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13. ABSTRACT

In a recent paper, Armitage (Allocation of Sorties in Air Interdiction, Journal of the Operations Research Society of America, XVIII, No. 3, (May-June), pp. 483-496) has considered a deterministic model of air interdiction in a simplified "hostile country" through which material is supplied. In this report, an analogous stochastic model is developed.

Three important elements of a stochastic model for air interdiction are: (i) Stochastic processes describing the movement of trucks carrying the material, amount of material, etc., (ii) a class of methods of air interdiction and (iii) optional method of air interdiction. In this report, we have studied the movement of trucks and amount of material reaching the friendly country and have also considered the effect of air interdiction.

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A STOCHASTIC MODEL FOR AIR INTERDICTION

D. E. Crawford and R. C. Srivastava

Division of Statistics

The Ohio State University

June, 1971

Technical Report No. 71-2

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0. Introduction

In a recent paper, Armitage (1970) has considered a deterministic model of air interdiction in a simplified "hostile country" through which material is supplied. In this report, an analogous stochastic model is developed.

Three important elements of a stochastic model for air interdiction are: (i) Stochastic processes describing the movement of trucks carrying the material, amount of material, etc., (ii) a class of methods of air interdiction and (iii) optional method of air interdiction. In this report, we have studied the movement of trucks and amount of material reaching the friendly country and have also considered the effect of air interdiction.

1. A Deterministic Model

Armitage (1970) has considered a deterministic model of air interdiction in a simplified hostile country. Among others, the following assumptions are made:

(A1) The country has a rectangular shape.

(A2) A quantity of material enters at one of the short sides and is transported to the other short side.

(A3) The material is distributed uniformly across the short sides, so only the distribution in the direction of travel is important.

(A4) The material is depleted by consumption within the country and by destruction due to air interdiction.

(A5) The air interdiction does not depend on time.

(A6) All quantities pertinent to the air interdiction are either constant or are defined by functions of a single variable.

(A7) There is no within-country source of target material.

The following notation will be needed. Let x be the distance from the boundary that the material exits from the country. x ranges from zero to one. $T(x)$ be the amount of material flowing past point x in one day.

$C(x)\Delta x$ be the amount of material consumed in a strip of width Δx in one day.

$D(x)\Delta x$ be the amount of material destroyed in a strip of width Δx in one day.

then $\frac{dT}{dx} = C(x)+D(x)$.

Now let $C(x) = k_1 T(x)+a(x)$, where $a(x)$ is the amount of material consumed within the country, and $k_1 T(x)$ is the amount of material consumed by the transportation process. k_1 is a constant determined by observations of the real process.

Also let $D(x) = k_2 N(x)T(x)$. Where $N(x)$ is the density of aircraft applied to point x in planes per day. k_2 is a constant determined from observations. This assumption says that the number of tons of material destroyed at point x is proportional to the number of aircraft applied to x , times the amount of material at x .

Inserting this information into the first equation yields the differential equation:

$$\frac{dT}{dx} = k_1 T(x) + a(x) + k_2 T(x) N(x)$$

Armitage solves this equation so that $T(0)$ is an explicit function of $g(x) = \int_0^x N(z)dz$. $T(0)$ is the function to be minimized over all forms of $N(z)$ in the interval $[0,1]$.

He further shows that if $g(x)$ is a step function then $T(0)$ is minimized in the function space of all functions $g(x)$.

The constraint is the tour survival probability. If each crew flied m sorties then the probability of surviving these m sorties must be greater than some preassigned number $0<\pi<1$.

Armitage applies his results to a specific example. He divides the hostile country into three zones, as follows:

He assumes the probability of pilot loss is a constant within each zone. That is:

$$p(x) = \begin{cases} p_1 , & 0 \le x \le x_1 \\ p_2 , & x_1 < x \le x_2 \\ p_3 , & x_2 < x \le 1 \end{cases}$$

and $0 < p_1 < p_2 < p_3 < 1$.

He then assumes $k_1 = \log(1+f)$ where f is the number of tons consumed on each round trip for each ton of material.

He also assumes $a(x)$ has the form:

$$a(x) = \begin{cases} a_1, & 0 \leq x \leq x_1 \\ a_2, & x_1 < x \leq x_2 \\ a_3, & x_2 < x \leq 1 \end{cases}$$

Then he assumes N_0 sorties are available for the interdiction action. He assigns N_1 sorties to $x = x_1$, N_2 sorties to $x = x_2$ and $N_0 - N_1 - N_2$ to $x=1$. The quantities N_1 and N_2 must satisfy:

$$N_1 + N_2 \leq N_0 \qquad 0 \leq N_1 \text{ and } 0 \leq N_2$$

$$N_1 \log(1-p_1)/(1-p_3) + N_2 \log(1-p_2)/(1-p_3) \geq N_0 \log(1-p)/(1-p_3)$$

The quantities N_1 and N_2 must be chosen to minimize $T(0)$. Since the sorties are applied to three points $N(x) = 0$ in the interior of each zone, then the original differential equation becomes:

$$\frac{dT}{dx} = k_1 T(x) + a(x)$$

This discontinuity of $T(x)$ is:

$$T(x_1^-) = T(x_1^+)(1 - k_2 N(x_1))$$

Because of the discontinuity a different constant of integration occurs in each zone.

Armitage then constructs an admissible region and checks the boundaries for the minimum of $T(0)$. Without more specific data, he is unable to decide between three possible points. If more specific data were available, the values of $T(0)$ at these points could be computed and the minimum value of $T(0)$ selected. This would determine the optimal values for N_1 and N_2.

Armitage assumes implicitly that the trucks all line up and do not pass, and they maintain the same spacing throughout their journey across the country. This is obviously not realistic. Also, since the model is deterministic, the solutions are, by implication, exact. This is not realistic because the constants cannot be evaluated exactly.

Armitage assumes that all of the quantities pertinent to the interdiction are not functions of time. This causes the model not to reflect the results of the interdiction when it is started. It also means that we cannot tell what the long run consequences of an interdiction policy are, unless they are constant. For example, it is conceivable that some interdiction policy might reduce the flow of material to zero for a short time. Results of this type will not be demonstrated by this model.

He also assumes that all quantities are functions of one variable or constant. This is not realistic under the conditions of the problem. For example, there might be weather conditions that would preclude air interdiction. This would make the number of aircraft applied to a point x depend on time as well as location.

By assuming the material is distributed uniformly across the country in the direction perpendicular to the direction of travel, he has ignored the possibility of mountains, lakes, rivers and forests impeding the flow of material. The air interdiction would be more effective if applied to places where the flow is dense.

2. A Stochastic Model

Assume a country shaped as shown in Figure 2.2.1.

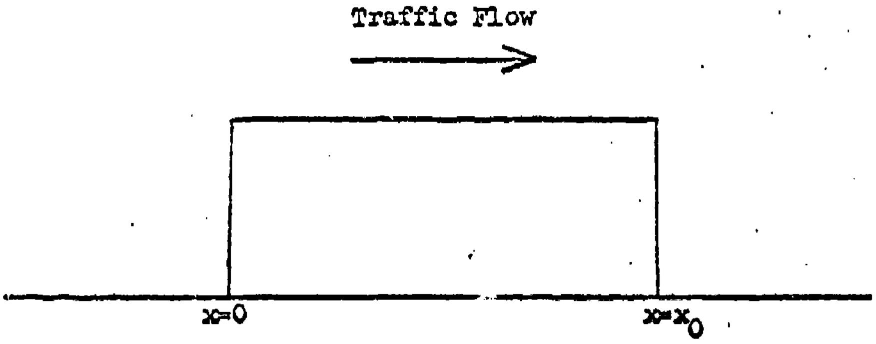

Figure .2.1

Trucks enter the country at $x=0$ and travel in the direction of the arrow to $x=x_0$.

During the trip the trucks may be subjected to bombing by aircraft. Some of the material on the trucks may be destroyed in this way. Also some of the material on the trucks is consumed within the country, either by the transportation process itself, by theft, or use by natives of the country.

We make the following assumptions:

(B1) All trucks enter the country at $x=0$ and travel to $x=x_0$, leaving the country only at this border.

(B2) A truck arriving at $x=0$ at time t_k, chooses a velocity v_k and then moves with this constant velocity across the country. The random variables v_k are independent, identically distributed with distribution function $F(v) = P(v_k \leq v)$. Sequences (t_k) and (v_k) are independent.

(B3) There is no interaction between trucks. That is, no time is lost in passing when one truck overtakes another.

(B4) The arrival times of the trucks at $x=0$ is a homogeneous Poisson process with parameter λ.

(B5) The material is distributed uniformly in the direction perpendicular to the direction of travel, so only the direction of travel is of interest.

(B6) There is no within country source of target material. That is, no material of interest is manufactured within the country.

(B7) The process has gone on for a considerable length of time so that transient effects due to the starting process have died out.

Theorem 2.1 The arrival times of trucks at $x=x_0$ satisfying assumptions (B1) through (B6) is a Poisson process with parameter $\lambda(\alpha)$ if trucks start arriving at $x=0$ at time $-\alpha$.

Proof: Suppose n trucks arrive at $x=0$ in the time interval $(-\alpha,t_2]$. Then their arrival times are distributed uniformly on $(-\alpha,t_2]$. If a truck arrives at $x=0$ at time t and chooses a velocity v, then to arrive at $x=x_0$ in time interval $(t_1,t_2]$ it must travel for time t such that

$$t_1 < t + \frac{x_0}{v} < t_2 \quad . \tag{2.1}$$

This can be rewritten as

$$\frac{x_0}{t_2-t} < v < \frac{x_0}{t_1-t} \; .$$

The P(n trucks arrive at $x=x_0$ in time $(t_1,t_2]|k$ trucks arrive at $x=0$ in time $(-\alpha,t_2]) =$

$$\frac{1}{t_2+\alpha} \int_{-\alpha}^{t_2} \int_{\frac{x_0}{t_2-t}}^{\frac{x_0}{t_1-t}} dF(v)dt \; .$$

Since the integral exists the order of integration may be reversed.

Rearranging (2.1), we get $t_1-\frac{x_0}{v} < t < t_2-\frac{x_0}{v}$ and the velocity clearly can vary from $\frac{x_0}{t_2+\alpha}$ to ∞. Using this information, we have

P(n trucks arrive at $x=x_0$ in time $(t_1,t_2]$ trucks arrive at $x=0$ in time $(-\alpha,t])$

$$= \frac{1}{t_2+\alpha} \int_{\frac{x_0}{t_2+\alpha}}^{\infty} \int_{t_1-\frac{x_0}{v}}^{t_2-\frac{x_0}{v}} dt \; dF(v)$$

$$= \frac{1}{t_2+\alpha} \int_{\frac{x_0}{t_2+\alpha}}^{\infty} [t_2 - \frac{x_0}{v} - (t_1 - \frac{x_0}{v})] dF(v)$$

$$= \frac{t_2-t_1}{t_2+\alpha} \int_{\frac{x_0}{t_2+\alpha}}^{\infty} dF(v)$$

We will call this $\frac{P(\alpha)}{t_2+\alpha}$.

Now we can find P(n trucks arrive at $x=x_0$ in time $(t_1,t_2]$) =

P(n trucks arrive at $x=x_0$ in time $(t_1,t_2]$ | k trucks arrive at $x=0$ in time $(-\alpha,t_2]$)$P(r=k)$

$$= \binom{k}{r} \frac{P(\alpha)^r}{t_2+\alpha} \left[1 - \frac{P(\alpha)}{t_2+\alpha}\right]^{k-r} \frac{(\lambda(t_2+\alpha))^k \exp(-\lambda(t_2+\alpha))}{k!}$$

$$= \frac{(\lambda P(\alpha))^r}{r!} \exp(-\lambda(t_2+\alpha)) \sum_{k=r} \frac{(\lambda(t_2+\alpha)-P(\alpha))^{k-r}}{(k-r)!}$$

Which by the definition of exp is

$$= \frac{(\lambda P(\alpha))^r}{r!} \exp(-\lambda(t_2+\alpha)) \exp(\lambda(t_2+\alpha)-P(\alpha))$$

$$= \frac{(\lambda P(\alpha))^r \exp(-\lambda P(\alpha))}{r!}$$

So the number of trucks that arrive at $x=x_0$ in time $(t_1,t_2]$ is distributed Poisson with parameter $\lambda P(\alpha)$ which we will call $\lambda(\alpha)$. This concludes the proof of Theorem .2.1.

Now assume that the material on each truck is divided into two classes. One class consists of materials that is needed for making war.

Examples of this kind of material are: weapons, ammunition, and grenades. This material will be called war material. The other class will consist of materials which will be called supplies. Examples of this kind of material are: food, gasoline, medicine and building materials.

2. . Next we find the distribution of both types of material that arrive $x=x_0$ in time t when there is no consumption or air interdiction.

The amount of material arriving at $x=0$ in truck i will be denoted (X_i, Y_i). Where X_i is the number of tons of war material and Y_i is the number of tons of supplies. The total amount of material will be $\left(\sum_{i=1}^{N(T)} X_i , \sum_{i=1}^{N(T)} Y_i\right)$ where $N(T)$ is a Poisson process with parameter $\lambda(\alpha)$. Here $N(t)$ denotes the number of trucks arriving at $x = x_0$ in time t.

The quantities X_i and Y_i may be constants or random variables from a number of different distributions. Several cases are considered.

Case I Assume that $X_i = X_0$ and $Y_i = Y_0$ for all i.

Since the amount of each material arriving at $x = 0$ is a constant, the probability generating function of the amount of material of each kind on each truck is;

$$G(s_1, s_2) = s_1^{X_0} s_2^{Y_0} .$$

The probability generating function of the amount of material arriving at $x = x_0$ in time t is:

$$f(s_1, s_2; t) = \exp(-\lambda(\alpha)+\lambda(\alpha)(s_1^{X_0} s_2^{Y_0}))t .$$

Next we find mean and variance of the material reaching $x = x_0$.

$$E(X) = \frac{\partial f(1,1;t)}{\partial s_1} = f(s_1, s_2; t)(\lambda(\alpha)\, X_0 s_1^{X_0-1} s_2^{Y_0})t \Big|_{s_1=s_2=1}$$

$$= f(1,1;t)\ \lambda(\alpha) X_0 t .$$

But $f(1,1;t) = \exp(-\lambda(\alpha)+\lambda(\alpha))t = 1$. So

$E(X) = \lambda(\alpha) X_0 t$.

Similarly, $E(Y) = \lambda(\alpha)\, Y_0 t$.

Next, we find the variance.

$$\frac{\partial^2 f(1,1;t)}{\partial s_1^2} = \frac{\partial}{\partial s_1}\left(\frac{\partial f(s_1, s_2; t)}{\partial s_1}\right)$$

$$= \frac{\partial}{s_1}\left(f(s_1, s_2; t)(\lambda(\alpha)\, X_0 s_1^{X_0-1} s_2^{Y_0} t\right)\Big|_{s_1=s_2=1}$$

$$= (f(s_1, s_2; t)\lambda^2(\alpha) X_0^2 s_1^{2(X_0-1)} s_2^{2Y_0} t^2$$

$$+ f(s_1, s_2; t) t \lambda(\alpha) X_0 (X_0-1) s_1^{X_0-2} s_2^{Y_0})\Big|_{s_1=s_2=1}$$

$$= f(1,1;t)\lambda^2(\alpha) X_0^2 t^2 + f(1,1;t)\lambda(\alpha) X_0 (X_0-1) t$$

Recall that $f(1,1;t) = 1$. So

$$\frac{\partial^2 f(1,1;t)}{\partial s_1^2} = \lambda^2(\alpha) X_0^2 t^2 + \lambda(\alpha) X_0^2 t - \lambda(\alpha) X_0 t .$$

Then $VAR(X) = \lambda^2(\alpha)X_0^2 t^2 + \lambda(\alpha)X_0^2 t - \lambda(\alpha)X_0 t - \lambda^2(\alpha)X_0^2 t^2 = \lambda(\alpha)X_0^2 t$.

Similarly $VAR(Y) = \lambda(\alpha)Y_0^2 t$.

To find the covariance, first a partial derivative is found then the covariance is found.

$$\frac{\partial^2 f(1,1;t)}{\partial s_1 \partial s_2} = \frac{\partial}{\partial s_2}\left(\frac{\partial f(1,1;t)}{\partial s_1}\right)$$

$$= \frac{\partial}{\partial s_2}\left(f(s_1,s_2;t)(\lambda(\alpha)X_0 s_1^{X_0-1} \lambda(\alpha)Y_0 s_2^{Y_0})t\right)\Big|_{s_1=s_2=1}$$

$$= [f(s_1,s_2;t)(\lambda(\alpha)X_0 s_1^{X_0-1} \lambda(\alpha)Y_0 s_2^{Y_0-1} t + f(s_1,s_2;t)$$

$$\lambda(\alpha)X_0 + f(s_1,s_2;t)\lambda(\alpha)X_0 s_1^{X_0-1} s_2^{Y_0-1} t]\Big|_{s_1=s_2=1}$$

$$= f(1,1;t)\lambda^2(\alpha)X_0 Y_0 t^2 + f(1,1;t)\lambda(\alpha)X_0 Y_0 t$$

$$= \lambda^2(\alpha)X_0 Y_0 t^2 + \lambda(\alpha)X_0 Y_0 t$$

So, $COV(X,Y) = \lambda^2(\alpha)X_0 Y_0 t^2 + \lambda(\alpha)X_0 Y_0 t - E(X)E(Y)$

$$= \lambda^2(\alpha)X_0 Y_0 t^2 + \lambda(\alpha)X_0 Y_0 t - \lambda^2(\alpha)X_0 Y_0 t^2$$

$$= \lambda(\alpha)X_0 Y_0 t .$$

EXAMPLE II. Assume that the distribution of the two different materials is a bivariate Poisson distribution with the following probability generating function:

$$g(s_1,s_2) = \exp(-\lambda_1-\lambda_2-\lambda_{12}+\lambda_1 s_1+\lambda_2 s_2+\lambda_{12} s_1 s_2) .$$

Then the probability generating function of the amount of material arriving at the border in time t is;

$$f(s_1,s_2;t) = \exp\{[-\lambda(\alpha)+\lambda(\alpha)\ \exp(-\lambda_1-\lambda_2-\lambda_{12}+\lambda_1 s_1+\lambda_2 s_2+\lambda_{12}s_1 s_2)]\,t\}.$$

Now we find the marginal expectations, variances and the covariance.

$$\frac{\partial f(s_1,s_2;t)}{\partial s_1} = f(s_1,s_2;t)\ \frac{\partial[-\lambda(\alpha)+\lambda(\alpha)g(s_1,s_2))t]}{\partial s_1}$$

$$= f(s_1,s_2;t)\ \lambda(\alpha)g(s_1,s_2)\ (\lambda_1+\lambda_{12}s_2)t$$

$$f(1,1;t) = f(1,1;t)\ \lambda(\alpha)g(1,1)\lambda_1+\lambda_{12})t$$

$$= \lambda(\alpha)g(1,1)f(1,1,;t)\lambda_1+\lambda_{12})t\ .$$ But

$$f(1,1;t) = \exp\ (-\lambda(\alpha)+\lambda(\alpha)g(1,1)t)$$

So we must find $g(1,1)$.

$$g(1,1) = \exp(-\lambda_1-\lambda_2-\lambda_{12}+\lambda_1+\lambda_2+\lambda_{12}) = 1\ .$$ So

$$f(1,1;t) = \exp[(-\lambda(\alpha)+\lambda(\alpha))t] = 1\ .$$ So

$$E(X) = (\lambda_1+\lambda_{12})\lambda(\alpha)\ t\ .$$

Similarly, $E(Y) = (\lambda_2+\lambda_{12})\lambda(\alpha)t$.

Now to find the variance of X. We first compute a required partial derivative.

$$\frac{\partial^2 f(1,1;t)}{\partial s_1^2} = \frac{\partial}{\partial s_1}\,[f(s_1,s_2;t)\lambda(\alpha)t(\lambda_1+\lambda_{12}s_2)g(s_1,s_2)]\Big|_{s_1=s_2=1}$$

$$= [f(s_1,s_2;t)\lambda^2(\alpha)t^2(\lambda_1+\lambda_{12}s_2)^2 g(s_1,s_2)^2 +$$

$$+ f(s_1,s_2;t)\lambda(\alpha)t(\lambda_1+\lambda_{12})^2 g(s_1,s_2)]\Big|_{s_1=s_2=1}$$

$$= \lambda^2(\alpha)t^2(\lambda_1+\lambda_{12})^2+\lambda(\alpha)t(\lambda_1+\lambda_{12})^2$$

So $$\mathrm{VAR}(X) = \lambda^2(\alpha)t^2(\lambda_1+\lambda_{12})^2+\lambda(\alpha)t(\lambda_1+\lambda_{12})^2+\lambda(\alpha)t(\lambda_1+\lambda_{12})$$

$$-\lambda^2(\alpha)t^2(\lambda_1+\lambda_{12})^2$$

$$= \lambda(\alpha)t(\lambda_1+\lambda_{12}^2+\lambda(\alpha)t(\lambda_1+\lambda_{12})$$

$$= \lambda(\alpha)t(\lambda_1+\lambda_{12})(\lambda_1+\lambda_{12}+1) .$$

Similarly, $\mathrm{VAR}(Y) = \lambda(\alpha)t(\lambda_2+\lambda_{12})(\lambda_2+\lambda_{12}+1)$.

Now to find the covariance of X and Y a required partial derivative is found, then this is used to find the covariance.

$$\frac{\partial^2 f(1,1;t)}{\partial s_1 \partial s_2} = \frac{\partial}{\partial s_2}[f(s_1,s_2,t;)\ (\alpha)t(\lambda_1+\lambda_{12}s_2)g(s_1,s_2)]\Big|_{s_1=s_2=1}$$

$$= [f(s_1,s_2;t)\lambda^2(\alpha)t^2(\lambda_1+\lambda_{12}s_2)(\lambda_2+\lambda_{12}s_1)g(s_1,s_2)$$

$$+ f(s_1,s_2;t)\lambda(\alpha)t\lambda_{12}g(s_1,s_2)$$

$$+ f(s_1,s_2;t)\lambda(\alpha)t(\lambda_1+\lambda_{12}s_2)(\lambda_2+\lambda_{12}s_1)g(s_1,s_2)]\Big|_{s_1=s_2=1}$$

$$= \lambda^2(\alpha)t^2(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})+\lambda(\alpha)t\lambda_{12}+\lambda(\alpha)t(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12}) .$$

$$\mathrm{COV}(X,Y) = \lambda^2(\alpha)t^2(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})+\lambda(\alpha)t\lambda_{12}+\lambda(\alpha)t(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})$$

$$-\lambda^2(\alpha)t^2(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})$$

$$= \lambda(\alpha)t\lambda_{12} + \lambda(\alpha)t(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12}) .$$

Case III Let n be the common capacity of the trucks and let (X,Y) denote to amount of war material and supplies in a truck. We assume that

$$P(X = r, Y = n-r) = \binom{n}{r} \pi^r \theta^{n-r} ;$$

when $\theta = 1 - \pi$. Here π is the probability of loading one ton of war material and $\theta = 1 - \pi$ is the probability of loading one ton of supplies.

The amount of material reaching $x = x_0$ in time t is given by $\left(\sum_{i=1}^{N(+)} X_i , \sum_{i=1}^{N(+)} Y_i \right)$ and its probability generating function is given by

$$f(s_1, s_2; t) = \exp[-\lambda(\alpha)t + \lambda(\alpha)t \ (\pi s_1 + \theta s_2)^n]$$

The mean of X is given by,

$$E(X) = \frac{f(1,1;t)}{\partial s_1} = f(1,1;t)n\lambda(\alpha)[\pi(1)+\theta(1)]^{n-1}\pi$$

$$= n\pi\lambda(\alpha)t .$$

Similarly $E(Y) = n\theta\lambda(\alpha)t$.

Now we find the variances of X and Y . First we find

$$\partial^2 f(1,1;t) = [f(s_1,s_2;t)n^2\lambda^2(\alpha)(\pi s_1+\theta s_2)^{2(n-1)}\pi^2 t^2$$

$$+ f(s_1,s_2;t)n(n-1)\lambda(\alpha)(\pi s_1+\theta s_2)^{n-2}\pi^2 t]_{s_1 = s_2 = \pi}$$

$$= n^2\lambda^2(\alpha)\pi^2 t^2 + n(n-1)\lambda(\alpha)\pi^2 t^2 .$$

The variance is

$$\mathrm{VAR}(X) = n^2\lambda(\alpha)\pi^2t^2 + n^2\lambda(\alpha)\pi^2t - n\lambda(\alpha)\pi^2t + n\lambda(\alpha)\pi t - n^2\pi^2\lambda^2(\alpha)t^2$$

$$= [n^2\lambda(\alpha)\pi^2 - n\lambda(\alpha)\pi^2 + n\pi\lambda(\alpha)]t$$

$$= [n^2\lambda(\alpha)\pi^2 + n\pi\lambda(\alpha)(1-\pi)]t$$

$$= [n^2\lambda(\alpha)\pi^2 + n\pi\theta\lambda(\alpha)]t \ .$$

Similarly $\mathrm{VAR}(Y) = [n^2\lambda(\alpha)\theta^2 + n\pi\theta\lambda(\alpha)]t$.

The covariance of X and Y can now be computed.

$$\mathrm{COV}(X,Y) = \frac{\partial^2 f(1,1;t)}{\partial s_1 \partial s_2} - E(X)E(Y)$$

$$\frac{\partial^2 f(1,1;t)}{\partial s_1 \partial s_2} = \frac{\partial}{\partial s_2}[f(s_1,s_2;t)n\lambda(\alpha)(\pi s_1 + s_2)^{n-1}\pi t]\Big|_{s_1=s_2=1}$$

$$= [f(s_1,s_2;t)n^2\lambda^2(\alpha)(\pi s_1+\theta s_2)^{2(n-1)}\pi\theta t^2$$

$$+ f(s_1,s_2;t)n(n-1)\lambda(\alpha)(\pi s_1+\theta s_2)^{n-2}\pi\theta t]\Big|_{s_1=s_2=1}$$

$$= n^2\lambda^2(\alpha)\pi\theta t^2 + n(n-1)\lambda(\alpha)\pi\theta t$$

$$\mathrm{COV}(X,Y) = n^2\lambda^2(\alpha)\pi\theta t^2 + n^2\lambda(\alpha)\pi\theta t - n\lambda(\alpha)\pi\theta t - n^2\lambda^2(\alpha)\pi\theta t^2$$

$$= n\lambda(\alpha)\pi\theta(n-1)t \ .$$

.2.2 Finally, we study the effect of consumption and air interdiction. We assume that each truck has (X_0,Y_0) tons of material in it. Then the amount of material arriving at $x = x_0$ in time t is distributed as bivariate Poisson with parameters $(\lambda(\alpha)X_0, \lambda(\alpha)Y_0, 0)$. Also assume that some of each of the materials on each truck is consumed, some is used, some might be damaged, and some might be stolen. First we assume that the consumption is binomial. That is, if there were (n_1,n_2) tons of material, then there will be (r_1,r_2) tons after consumption. The probability generating function of the consumption is,

$$g(s_1,s_2) = (p_1's_1+1-p_1')^{n_1}(p_2'+1-p_2')^{n_2} .$$

It follows that the probability generating function of the resulting process is,

$$f(s_1,s_2;t) = \exp\ (\lambda(\alpha)p_1'(s_1-1)+\lambda(\alpha)p_2'(s_2-1))t .$$

But this is the probability generating function of the Poisson process with parameters $(\lambda(\alpha)p_1't,\ \lambda(\alpha)p_2',0)$. Where p_1' is the probability of consuming one ton of war material, and p_2' is the probability of consuming one ton of supplies.

Now assume that the consumption is distributed as Poisson with parameters $(\lambda_1,\lambda_2,\lambda_{12})$, then the generating function of the resulting process is,

$$f(s_1,s_2;t) = \exp[-\lambda(\alpha)+\lambda(\alpha)\ \exp(-\lambda_1-\lambda_2-\lambda_{12}+\lambda_1 s_1+\lambda_{12}s_1 s_2)]t$$

Now we should like to consider what happens when aircraft are dispatched to bomb this country in an attempt to destroy some of the trucks. We will investigate different types of bombing.

We assume that the probability of destruction is directly proportional to the number of trucks present and to the number of aircraft dispatched to bomb the trucks. We will also assume that the destruction process is binomial. That is,

$$P(Y=r) = \binom{n}{r} p^r(1-o)^{n-r}\ 0<p<1 .$$

Or in other words, the probability that r trucks remain after bombing given that there were n trucks before the bombing, is binomial.

Now if the arrival of trucks at $x = x_0$ in time t is distributed as a Poisson distribution with parameter $\lambda(\alpha)$, then the probability distribution of the arrival of undamaged trucks in time t is,

$$P(Y=r) = e^{-\lambda(\alpha)pt} \frac{(\lambda(\alpha)pt)^r}{r!} .$$

So if we know that the distribution of trucks at $x = x_0$ is Poisson with parameter $\lambda(\alpha)$, then the distribution after the bombing is Poisson with parameter $\lambda(\alpha)pt$, where p is the probability of destruction of a single truck.

Now we consider what happens if we allow each type of material to be damaged at a different rate. That is, some of the material on each truck could be damaged and some undamaged.

Then the distribution of material arriving at the border is bivariate Poisson distribution with parameters $(\lambda(\alpha)X_0p_1t , \lambda(\alpha)Y_0p_2t , 0)$. Here p_1 is the probability of destroying one ton of war material, and p_2 is the probability of destroying one ton of supplies. This follows from Theorem 1.3.2.

Now consider the problem of having both consumption and interdiction. Assume that both are binomial, and the loading is constant. That is, assume each truck has (X_0,Y_0) tons of material on it. Let p_1' be the probability of consuming one ton of war material and p_1'' be the probability of consuming one ton of supplies and p_2' be the probability of consuming one ton of supplies and p_2'' be the probability of destroying one ton of supplies. Then by Theorem 1.3.2 and the previous result for consumption only, the distribution of material arriving at $x = x_0$ in time t undamaged is Poisson with parameters, $(\lambda(\alpha)X_0p_1'p_1''t, \lambda(\alpha)Y_0p_2'p_2''t, 0)$.

3 General Discussion

The purpose of this paper is to construct a stochastic model of an air interdiction process.

This model shares some shortcomings with the deterministic model. For example, the stochastic model assumes that the flow of trucks is uniform across the country in a direction perpendicular to the traffic flow. This assumption is not realistic. Also, it was assumed that enough time has elapsed for the process to become stable. This may also hide a policy of air interdiction. We also assumed that there was no interaction between passing trucks. This also is not realistic.

There is also the practical problem of determining the parameters of the process. In an actual situation the data to determine these parameters would be difficult to obtain.

There are some questions raised by this study that are not answered here. First there is the problem of choosing a strategy of bombing that will minimize, in some sense, the amount of material arriving in $x = x_0$ in time t . This might be done by attempting $E(X+Y)$ as a function of the number of aircraft dispatched or the number of trucks in the target area and attempt to minimize $E(X+Y)$.

This might be done by the standard methods, or other methods. There is also the question of whether these are the only combinations of transportation process and bombing and consumption processes that yield well-known distributions.

BIBLIOGRAPHY

Armitage, John V. "Allocation of Sorties in Air Interdiction," Journal of the Operations Research Society of America, XVIII, No. 3, (May - June 1970), pp. 483-496.