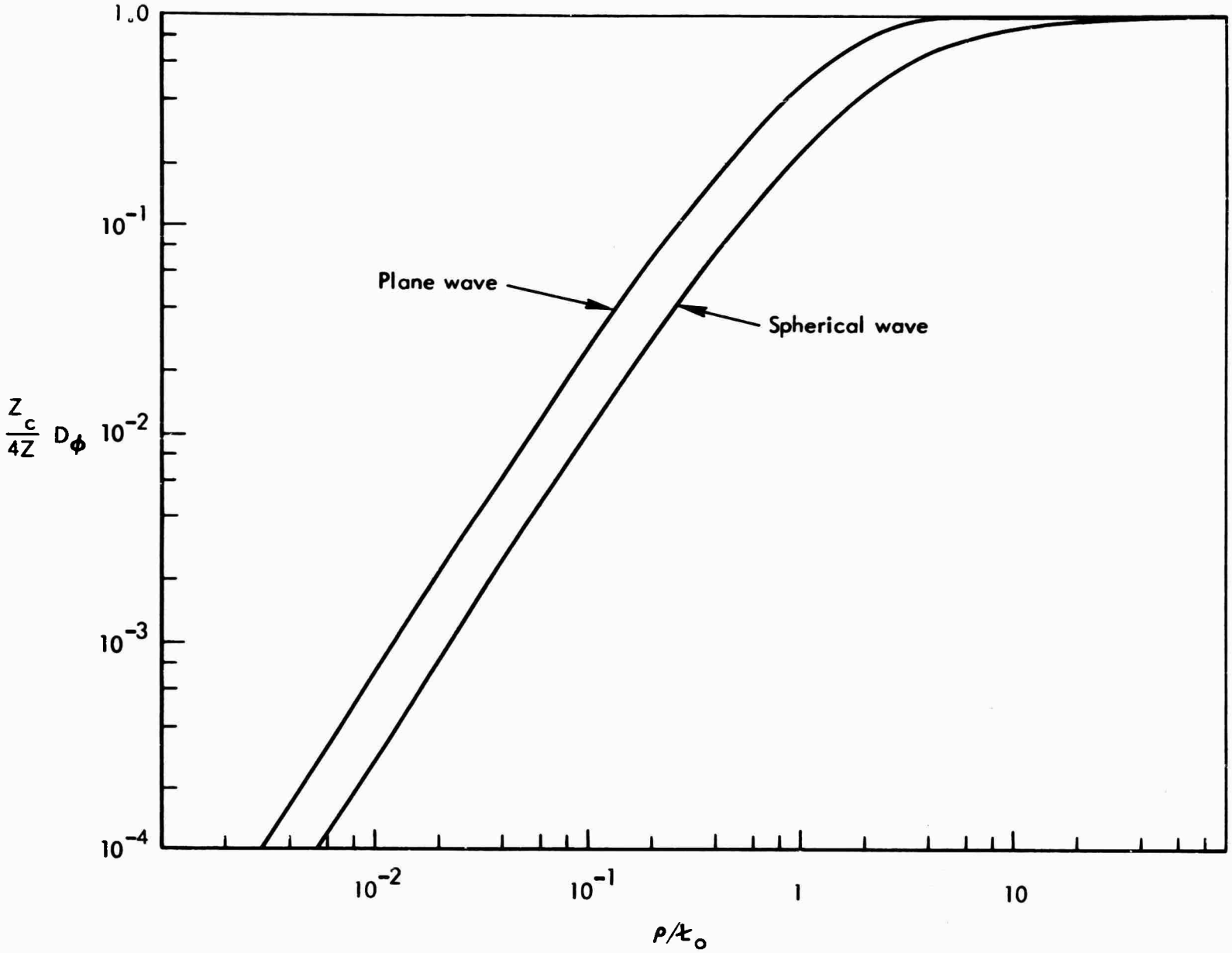

Fig. 2—The normalized wave structure function as a function of ρ/ℓ_o

III. APPROXIMATE FORMULAE FOR THE MCF

The mutual coherence function is defined as the cross-correlation of the complex fields in a direction transverse to the direction of propagation: $M(\rho,z) \equiv \langle U(\underline{r}_1,z)U^*(\underline{r}_2,z)\rangle$, where $\rho = |\underline{r}_1 - \underline{r}_2|$. To second order in the refractive index fluctuations in general, and to all orders when the index fluctuations are a gaussian process, $M(\rho,z) = \exp\{-\frac{1}{2}D(\rho,z)\}$, where D is given by Eqs. (1) or (2).

While it is a simple numerical calculation to compute the MCF directly using the spectrum of Eq. (7) in Eq. (1), it is useful to have approximate formulae for the MCF for estimating the coherence at various ranges.

From Eq. (2) it follows that the MCF decreases monotonically from $M(0,z) = 1$ to $M(\infty,z) = e^{-2z/z_c}$. This behavior is in accord with the physical picture of the light arriving at the two points $\underline{r}_1$, $\underline{r}_2$ being scattered through statistically independent media when $\rho = |\underline{r}_1 - \underline{r}_2|$ is sufficiently large that $M(\rho,z) = \langle U(\underline{r}_1)U^*(\underline{r}_2)\rangle \to \langle U(r_1)\rangle \langle U^*(r_2)\rangle$ Hence for $z \ll z_c$ the MCF is essentially unity for all values of ρ, i.e.,

$$M(\rho,z) = 1, \qquad z \ll z_c \tag{10}$$

Considering the inertial subrange, substituting Eq. (7) into Eq. (1), assuming $\rho \gg \ell_o$, and expanding to lowest order in $(\rho/\mathcal{L}_o)$, we obtain

$$M_1(\rho,z) = \exp\left\{-\frac{2.91}{2}k^2C_n^2z\rho^{5/3}\left[1 - 0.80\left(\frac{\rho}{\mathcal{L}_o}\right)^{1/3}\right]\right\}, \qquad \text{for } \ell_o \ll \rho \ll L_o \tag{11}$$

In order that, at a given range, all of the transverse separations of interest lie in the inertial subrange, it is necessary that $M(\ell_o,z) \simeq 1$, and $M(L_o,z) \ll 1$, which is essentially the condition

$$z_c \ll z \ll z_i \tag{12}$$

where z_c is given by Eq. (8), and z_i, defined by replacing $\mathcal{L}_o$ by ℓ_o in

the formula for z_c, is a distance at which the coherence length of the field is of the order of the inner scale.

For ranges greater than z_i, all of the ρ's of interest are small compared with the inner scale. The Bessel function in Eq. (1) can then be expanded in powers of ρ/ℓ_o to yield

$$M(\rho,z) \cong \exp\left(-\ 1.72k^2C_n^2\ell_o^{-1/3}z\rho^2\right), \qquad z >> z_i \tag{13}$$

It follows from the above discussion that, for $z << z_i$, the plane wave MCF does not depend on the inner scale and can be written as

$$M_p(\rho,z) = \exp\left[-\ \tfrac{1}{2}D_p(\rho,z)\right] \equiv F\left(\frac{\rho}{L_o},\frac{z}{z_c}\right) \tag{14}$$

where D_p is given by Eq. (9a). In Fig. 3 we compare the approximate expressions of Eqs. (5) and (10) with Eq. (14), taking $z/z_c = 10$ and plotting the results as a function of ρ/L_o. For this comparison, the MCF corresponding to Eq. (5) has been written as

$$M_o(\rho,z) = \exp\left[-\ 3.72(z/z_c)(\rho/L_o)^{5/3}\right] \tag{15}$$

and Eq. (11) as

$$M_1(\rho,z) = \exp\left\{-\ 3.72\ \frac{z}{z_c}\left(\frac{\rho}{L_o}\right)^{5/3}\left[1 - 0.80\left(\frac{\rho}{L_o}\right)^{1/3}\right]\right\} \tag{16}$$

The comparison reveals that for $z << z_i$, the field from a plane or spherical wave retains its transverse coherence to a greater degree than previously predicted. The approximate expressions for the MCF with respective ranges of validity are summarized in the Table on p. 12.

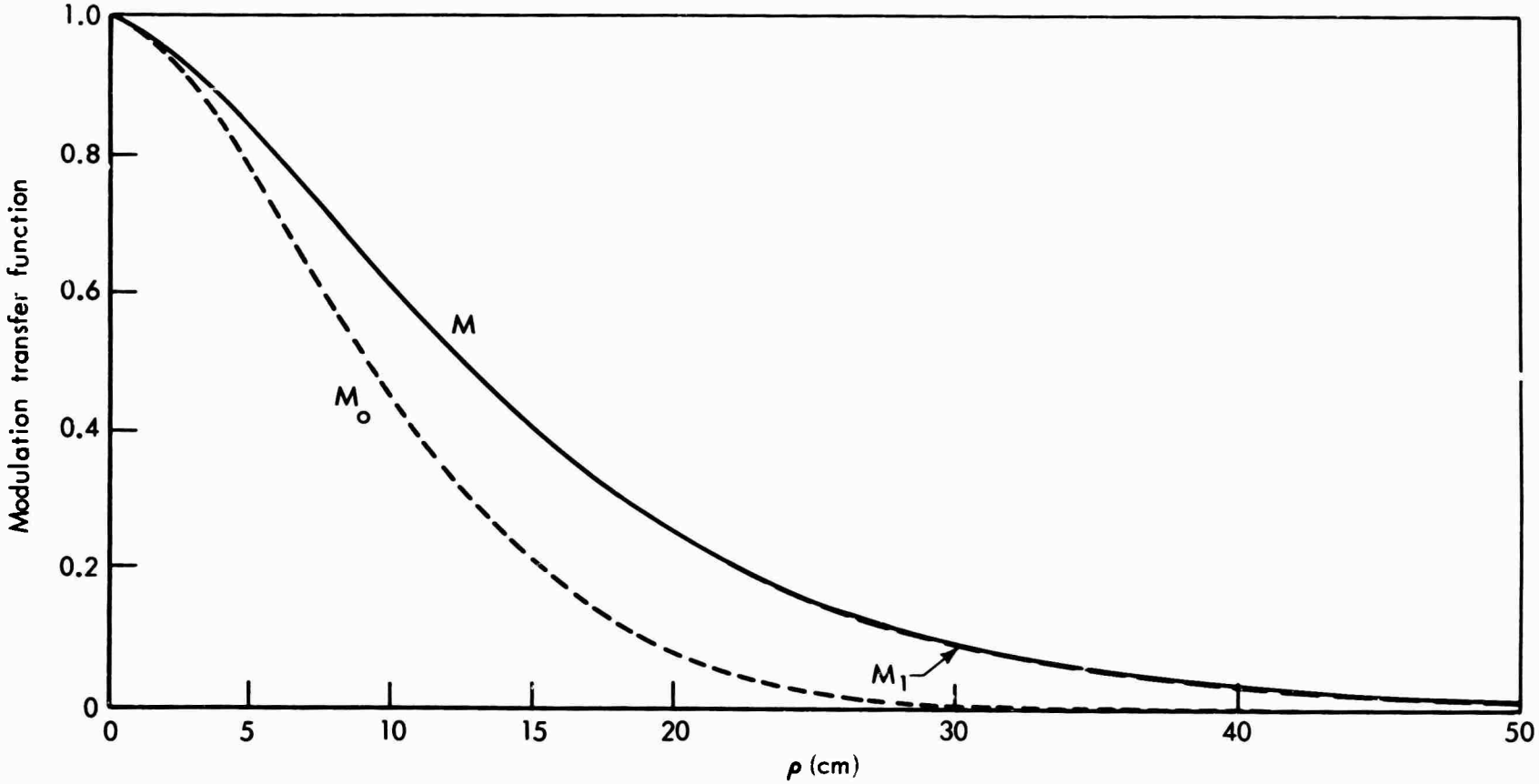

Fig. 3—Comparison of plane wave MCFs of Eqs. (14) - (16) for $z/z_c = 10$

APPROXIMATE EXPRESSIONS FOR THE PLANE WAVE MODULATION TRANSFER FUNCTION

Range[a]	MCF
$z \ll z_c$	1
$z_c \ll z \ll z_i$	$\exp\left\{-\frac{2.91}{2} k^2 C_n^2 z\rho^{5/3}\left[1 - 0.80(\rho/\ell_o)^{1/3}\right]\right\} =$ $\exp\left\{-3.72(z/z_c)(\rho/\ell_o)^{5/3}\left[1 - 0.80(\rho/\ell_o)^{1/3}\right]\right\}$
$z \gg z_i$	$\exp\left[-1.72k^2 C_n^2 \ell_o^{-1/3} z\rho^2\right] =$ $\exp\left[-2.4(z/z_i)(\rho/\ell_o)^2\right]$

[a]The quantities z_c (the distance where the average field is down by e^{-1}) and z_i (the distance where the coherence length of the field is of the order of the inner scale of turbulence) are given by $(0.39k^2C_n^2\ell_o^{5/3})^{-1}$ and $(0.39k^2C_n^2\ell_o^{5/3})^{-1}$, respectively.

IV. RESOLUTION AND BEAM SPREADING

Two important consequences of our more coherent MCF are that, for given turbulence parameters along a uniform atmospheric path, the mean visibility is better, and the average amount by which a finite beam spreads is smaller, than predicted by Fried.[7] Defining ρ_o as the transverse separation at which the atmospheric MCF is reduced by e^{-1}, then the minimum resolvable length at a distance z from an observer is well known to be $\sim z/k\rho_o$. It can be shown that when ρ_o is small compared to the size of the transmitting aperture, the angular spread of a finite beam due to the atmosphere is $\sim 1/k\rho_o$.

As shown in the previous section, for distances small compared with z_1 ($= 10^5\ z_c$ for $L_o/\ell_o = 10^3$), the plane wave MCF does not depend on the inner scale, and can be written as

$$M_p(\rho,z) = \exp\left[-\tfrac{1}{2}D_p(\rho,z)\right] \equiv F_p\left(\frac{\rho}{\mathcal{L}_o},\frac{z}{z_c}\right), \qquad z \ll z_1 \tag{17}$$

where D_p is given by Eq. (9a). The modification of Eq. (14) for spherical waves is given by

$$M_s(\rho,z) = \exp\left[-\tfrac{1}{2}D_s(\rho,z)\right] \equiv F_s\left(\frac{\rho}{\mathcal{L}_o},\frac{z}{z_c}\right), \qquad z \ll z_1 \tag{18}$$

where D_s is given by Eq. (9b). In Figs. 4a and 4b we plot $\rho_o/\mathcal{L}_o$ versus z/z_c for plane and spherical waves, obtained by inverting $F(\rho_o/\mathcal{L}_o,\ z/z_c) = e^{-1}$ and $F_s(\rho_o/\mathcal{L}_o,\ z/z_c) = e^{-1}$, respectively. On the same graph in each figure, we compare our results with the MCF obtained from the structure function of Eq. (5) and the corresponding expression for spherical waves (obtained by multiplying the exponent in Eq. (5) by 3/8). For the purposes of this computation, we have written Eq. (5) in the form

$$D_o(\rho,z) = 7.44\left(\frac{z}{z_c}\right)\left(\frac{\rho}{\mathcal{L}_o}\right)^{5/3}$$

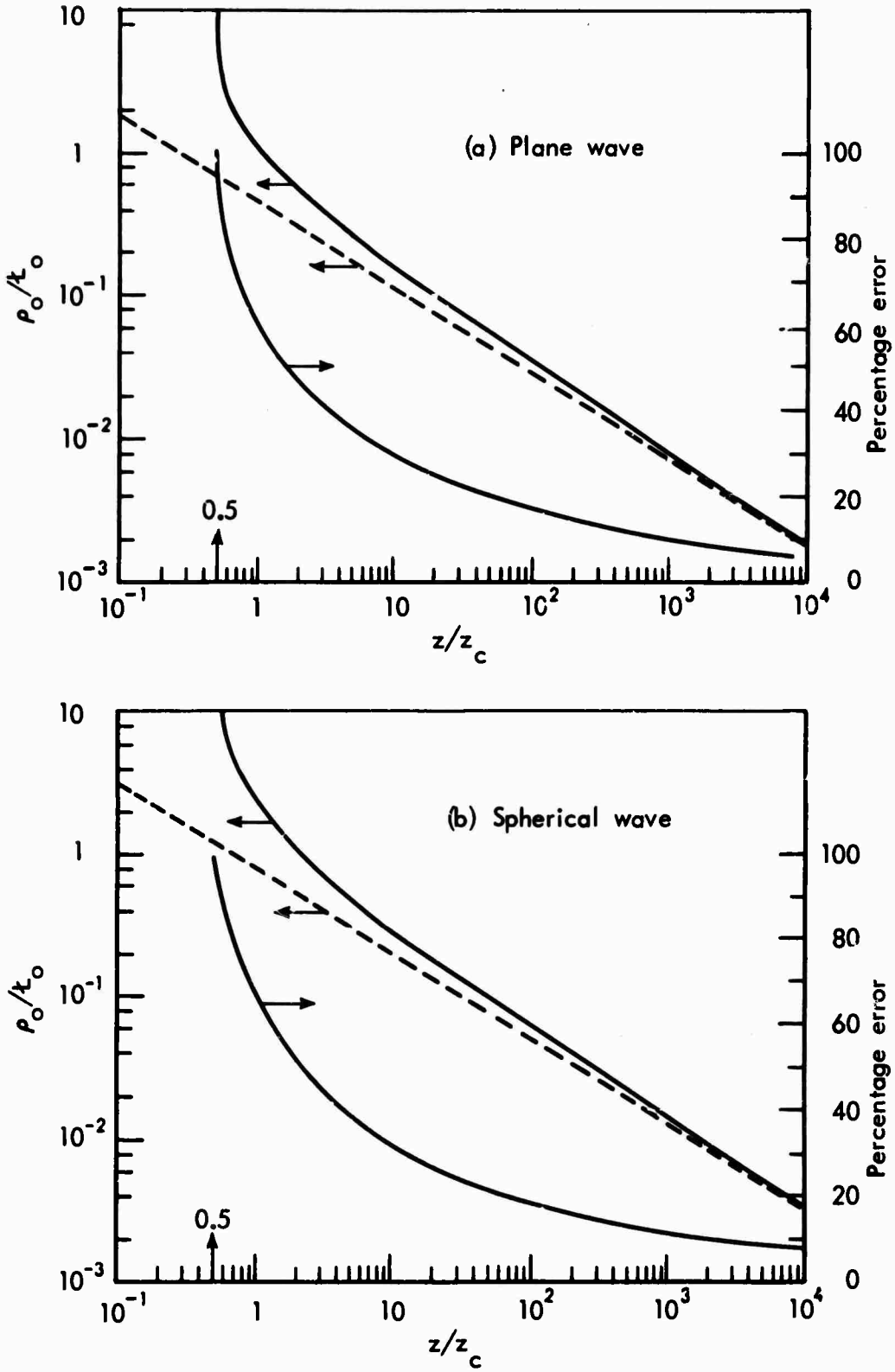

Fig. 4—The normalized coherence lengths, ρ_o/ℓ_o, obtained by inverting $M = e^{-1}$ and $M_o = e^{-1}$, respectively, versus z/z_c

The percentage error in each case, defined as $[(\rho_o - \rho_o')/\rho_o] \times 100$, where ρ_o' satisfies $M_o(\rho_o', z) = e^{-1}$, is indicated on the scale at the right.

As the plane or spherical wave source is approached, the present analysis indicates that the transverse coherence of the field increases at a greater rate than previously predicted. The errors increase from ~ 10 percent at $10^4 z_c$ to 100 percent at $0.5 z_c$, whereas for distances $< 0.5 z_c$, the new MCF is never down to e^{-1}. For $z > z_i$ ($= 10^5 z_c$ for $L_o/\ell_o = 10^3$), the use of M_o again gives a poor approximation to the new MCF, which in this region is given by Eq. (12).

V. SIGNAL-TO-NOISE RATIO IN HETERODYNE DETECTION

As an additional illustration of the implications of our more coherent MCF, we use it to compute the signal-to-noise ratio in a coherent detection system and compare the results with those computed by Fried.[3] The signal-to-noise ratio is given by[3]

$$\frac{\langle S\rangle}{N} = 4\left(\frac{\eta}{e}\right)\langle A_S^2\rangle\, D^2 \int_0^1 dx\; x\; K_o(x) M(Dx,z) \tag{19}$$

where A_S is the signal amplitude, η/e is the quantum efficiency measured in electrons per unit energy, D is the diameter of the collecting aperture, and

$$K_o(x) = \frac{1}{2}\left[\cos^{-1}(x) - x(1 - x^2)^{1/2}\right] \tag{20}$$

Then Eq. (19) can be written

$$\frac{\langle S\rangle}{N} = 4\left(\frac{\eta}{e}\right)\langle A_S^2\rangle\, \ell_o^2\, \psi\left(\vartheta, \frac{z}{z_c}\right) \tag{21}$$

for $0 \le z \ll z_1$, where $\vartheta = D/\ell_o$, and

$$\psi\left(\vartheta, \frac{z}{z_c}\right) = \vartheta^2 \int_0^1 dx\; x\; K_o(x) F\left(\vartheta x, \frac{z}{z_c}\right) \tag{22}$$

where F is given by Eq. (14). The function ψ contains the dependence of signal-to-noise ratio on the collector diameter in the presence of atmospheric distortion, and is plotted as a function of ϑ in Figs. 5a and 5b for various values of z/z_c. The reduced signal-to-noise ratio ψ, as computed from Eq. (22) (solid curve), is compared with the results of Fried,[1] who used the MCF of Eq. (15) (dashed curve). In general, the present analysis predicts a somewhat larger signal-to-noise ratio.

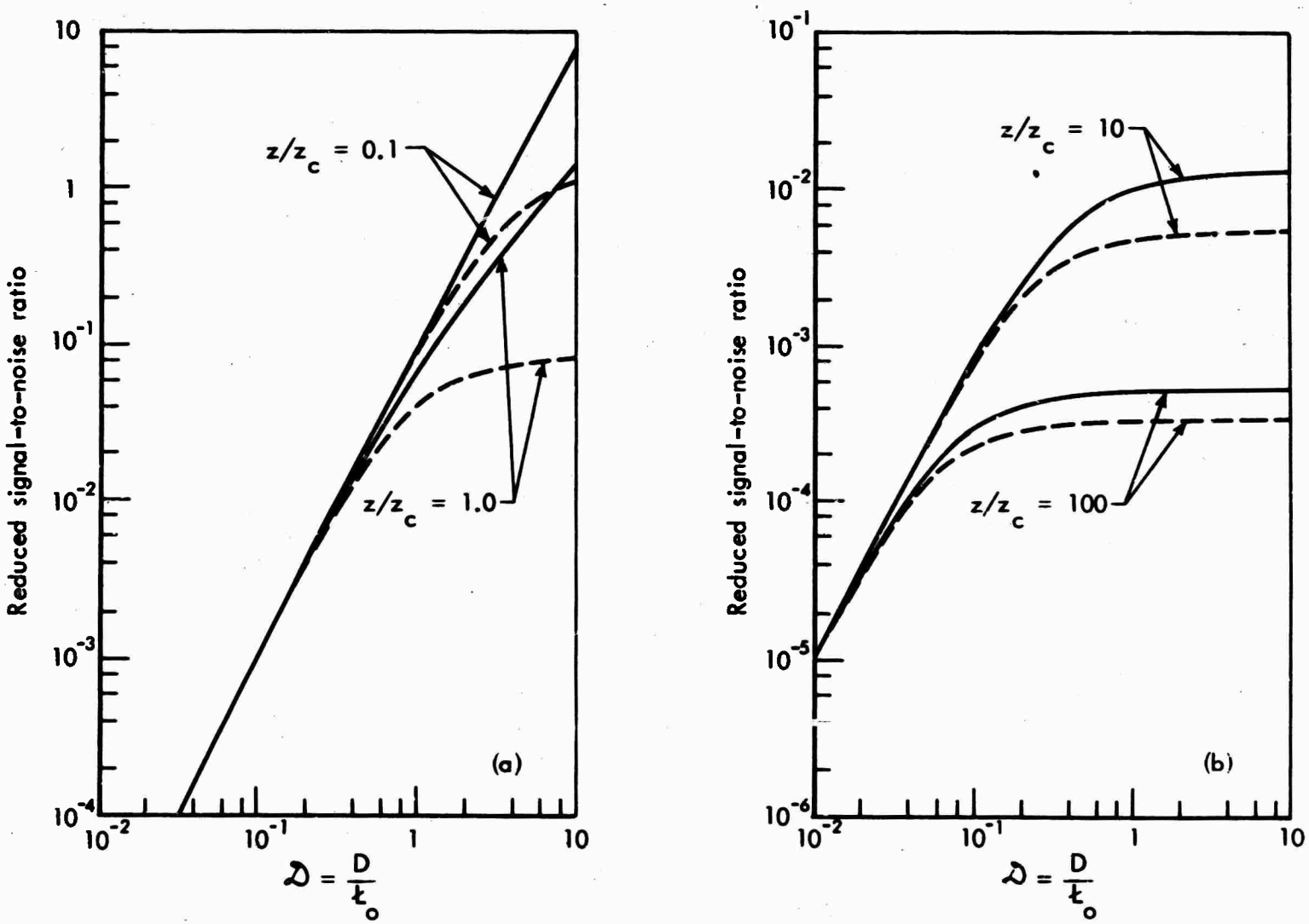

Fig. 5—Reduced signal-to-noise ratios at 10.6 μ in a heterodyne detection system as a function of collector diameter (measured in units of ℓ_o) for ℓ_o = 1 m and various values of z/z_c

The difference is more pronounced for distances small compared with z_c, where $\langle S \rangle / N$ as derived here increases indefinitely with aperture size. This result is in contrast with that of Fried, who derived an effective limiting diameter beyond which increasing the diameter results in very little improvement in $\langle S \rangle / N$ for all ranges. For ranges very large compared with z_c (but still small compared with z_i), the MCF obtained from Eq. (1) approaches that obtained from Eq. (5), and the difference between the two signal-to-noise ratios tends to zero. This trend is illustrated in Fig. 6, in which the reduced signal-to-noise ratio is plotted as a function of z/z_c for $\vartheta = 1$.

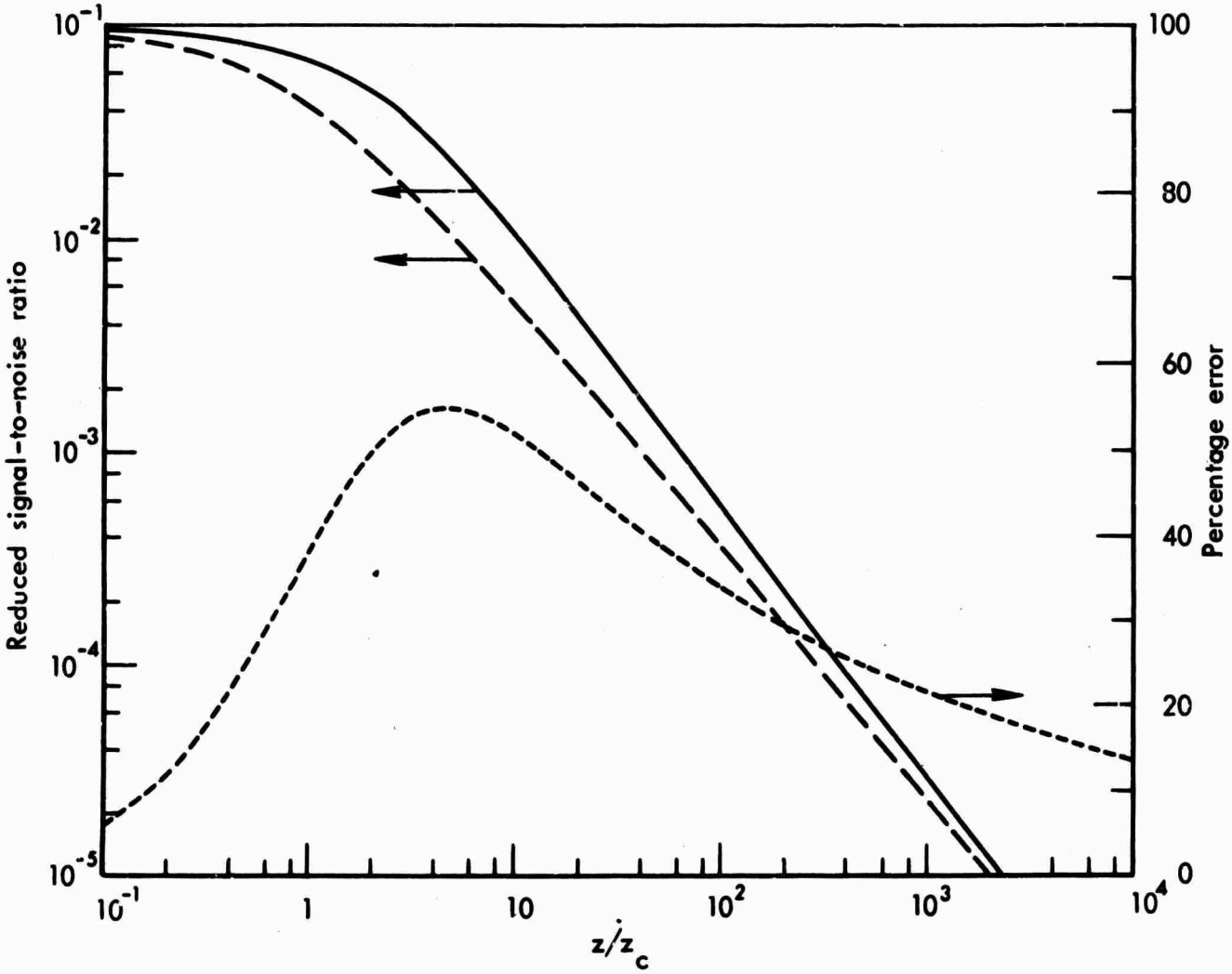

Fig. 6—Reduced signal-to-noise ratio at 10.6 μ in a heterodyne detection system as a function of z/z_c for the diameter of the collector equal to ℓ_o

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