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13. ABSTRACT

A theory for the motion of two-dimensional turbulent vortex pairs in homogeneous media has been developed based on separate velocity scaling of the internal and external flow fields involved in the motion and taking into account variations in volume, circulation, momentum, and energy. Based on the results obtained from this theory, ~~(I)~~ a simplified theory ~~(II)~~ is derived to deal with the rising motion of turbulent vortex pairs in stratified media. The theoretical results are compared with systematic experimental observations.

In theory (I) the ratio of internal to external velocity scales, ψ, is introduced as an important variable and the theory is specifically derived for the two limiting cases of weak ($\psi \rightarrow 1$) and strong ($\psi >> 1$) circulation. The weak circulation theory leads to results similar to those obtained in the past using theory based on complete similarity and momentum conservation; i.e., $z \sim t^{\frac{1}{3}}$. The strong circulation theory leads to results which depend very much on the way in which vorticity from the shear layer is ingested into the vortex pair. When ingested so as to cause annihilation (cancellation) of the ingested vorticity, the asymptotic trajectory is $z \sim t^{\frac{1}{2}}$. Under the same conditions the velocity ratio, ψ, increases toward an asymptotic value, and the virtual momentum coefficient for the motion

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tends to zero. As a result, the asymptotic motion (assuming vorticity annihilation) corresponds to a motion with complete similarity and with energy conservation.

A comparison of experimental observations of rise versus time and radius versus height with theory (I) lend strong support to the strong circulation theory and suggest that ingested vorticity may be largely annihilated.

Based on these finding for homogeneous flows, a simplified theory (II) for stratified media was developed upon the assumptions: (i) the motion is determined by conservation of volume, mass, and energy (neglecting vorticity and momentum); (ii) complete similarity ($dR/dz = \beta$, a constant). Good agreement was found between the predictions of this theory and the results of systematic experiments, and particularly for the maximum rise of height.

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THE MOTION OF TURBULENT VORTEX-PAIRS IN HOMOGENEOUS AND DENSITY STRATIFIED MEDIA

By

Marshall P. Tulin and Josef Shwartz

April 1971

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NOTATION

a	Density gradient in surrounding fluid, $a = \frac{1}{\rho_e}\frac{d\rho_e}{dz}$
A	Initial buoyancy parameter (theory), Equation [44]
A_1	Vorticity mixing coefficient, Equation [9]
b	Half distance between cores of vortex-pair
B	Stratification parameter (theory), Equation [45]
c	Constant
C_D	Energy dissipation coefficient
D	Energy dissipation parameter, Equation [46]
E	Total energy of convected mass
G	Experimental buoyancy parameter, Equation [44]
j	Geometrical parameter, $j = 0$ for planar geometry, $j = 1$ for axial symmetry
k	Virtual potential energy coefficient, Equation [38]
k_m	Virtual mass coefficient
K	Virtual kinetic energy coefficient, Equation [37]
M_z	Vertical component of total momentum
n	Parameter defined by Equation [47]
r	Radial distance from center of rising mass, $r^2 = \xi^2+\eta^2+\zeta^2$
R	Mean radius of rising mass
$\bar{R}$	Non-dimensional mean radius of rising mass, $\bar{R} = R/R_o$
S	Experimental density stratification parameter, Equation [53]
t	Time
$\bar{t}$	Non-dimensional time, $\bar{t} = W_o t/z_o$
t_{max}	Time at which the maximum height of rise is reached

u, v, w	Velocity components, see Figure 14
W	Vertical velocity of rising mass
$\bar{W}$	Non-dimensional vertical velocity of rising mass, $\bar{W} = W/W_o$
W^*	Vertical velocity of ideal vortex-pair
z	Height of rising mass center above its virtual origin
$\bar{z}$	Non-dimensional height of rising mass center, $\bar{z} = z/z_o$
z_{max}	Maximum height reached by rising mass
β	Modified entrainment coefficient
γ	Local vorticity, Equation [7]
Γ	Total circulation about a single vortex
ξ, η, ζ	Coordinate system, see Figure 14
ρ	Local density inside convected mass
ρ_e	Density of surrounding fluid
ρ_i	Average density of convected mass, Equation [43]
$\Delta\rho$	Density difference, $\Delta\rho = \rho_i - \rho_e$
ψ	Velocity ratio, W_i/W_e

<u>Subscripts</u>

$(\ )_o$	Initial conditions
$(\ )_i$	Internal
$(\ )_e$	External

ABSTRACT

A theory for the motion of two-dimensional turbulent vortex pairs in homogeneous media has been developed based on separate velocity scaling of the internal and external flow fields involved in the motion and taking into account variations in volume, circulation, momentum, and energy. Based on the results obtained from this theory (I) a simplified theory (II) is derived to deal with the rising motion of turbulent vortex pairs in stratified media. The theoretical results are compared with systematic experimental observations.

In theory (I) the ratio of internal to external velocity scales, ψ, is introduced as an important variable and the theory is specifically derived for the two limiting cases of weak ($\psi \rightarrow 1$) and strong ($\psi >> 1$) circulation. The weak circulation theory leads to results similar to those obtained in the past using theory based on complete similarity and momentum conservation; i.e., $z \sim t^{\frac{1}{2}}$. The strong circulation theory leads to results which depend very much on the way in which vorticity from the shear layer is ingested into the vortex pair. When ingested so as to cause annihilation (cancellation) of the ingested vorticity, the asymptotic trajectory is $z \sim t^{\frac{1}{2}}$. Under the same conditions the velocity ratio, ψ, increases toward an asymptotic value, and the virtual momentum coefficient for the motion tends to zero. As a result, the asymptotic motion (assuming vorticity annihilation) corresponds to a motion with complete similarity and with energy conservation.

A comparison of experimental observations of rise versus time and radius versus height with theory (I) lend strong support to the strong circulation theory and suggest that ingested vorticity may be largely annihilated.

Based on these finding for homogeneous flows, a simplified theory (II) for stratified media was developed upon the assumptions: (i) the motion is determined by conservation of volume, mass, and energy (neglecting vorticity and momentum); (ii) complete similarity ($dR/dz = \beta$, a constant). Good agreement was found between the predictions of this theory and the results of systematic experiments, and particularly for the maximum rise of height.

I. INTRODUCTION

Ideal Vortex-Pairs. Flow visualization studies carried out by Scorer (1957), Woodward (1959) and Richards (1965), indicate that the shear layer which is formed between a moving isolated mass of fluid and the stationary surrounding medium tends to roll up and create a flow field which resembles (in two dimensions) the one associated with two line vortices of equal strength but opposite sign, separated by a distance 2b, so-called "vortex-pairs." The possibility of vortex-pair motions in an inviscid fluid was considered and analyzed over 100 years ago by Sir. W. Thomson (1867). His analysis applies only to an idealized vortex-pair in which each vortex has a highly concentrated core which is set into motion only by the influence of the other vortex. Such an ideal vortex-pair moves through the surrounding fluid in a direction perpendicular to the plane joining the vortex cores and with a velocity W^* determined only by the pair separation, 2b, and the circulation about a single vortex, Γ, according to the relation

$$W^* = \frac{\Gamma}{4\pi b} \qquad [1]$$

A unique feature of this idealized vortex-pair motion is the existence of a closed streamline and a finite captured mass, as indicated by the oval in Figure 2a. Thomson (1867) calculated the semi-axes of the oval-shaped captured mass to be 2.09b and 1.73b so that the cross-sectional area is approximately $3.62\pi b^2$ and the ratio of width to thickness is 1.21.

Under certain circumstances it is entirely possible that carefully balanced vortex-pairs, approximating Thomson's idealization, can be formed. The motion around the vortex centers must be affected by viscosity in a real fluid, but as long as the viscous cores do not extend close to the bounding closed streamline, the flow within and without this streamline may be so closely matched that no large shearing motions or accompanying drag are associated with the motion of the captured mass. In fact, nearly ideal vortex-pairs are sometimes found in the wakes of lifting surfaces, Figure 1a, and are known as "contrails", see Scorer (1958) and Spreiter and Sachs (1951). Of course, the concentrated vorticity in the vortex cores tends to diffuse, and does so rapidly when the flow in the core is turbulent.

Turbulent Vortex-Pairs. The probable short lifetime of ideal vortex-pairs under turbulent conditions gives special importance to vortex-pairs whose behavior is governed by turbulent entrainment; indeed, it is these kinds of motions which are most commonly observed in nature, as in the case of a mass of fluid forced out rapidly through an aperture, Figure 1b, or in the convection of isolated masses in nature, Figure 1c, or in the bent-over and rising chimney plume.

Turbulent vortex-pairs are characterized by the fact that the interior motion does not match the outer flow at the boundary of the captured mass, so that a region of high shear exists there, accompanied by the production of vorticity and by turbulent entrainment. In other words, these vortex-pairs move with a velocity W not equal to the velocity W^* derived from Thomson's model, Equation [1].

We may, in principle, generalize Thomson's model to consider those cases where the velocity of translation, W, has a more general relation to the velocity W^* which characterizes the internal, rotational motions of the vortex-pair. Two distinct cases suggest themselves, in theory. In one case, $W > W^*$, and the convected mass loses volume to the surrounding fluid and continually shrinks in size; we denote this vortex-pair as "underdeveloped", see Figure 2b. In the other case, $W < W^*$, and the "overdeveloped" vortex-pair gains mass through the entrainment of exterior fluid, Figure 2c. Of these it is the latter motion which is most commonly observed in nature and forms the main subject of this work.

Within an overdeveloped motion, the velocity at the boundary inside the vortex-pair, as seen by an observer moving with it, will be larger than the velocity of the surrounding fluid just outside the boundary of the pair. Accompanying the velocity gradients thus created across this boundary, shear stresses are exerted by the vortex-pair on the surrounding fluid, resulting in the entrainment of outer fluid and a general increase in the volume of the convected mass, see Figure 3. Within the high shear zones at the boundary on either side vorticity of sign

opposite to that within the respective interior is created, and is pulled down around the bottom of the rising mass toward the plane of symmetry. To the extent that the ingested vorticity remains on its own side of this plane, the vorticity within the interior will be steadily reduced; of course, ingested vorticity of opposite sign does have a chance to mix and thus to annul itself, depending on the efficiency of mixing. Should effective annihilation of ingested vorticity occur, then the initial total vorticity within one side of the pair would be conserved in time.

As for the kinetic energy implicit in the motion of the vortex pair, it must be continuously reduced with time due to turbulent dissipation.

Self-Similarity in Vortex-Pair Motions. It is a striking characteristic of free turbulent flows in homogeneous media at sufficiently high Reynolds numbers that, under similar circumstances, the flows at different points in space or time can usually be reduced from one to another upon normalization by an appropriate length and velocity scale (self-similarity). This is true, for example, of the flow at different downstream sections of turbulent jets and wakes. It is therefore natural to expect that a turbulent vortex-pair exhibits complete self-similarity during its life time, and this assumption has been made in all theoretical treatments of the subject, starting with Morton, Taylor, and Turner (1956). Two important consequences of this complete similarity are: (1) conservation of the ratio of internal and external velocity scales, W/W^*, during the motion; (2) linearity of the length scale of the convected mass with the distance traveled from a virtual origin.

This latter result, predicting that the traces of the side boundaries of the convected mass form a wedge, is independent of the dynamics of the motion and serves to provide a check on self-similarity. In fact, a number of previous experiments on self-convecting masses claim to confirm this behaviour to a reasonable approximation, see, e.g., Scorer (1958), Woodward (1959) and Richards (1965).

It is obvious to ask whether a "natural" value of the velocity ratio W/W^*, or the same thing, of the constant $\beta = dR/dz$ is observed, independent of the original circumstances giving rise to the convected mass. The answer seems to be no. In the present experiments, two distinctly different ranges of value of dR/dz differing by a factor 2, have been repeatedly measured; these correspond to two different stroke lengths in the apparatus used to originate the vortex motions. Furthermore, although the present data may be claimed to correspond "in a reasonable approximation" to a constant value of dR/dz, yet quite consistent deviations from linearity exist between the traces of part radius and distance traveled, see Figure 4. These deviations are such that dR/dz seems actually to <u>increase</u> throughout the observed motions.

In view of these facts, and for other reasons, it seems desirable to attempt a more general theory of the motion of turbulent vortex pairs, based on the assumption of separate velocity scaling of the internal and external motions; i.e. allowing W/W^* to vary continuously. Afterwards, a simplified theory pertaining to motions in stratified media will be developed and the results compared to experimental observations.

II. THEORY (HOMOGENEOUS FLOWS)

Separate Similarity of Internal and External Flows. We visualize the vortex-pair motion to be divided into internal and external flow fields, separated by a thin region of high shear, which also forms the boundary of the captured mass, see Fig. 3. We assume that each flow field is itself self-similar.

Internal.

$$w_i(x,y,t) = W_i(t)\cdot\bar{w}_i\left(\frac{x}{R}\,;\,\frac{y}{R}\right) \qquad [2]$$

External.

$$w_e(x,y,t) = W_e(t)\cdot\bar{w}_e\left(\frac{x}{R}\,;\,\frac{y}{R}\right) \qquad [3]$$

and similarly for the other velocity components.

We let $\psi = W_i/W_e$, where ψ is, in general, not constant in time as it is in the case of complete similarity.

We choose W_i as the circumferental velocity averaged over the inner boundary of one-half of the vortex pair and W_e the same except averaged over the outer boundary. (The inner and outer boundaries are separated by a thin shear layer).

Volume Changes. The volume of fluid comprising the vortex pair increases continuously with time due to entrainment into it. Because of the similarity assumed, the rate of entrainment of volume must (in two dimensions) be proportional to a characteristic velocity and a characteristic length. We take for the former, the velocity difference $W_i - W_e$

$$\frac{d(\pi R^2)}{dt} = 2\pi R(W_i - W_e)\cdot\alpha(\psi) \qquad [4]$$

or,

$$\frac{dR}{dz} = \frac{W_e}{W}(\psi-1)\cdot\alpha(\psi) = (\psi-1)\alpha'(\psi) \qquad [5]$$

Note that the dependence of the proportionality constant α' on the velocity ratio ψ has been left unspecified.

Circulation Changes. The circulation Γ about one half of the vortex pair is, on account of the inner similarity, proportional to the length scale and inner velocity scale. On account of the way in which W_i was defined,

$$\Gamma \propto RW_i \qquad [6]$$

The fluid entrained into the vortex pair from the surrounding high shear layer carries vorticity of opposite sign to that already within the interior, see Fig. 3 and thus reduces it to the extent it is not annihilated through mixing with vorticity being entrained in the opposite lobe. The strength of entrained vorticity must, on account of similarity, be proportional to the ratio of a pertinent velocity and length scale. In particular:

$$\gamma(\text{entrained}) \propto \frac{(W_e - W_i)}{R} \qquad [7]$$

The flux of entrained vorticity takes the form:

$$\text{Vorticity Flux} \propto \text{Volume Flux} \cdot \gamma(\text{entrained})$$

$$\propto [2\pi R(W_i - W_e)\cdot\alpha(\psi)]\left[\frac{W_e - W_i}{R}\right] \qquad [8]$$

Finally, the change in circulation may be related to the flux of vorticity:

$$\frac{d\Gamma}{dt} = -A_1 \cdot 2\pi\alpha(\psi)\cdot(W_i - W_e)^2 \qquad [9]$$

or,

$$\frac{d(W_i R)}{dz} = -A_1\ \alpha'\cdot(W_i - W_e)\cdot(\psi - 1) \qquad [10]$$

The value of the parameter A_1 will depend in part upon the extent to which entrained vorticity from each side mixes together causing annihilation. In the case of complete annihilation, $A_1 \equiv 0$.

<u>Momentum Conservation</u>. The momentum, M_z, of the vortex pair is conserved in motion through a homogeneous medium. It may on account of similarity be expressed in the form,

$$\frac{M_z}{\rho} = K_m(\psi)\cdot W_e R^2 = \text{const.} \qquad [11]$$

where $K(\psi)$ is a momentum coefficient. Its form may be deduced through use of the identity, Lamb (1945), pg. 229,

$$\frac{M_z}{\rho} = \iint \gamma\zeta d\xi d\zeta \qquad [12]$$

The latter integral can be taken separately over the interior and shear layer of the vortex pair, which yields terms upon making use of similarity,

$$\iint_{\text{int.}} \gamma\zeta d\xi d\zeta \propto W_i R^2 \qquad [13]$$

$$\iint_{\text{shear layer}} \gamma\zeta d\xi d\zeta \propto (W_e - W_i)R^2 \qquad [14]$$

As a result, the form of $K_m(\psi)$ consistent with our assumptions, is seen to be,

$$K_m(\psi) = K_1 - K_2\psi \qquad [15]$$

where K_1 and K_2 are undetermined constants.

Energy. The total kinetic energy in the vortex pair motion, taking account of similarity, may be expressed as the sum of two terms,

$$\frac{K.E.}{\rho} = K_i W_i^2 R^2 + K_e W_e^2 R^2 \qquad [16]$$

while the dissipation takes the form,

$$\frac{d\,\frac{KE}{\rho}}{dt} = - W_i^3 R \cdot C_D(\psi) \qquad [17]$$

where for large values of ψ, C_D must approach some limiting value C_D', while for small values of $\psi(\psi \to 1)$, $C_D \to C_D(1) \cdot \left(\frac{W_e}{W_i}\right)^3$.

Laws of Motion. Limiting Cases; Weak Circulation ($\psi \to 1$). In this case the inner and outer flows are almost matched and the deviation from the ideal vortex motion is small. The laws of motion in their appropriate form become,

Volume. $$\frac{dR}{dz} = (\psi-1) \cdot \alpha'(1) \qquad [5a]$$

Vorticity. $$\frac{d(W_e R)}{dz} = - A_1 \alpha'(1) \cdot (\psi-1)^2 \cdot W_e \qquad [10a]$$

Momentum. $$K_m(1) \cdot W_e R^2 = \text{const.} \qquad [11a]$$

Energy. $$K(1)\,\frac{d(W_e{}^2R^2)}{dz} = -\left(\frac{W_e}{W}\right) W_e{}^2R\cdot C_D(1) \qquad [17a]$$

Combining [5a] and [10a] leads to the result,

$$W_e \propto \frac{1}{R^{1+A_1(\psi-1)}} \qquad A_1 \neq 0 \qquad [18]$$

Whereas, [11a] requires $$W_e \propto \frac{1}{R^2} \quad \text{or } z \propto t^{\frac{1}{3}} \qquad [19]$$

so that the presumed motion can take place only if $A_1 = 0$ or $A_1 = (\psi-1)^{-1}$. Combining [17a] and [10a] leads to the requirement that the velocity ratio be constant and have the value

$$\psi = 1 + \left[\frac{C_D(1)}{2K(1)\cdot\alpha(1)}\right] \qquad [20]$$

At the same time, dR/dz is required to be constant and to have the value,

$$\psi \to 1 \qquad \frac{dR}{dz} = \left(\frac{W_e}{W}\right)\cdot\frac{C_D(1)}{2K_e(1)} \qquad [21]$$

We leave till later a discussion of comparison with experiment, but we may note now that the prediction [19] is similar to that of the previous theory based on complete similarity.

<u>Strong Circulation ($\psi \gg 1$)</u>. In this case the interior circulation is very strong relative to the ideal value and the deviation from the ideal vortex motion is large. The appropriate laws of motion are,

Volume. $$\frac{dR}{dz} = \psi\cdot\alpha'(\psi) \qquad [5b]$$

Vorticity. $$\frac{d(W_1R)}{dz} = -A_1\alpha'(\psi)\cdot W_1\cdot\psi \qquad [10b]$$

Momentum. $$(K_1 - K_2\psi)\cdot W_1R^2 = \psi\cdot(M_z/\rho) \qquad [11b]$$

Energy. $$K_eW\frac{d}{dz}(W_1R)^2 = -W_1^3\cdot R\cdot C_D' \qquad [17b]$$

Combining [10b] and [17b] leads to the requirement that $\alpha(\psi)$ be constant and equal to,

$$\alpha = \frac{C_D'}{2K_e\cdot A_1} \qquad [21]$$

Since similarity requires that W_e/W be constant, we may hereafter take α' constant in [5b] and [10b]. Combining [5b] and [10b[leads to the result

$$W_1 \propto \frac{1}{R^{(1+A_1)}} \quad \text{or} \quad W_1/W_{1_o} \propto \left(\frac{R_o}{R}\right)^{1+A_1} \qquad [22]$$

and, substituting [22] and [5b] into [11b] leads to the differential relation,

$$\left(K_1 - \frac{K_2}{\alpha'}\frac{d\bar{R}}{d\bar{z}}\right)\bar{R}^{(1-A_1)} = \frac{1}{\alpha'}\cdot\frac{d\bar{R}}{d\bar{z}} \qquad [23]$$

which has the solution,

$$\alpha' K_1\cdot z = \frac{\bar{R}^{A_1}}{A_1} + K_2\bar{R} + \text{const.} \qquad (A_1 \neq 0) \qquad [24]$$

and $\alpha' K_1 \bar{z} = \ln \bar{R} + K_2 \bar{R} + \text{const.}$ $(A = 0)$

or,

$$\alpha' K_1 \frac{(z-z_o)}{R_o} = \ln \bar{R} + K_2(\bar{R} - 1) \qquad [24a]$$

Substituting dR/dz derived from [23] into [5b] yields a relation between ψ and R,

$$\psi = \frac{K_1}{\bar{R}^{(A_1-1)} + K_2} \qquad [25]$$

and, finally, it may be shown that,

$$\frac{W_e}{W_{i_o}} = \frac{K_1}{\bar{R}^{2A_1} + K_2 \bar{R}^{(1+A_1)}} \qquad [26]$$

The type of motions which ensue from this theory in the case of strong circulation are seen to depend very much on the value of A_1, the constant appearing in the relatic for circulation change, and which depends in part on the way in which vorticity is ingested into the vortex pair. In fact, the asymptotic behaviour of the vortex pair changes radically as A_1 varies around the value unity. This is demonstrated in the table below.

Asymptotic Behaviour ($\psi >> 1$)				
	$\bar{z}$	ψ	W_e/W_{i_o}	$\bar{z}$
$A_1 > 1$	$\dfrac{\bar{R}^{A_1}}{A_1\alpha' K_1}$	$K_1\ \bar{R}^{-A_1}$	$K_1\ \bar{R}^{-2A_1}$	$\sim t^{\frac{1}{3}}$
$A_1 = 1$	$\bar{R}\left(\dfrac{1+A_1K_2}{A_1\alpha' K_1}\right)$	$K_1/1+K_2$	$K_1\ \bar{R}^{-2}$	$\sim t^{\frac{1}{3}}$
$A_1 < 1$	$\bar{R}\ (K_2/\alpha' K_1)$	K_1/K_2	$K_1/K_2 \cdot \bar{R}^{-(1+A_1)}$	$\sim t^{\frac{1}{2+A_1}}$
$A_1 = 0$	$\bar{R}\ (K_2/\alpha' K_1)$	K_1/K_2	$K_1/K_2 \cdot \bar{R}^{-1}$	$\sim t^{\frac{1}{2}}$

The values of $A_1 > 0$, the velocity ratio is seen to decline, so that the strong circulation assumption must eventually become invalid. The case $A_1 = 1$ yields results qualitatively similar to the weak circulation case. In the case where $A_1 < 1$, however, the velocity ratio <u>increases</u> the asymptotic value shown and, most interesting, the added momentum coefficient $(K_1 - K_2\psi)$ vanishes asymptotically, so that the motion becomes determined by volume, vorticity, and energy balances alone. Finally, in the case where $A_1 << 1$ (effective annihilation of ingested vorticity), then asymptotically the motion becomes determined by volume and energy balances alone, yielding $\bar{z} \sim t^{\frac{1}{2}}$.

III. COMPARISON WITH EXPERIMENT (HOMOGENEOUS MEDIA)

In Figures 5 and 6 are shown data from actual experiments on two-dimensional vortex pair motions in homogeneous fluids. Most of the data shown were obtained in experiments carried out in our own laboratory. Suffice it here to show a schematic of the facility which was used, Fig. 7, and to show Table 1, in which the properties and characteristics of the experimental vortex pairs are listed.

Most significant, we found in our experiments and from the data of Richards (1965) that the measured variation of vertical velocity and pair radius (two-dimensions) conformed more closely to the law $W \sim R^{-1}$ or $z \sim t^{\frac{1}{2}}$ than to the law derived in the past by others and which is based on complete similarity and momentum conservation; i.e. $W \sim R^{-2}$ or $z \sim t^{\frac{1}{3}}$. A test of the simple conservation of momentum, $W \sim R^{-2}$, using a typical trajectory is illustrated in Fig. 5 and, similarly, in Fig. 6 it is shown that the trajectories, so far as they have been observed experimentally, conform more closely to the asymptotic law derived earlier for the case of strong circulation, utilizing a small value of A_1 $(0 < A_1 < 2)$.

In the case of strong circulation, the radius grows in a linear fashion asymptotically, but the theory predicts that during the initial phases of the motion the quantity dR/dz is less than its asymptotic value. A similar behaviour was observed in our experiments, see Fig. 4. The matching-up of these observed trajectories with the theory offers an opportunity to determine some of the constants of the theory. For this purpose

we assume to begin with that $A_1 = 0$, since the comparison between observed and theoretical trajectories suggests a small value. In this case,

$$(\alpha' K_1) \left[\frac{z-z_o}{R_o}\right] = \ln \bar{R} + K_2 [\bar{R}-1] \qquad [24a]$$

The trajectory according to Eq. [24a] with $K_2 = 1/2$ and $\alpha' K_1 = .38$ is shown in Fig. 4. A fair fit with the experiments has been achieved, and noticeably better than is possible with any linear trajectory.

The strong circulation theory thus explains the two important features of vortex pair behaviour which cannot be explained by the usual theory of complete similarity. These features are: (i) the tendency for forced vortex trajectories (homogeneous flow) to more nearly follow the law $z \sim t^{\frac{1}{2}}$ rather than $z \sim t^{\frac{1}{3}}$, and (ii) the tendency for the entrainment coefficient (dR/dz) to grow during the initial phase of the motion. These results suggest that in vortex motions in homogeneous flows the internal velocity scale grows steadily relative to the translational (external) velocity, the ratio approaching a value considerably larger than unity, while at the same time, the virtual momentum <u>coefficient</u> associated with the vortex motion approaches the value zero. The data also suggest that vorticity ingested from around one half of the vortex pair is almost annihilated through mixing with vorticity ingested from the opposite side.

IV. SIMPLIFIED THEORY (VORTEX PAIR MOTION IN STRATIFIED MEDIA)

Convected masses in nature are often rising or falling in a medium of varying density, as in the case of a chimney plume projected upwards into a stable atmosphere. The latter may be characterized by a characteristic time (the Vaissala period), $1/\sqrt{ag}$, where $a = -\frac{1}{\rho_e}\frac{d\rho_e}{dz}$ and ρ_e is the potential density of the atmosphere. The same definition can be used to characterize any density stratified media.

The motion of the convecting mass may also be characterized at any instant by the time, R/W. It is almost apparent that when the latter time is long in comparison to the Vaissala period that the effect of stratification will dominate, and conversely. That is,

Effect of Stratification Vanishes	decreasing $\left(\frac{R\sqrt{ag}}{W}\right)$	increasing	Stratification Dominates

Quite clearly, too, as the motion proceeds in time, the ratio R/W increases continuously, so that stratification <u>must</u> eventually dominate. When this happens, the vertical motion of vortex-pairs may become oscillatory, and is accompanied by the collapse and horizontal spreading of the convected mass, as illustrated in Figure 8. This behavior is, of course, not consistent with similarity either complete or of the kind assumed in the preceding section.

It is sometimes desirable to be able to estimate the trajectory of a vortex pair while it is rising in a stratified media and particularly to predict the maximum height of rise and the time

required to reach the maximum. For this purpose, we adopt here a simplified theory based essentially on the assumption of strong circulation and annihilation of ingested vorticity. In fact, the particular assumptions adopted would apply if the velocity ratio, ψ, had already closely approached its limiting value. These assumptions are: (i) the motion is determined by conservation of volume, mass, and energy (neglecting vorticity and momentum); (ii) complete similarity ($dR/dz = \beta$, a constant). For further justification of these assumptions we shall depend finally upon a comparison between theoretical predictions and the results of systematic experiments.

The energy balance is expressed as follows,

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t} \int \left[\frac{\rho}{2} (u^2+v^2+w^2) + (\rho-\rho_e)\, g\zeta \right] d\xi d\eta d\zeta = - \text{(Rate of Dissipation of Energy)} \qquad [27]$$

See Figure 14 for nomenclature.

For a self-similar, self-convecting flow, the dissipation of kinetic energy per unit volume which occurs due to the action of turbulent shear stresses must for dimensional reasons be of the form,

$$\frac{\text{Dissipation}}{\text{Unit Volume}} \propto \frac{\rho W^3}{R} \qquad [28]$$

As a result, the energy balance, Equation [27], for a self-convecting mass in a homogeneous medium of the same density takes the form,

$$\frac{K}{2} \frac{d(W^2 R^{2+j})}{dt} = - C_D \frac{W^3}{R} R^{2+j} \qquad [29]$$

where

$j = 0$ (two-dimensions)

$j = 1$ (axisymmetric)

and K is a constant of virtual energy, defined by the identity,

$$\int \frac{u^2 + v^2 + w^2}{2} \, d\xi d\eta d\zeta = \frac{K}{2} W^2 R^{2+j} \qquad [30]$$

and where C_D is a dissipation coefficient.

Making use of [29], together with the relationship $R = \beta z$, it may be shown that the height of rise follows the law,

$$z^{(2+j/2 + C_D/\beta K)} \propto t \qquad [31]$$

in a medium of uniform density with $\Delta\rho = 0$. The dissipation coefficient in nature, $D = C_D/\beta K$, may be determined by a comparison between theoretical trajectories such as given by [31] and observations of vortex pair rise in homogeneous media. As shown in Fig. 6, such a comparison leads to the conclusion that D is quite small ($D < 0.2$).

The trajectory given by [31] may be compared to the law which would apply if momentum were conserved,

$$z^{3+j} \propto t \qquad [32]$$

which coincides with [31] only if $D = 1$. The variance of observed trajectories from the momentum law [32] is clearly seen in Figs. 5 and 6.

Volume conservation in a self-similar flow leads to a linear relation between the nominal radius of the mass and the height of rise from the virtual origin z = 0:

$$R = \beta z \qquad [33]$$

Conservation of mass takes the form

$$\frac{d}{dt}\left[\left(\frac{4}{3}\right)^{j} \pi R^{2+j} \rho_i\right] = 2^j \, 2\pi R^{1+j} \rho_e W\beta \qquad [34]$$

where ρ_e, is the density of the surrounding fluid at any given height z and ρ_i is the average density within the rising mass, defined by

$$\left(\frac{4}{3}\right)^{j} \pi R^{2+j} [\rho_i(z) - \rho_e(z)] = \int (\rho-\rho_e) \, d\xi (d\eta)^j d\zeta \, , \qquad [35]$$

where the integration is taken over the entire volume of the rising mass.

The formulation of conservation of energy is based upon [27] and [28]

$$\frac{d}{dt}\left[\frac{K}{2} \rho_i W^2 R^{2+j} + k(\rho_i - \rho_e) g z R^{2+j}\right] = - C_D \rho_i \frac{W^3}{R} R^{2+j} \qquad [36]$$

where W and R are the observed and measured gross properties of the rising mass while K and k are the coefficients of the virtual kinetic and potential energies, respectively, defined in two dimensions, e.g., by

$$K\rho_i \frac{W^2R^2}{2} = \int \frac{\rho}{2} (u^2 + w^2)\, d\xi d\zeta \qquad [37]$$

and

$$k(\rho_i - \rho_e)\, zR^2 = \int (\rho - \rho_e)(z+\zeta)\, d\xi d\zeta \qquad [38]$$

In most practical instances where one is dealing with a mass of fluid convected through a homogeneous or stratified medium such as the ocean or the atmosphere, the difference between the densities of the convected and surrounding masses is very small; thatis $\Delta\rho/\rho_e << 1$, being usually of the order of 10^{-3}, and therefore ρ_i/ρ_e can be taken as 1. This assumption, frequently referred to as the Boussinesq approximation, see Phillips (1966), will be used throughout the analysis presented herein.

Using the Boussinesq approximation and the identity dz = wdt, the three conservation statements, Equations [33], [34] and [36], may be reduced to the following form in the case of a planar motion:

$$R = \beta z \qquad [39]$$

$$\left.\begin{aligned} &\frac{d\rho_i}{dz} + \frac{2\Delta\rho}{z} = 0, \text{ or} \\ &\frac{d}{dz}\left(\frac{\Delta\rho}{\rho_e}\right) + \frac{2}{z}\left(\frac{\Delta\rho}{\rho_e}\right) = a \end{aligned}\right\} \qquad [40]$$

$$z\frac{dW^2}{dz} + 2\left(1 + \frac{C_D}{\beta K}\right) W^2 + \frac{2kg}{K}\left(\frac{\Delta\rho}{\rho_e} + az\right) z = 0 \qquad [41]$$

where $a = -\frac{1}{\rho_e}\frac{d\rho_e}{dz}$ and $\Delta\rho = (\rho_i - \rho_e)$.

Finally, explicit general solutions of [40] and [41] may be found. They are:

$$\frac{\Delta\rho}{\rho_e} = \left[\left(\frac{\Delta\rho}{\rho_e}\right)_o - \frac{az_o}{3}\right]\bar{z}^{-2} + \frac{az_o}{3}\bar{z} \qquad [42]$$

$$\left(\frac{w}{w_o}\right)^2 = \left[1 - \frac{A}{(1+2D)} + B\left(\frac{1}{1+D/2} - \frac{1}{1+2D}\right)\right]\bar{z}^{-n}$$

$$+ \left[\frac{(A+B)}{(1+2D)}\right]\bar{z}^{-1} - \left[\frac{B}{(1+D/2)}\right]\bar{z}^{2} \qquad [43]$$

where

$$A \equiv \left(\frac{2k}{K}\right) G \quad \text{and} \quad G \equiv \frac{z_o g}{W_o^2}\left(\frac{\Delta\rho}{\rho_e}\right)_o \qquad [44]$$

$$B \equiv \left(\frac{2k}{3K}\right) S \quad \text{and} \quad S \equiv \frac{z_o^2 ag}{W_o^2} \qquad [45]$$

$$D \equiv \frac{C_D}{\beta K} \qquad [46]$$

$$n \equiv 2(1+D) \qquad [47]$$

and

$$\bar{z} \equiv z/z_o \qquad [48]$$

The parameter G is a measure of the initial buoyancy of the convected mass relative to its initial momentum. G is taken as positive when a net buoyancy force is acting on the mass, i.e. $\Delta\rho < 0$. S is a measure of the added buoyancy which would result from moving the convected mass a vertical distance z_o through a stratified medium. Since $\sqrt{ag}$ is the frequency with which a finite volume of fluid of given density would oscillate in a stratified medium, often referred to as the Vaisala frequency, the parameter S can also be considered as the square of the ratio of the characteristic time of the convected motion, z_o/W_o, and the reciprocal of the Vaisala frequency.

The maximum height of the rising mass, reached at the point where W = 0, is according to [43], given by the solution of the following:

$$\left[\frac{B}{(1 + D/2)}\right]\left(\frac{z_{max}}{z_o}\right)^{2+n} - \left[\frac{A + B}{1 + 2D}\right] \cdot \left(\frac{z_{max}}{z_o}\right)^{1+n} - \left[1 - \frac{A}{(1 + 2D)} + B\left(\frac{1}{1 + D/2} - \frac{1}{1 + 2D}\right)\right] = 0 \qquad [49]$$

It is of interest to consider certain special cases:

1. A mass rising in a homogeneous medium with the same density as itself; i.e., A = 0 and B = 0. Then,

$$\left(\frac{W}{W_o}\right) = \left(\frac{z}{z_o}\right)^{-n/2} \qquad [50]$$

or,

$$\left(\frac{z}{z_o}\right)^{1+n/2} = 1 + \left(1 + \frac{n}{2}\right)\frac{W_o t}{z_o} \qquad [51]$$

This result suggests how to estimate the dimensionless quantity n (or D) through the analysis of the trajectories of rising masses in this special case.

2. A mass with initial density difference rising in a homogeneous medium; i.e., B = 0. Then,

$$\left(\frac{W}{W_o}\right)^2 = \left[1 - \frac{A}{(1 + 2D)}\right] \bar{z}^{-n} + \left[\frac{A}{(1 + 2D)}\right] \bar{z}^{-1} \qquad [52]$$

If A > 0, then no maximum height is reached, but if A < 0,

$$\frac{z_{max}}{z_o} = \left[\frac{1 + 2D + |A|}{|A|}\right]^{\frac{1}{(1+2D)}} \qquad [53]$$

The predicted rise of the mass as a function of time and the maximum rise of the mass for a range of values of A(<0) and D, as obtained from Equations [52] and [53], are presented in Figures 9 and 10. These figures demonstrate clearly the effect of the (negative) initial buoyancy and energy dissipation parameters on the time history of an impulsively started rising mass moving through a uniform surrounding fluid of smaller density.

2a. The same case as above but for $W_o = 0$ and A > 0. First of all, [43] may be rewritten:

$$W^2 = W_o^2 \left(\frac{z_o}{z}\right)^n + \frac{2kgz_o}{(1 + 2D)K} \left(\frac{\Delta\rho}{\rho_e}\right)_o \left(\frac{z_o}{z}\right)^n - \frac{2kgz_o}{(1 + 2D)K} \left(\frac{\Delta\rho}{\rho_e}\right)_o \cdot \frac{z_o}{z}$$

or in this case

$$W^2 \cdot z = \left[\frac{-2kg}{\beta^2(1 + 2D)K}\right]\left(\frac{\Delta\rho}{\rho_e}\right) \cdot R_o^{\,2}\left[1 - \left(\frac{R_o}{\beta z}\right)^{n-1}\right] \qquad [54]$$

3. A mass with no initial buoyancy rising in a stratified medium, i.e., A = 0. Then,

$$\left(\frac{W}{W_o}\right)^2 = \left[1 + B\left(\frac{1}{1 + D/2} - \frac{1}{1 + 2D}\right)\right]\bar{z}^{\,-n} + \frac{B}{(1 + D/2)} \cdot \bar{z}^{-1} - \frac{B}{(1 + D/2)} \cdot \bar{z}^2 \qquad [55]$$

and the maximum rise of the mass, as a function of B, is obtained from Equation [55] by setting W = 0.

Approximate integrated solution for the height of the convected mass, $\bar{z}$, as a function of time can be readily obtained from Equation [55] whenever $D \ll 1$ and $B \ll 1$. When these two requirements are satisfied, Equation [55] can be rewritten as

$$\left(\frac{W}{W_o}\right)^2 = \bar{z}^{-2} - B\bar{z}^2 \qquad [56]$$

and upon integration we obtain

$$\sqrt{B}\bar{z}^2 = \sin\left[\sin^{-1}\sqrt{B} + 2\sqrt{B}\bar{t}\right] \qquad [57]$$

where we have normalized the time t according to

$$t \equiv \frac{W_o t}{z_o} \qquad [58]$$

Equation [57] is particularly useful for the approximate determination of the maximum rise of a mass convected in a stratified medium and the time at which this maximum height is reached, $\bar{t}_{max}$. For the maximum height we find

$$\bar{z}_{max} = \left(\frac{1}{B}\right)^{\frac{1}{4}} \qquad [59]$$

and the time required to reach this height is given by

$$\bar{t}_{max} = \frac{\pi}{4\sqrt{B}} - \frac{1}{2} \qquad [60]$$

This last result is especially interesting. In experiments on convected masses of fluid moving through a density stratified medium it was frequently observed that the time it takes the mass to reach its maximum height is inversely proportional to the Vaisala frequency of the stratified medium, i.e., $t_{max}\sqrt{ag}$ = const. Equation [60] is just a statement of this same fact for small values of B (as usually exist in nature), since $\bar{t}_{max}\sqrt{B} = \sqrt{2k/3K}\,(t_{max}\sqrt{ag})$.

V. COMPARISON OF EXPERIMENTAL AND THEORETICAL RESULTS (STRATIFIED MEDIA)

In our experimental investigation we have studied the motion of impulsively started rising masses (or vortex-pairs) in both uniform and density-stratified surroundings. According to the theoretical considerations presented in the previous section, the time-histories of these motions, expressed in appropriate non-dimensional terms, are determined by three parameters, A, B and D; i.e., for given values of these parameters the rise and

growth of the convected mass as a function of time can be predicted. A and B are determined by G and S and by a third parameter which is the ratio of the virtual kinetic and potential energy coefficients, K/k, according to Equations [44] and [45].

While the parameters G and S are determined in each case by the initial conditions of the rising vortex-pair, there is no practical way for determining a priori the values of D and K/k. These latter parameters can be determined only by comparing certain sets of experimental results with corresponding theory.

A series of experiments on the motion of vortex-pairs in a homogeneous medium of the same density, Series III (see Table 1), where the parameters G and S (and therefore also A and B) are identically equal zero, may be used for determining the dissipation parameter D. The rise of the vortex-pairs in this case is predicted by Equation [51] and is graphically depicted in Figure 9 (with A = 0). The actual predicted rise of the convected mass depends on the numerical value of the parameter D (or n).

In Figure 6 are shown a comparison between experimental and theoretical results on the rise of impulsively started masses in a uniform medium of the same density. In a log-log plot, the slope of the trajectory for large values of $(W_o t/z_o)$ should be equal, according to our analysis, to 1/2 + D, and it can be used therefore for determining the value of the dissipation parameter D associated with the motion of the rising mass. Included in Figure 6 are the experimental results of Richards (1965), on the rise of two-dimensional puffs in homogeneous surroundings. The best agreement with all experimental results is obtained when we choose D = 0.2.

The numerical value of K/k enters into the analysis only when there is an initial difference between the rising and surrounding fluid densities or when the surrounding fluid is stratified. This value will be also determined from a comparison of some experimental and theoretical results. For a vortex-pair convected in a density-stratified medium we found earlier that, for sufficiently small values of the parameters A, B and D, the time it takes the mass to reach its maximum height is inversely proportional to the Vaisala frequency and is given by

$$\left(t_{max}\sqrt{ag}\right) \approx \sqrt{\frac{3K}{2K}}\,\frac{\pi}{4} \qquad [61]$$

This value decreases only very gradually as the value of B (or S) so that Eq. [61] is very useful for the experimental determination of K/k.

In Figure 11 the value of the product $(t_{max}\sqrt{ag})$, as measured in the experiments of Series I and II, is presented as a function of the stratification parameters S; the initial conditions for each experiment presented in the Figure are included in Table 1. There are certain inherent inaccuracies in the experimental determination of t_{max} which explain the scatter. Also shown in Figure 11 are the asymptotic solution for the maximum rise time, Equation [60], and the exact solution, according to Equation [43], with D = 0.2 and G/S = -0.715. Close agreement between theoretical and experimental results was obtained when we used 3K/2K = 6 or K/k = 4. We have included in the same figure the experimental results from Series III which had a markedly different value of β; these too were found to agree very closely with the theoretical prediction based on D = 0.2 and K/k = 4,

indicating that the dependence of these latter two parameters on β is probably very weak. We have used these values of D and K/k for all subsequent comparisons of experimental and theoretical results. The coefficient of virtual potential energy cannot be much different from unity, according to its definition in Equation [38]. The total kinetic energy of a rising vortex-pair was thus found to be about four times larger than the kinetic energy associated with its linear convection alone, an indication of the intensity of motion inside the vortex-pair, which contributes to its total kinetic energy. This finding lends important support to the strong circulation assumption.

A comparison between the predicted and actual maximum heights of rise of a vortex-pair convected in a linearly density-stratified medium is shown as Figure 12. The figure includes a prediction based on the simplified asymptotic solution for small B (or S), Equation [59], and a prediction obtained from the exact solution of Equation [49] for $\bar{z}_{max}$. Generally there is good agreement between the experimentally measured maximum height of vortex-pairs and the exact theoretical solution.

Finally, in Figure 13, we compare the measured trajectories (height versus time) of vortex-pairs with their theoretically predicted trajectories. The vortex-pairs included in the Figure all had different starting conditions and they were moving through media with different density-stratifications. However, their trajectories, as depicted in the figure are shown to depend only on the values of the two lumped parameters G and S which combine their starting conditions with the properties of the surrounding medium.

The fact that trajectories of different vortex-pairs are grouped according to their G and S values confirms the validity of the scaling laws and scaling parameters used herein, while the agreement obtained between the experimental and theoretical trajectories lends further support to the validity of the simplified theory presented here for the motion of vortex-pairs in stratified media.

VI. SUMMARY AND CONCLUSIONS

A theory for the motion of two-dimensional turbulent vortex-pairs in homogeneous media has been developed based on separate velocity scaling of the internal and external flow fields involved in the motion. These two flow fields are depicted to be separated by a thin region of high shear, which also forms the boundary of the captured mass. The theory takes into consideration variations in volume, circulation, momentum, and energy in the flow field. The ratio of internal to external velocity scales, ψ, is introduced as an important variable. The virtual momentum coefficient is shown to be linear in ψ, of the form $K_1 - K_2\psi$.

The theory is specifically derived for the two limiting cases of weak and strong circulation. In the former case, $\psi \rightarrow 1$ and the entrainment is weak; the asymptotic behavior of the trajectory is $z \sim t^{\frac{1}{3}}$ just as predicted by the usual theory based on complete similarity and momentum conservation.

In the case of strong circulation, $\psi \gg 1$, the asymptotic behavior of the trajectory depends very much on the way in which vorticity from the shear layer is ingested into the vortex pair.

In the case where the shear layer from opposite sides is ingested in such a way as to cause annihilation of the ingested vorticity, then the asymptotic trajectory is $z \sim t^{\frac{1}{2}}$. Under the same conditions, the velocity ratio, ψ, increases toward the asymptotic value K_1/K_2 so that the virtual momentum coefficient tends to zero. As a result, the asymptotic motion assuming vorticity annihilation corresponds to a motion with complete similarity and with energy conservation. The ratio of growth of the pair radius with height is shown to increase, approaching a linear relation asymptotically.

Systematic experiments have been carried out, and the results for rise versus time and radius versus height are compared with the theory. They lend strong support to the strong circulation theory and further suggest that ingested vorticity is to a large degree annihilated.

Based on these findings for the case of homogeneous flows, a simplified theory is derived for the rising motion of vortex pairs in stratified media. The assumptions of the theory are: (i) the motion is determined by conservation of volume, mass, and energy (neglecting vorticity and momentum); (ii) complete similarity ($dR/dz = \beta$, a constant). General laws of motion in stratified media have been derived and solutions given; particularly interesting cases are discussed in detail.

Motions in stratified media were shown to depend on four non-dimensional parameters. Two of these depend upon the initial conditions of the motion and the stratification of the media. The other two are inherent in the details of the motion and had to be determined from experiments; one of these, the dissipation

parameter $D = C_D/\beta K$ was found to be 0.2 while the other, the ratio of virtual kinetic and potential energy coefficients K/k was found to be 4. On the basis of these numbers it may be concluded that the dissipation rate is small and that the contribution of internal motions to the overall kinetic energy is large.

The experiments confirmed the environmental scaling parameters, which were used to collapse data taken under differing conditions. Good agreement was found between predicted and observed trajectories. Particularly good agreement was found for the maximum height of rise. The time required to reach maximum height was found to be inversely proportional to the Vaisala frequency, $\sqrt{ag}$, and was given approximately by $t_{max}\sqrt{ag} = 1.8$, in good agreement with the theory. In general the experiments confirmed the utility of the simplified theory for predictions of the motion of vortex pairs in stratified media. This theory has been utilized elsewhere for the prediction of the behavior of chimney plumes rising into a stable atmosphere, with very good agreement between the theory and full scale observations, Tulin and Shwartz (1970), and also with excellent correlation with experiments to the penetration of a density discontinuity by a turbulent vortex-pair, Birkhead, Shwartz, and Tulin (1969).

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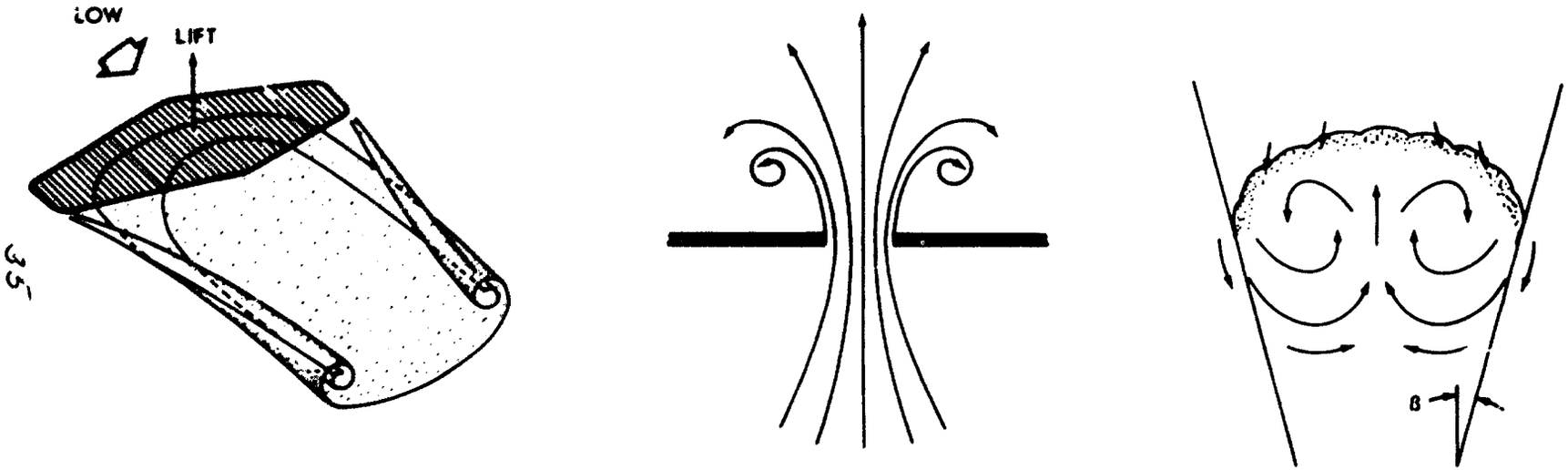
TABLE 1

Characteristic Properties and Parameters of Impulsively Generated Vortex-Pairs

Series	Run	L (ft.)	a (ft^{-1})	W_o (ft/sec)	R_o (ft)	β	$(\Delta\rho/\rho_e)_o$	z_o (ft)	G	S	$z_{max} - z_o$ (ft)	$t_{max}\sqrt{ag}$	G/S
I	101	0.50	0.0177	0.295	0.160	0.310	0.00631	0.515	-1.200	1.735	0.283	1.66	-0.700
	201	0.50	0.0081	0.480	0.134	0.280	0.00269	0.478	-0.180	0.258	0.656	1.62	-0.700
	202	0.50	0.0081	0.532	0.133	0.320	0.00265	0.416	-0.125	0.160	0.621	1.48	-0.731
	203	0.50	0.0081	0.570	0.136	0.306	0.00275	0.495	-0.121	0.159	0.619	1.75	-0.758
	204	0.50	0.0081	0.270	0.146	0.320	0.00276	0.456	-0.555	0.743	0.416	1.51	-0.746
	205	0.50	0.0081	0.338	0.143	0.300	0.00283	0.476	-0.374	0.509	0.441	1.84	-0.735
	206	0.50	0.0081	0.296	0.148	0.266	0.00291	0.556	-0.595	0.920	0.386	1.75	-0.650
	208	0.50	0.0081	0.112	0.150	0.304	0.00265	0.493	-3.260	4.910	0.159	1.69	-0.667
	209	0.50	0.0081	0.122	0.150	0.321	0.00277	0.467	-2.780	3.790	0.139	1.75	-0.736
	401	0.50	0.0020	0.138	0.148	0.294	0.00067	0.503	-0.612	0.904	0.421	1.79	-0.676
	403	0.50	0.0020	0.296	0.143	0.274	0.00075	0.522	-0.143	0.199	0.686	1.56	-0.720
	404	0.50	0.0020	0.154	0.151	0.277	0.00072	0.546	-0.534	0.810	0.454	1.52	-0.658
	405	0.50	0.0020	0.318	0.143	0.300	0.00077	0.476	-0.117	0.145	0.661	1.73	-0.807
	406	0.50	0.0020	0.455	0.158	0.320	0.00079	0.494	-0.061	0.076	0.809	1.75	-0.800
	407	0.50	0.0020	0.110	0.162	0.244	0.00074	0.663	-1.300	2.330	0.364	1.80	-0.569
	408	0.50	0.0020	0.098	0.160	0.256	0.00075	0.624	-1.568	2.600	0.316	1.70	-0.600
	410	0.50	0.0020	0.498	0.157	0.311	0.00077	0.505	-0.051	0.066	0.844	1.63	-0.770
	411	0.50	0.0020	0.336	0.155	0.320	0.00073	0.484	-0.101	0.133	0.659	1.80	-0.758
	501	0.50	0.0231	0.546	0.160	0.321	0.00846	0.498	-0.454	0.618	0.418	1.42	-0.735
	502	0.50	0.0231	0.545	0.156	0.332	0.00800	0.470	-0.408	0.554	0.434	1.52	-0.735
	503	0.50	0.0231	0.528	0.156	0.282	0.00827	0.553	-0.528	0.814	0.419	1.57	-0.650
II	412	0.25	0.0020	0.266	0.127	0.560	-0.00041	0.227	-0.042	0.047	0.464	1.65	-0.885
	413	0.25	0.0020	0.308	0.124	0.562	-0.00046	0.223	-0.035	0.034	0.523	1.68	-1.030
	414	0.25	0.0020	0.304	0.125	0.680	-0.00037	0.184	-0.024	0.024	0.459	1.75	-1.000
III	301	0.50	0	0.550	0.147	0.300	0	0.490	0	0	-	-	-
	302	0.50	0	0.525	0.147	0.286	0	0.514	0	0	-	-	-
	303	0.50	0	0.336	0.155	0.302	0	0.514	0	0	-	-	-
	304	0.50	0	0.358	0.155	0.300	0	0.517	0	0	-	-	-
	305	0.50	0	0.109	0.150	0.280	0	0.536	0	0	-	-	-

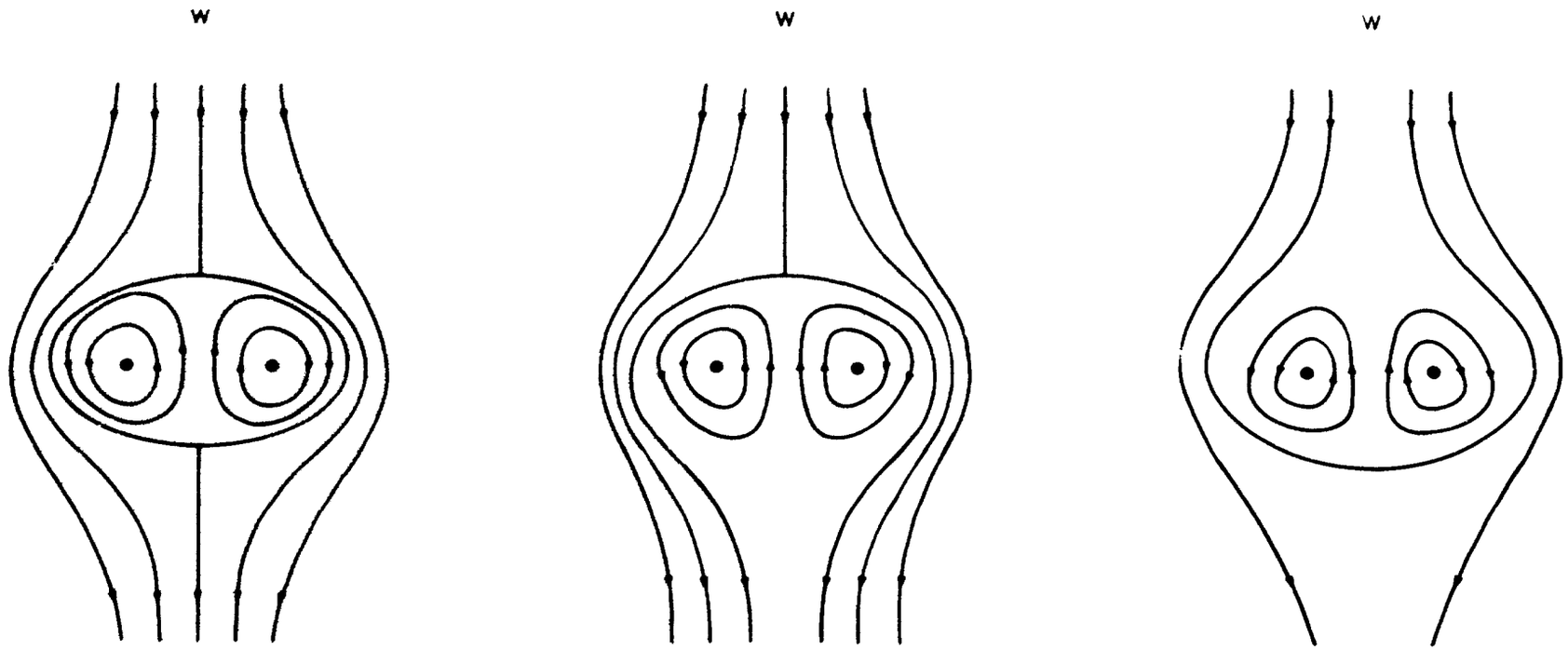


(a) TRAILING VORTEX SHEET

(b) IMPULSIVE MOTION; FLUID FORCED THROUGH AN APERTURE

(c) FLOW IN AND AROUND NATURALLY OCCURRING THERMALS

FIGURE 1 - EXAMPLES OF VORTEX-PAIRS



(a) IDEAL VORTEX PAIR
($W = W^*$)

(b) UNDERDEVELOPED VORTEX PAIR
($W > W^*$)

(c) OVERDEVELOPED VORTEX PAIR
($W < W^*$)

FIGURE 2 - GENERALIZED VORTEX-PAIRS (OBSERVER MOVING WITH THE CORES)

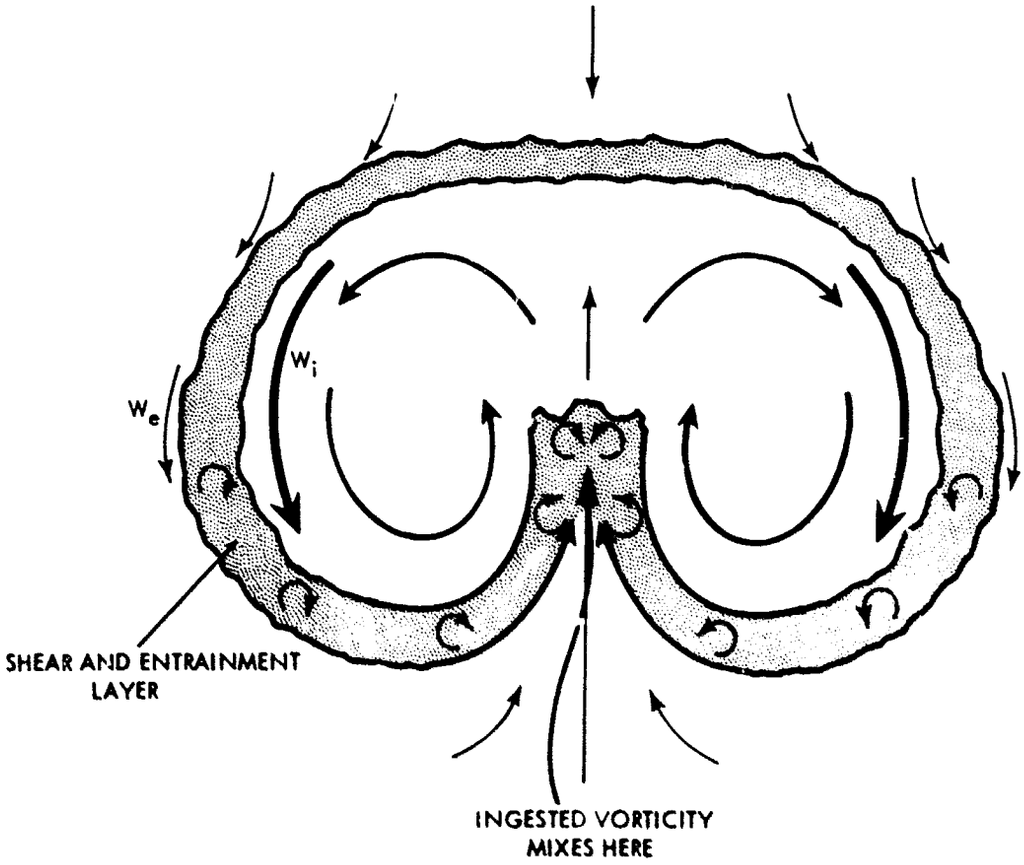


FIGURE 3 - ENTRAINING VORTEX PAIR ($W_i > W_e$).

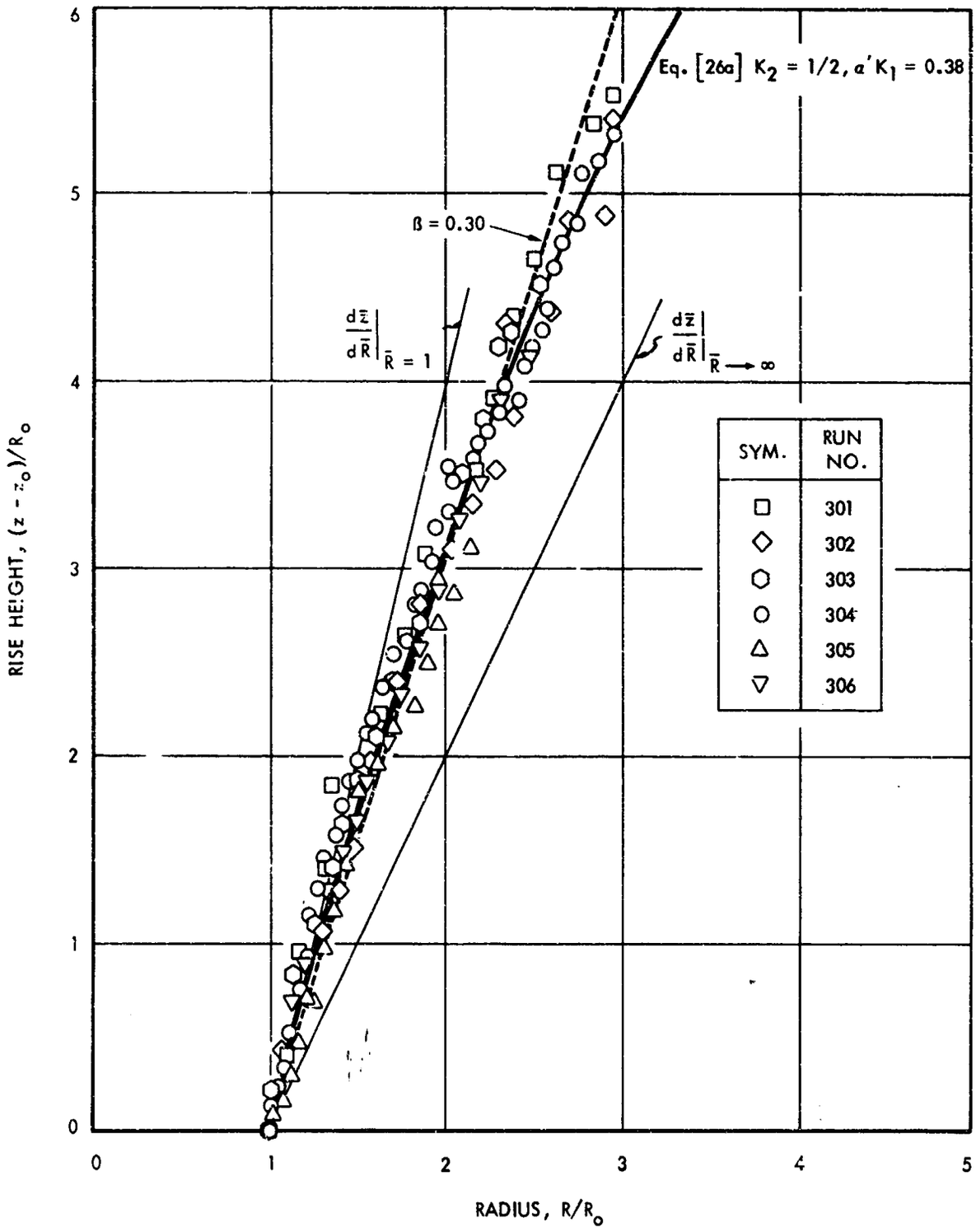


FIGURE 4 - THE VARIATION OF VORTEX-PAIR RADIUS WITH HEIGHT IN A HOMOGENEOUS MEDIUM, EXPERIMENT

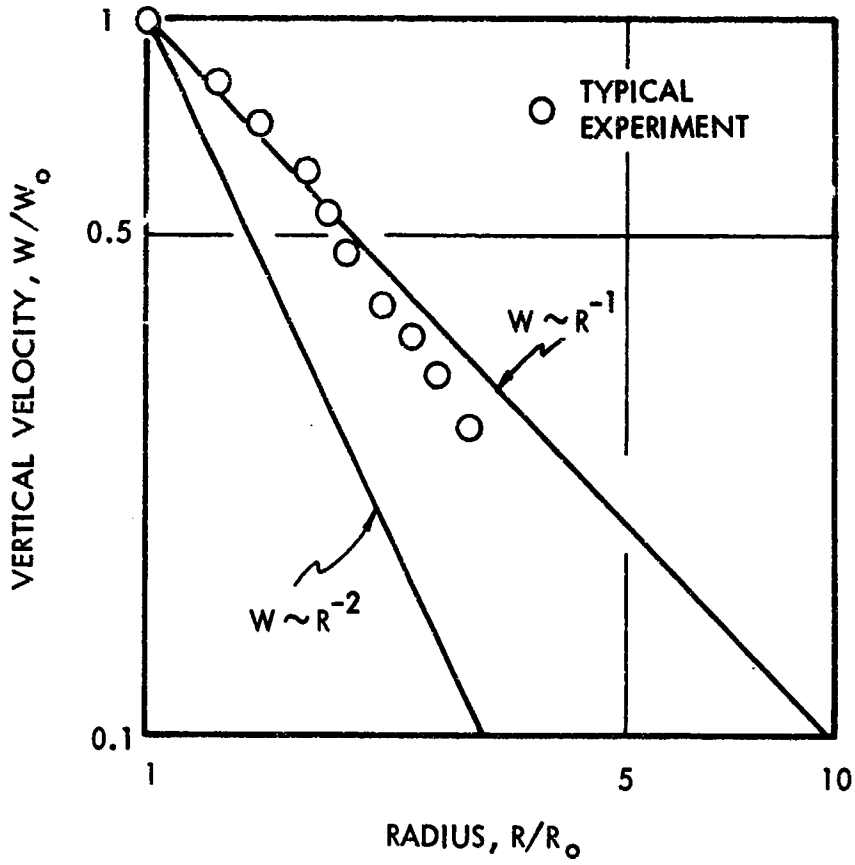


FIGURE 5 - THE VERTICAL VELOCITY VS. RADIUS OF A VORTEX-PAIR MOVING IN A HOMOGENEOUS FLUID

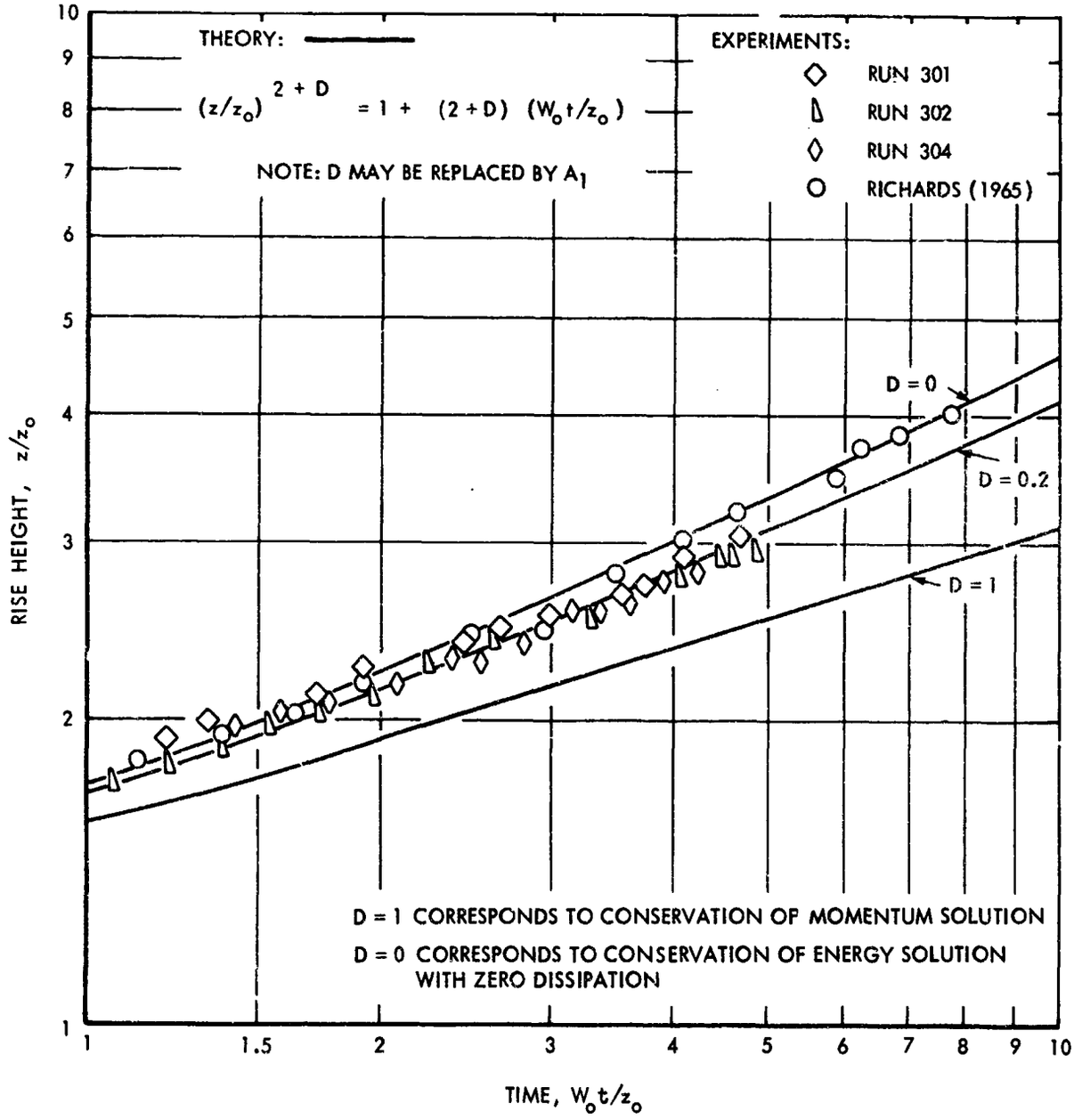


FIGURE 6 - THE RISE OF VORTEX-PAIRS IN A HOMOGENEOUS MEDIUM; COMPARISON OF EXPERIMENT AND THEORY

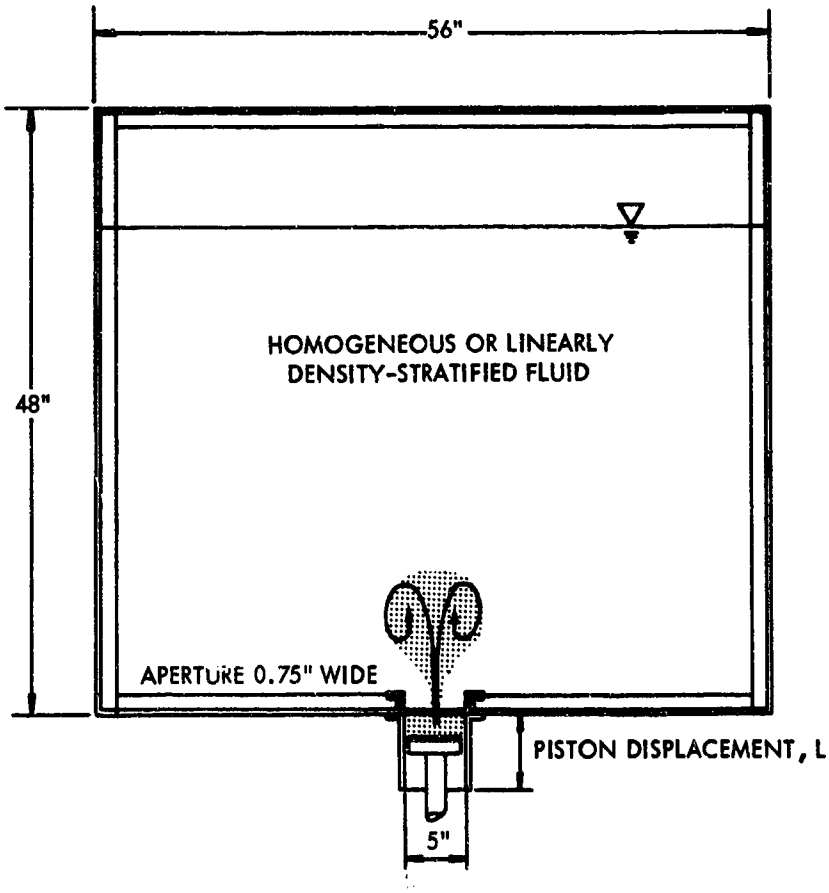


FIGURE 7 - EXPERIMENTAL FACILITY FOR STUDYING VORTEX PAIR MOTION.

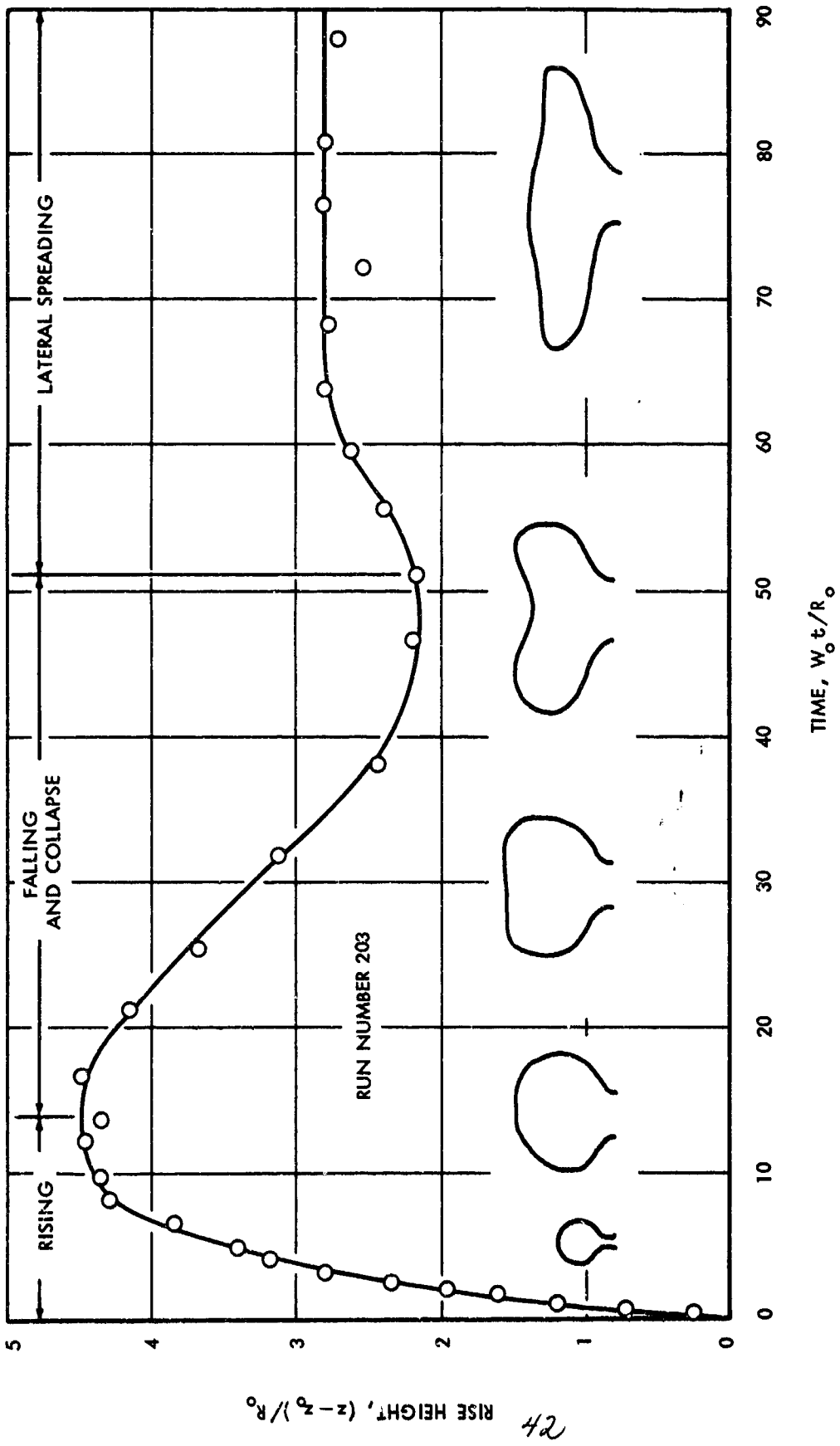


FIGURE 8 - TYPICAL TRAJECTORY OF A VORTEX-PAIR IN A LINEARLY DENSITY STRATIFIED MEDIUM

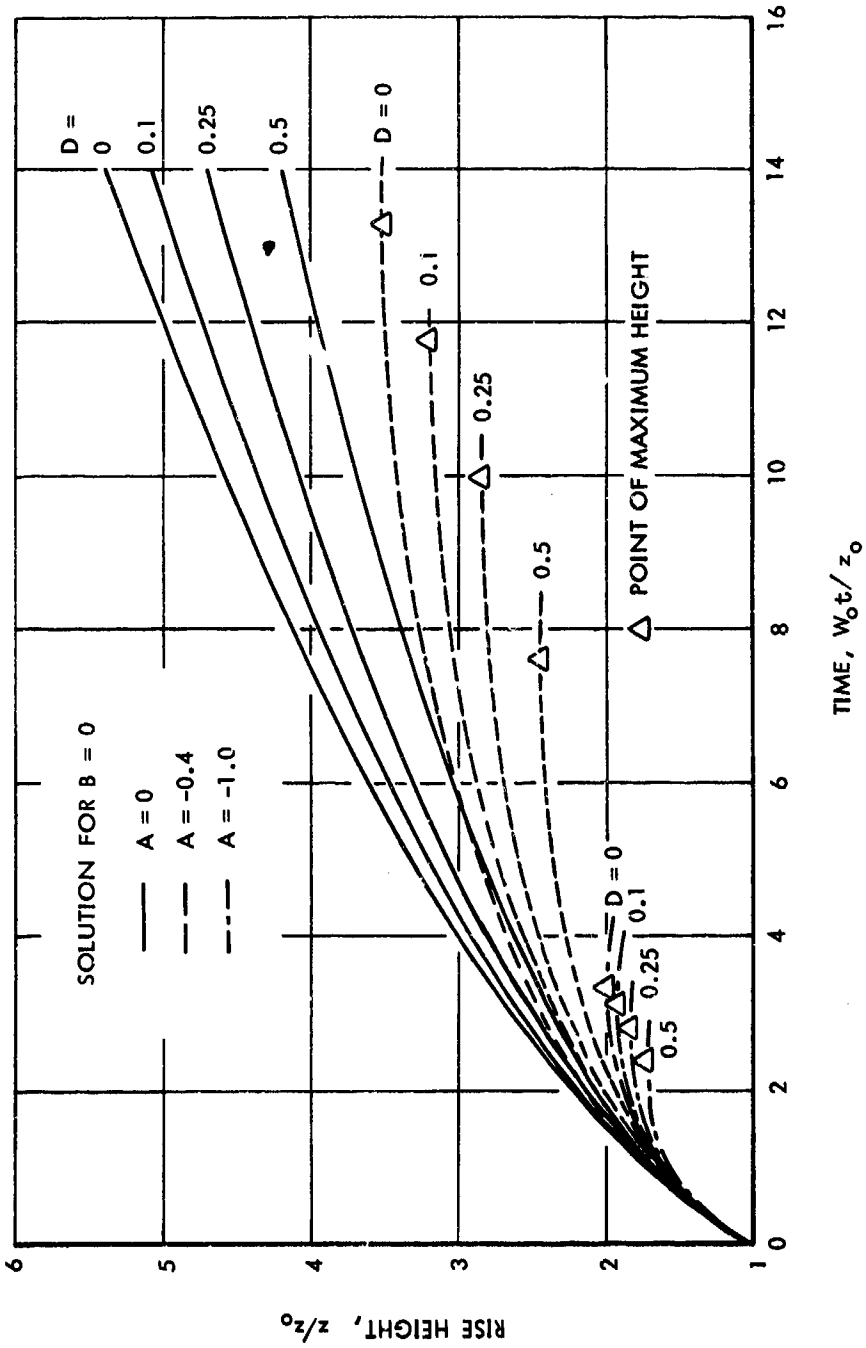


FIGURE 9 - THE RISE OF AN IMPULSIVELY STARTED HEAVY MASS OF FLUID IN UNIFORM SURROUNDINGS (B = 0), THEORY

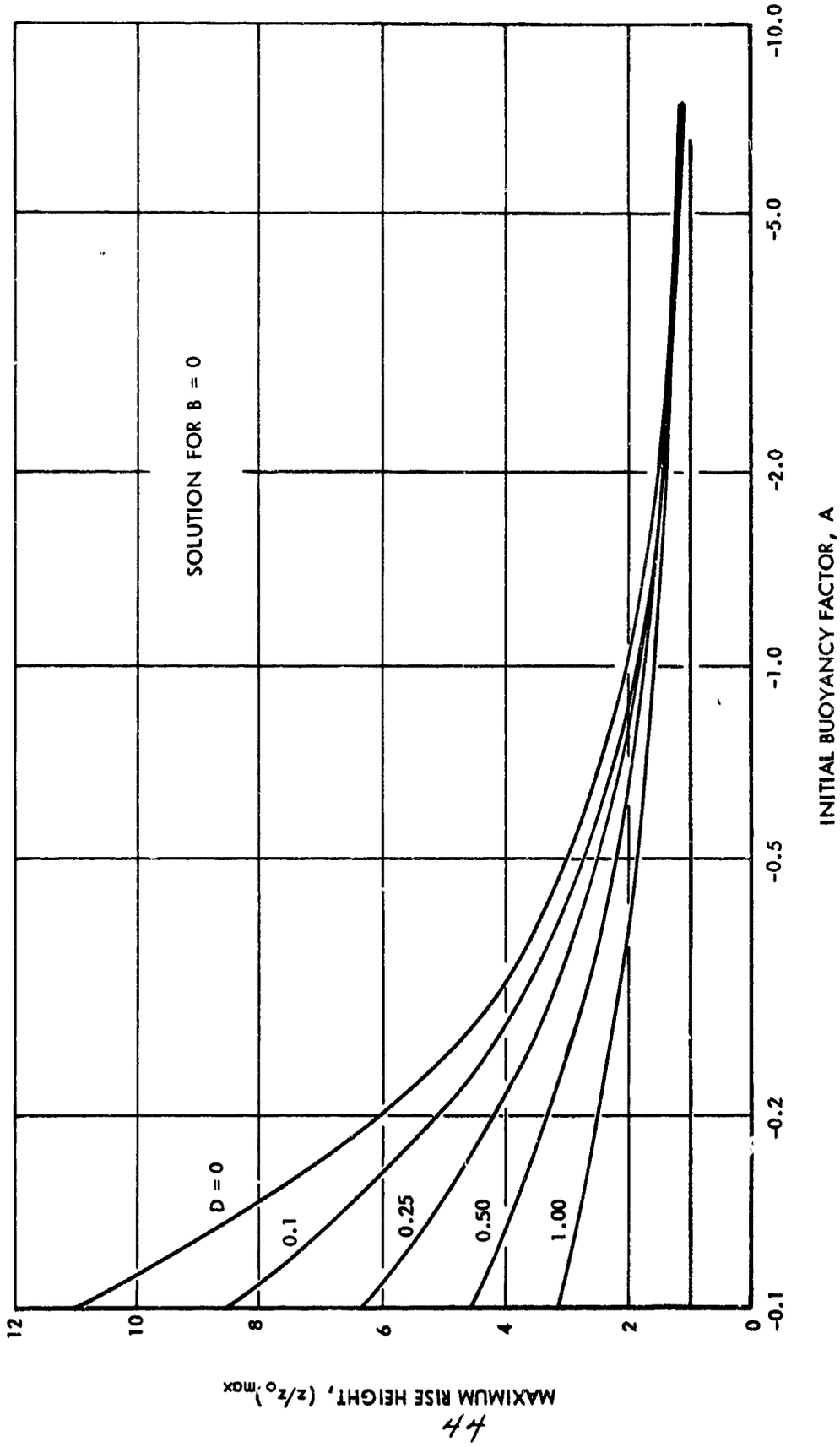


FIGURE 10 - THE MAXIMUM RISE OF AN IMPULSIVELY STARTED HEAVY MASS OF FLUID IN UNIFORM SURROUNDINGS (B = 0), THEORY

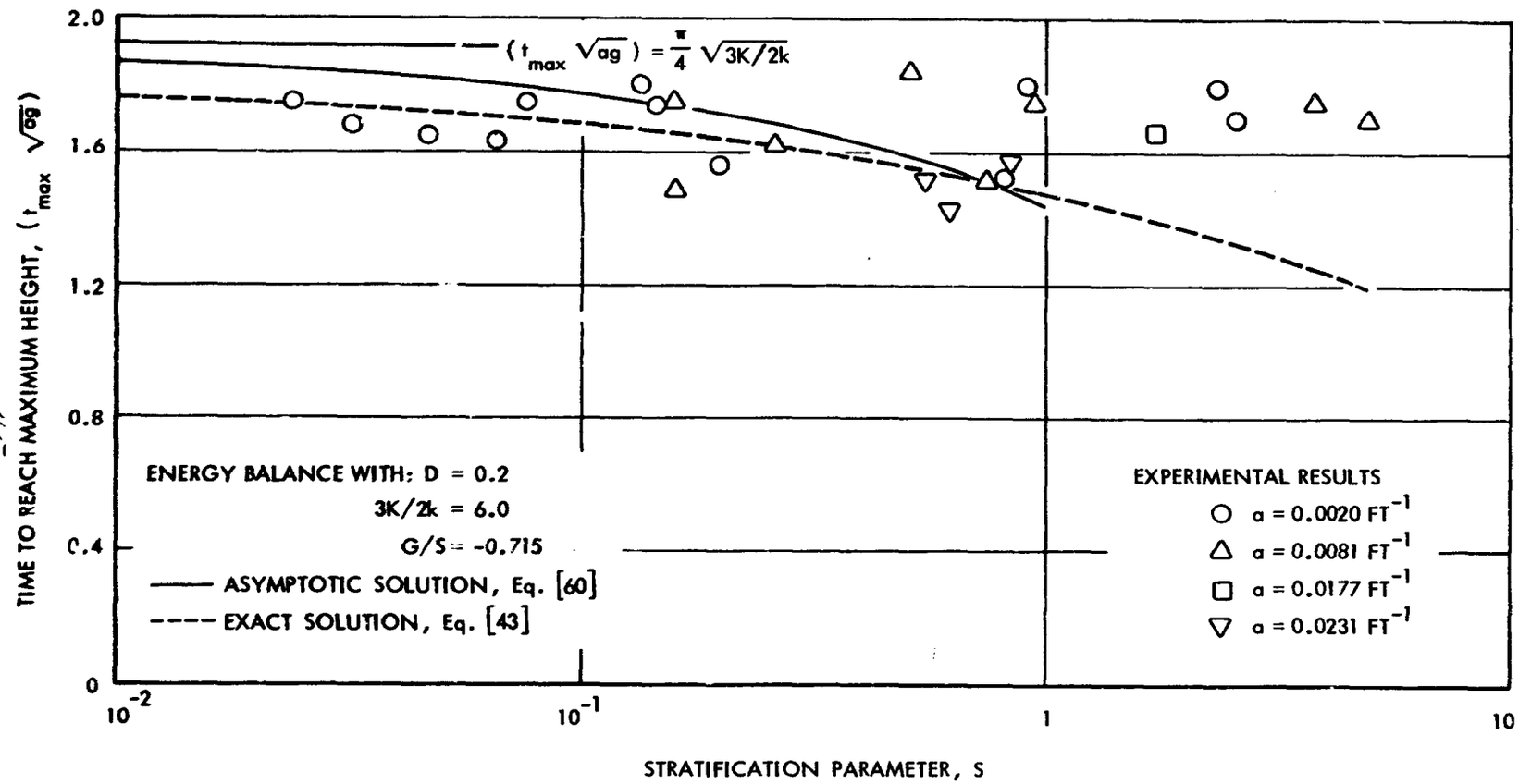


FIGURE 11 - TIME OF MAXIMUM RISE FOR VORTEX-PAIRS CONVECTED IN A DENSITY-STRATIFIED MEDIUM; COMPARISON OF EXPERIMENT AND THEORY

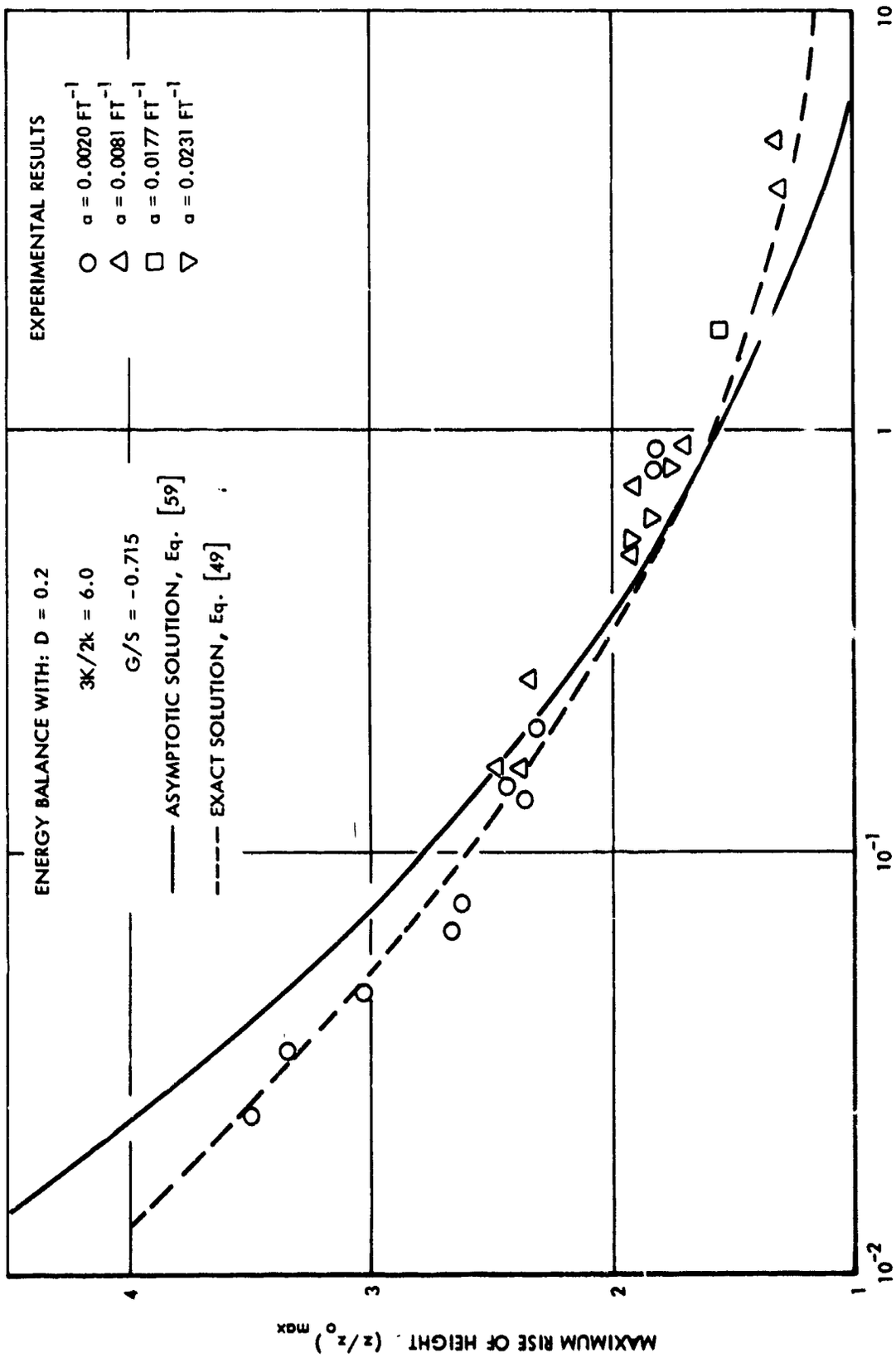


FIGURE 12 - THE MAXIMUM RISE OF VORTEX-PAIRS CONVECTED IN A DENSITY-STRATIFIED MEDI'JM; COMPARISON OF EXPERIMENT AND THEORY

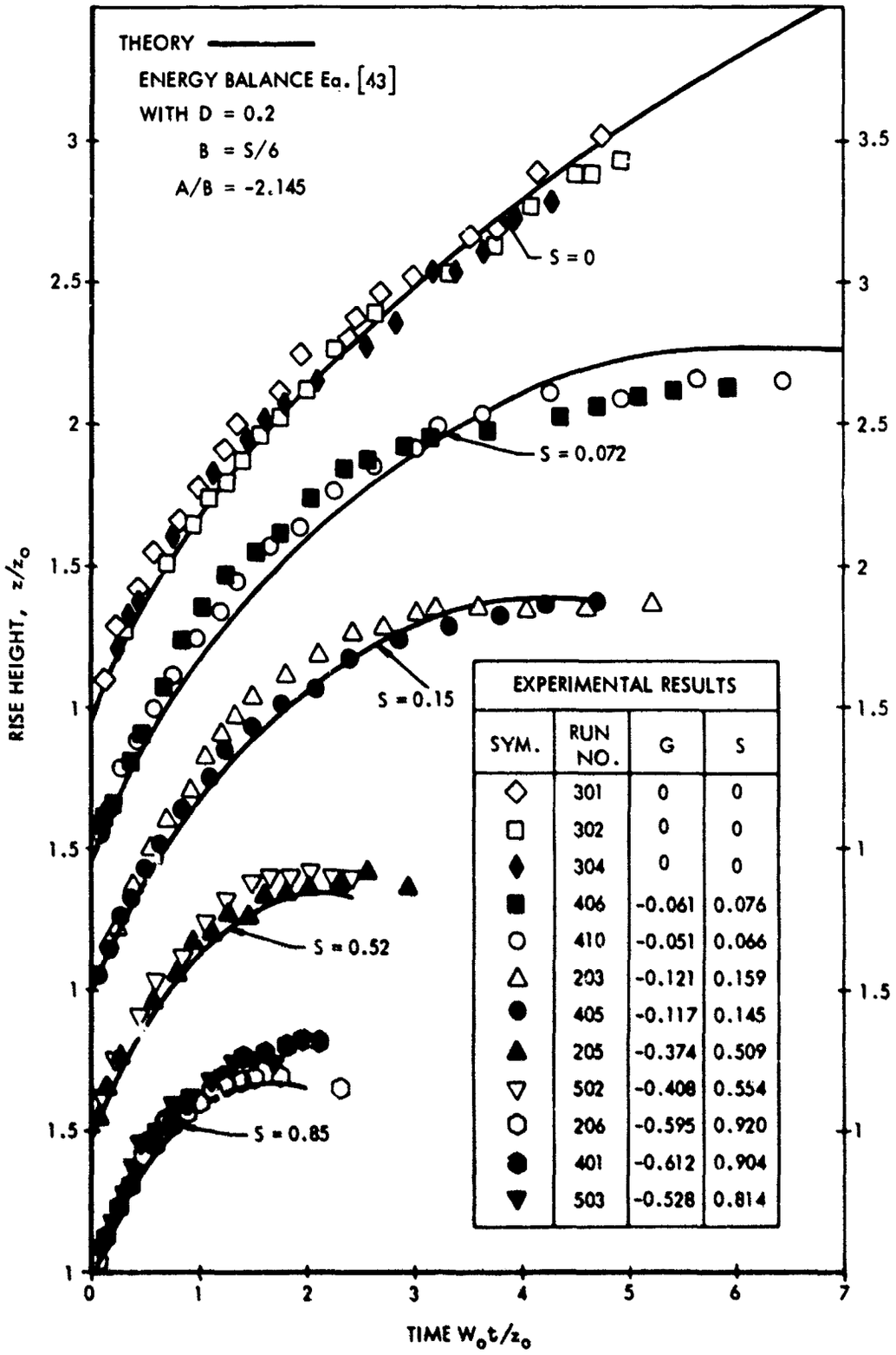


FIGURE 13 - THE TRAJECTORY OF VORTEX-PAIRS IN STRATIFIED MEDIA; COMPARISON OF EXPERIMENT AND THEORY.

ζ, η, ξ - CARTESIAN COORDINATES

u, v, w - CORRESPONDING VELOCITIES

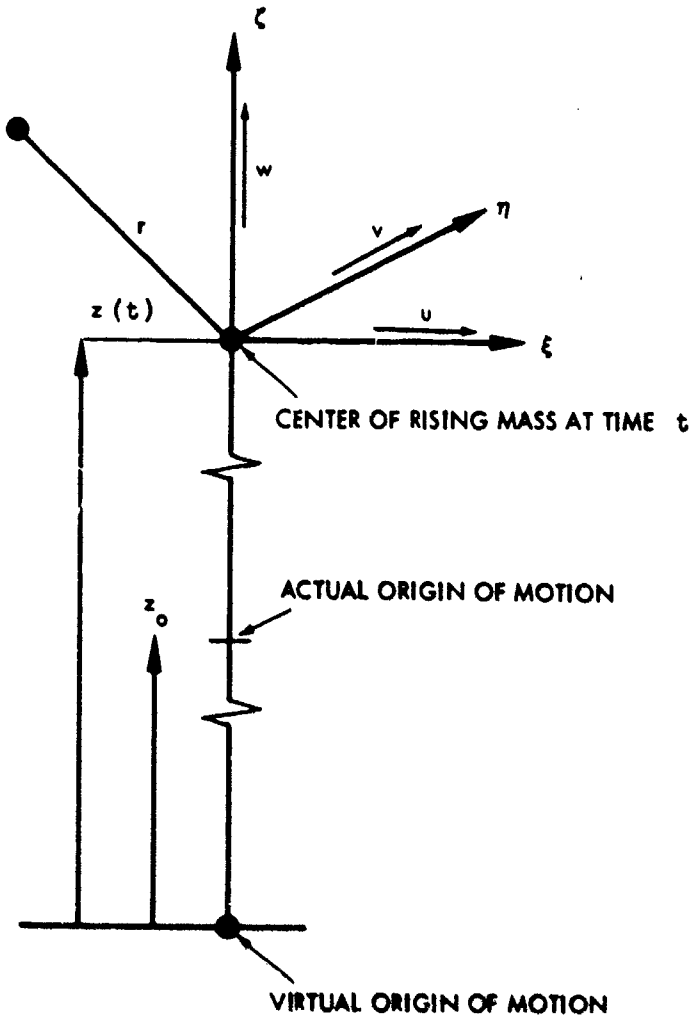


FIGURE 14 - NOMENCLATURE

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Corrigenda to Technical Report 231-15

"THE MOTION OF TURBULENT VORTEX-PAIRS IN HOMOGENEOUS AND DENSITY STRATIFIED MEDIA"

By

M.P. Tulin and J. Shwartz

April 1971

(1) Eq. 21, p. 12 should be renamed Eq. 21a. In this equation, in the left-hand side, α should be replaced by α', and $\alpha(\psi)$ in the preceeding sentence should be replaced by α'.

(2) Eq. 23, p. 12, Right-hand side should be multiplied by μ, where $\mu = (M_z/\rho)/(W_{i_o} R_o^2)$, i.e., Eq. 23 should now read

$$K_1 - \frac{K_2}{\alpha'} \frac{d\bar{R}}{d\bar{z}} \bar{R}^{(1-A_1)} = \frac{\mu}{\alpha'} \frac{d\bar{R}}{d\bar{z}} \qquad [23]$$

(3) Eq. 24, p. 12, should be replaced by

$$\frac{\alpha' K_1 \bar{z}}{\mu} = \frac{\bar{R}^{A_1}}{A_1} + \frac{K_2 \bar{R}}{\mu} + \text{const.} \qquad (A_1 \neq 0) \qquad [24]$$

And the following equation, on p. 13, by

$$\frac{\alpha' K_1 \bar{z}}{\mu} = \ln \bar{R} + \frac{K_2 \bar{R}}{\mu} + \text{const.} \qquad (A = 0)$$

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(4) Eq. 25, p. 13, should read

$$\frac{\alpha' K_1}{\mu} \frac{(z-z_o)}{R_o} = \ln \bar{R} + \frac{K_2}{\mu} (\bar{R} - 1) \qquad [25]$$

(5) Eq. 26, p. 13, should read

$$\psi = \frac{K_1}{\mu \bar{R}^{(A_1 - 1)} + K_2} \qquad [26]$$

(6) Eq. 27, p. 13, should read

$$\frac{W_e}{W_i} = \frac{K_1}{\bar{R}^{-2} + K_2 \bar{R}^{-(1 + A_1)}} \qquad [27]$$

(7) The table on p. 14 should be replaced by the following:

Asymptotic Behaviour ($\psi \gg 1$)

	$\bar{z}$	ψ	W_e/W_{i_o}	$\bar{z}$
$A_1 > 1$	$\frac{\mu \bar{R}^{A_1}}{A_1 \alpha' K_1}$	$K_1 \bar{R}^{(-A_1+1)}$	$K_1 \bar{R}^{-2}$	$\sim t^{\frac{1}{1+2/A_1}}$
$A_1 = 1$	$\bar{R} \frac{\mu' + A_1 K_2}{A_1 \alpha' K_1}$	$K_1/(\mu + K_2)$	$K_1 \bar{R}^{-2}$	$\sim t^{1/3}$
$A_1 < 1$	$\bar{R}(K_2/\alpha' K_1)$	K_1/K_2	$K_1/K_2 \bar{R}^{-(1+A_1)}$	$\sim t^{\frac{1}{2+A_1}}$
$A_1 = 0$	$\bar{R}(K_2/\alpha' K_1)$	K_1/K_2	$(K_1/K_2)\bar{R}^{-1}$	$\sim t^{\frac{1}{2}}$

(8) Eq. 25a, p. 16, should be replaced by

$$\frac{\alpha' K_1}{\mu} = -\frac{z-z_o}{R_o} = \ln \bar{R} + \frac{K_2}{\mu} [\bar{R} - 1] \qquad [25a]$$