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Technical Note

1970-45

B. R. Julian

Ray Tracing
in Arbitrarily
Heterogeneous Media

31 December 1970

Prepared for the Advanced Research Projects Agency
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Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts



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RAY TRACING IN ARBITRARILY HETEROGENEOUS MEDIA

B. R. JULIAN

Group 22

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ABSTRACT

The study of lateral variations of earth structure has been stimulated recently by several factors, especially the theory of plate tectonics and the increasing use of large seismic arrays. The extension of seismic ray theory to two and three dimensional structures is thus of great practical importance.

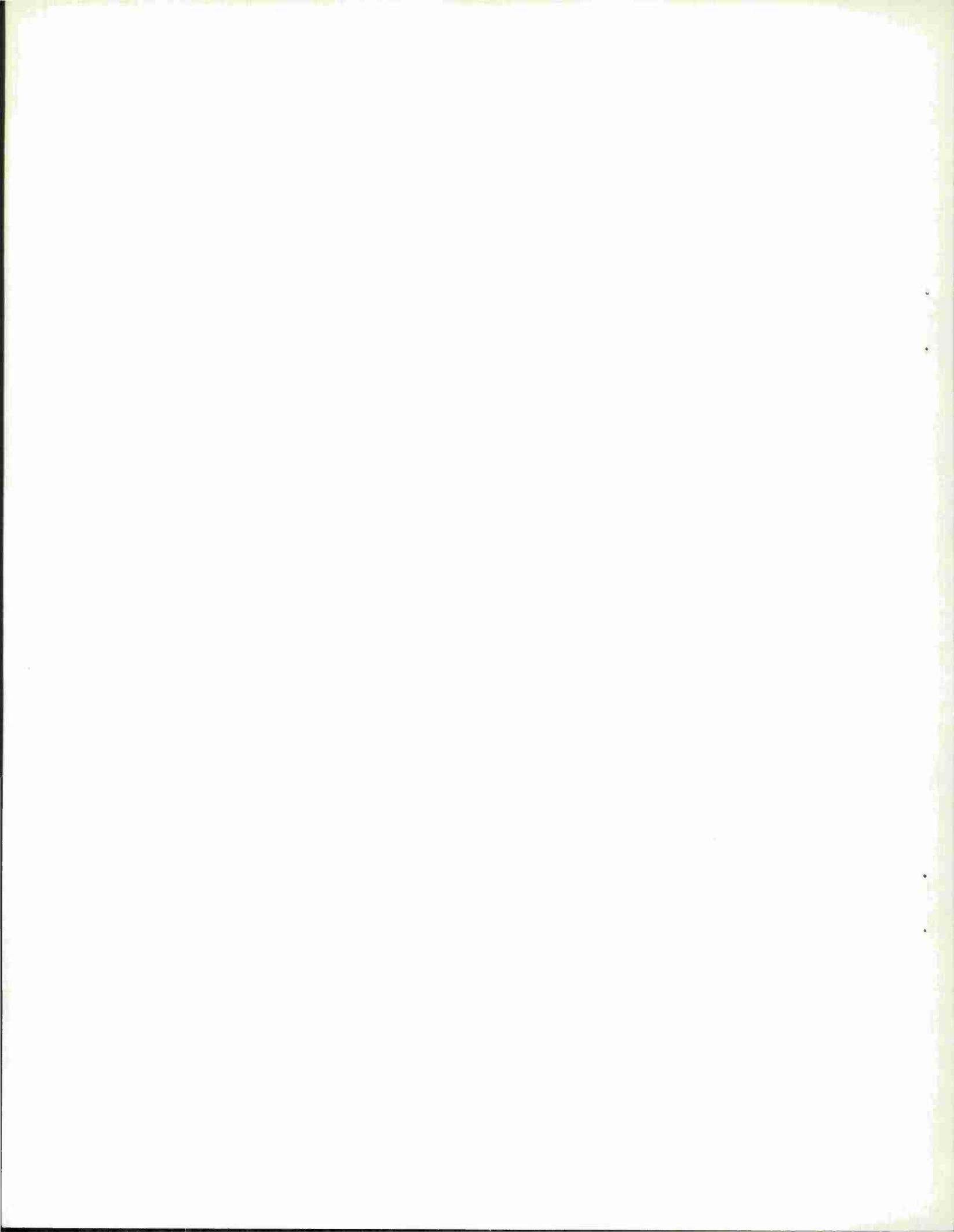
The problem of ray tracing in a generally heterogeneous medium is treated, using the calculus of variations and Fermat's principle of stationary time. The solution is expressed in terms of a system of five simultaneous first order differential equations giving the variation with time of the position and direction of motion of a point on a ray in terms of the wave speed and its spatial derivatives in the medium. The form of the equations is particularly convenient for solution on a digital computer.

The effect upon the wave amplitudes of geometric spreading can also be calculated by the inclusion of ten additional equations in the system, although this greatly increases the computational labor.

If the earth model has certain symmetry properties, then constants of the motion along each ray can be found which simplify the calculations. For example, for two-dimensional models in which the velocity depends only on the coordinates r (radius) and θ (colatitude) one equation can be eliminated from the ray tracing system, and two more can be eliminated from the amplitude equations.

The propagation of surface waves on an earth with geographical variations can be treated by a simplified special case of the method presented here.

Accepted for the Air Force
Joseph R. Waterman, Lt. Col., USAF
Chief, Lincoln Laboratory Project Office



Introduction

In the past, most studies of seismic waves have been based upon the assumption that the earth is spherically symmetrical, with physical properties depending only on radius. Recently, however, it has become apparent that this approximation is often not justified (Julian, 1969) and more attention has been directed by seismologists to the study of lateral variations of structure. Two developments in particular have stimulated interest in wave propagation through two and three dimensional structures: the existence of large seismic arrays, which have yielded clear evidence of azimuthally dependent travel time anomalies¹, and the theory of plate tectonics which hypothesizes the existence of relatively systematic large scale motions in the upper part of the earth (Isacks *et. al.*, 1968; Davies and McKenzie, 1969). This paper presents a formulation of seismic ray theory which is applicable to such problems. The system of differential equations derived is easily soluable numerically to trace rays through complicated structures.

Ray Tracing

Recently V. A. Eliseevnin (1965) has formulated the ray problem for an arbitrarily inhomogeneous medium. Starting with the Eikonal equation, he derived the following system of six simultaneous differential equations for the motion of a disturbance along a ray:

¹Niazi, 1966; Bolt and Nuttli, 1966; Nuttli and Bolt, 1969; Greenfield and Sheppard, 1969.

$$\begin{aligned}
\dot{x} &= v \cos \alpha \\
\dot{y} &= v \cos \beta \\
\dot{z} &= v \cos \gamma \\
\dot{\alpha} &= \frac{\partial v}{\partial x} \sin \alpha - \frac{\partial v}{\partial y} \cot \alpha \cos \beta - \frac{\partial v}{\partial z} \cot \alpha \cos \gamma \\
\dot{\beta} &= -\frac{\partial v}{\partial x} \cos \alpha \cot \beta + \frac{\partial v}{\partial y} \sin \beta - \frac{\partial v}{\partial z} \cot \beta \cos \gamma \\
\dot{\gamma} &= -\frac{\partial v}{\partial x} \cos \alpha \cot \gamma - \frac{\partial v}{\partial y} \cos \beta \cot \gamma + \frac{\partial v}{\partial z} \sin \gamma
\end{aligned} \tag{1}$$

where:

x, y, z are the cartesian coordinates of the disturbance at a particular time.

α, β, γ are the direction angles of the tangent to the ray.

$v(x, y, z)$ is the wave speed.

The partial derivatives $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, and $\frac{\partial v}{\partial z}$ indicate spatial derivatives at x, y, z , not derivatives along the ray path.

Only five of these equations are independent, because the angles α, β and γ are connected by the relation $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. A much simpler way of stating these equations follows if we define a slowness vector, \bar{S} , such that

$$\dot{\bar{r}} = v^2 \bar{S} . \tag{1a}$$

The components of \bar{S} are

$$S_x = \frac{\cos \alpha}{v}$$

$$S_y = \frac{\cos \beta}{v}$$

$$S_z = \frac{\cos \gamma}{v} .$$

The rate of change of \bar{S} is

$$\dot{\bar{S}} = -\frac{1}{v} \{ \sin \alpha \dot{\alpha} \hat{e}_x + \sin \beta \dot{\beta} \hat{e}_y + \sin \gamma \dot{\gamma} \hat{e}_z \}$$

$$- (\bar{S} \cdot \bar{\nabla} v) v \bar{S}$$

where e_x , e_y and e_z are unit vectors in the direction of the coordinate axes and, using (1) this becomes

$$\dot{\bar{S}} = -\frac{\bar{\nabla} v}{v} . \quad (1b)$$

Despite the simplicity of (1a) and (1b), for computational purposes expressions such as (1) in terms of angles instead of slowness are usually more useful, particularly in curvilinear coordinate systems. We shall give a different derivation of equations (1), based on Fermat's principle of least time, and carry out the derivation in spherical coordinates, so that the result will be in a seismologically useful form.

Let r , θ , ϕ , be the spherical coordinates, at time t , of a point on a ray. Further, letting \hat{e}_r , \hat{e}_θ , \hat{e}_ϕ , be the conventional unit vectors for spherical coordinates, define:

$$i_r = \text{angle between ray direction and } \hat{e}_r .$$

$$i_\theta = \text{angle between ray direction and } \hat{e}_\theta .$$

$$i_\phi = \text{angle between ray direction and } \hat{e}_\phi .$$

The first three differential equations follow geometrically:

$$\dot{r} = v \cos i_r \quad (2)$$

$$\dot{\theta} = \frac{v}{r} \cos i_\theta \quad (3)$$

$$\dot{\phi} = \frac{v}{r \sin \theta} \cos i_\phi \quad (4)$$

To find the differential equations for \dot{i}_r , \dot{i}_θ , and \dot{i}_ϕ , we consider conditions for the travel time to be stationary with respect to small changes in the ray path. Using θ as a parameter, let the ray path, c , be specified in terms of $r(\theta)$ and $\phi(\theta)$. The travel time of a ray between two points where $\theta = \theta_1$ and $\theta = \theta_2$ is

$$T = \int_c \frac{d\sigma}{v} = \int_{\theta_1}^{\theta_2} \frac{r d\theta}{v \cos i_\theta}.$$

where $d\sigma$ is an element of length of the ray path. Consider a small change in the ray path specified by $\delta r(\theta)$, $\delta \phi(\theta)$ with $\delta r(\theta_1) = \delta r(\theta_2) = \delta \phi(\theta_1) = \delta \phi(\theta_2) = 0$, that is, with the end points of the ray fixed. The change in the travel time is

$$\delta T = \int_{\theta_1}^{\theta_2} \frac{\delta r d\theta}{v \cos i_\theta} + \int_{\theta_1}^{\theta_2} \delta \left(\frac{1}{v} \right) \frac{r d\theta}{\cos i_\theta} + \int_{\theta_1}^{\theta_2} \delta \left(\frac{1}{\cos i_\theta} \right) \frac{r d\theta}{v} \quad (5)$$

Performing some algebraic manipulations, and integrating by parts, this can be written

$$\begin{aligned} \delta T = \int_{\theta_1}^{\theta_2} \left\{ \left[\frac{1}{v \cos i_\theta} - \frac{r}{v^2 \cos i_\theta} \frac{\partial v}{\partial r} - \frac{\cos i_\theta}{v r^2} \left(\frac{dr}{d\theta} \right)^2 \right. \right. \\ \left. \left. - \frac{d}{d\theta} \left(\frac{\cos i_\theta}{r v} \frac{dr}{d\theta} \right) \right] \delta r + \left[- \frac{r}{v^2 \cos i_\theta} \frac{\partial v}{\partial \phi} \right. \right. \end{aligned}$$

$$- \frac{d}{d\theta} \left(\frac{r \cos i_\theta}{v} \sin^2 \theta \frac{d\phi}{d\theta} \right) \delta \phi \} d\theta \quad (6)$$

As before, $\frac{\partial v}{\partial r}$, $\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial \phi}$ are ordinary derivatives with respect to position, not derivatives along the ray. Since we want $\delta T = 0$ for arbitrary δr and $\delta \phi$, the coefficients of δr and $\delta \phi$ in (6) must vanish. Noting, from (2), (3), and (4) that

$$\frac{dr}{d\theta} = \frac{r \cos i_r}{\cos i_\theta}$$

and

$$\frac{d\phi}{d\theta} = \frac{\cos i_\phi}{\sin \theta \cos i_\theta}$$

these conditions can be written as

$$\begin{aligned} \frac{v}{r} \cos i_\theta \frac{di_r}{d\theta} &= \sin i_r \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \\ - \cot i_r \left[\frac{\cos i_\theta}{r} \frac{\partial v}{\partial \theta} + \frac{\cos i_\phi}{r \sin \theta} \frac{\partial v}{\partial \phi} \right] & \end{aligned} \quad (7)$$

and

$$\begin{aligned} \frac{v}{r} \cos i_\theta \frac{di_\phi}{d\theta} &= - \cot i_\phi \left[\cos i_r \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \right. \\ & \left. + \frac{\cos i_\theta}{r} \frac{\partial v}{\partial \theta} - v \cot \theta \right] + \frac{\sin i_\phi}{r \sin \theta} \frac{\partial v}{\partial \phi} \end{aligned} \quad (8)$$

Using (3), we see that these are the expressions for \dot{i}_r and \dot{i}_ϕ . The expression for \dot{i}_θ can be found from them using the fact that $\cos^2 i_r + \cos^2 i_\theta + \cos^2 i_\phi = 1$.

Instead of i_ϕ or i_θ , it is simpler to use the angle ζ between the vertical plane

tangent to the ray and the meridional plane. We have

$$\cos \zeta = \frac{\cos i_{\theta}}{\sin i_r} \quad \sin \zeta = \frac{\cos i_{\phi}}{\sin i_r}$$

so

$$\begin{aligned} \sin \zeta \dot{\zeta} &= \frac{\sin i_{\theta}}{\sin i_r} \frac{d i_{\theta}}{d t} + \frac{\cos i_{\theta} \cos i_r}{\sin^2 i_r} \frac{d i_r}{d t} \\ &= \frac{\sin^2 \zeta}{r \sin i_r} \frac{\partial v}{\partial \theta} - \frac{\cos \zeta \sin \zeta}{\sin i_r} \cdot \frac{1}{r \sin \theta} \frac{\partial v}{\partial \phi} - \frac{v}{r} \sin i_r \sin^2 \zeta \cot \theta \end{aligned}$$

and we can write the five equations for the ray as

$$\dot{r} = v \cos i_r \quad (9)$$

$$\dot{\theta} = \frac{v}{r} \sin i_r \cos \zeta \quad (10)$$

$$\dot{\phi} = \frac{v}{r \sin \theta} \sin i_r \sin \zeta \quad (11)$$

$$\dot{i}_r = \sin i_r \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) - \frac{\cos i_r}{r} \left[\cos \zeta \frac{\partial v}{\partial \theta} + \frac{\sin \zeta}{\sin \theta} \frac{\partial v}{\partial \phi} \right] \quad (12)$$

$$\dot{\zeta} = \frac{\sin \zeta}{r \sin i_r} \frac{\partial v}{\partial \theta} - \frac{\cos \zeta}{\sin i_r} \cdot \frac{1}{r \sin \theta} \frac{\partial v}{\partial \phi} - \frac{v}{r} \sin i_r \sin \zeta \cot \theta \quad (13)$$

Amplitudes - Geometric Spreading

Two phenomena affect the amplitudes of body waves: geometric spreading of the rays and attenuation due to anelasticity. We will direct our attention to geometric spreading first, assuming the earth is non-dissipative.

Let

$I(i_{ro}, \zeta_o) =$ power/unit solid angle radiated at the focus

$E(\theta, \phi) =$ power/unit area of wavefront at the point of observation

where i_{ro}, ζ_o are the initial values of i_r , and θ, ϕ are the values of θ, ϕ at the point of observation. In a non-dissipative earth

$$I(i_{ro}, \zeta_o)d\Omega = E(\theta, \phi)dA \quad (14)$$

where $d\Omega$ and dA are the corresponding elements of solid angle at the source and surface area of wavefront at the receiver and are given by

$$\begin{aligned} d\Omega &= \sin i_{ro} di_{ro} d\zeta_o \\ dA &= \frac{R^2 \sin \theta}{\cos i_r} d\theta d\phi \end{aligned} \quad (15)$$

R is the earth's radius. Here i_r refers to the value at the observation point. $d\theta$ and $d\phi$ refer to changes with r held fixed, along the earth's surface, not along a wavefront. $di_{ro}, d\zeta_o, d\theta$ and $d\phi$ are related by the Jacobian of the transformation from θ, ϕ to i_{ro}, ζ_o defined by the rays:

$$\frac{\partial(\theta, \phi)}{\partial(i_{ro}, \zeta_o)} = \begin{vmatrix} \frac{\partial \theta}{\partial i_{ro}} & \frac{\partial \phi}{\partial i_{ro}} \\ \frac{\partial \theta}{\partial \zeta_o} & \frac{\partial \phi}{\partial \zeta_o} \end{vmatrix} \quad (16)$$

From (14), (15), and (16) we get

$$E = I \frac{\sin i_{ro} \cos i_r}{R^2 \sin \theta \frac{\partial(\theta, \phi)}{\partial(i_{ro}, \zeta_o)}} \quad (17)$$

To evaluate the partial derivatives in (16) we must solve ten more differential equations, for $\frac{\partial r}{\partial i_{r0}}, \frac{\partial r}{\partial \zeta_0}, \dots, \frac{\partial \zeta}{\partial i_{r0}}, \frac{\partial \zeta}{\partial \zeta_0}$ simultaneously with (9)-(13). These equations are obtained by differentiating equations (9)-(13) with respect to i_{r0} and ζ_0 and reversing the order of differentiation ($\frac{\partial}{\partial i_{r0}} \left[\frac{dr}{dt} \right] = \frac{d}{dt} \left[\frac{\partial r}{\partial i_{r0}} \right]$, etc.), yielding

$$\frac{d}{dt} \left[\frac{\partial r}{\partial q} \right] = \frac{\partial \dot{r}}{\partial q} = \frac{Dv}{Dq} \cos i_r - v \sin i_r \frac{\partial i_r}{\partial q} \quad (18)$$

$$\begin{aligned} \frac{d}{dt} \left[\frac{\partial \theta}{\partial q} \right] &= \frac{\partial \dot{\theta}}{\partial q} = \frac{1}{r} \frac{Dv}{Dq} \sin i_r \cos \zeta - \frac{v}{r^2} \sin i_r \cos \zeta \frac{\partial r}{\partial q} + \frac{v}{r} \cos i_r \cos \zeta \frac{\partial i_r}{\partial q} \\ &\quad - \frac{v}{r} \sin i_r \sin \zeta \frac{\partial \zeta}{\partial q} \end{aligned} \quad (19)$$

$$\begin{aligned} \frac{d}{dt} \left[\frac{\partial \phi}{\partial q} \right] &= \frac{\partial \dot{\phi}}{\partial q} = \frac{1}{r \sin \theta} \sin i_r \sin \zeta \frac{Dv}{Dq} - \frac{v}{r^2 \sin \theta} \sin i_r \sin \zeta \frac{\partial r}{\partial q} - \frac{v}{r \sin^2 \theta} \\ &\quad \sin i_r \sin \zeta \cos \theta \frac{\partial \theta}{\partial q} + \frac{v}{r \sin \theta} \cos i_r \sin \zeta \frac{\partial i_r}{\partial q} + \frac{v}{r \sin \theta} \\ &\quad \sin i_r \cos \zeta \frac{\partial \zeta}{\partial q} \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{d}{dt} \left[\frac{\partial i_r}{\partial q} \right] &= \frac{\partial \dot{i}_r}{\partial q} = \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \cos i_r \frac{\partial i_r}{\partial q} + \sin i_r \left\{ \frac{D}{Dq} \left(\frac{\partial v}{\partial r} \right) - \frac{1}{r} \frac{Dv}{Dq} + \frac{v}{r^2} \frac{\partial r}{\partial q} \right\} \\ &\quad - \frac{\cos i_r}{r} \left\{ \cos \zeta \frac{D}{Dq} \left(\frac{\partial v}{\partial \theta} \right) - \frac{\partial v}{\partial \theta} \sin \zeta \frac{\partial \zeta}{\partial q} + \frac{\sin \zeta}{\sin \theta} \frac{D}{Dq} \left(\frac{\partial v}{\partial \phi} \right) \right. \\ &\quad \left. + \frac{\partial v}{\partial \phi} \left(\frac{\cos \zeta}{\sin \theta} \frac{\partial \zeta}{\partial q} - \frac{\sin \zeta \cos \theta}{\sin^2 \theta} \frac{\partial \theta}{\partial q} \right) \right\} + \left(\frac{\sin i_r}{r} \frac{\partial i_r}{\partial q} + \frac{\cos i_r}{r^2} \frac{\partial r}{\partial q} \right) \\ &\quad \left(\cos \zeta \frac{\partial v}{\partial \theta} + \frac{\sin \zeta}{\sin \theta} \frac{\partial v}{\partial \phi} \right) \end{aligned} \quad (21)$$

$$\begin{aligned}
\frac{d}{dt} \left[\frac{\partial \zeta}{\partial q} \right] &= \frac{\dot{\zeta}}{\partial q} = \frac{\sin \zeta}{r \sin i_r} \frac{D}{Dq} \left(\frac{\partial v}{\partial \theta} \right) + \frac{\partial v}{\partial \theta} \left\{ \frac{\cos \zeta}{r \sin i_r} \frac{\partial \zeta}{\partial q} - \frac{\sin \zeta}{r^2 \sin i_r} \frac{\partial r}{\partial q} - \right. \\
&\quad \left. \frac{\sin \zeta \cos i_r}{r \sin^2 i_r} \frac{\partial i_r}{\partial q} \right\} - \frac{\cos \zeta}{r \sin \theta \sin i_r} \frac{D}{Dq} \left(\frac{\partial v}{\partial \phi} \right) + \frac{\partial v}{\partial \phi} \left\{ \frac{\sin \zeta}{r \sin i_r \sin \theta} \right. \\
&\quad \left. \frac{\partial \zeta}{\partial q} + \frac{\cos \zeta \cos i_r}{r \sin \theta \sin^2 i_r} \frac{\partial i_r}{\partial q} + \frac{\cos \zeta}{r^2 \sin \theta \sin i_r} \frac{\partial r}{\partial q} + \frac{\cos \zeta \cos \theta}{r \sin^2 \theta \sin i_r} \frac{\partial \theta}{\partial q} \right\} \\
&\quad - \frac{1}{r} \sin i_r \sin \zeta \cot \theta \frac{Dv}{Dq} + \frac{v}{r^2} \sin i_r \sin \zeta \cot \theta \frac{\partial r}{\partial q} - \frac{v}{r} \cos i_r \sin \zeta \\
&\quad \cot \theta \frac{\partial i_r}{\partial q} - \frac{v}{r} \sin i_r \cos \zeta \cot \theta \frac{\partial \zeta}{\partial q} + \frac{v}{r \sin^2 \theta} \sin i_r \sin \zeta \frac{\partial \theta}{\partial q}
\end{aligned} \tag{22}$$

where

$$\frac{D}{Dq} = \frac{\partial}{\partial r} \frac{\partial r}{\partial q} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial q} + \frac{\partial}{\partial \phi} \frac{\partial \phi}{\partial q} \tag{23}$$

and we have used q to represent either of the initial angles i_{r0} or ζ_0 . The derivatives $\frac{\partial r}{\partial i_{r0}}$, $\frac{\partial r}{\partial \zeta_0}$, etc. thus obtained are those which apply when the travel time is held fixed; that is they apply to values on a particular wavefront. Derivatives along the earth's surface (r constant) needed in equation (16) can be obtained using the identity

$$\frac{\partial}{\partial q} \Big|_r = \frac{\partial}{\partial q} \Big|_t - \left(\frac{\partial r}{\partial q} \Big|_t / \frac{\partial r}{\partial t} \Big|_q \right) \frac{\partial}{\partial t} \Big|_q. \tag{24}$$

The constant q derivatives are just those given in equations (9)-(13).

Amplitudes - Attenuation

The attenuation of seismic waves due to anelasticity is quite easy to calculate if the intrinsic quality factor, Q , is known as a function of position and the ray path has been calculated. The power in the wave is simply reduced by the factor

$$\exp \left(- \omega \int \frac{dt}{Q} \right)$$

where ω is the angular frequency of the wave and the integral is evaluated along the ray path. The power per unit area of wavefront, P , is related to the amplitude, A , by

$$P = \frac{1}{2} \rho v \omega^2 A^2$$

for both compressional and shear waves, where ρ is the density of the medium.

Symmetry relations

In special cases in which the earth model possesses certain symmetry properties, the 5 ray variables are no longer independent and one or more of the equations (9)-(13) may be eliminated from the system to be solved. For example, in cartesian coordinates, if the velocity is independent of one of the coordinates then by (1b) the corresponding component of the slowness vector \bar{S} is constant along a ray. In spherical coordinates the situation is more complicated. The rate of change of \bar{S} is

$$\begin{aligned} \dot{\bar{S}} &= [\dot{S}_r - S_\theta \dot{\theta} - S_\phi \sin \theta \dot{\phi}] \hat{e}_r \\ &+ [\dot{S}_\theta + S_r \dot{\theta} - S_\phi \cos \theta \dot{\phi}] \hat{e}_\theta \end{aligned}$$

$$+ [\dot{S}_\phi + S_r \sin \theta \dot{\phi} + S_\theta \cos \theta \dot{\phi}] \hat{e}_\phi$$

and, with (1b) this gives

$$\dot{S}_r = -\frac{1}{v} \frac{\partial v}{\partial r} + \frac{v^2}{r} (S_\theta^2 + S_\phi^2)$$

$$\dot{S}_\theta = -\frac{1}{vr} \frac{\partial v}{\partial \theta} - \frac{v^2}{r} (S_r S_\theta - \cot \theta S_\phi^2)$$

$$\dot{S}_\phi = -\frac{1}{vr \sin \theta} \frac{\partial v}{\partial \phi} - \frac{v^2}{r} (S_r S_\phi + \cot \theta S_\theta S_\phi)$$

From (1a), (9), (10), and (11) the components of \bar{S} are

$$S_r = \frac{\cos i_r}{v}$$

$$S_\theta = \frac{\sin i_r \cos \zeta}{v}$$

$$S_\phi = \frac{\sin i_r \sin \zeta}{v}$$

These equations can be used to verify the following relations:

$$\underline{v = f(r):} \quad (rS_\theta)^2 + (rS_\phi)^2 = \left(\frac{r}{v} \sin i_r\right)^2 = \text{const.} \quad (25)$$

$$\underline{v = f(r, \theta):} \quad r \sin \theta S_\phi = \frac{r}{v} \sin i_r \sin \zeta \sin \theta = \text{const.} \quad (26)$$

The amplitude calculations, also, can be simplified by using symmetry relations. For example, when the velocity depends only on r and θ , from equation (26) the quantity

$$p = \frac{r}{v} \sin i_r \sin \zeta \sin \theta$$

is constant along a ray, and thus so are its derivatives

$$\frac{\partial p}{\partial q} = p \left[\frac{1}{r} \frac{\partial r}{\partial q} - \frac{1}{v} \frac{Dv}{Dq} + \cot i_r \frac{\partial i_r}{\partial q} + \cot \zeta \frac{\partial \zeta}{\partial q} + \cot \theta \frac{\partial \theta}{\partial q} \right] \quad (27)$$

where, as before, q represents either i_{r0} or ζ_0 and $\frac{Dv}{Dq}$ is defined in (23). It further follows that the quantity in brackets in (27) is constant and this fact can be used to eliminate two of the equations (18)-(22) from the amplitude calculations.

Ray Tracing for Surface Waves

It should be pointed out that a very similar approach can be used to calculate surface wave paths on an earth model in which the surface wave velocity is a function of geographic position. The ray tracing equations are obtained from equations (10), (11) and (13) by setting $r = R$ and $i_r = \pi/2$:

$$\dot{\theta} = \frac{v}{R} \cos \zeta \quad (28)$$

$$\dot{\phi} = \frac{v}{R \sin \theta} \sin \zeta \quad (29)$$

$$\dot{\zeta} = \frac{\sin \zeta}{R} \frac{\partial v}{\partial \theta} - \frac{\cos \zeta}{R \sin \theta} \frac{\partial v}{\partial \phi} - \frac{v}{R} \sin \zeta \cot \theta \quad (30)$$

The calculation of amplitudes, too, follows a development similar to that given above. By analogy with equation (14) we have

$$I(\zeta_0) d\zeta_0 = E(\theta, \phi) d\ell \quad (31)$$

where $I(\zeta_0) =$ power/unit angle radiated at the source

$E(\theta, \phi) =$ power/unit length of wavefront, $d\ell$, at θ, ϕ .

Using the fact that

$$\frac{\partial \ell}{\partial \xi_0} = R \sqrt{\left(\frac{\partial \theta}{\partial \xi_0}\right)^2 + \sin^2 \theta \left(\frac{\partial \phi}{\partial \xi_0}\right)^2}$$

equation (31) gives

$$E = \frac{I}{R \sqrt{\left(\frac{\partial \theta}{\partial \xi_0}\right)^2 + \sin^2 \theta \left(\frac{\partial \phi}{\partial \xi_0}\right)^2}}$$

and the additional differential equations needed to calculate $\frac{\partial \theta}{\partial \xi_0}$ and $\frac{\partial \phi}{\partial \xi_0}$ can be obtained from equations (28)-(30) the same way equations (18)-(22) were derived from (9)-(13).

Examples

As examples of the use of the ray tracing technique, we will show ray paths calculated from two-dimensional models of dipping high-velocity slabs within the earth's upper mantle, such as are present beneath island arcs. The numerical solution of the differential equations has been carried out using the step size extrapolation method of Bulirsch and Stoer (1966), which is particularly fast and accurate.

Figure 1 shows ray paths and wavefronts for a very simple model, in which the velocity is specified by an analytical function

$$v = v_0 + A \exp \left[-\left(\frac{x}{h}\right)^2 - \frac{z}{d} \right]$$

where x is the distance from the slab axis (assumed to dip at a 45° angle, and z is depth from the earth's surface. v_0 , A , h , and d are constants. In this example, the earthquake focus is located near the edge of the slab, thus producing, in addition to a

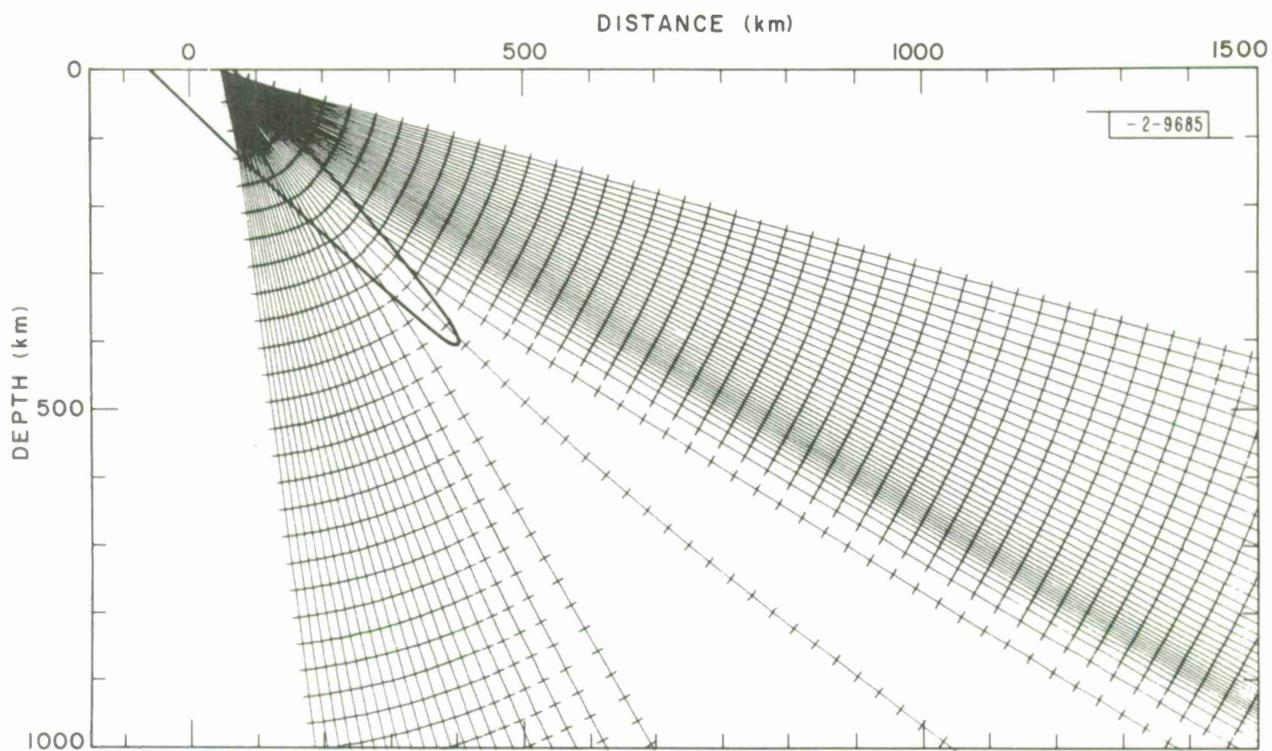


Fig. 1. P wave ray paths and wavefronts calculated for a focus near the edge of a simple analytical model of a high-velocity lithospheric slab (see text). The slab parameters are: dip = 45° ; velocity outside slab, $v_0 = 8.0$ km/sec; maximum velocity contrast, $A = 0.8$ km/sec; slab thickness, $2h = 80$ km; depth of penetration, $d = 300$ km.

large shadow zone, a region of strong focusing of energy.

Figure 2 shows ray paths and wavefronts for a wave emerging from the mantle beneath a more detailed slab model. The velocity model is derived from the theoretical temperature field calculated numerically by Toksoz et al (1971). In this case, the velocity is specified at a grid of points, and interpolation between these points is accomplished using cubic splines, so that the velocity and its first spatial derivatives are continuous. Again, a shallow zone is produced, along with regions of focusing. Seismic stations on or near island arcs would thus be expected to have "blind spots" for earthquakes in certain regions.

Conclusions

The formulation of ray theory given here is applicable to a medium with any type of inhomogeneity, and is well suited for numerical computations. It allows the computation of amplitudes, considering both geometric and anelastic effects, though the computational labor may be greatly increased. Furthermore, it can be extended easily to problems such as surface wave propagation on an earth with geographical variations.

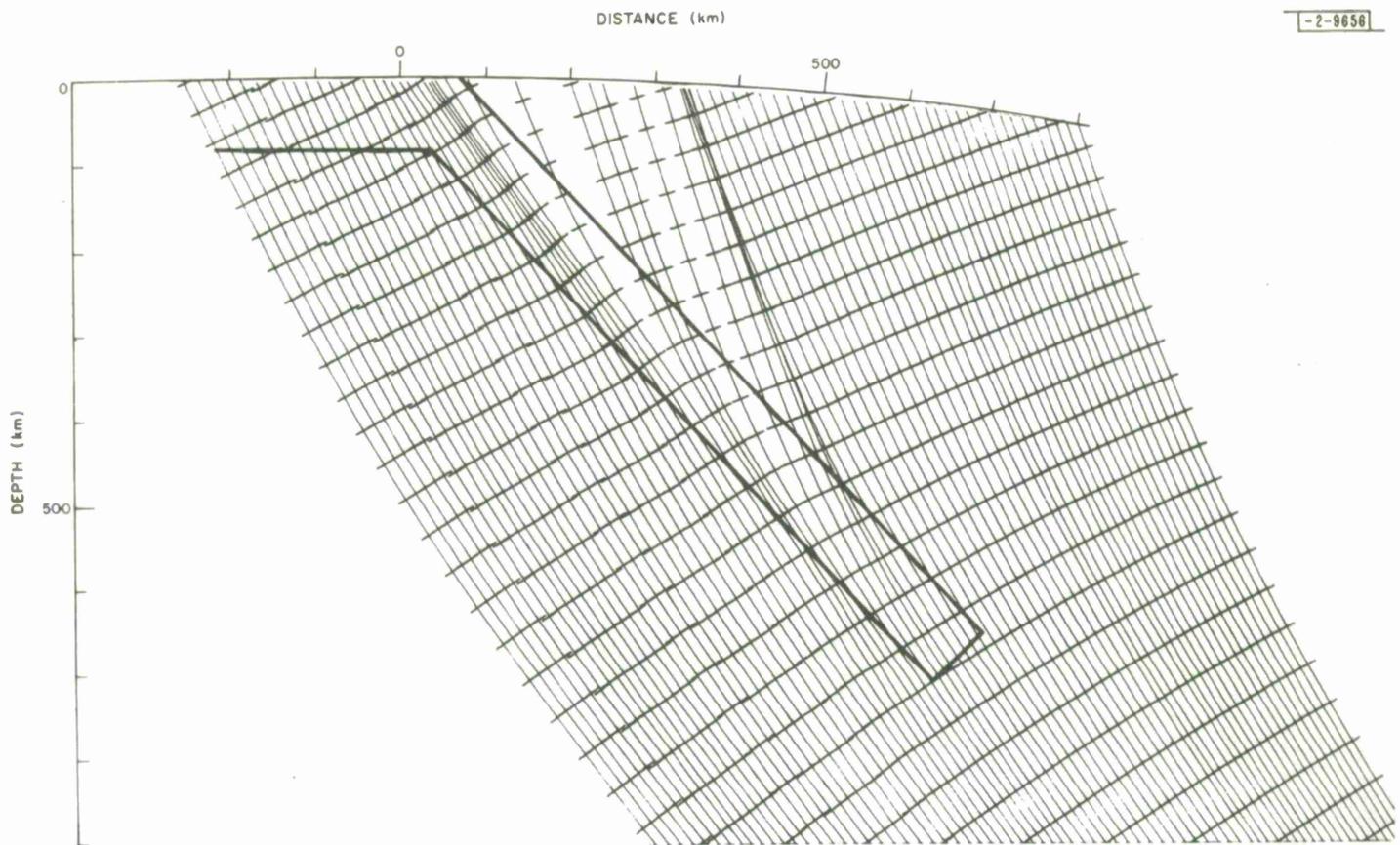


Fig. 2. Ray paths and wavefronts for P wave emerging beneath an island arc. The slab model is derived from numerical temperature field calculations.

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<p>The study of lateral variations of earth structure has been stimulated recently by several factors, especially the theory of plate tectonics and the increasing use of large seismic arrays. The extension of seismic ray theory to two and three dimensional structures is thus of great practical importance. The problem of ray tracing in a generally heterogeneous medium is treated, using the calculus of variations and Fermat's principle of stationary time. The solution is expressed in terms of a system of five simultaneous first order differential equations giving the variation with time of the position and direction of motion of a point on a ray in terms of the wave speed and its spatial derivatives in the medium. The form of the equations is particularly convenient for solution on a digital computer. The effect upon the wave amplitudes of geometric spreading can also be calculated by the inclusion of ten additional equations in the system, although this greatly increases the computational labor. If the earth model has certain symmetry properties, then constants of the motion along each ray can be found which simplify the calculations. For example, for two dimensional models in which the velocity depends only on the coordinates r (radius) and θ (colatitude) one equation can be eliminated from the ray tracing system, and two more can be eliminated from the amplitude equations. The propagation of surface waves on an earth with geographical variations can be treated by a simplified special case of the method presented here.</p>			
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