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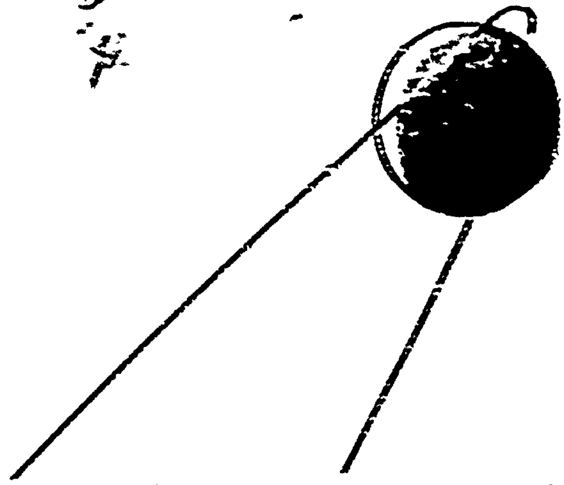

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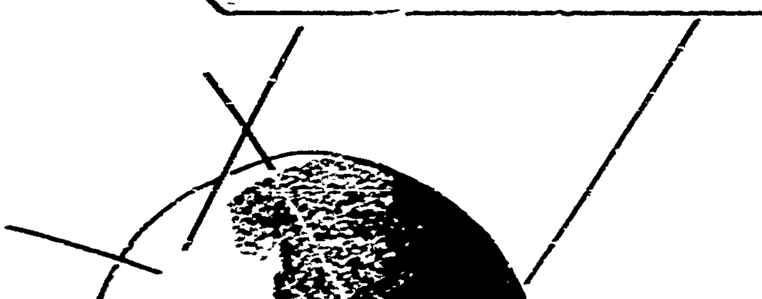

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Austin, Texas

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THE LEVI-CIVITA PROBLEM AND THE KUSTAANHEIMO-STIEFEL TRANSFORMATION

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Austin

Abstract

By imposing regularity of the new accelerations and velocities we introduce a time and a coordinate transformation which generate as particular cases the Levi-Civita and the KS-transformations.

Introduction

Consider the differential equation

$$\ddot{\underline{x}} = \left(\frac{\partial W}{\partial \underline{x}}\right)^T \tag{1}$$

where $\underline{x}$ is an n-column in R^n, $W = W(\underline{x})$ is a scalar function and $\partial W/\partial \underline{x}$ its row gradient with respect to $\underline{x}$. The superscript T indicates the transpose of a vector or matrix. Let x be the absolute value of $\underline{x}$, $x = (\sum_{j=1}^{n} x_j^2)^{\frac{1}{2}}$. We consider the case

$$W = 1/x + R(\underline{x}) + E \tag{2}$$

and assume that $R(\underline{x})$ is C^2 in an open region Ω of R^n containing the origin $x = 0$. It follows that for every $\underline{x}$ in Ω except $\underline{x} = \underline{0}$ and for every finite $\dot{\underline{x}}$ there exists and is unique a

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solution of Equation (1) which can be continued in Ω except through the origin. We wish to study the solutions which approach the singularity $x = 0$. Assuming E to be an arbitrary constant, the energy integral can be written

$$\underline{\dot{x}}^T \underline{\dot{x}} = 2W(\underline{x}) \tag{3}$$

Let us consider the tangential transformation

$$d\underline{x} = 2M(\underline{u})\, d\underline{u} \tag{4}$$

where $\underline{u}$ is an n-column in R^n and M an n x n matrix. Necessary properties are that in an open region Ω_u containing the origin $\underline{u} = \underline{0}$ and, except at a finite number of points, M^{-1} exists and also

$$\frac{\partial M_{k\ell}}{\partial u_s} = \frac{\partial M_{ks}}{\partial u_\ell} \tag{5}$$

for $k,\ell = 1,2,\ldots,n$ and $s > \ell$, which are the integrability conditions of Equation (4).

We also introduce a new independent variable τ by

$$d\tau = \sigma(\underline{u})\, dt \tag{6}$$

where $\sigma(\underline{u})$ is a scalar function, at least C^2 in Ω_u except at a finite number of points. With the use of Equations (4) and (5) into Equations (1) and (3), we obtain (Giacaglia, 1968) the following relations

$$\underline{u}'' = -M^{-1}M\underline{u}' - (\sigma'/\sigma)\underline{u}' + (1/4\sigma^2)\, M^{-1}(M^{-1})^T\, (\partial W/\partial \underline{u})^T \tag{7}$$

and

$$\underline{u}'^T M^T M\underline{u}' = W/2\sigma^2 \tag{8}$$

where primes indicate derivatives with respect to τ.

Regularization

We require that at $x = 0$, $|u_k'| < A$, A finite. We also restrict ourselves to orthogonal transformations, that is, such that

$$g(\underline{u})\sigma^2(\underline{u})M^T M = I \tag{9}$$

where I is the identity $n \times n$ matrix and $g(\underline{u})$ is a scalar C^2 function in Ω_u except at a finite number of points. Equation (8) becomes

$$\underline{u}'^2 = \frac{1}{2}gW \tag{10}$$

while (7) can be written as

$$\underline{u}'' = \underline{F}(\underline{u},\underline{u}') + \frac{1}{4}\,[\frac{\partial}{\partial \underline{u}}(gW)]^T \tag{11}$$

where

$$\underline{F}(\underline{u},\underline{u}') = -g\sigma^2 M^T M'\underline{u}' - (\sigma'/\sigma)\underline{u}' - (1/2g)(\partial g/\partial \underline{u})^T \underline{u}'^T \underline{u}' \tag{12}$$

It follows from (10) that if $|u_k'| < A$ as $x \to 0$, then g must be infinitesimal of order x, that is, in the neighborhood of $x = 0$ we may write $g = xf(\underline{x})$ where $f(\underline{0})$ is finite and not zero.

For the transformation to be canonical it is now sufficient to have $\underline{F} = \underline{0}$ whatever $\underline{u}'$. The p-th component of $\underline{F}$ is

$$F_p = -g\sigma^2 \sum_{j,s,q} M_{jp}(\partial M_{js}/\partial u_q)u'_q u'_s - \frac{1}{\sigma}\sum_q(\partial\sigma/\partial u_q)u'_q u'_p$$

$$-(1/2g)\ (\partial g/\partial u_p)\ \sum_q u'^2_q$$

For F_p to be identically zero, the coefficients of $(u'_\alpha u'_\beta)$ must vanish whatever (α,β) are, so that we obtain

$$2g\sigma^2 \sum_j M_{jp} \frac{\partial M_{i\beta}}{\partial u_\alpha} + \frac{1}{\sigma}\left(\frac{\partial\sigma}{\partial u_\alpha}\delta_{\beta p} + \frac{\partial\sigma}{\partial u_\beta}\delta_{\alpha p}\right) + \frac{1}{g}\ \frac{\partial g}{\partial u_p}\delta_{\alpha\beta} = 0 \qquad (13)$$

for all values of p,α,β. If one multiplies Equation (13) by $(M^{-1})_{pi}$ and makes use of Equation (9) the result is

$$\frac{\partial M_{i\beta}}{\partial u_\alpha} + \frac{1}{2g}\left(\frac{\partial\sigma}{\partial u_\alpha}\ M_{i\beta} + \frac{\partial\sigma}{\partial u_\beta} M_{i\alpha}\right)$$

$$+ \frac{1}{2g}\ \delta_{\alpha\beta}\ \sum_p M_{ip}\ \frac{\partial g}{\partial u_p} = 0 \qquad (14)$$

for all (i,α,β), the equation being symmetric with respect to the pair (α,β). This is a system, therefore, of $n^2(n+1)/2$ partial differential equations which define M, g,σ such that the vector $\underline{F}$ is identically zero.

Getting a new system independent of the velocity $\underline{u}'$ and performing an orthogonal transformation will reduce the equations to the canonical form

$$\underline{u}'' = \frac{1}{4}\left[\frac{\partial}{\partial \underline{u}}(gW)\right]^T \tag{15}$$

and this can be termed to be Levi-Civita's problem. In principle, the problem is now reduced to the choice of the scalar function $g(\underline{u})$.

<u>Connection between time and coordinates transformations</u>

A simple relation between g and σ can be obtained from Equation (13). In fact, considering the case $\beta = p \neq \alpha$, one finds

$$2g\sigma^2 \sum_j M_{jp} \frac{\partial M_{jp}}{\partial u_\alpha} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial u_\alpha} = 0$$

or

$$-\frac{1}{g\sigma^3}\frac{\partial \sigma}{\partial u} = \frac{\partial}{\partial u} \sum_j M_{jp}^2$$

or, considering Equation (9),

$$-\frac{1}{g\sigma^3}\frac{\partial \sigma}{\partial u_\alpha} = \frac{\partial}{\partial u_\alpha}\left(\frac{1}{g\sigma^2}\right)$$

so that

$$\frac{\partial}{\partial u_\alpha}(\ln g\sigma) = 0$$

for all values of α. Thus, by proper normalization

$$g\sigma = 1 \tag{16}$$

independently of the form of matrix M but its orthogonality.

This result indicates the fact that if g is infinitesimal with x as resulted from Equation (10), then $1/\sigma$ defining the time transformation must have exactly the same character as $x \to 0$. It also follows from Equation (9) that $|\det M| = |g|^{n/2}$, so that if and only if $n = 2$ the $|\det M|$ is infinitesimal of the same order of g as $x \to 0$.

The choice of $g(\underline{u}(\underline{x}))$ and the connection between $\underline{u}$ and $\underline{x}$

Use of (16) into (14) gives

$$\frac{\partial M_{i\beta}}{\partial u_\alpha} - \frac{1}{2g}\left(M_{i\beta}\frac{\partial g}{\partial u_\alpha} + M_{i\alpha}\frac{\partial g}{\partial u_\beta} - \delta_{\alpha\beta}\sum_p M_{ip}\frac{\partial g}{\partial u_p}\right) = 0 \qquad (17)$$

The simplest choice of g, satisfying the above derived regularization condition is

$$g = Kx$$

where K is a constant, not zero.
It follows that

$$\frac{\partial g}{\partial u_p} = K\sum_\ell \frac{\partial x}{\partial x_\ell}\frac{\partial x_\ell}{\partial u_p} = 2K\sum_\ell \frac{x_\ell}{x} M_{\ell p}$$

or

$$\frac{1}{2K^2}\, g\frac{\partial g}{\partial u_p} = \sum_\ell x_\ell M_{\ell p}$$

Multiplying this last by $(M^{-1})_{pm}$ and taking Equation (9) into account, summation of p gives

$$\frac{1}{2K^2} \sum_p M_{mp} \frac{\partial g}{\partial u_p} = x_m \tag{18}$$

Differentiating again with respect to u_ℓ and multiplying by $M_{m\ell}$ we obtain, after summation over m, and together with the integrability condition,

$$\frac{\partial^2 g}{\partial u_\ell^2} + \frac{1}{2g} \sum_p \left(\frac{\partial g}{\partial u_p}\right)^2 - 4K^2 = 0 \tag{19}$$

This last set of n partial differential equations defines the function $g(\underline{u}) = Kx$, explicitely in terms of $u_1, u_2, \ldots, u_n$. Evidently this is not enough to define the transformation from $\underline{x}$ to $\underline{u}$, or, better say, the matrix M.

Up to this moment one can conclude that for an orthogonal and integrable matrix $M(\underline{u})$, such that $M^T M = gI$, if $g = Kx$ and the new equations are independent of the velocity $\underline{u}'$, then the possible transformations M should be such that $g(\underline{u})$ satisfies Equation (19). Some of the results of these sections were presented in a previous work (Giacaglia, 1968) related to the n-body problem.

Two-dimensional case. Levi-Civita's Transformation.

The orthogonality condition

$$M^T M = gI$$

can be written, in general, as

$$M_{11} = \sqrt{g} \cos \phi$$
$$-M_{12} = \sqrt{g} \sin \phi$$
$$M_{21} = \sqrt{g} \sin \phi \qquad (20)$$
$$M_{22} = \sqrt{g} \sin \phi$$

where $\phi = \phi(\underline{u})$ has to be determined.

By considering the integrability conditions $\partial M_{11}/\partial u_2 = \partial M_{12}/\partial u_1$, $\partial M_{21}/\partial u_2 = \partial M_{22}/\partial u_1$, we obtain

$$\frac{\partial\phi}{\partial u_1} = -\frac{1}{2g}\frac{\partial g}{\partial u_2} , \quad \frac{\partial\phi}{\partial u_2} = \frac{1}{2g}\frac{\partial g}{\partial u_1}$$

or, since one must have $\partial^2\phi/\partial u_1\partial u_2 = \partial^2\phi/\partial u_2\partial u_1$, the result is

$$2g\frac{\partial^2 g}{\partial u_1^2} - \left[\left(\frac{\partial g}{\partial u_1}\right)^2 + \left(\frac{\partial g}{\partial u_2}\right)^2\right] = 0 \qquad (21)$$

But, for n = 2, Equation (19) gives

$$2g\frac{\partial^2 g}{\partial u_1^2} + \left[\left(\frac{\partial g}{\partial u_1}\right)^2 + \left(\frac{\partial g}{\partial u_2}\right)^2\right] = 8K^2 g$$

so tnat, necessarily,

$$\frac{\partial^2 g}{\partial u_1^2} = 2K^2$$

and similarly

$$\frac{\partial^2 g}{\partial u_2^2} = 2K^2 \quad .$$

By making use of Equation (19) we finally obtain

$$g = K^2(u_1^2 + u_2^2) + \alpha u_1 + \beta u_2 + \frac{\alpha^2 + \beta^2}{4K^2}$$

with α,β arbitrary constants. It is worth noting that the following solutions of Equation (21)

$$g = \frac{2}{3} K^2 \ (u_1 \pm u_2)^2 + 2\alpha K\sqrt{\frac{2}{3}} \ (u_1 \pm u_2) + \alpha^2$$

do not lead to orthogonality and integrability of M so they must be discarded. There is no other possible solution.

For the angle ϕ we obtain

$$\frac{\partial\phi}{\partial u_1} = - \frac{1}{2g} (2K^2u_2 + \beta)$$

$$\frac{\partial\phi}{\partial u_2} = + \frac{1}{2g} (2K^2u_1 + \alpha)$$

or, integrating,

$$\phi = \arctan \frac{2K^2u_2 + \beta}{2K^2u_1 + \alpha} - \gamma \tag{22}$$

where γ is an arbitrary constant. We now obtain

$$\sqrt{g} \cos \phi = [(2K^2u_1 + \alpha) \cos \gamma + (2K^2u_2 + \beta) \sin\gamma]/2K$$

$$\sqrt{g} \sin \phi = [-(2K^2u_1 + \alpha) \sin \gamma + (2K^2u_2 + \beta) \cos\gamma] /2K$$

so that the transformation matrix M can be written

$$M = R(\gamma) \left[K \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} + \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \right] \tag{23}$$

where

$$R(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}$$

In complex form, the transformation can be written

$$dx^* = 2e^{i\gamma}(Ku^* + a^*)du^* \tag{24}$$

where

$$x^* = x_1 + ix_2 \ , \ u^* = u_1 + iu_2 \ , \ a^* = \alpha + i\beta \ .$$

Equation (24) gives the most general form of Levi-Civita's transformation which corresponds to the particular case $\gamma = 0$, $K = 1$, $\alpha = \beta = 0$. In the same notation of Equation (24) the function g is

$$\frac{dt}{d\tau} = g = (Ku_1 + \alpha)^2 + (Ku_2 + \beta)^2 \tag{25}$$

with the corresponding Levi-Civita case

$$g = u_1^2 + u_2^2 \ .$$

We also remember that the transformation is

$$d\underline{x} = 2Md\underline{u}$$

so that one finds

$$\underline{x} = R(\gamma)\left[K\begin{pmatrix} u_1^2 - u_2^2 \\ 2u_1u_2 \end{pmatrix} + 2\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\underline{u}\right] + \underline{b} \tag{26}$$

or, in complex notation

$$x^* = e^{i\gamma}(Ku^{*2} + a^*u^*) + b^* \quad ,$$

where $\underline{b}$ is an arbitrary vector. It can also be written as

$$\underline{x} = R(\gamma)\left[K\begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} + 2\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\right]\underline{u} + \underline{b} \tag{27}$$

<u>The n-dimensional case</u>

Consider Equation (19) for $n > 2$. Let (u_k, u_ℓ), $k \neq \ell$, be a pair chosen however and consider the necessary conditions

$$\frac{\partial^2 g}{\partial u_k^2} = \frac{\partial^2 g}{\partial u_\ell^2} \tag{28}$$

that results from Equation (19). The general solution is

$$\begin{aligned} g &= f(u_k + u_\ell, u_\alpha) + h(u_k - u_\ell, u_\alpha) = \\ &= f(x,u_\alpha) + h(y,u_\alpha) \end{aligned} \tag{29}$$

where u_α is the set of all u_α $(\alpha=1,\ldots,n;\ \alpha \neq k,\ell)$ except the elements u_k and u_ℓ. By substitution into Equation (19) we find

$$\begin{aligned} 2\left(f\,\frac{\partial^2 h}{\partial y^2} + h\,\frac{\partial^2 f}{\partial x^2}\right) + 2\sum_\alpha \frac{\partial f}{\partial u_\alpha}\,\frac{\partial h}{\partial u_\alpha} &= -\left[2f\,\frac{\partial^2 f}{\partial x^2} + \right. \\ \left. + 2\left(\frac{\partial f}{\partial x}\right)^2 + \sum_\alpha \left(\frac{\partial h}{\partial u_\alpha}\right)^2 - 8K^2 f\right] &- 2h\left[\frac{\partial^2 h}{\partial x^2} + 2\left(\frac{\partial h}{\partial y}\right)^2 + \right. \\ \left. + \sum_\alpha \left(\frac{\partial h}{\partial u_\alpha}\right)^2 - 8K^2 h\right] &\equiv F(x,u_\alpha) + H(y,u_\alpha) \end{aligned}$$

Differentiating with respect to x and y successively we obtain

$$\frac{\partial f}{\partial x}\frac{\partial^3 h}{\partial y^3} + \frac{\partial^3 f}{\partial x^3}\frac{\partial h}{\partial y} + \sum_\alpha \frac{\partial^2 f}{\partial x \partial u_\alpha}\frac{\partial^2 h}{\partial y \partial u_\alpha} = 0 \ .$$

Admitting that $(\partial f/\partial x)\cdot(\partial f/\partial y) \neq 0$, division by this quanti⁺y leads to

$$\frac{\partial^3 f/\partial x^3}{\partial f/\partial x} + \frac{\partial^3 h/\partial y^3}{\partial h/\partial y} \equiv K(x,u_\alpha) + L(y,u_\alpha) =$$

$$= - \sum_\alpha \left(\frac{\partial^2 f}{\partial x \partial u_\alpha}\ \frac{\partial^2 h}{\partial y \partial u_\alpha}\right) \Big/ \left(\frac{\partial f}{\partial x}\ \frac{\partial h}{\partial y}\right)$$

Again, successive differentiation with respect to x and y, gives

$$\sum_\alpha \left(\frac{\partial h}{\partial y}\ \frac{\partial^3 h}{\partial y^2 \partial u_\alpha} - \frac{\partial^2 h}{\partial y \partial u_\alpha}\ \frac{\partial^2 h}{\partial y^2}\right)\left(\frac{\partial f}{\partial x}\ \frac{\partial^3 f}{\partial x^2 \partial u_\alpha} - \frac{\partial^2 f}{\partial x \partial u_\alpha}\ \frac{\partial^2 f}{\partial x^2}\right) = 0$$

and this relation has to be valid for all pairs (k,ℓ), that is, whatever u_α is.

The only way to satisfy both the last equation and Equation (19) is then given by the 2n equations

$$\frac{\partial h}{\partial y}\ \frac{\partial^3 h}{\partial y^2 \partial u_\alpha} - \frac{\partial^2 h}{\partial y \partial u_\alpha}\ \frac{\partial^2 h}{\partial y^2} = 0$$

$$\frac{\partial f}{\partial x}\ \frac{\partial^3 f}{\partial x^2 \partial u_\alpha} - \frac{\partial^2 f}{\partial x \partial u_\alpha}\ \frac{\partial^2 f}{\partial x^2} = 0$$

and, more specifically,

$$\frac{\partial^2 f}{\partial x^2} = 2A = \text{const.}$$

$$\frac{\partial^2 h}{\partial y^2} = 2P = \text{const.}$$

for every pair (k,ℓ). It follows from

$$f = Ax^2 + B(u_\alpha,y)x + C(u_\alpha,y)$$

$$h = Py^2 + Q(u_\alpha,x)y + R(u_\alpha,x)$$

that the most general allowed form for g, under the above specified conditions, is given by

$$g = \sum_{j=1}^{n} (Ku_j + a_j)^2 \tag{30}$$

where K and the a_j are constants, and there remains to be verified whether this form leads to integrable relations of the matrix M. More details on these results were given by Giacaglia (1966).

<u>The definition of the transformation M</u>

Consider again Equation (18) and differentiate it with respect to u_ℓ

$$\frac{\partial}{\partial u_\ell} \left(\sum_p M_{mp} \frac{\partial g}{\partial u_p} \right) = 4K^2 M_{m\ell}$$

and making use of (30),

$$\sum_p (Ku_p + a_p) \frac{\partial M_{m\ell}}{\partial u_p} = KM_{m\ell} \ . \tag{31}$$

Let us define

$$\omega_p = Ku_p + a_p \qquad (p = 1,2,\ldots,n) \tag{32}$$

so that Equation (31) becomes

$$\sum_p \omega_p \frac{\partial M_{m\ell}}{\partial \omega_p} = M_{m\ell}$$

and therefore $M_{m\ell}$ is an homogeneous function of the first degree of the ω_p, that is, a linear function of the u_p.

Let

$$M_{k\ell} = \sum_s \alpha^s_{k\ell}\omega_s \tag{33}$$

where the matrix M must be orthogonal and lead to integrable relations.

Consider first the orthogonality

$$\sum_k M_{ki}M_{kj} = g\,\delta_{ij}$$

where, according to (30) and (32)

$$g = \sum_p \omega_p^2 \quad . \tag{34}$$

We obtain

$$\sum_s \sum_r \left(\sum_k \alpha^s_{ki}\alpha^r_{kj}\right)\omega_s\omega_r = \delta_{ij} \sum_p \omega_p^2$$

or

$$\alpha^s_{ki}\alpha^r_{kj} + \alpha^r_{ki}\alpha^s_{kj} = 2\delta_{ij}\varepsilon_{rs} \tag{35}$$

Consider the n matrices $\alpha_k = \{c^k_{ij}\}$, so that Equation (35) can be written

$$\alpha_s^T\alpha_r + \alpha_r^T\alpha_s = 2I\,\delta_{rs} \tag{36}$$

with the integrability conditions giving

$$\alpha^p_{k\ell} = \alpha^\ell_{kp} \tag{37}$$

The solution for u = 2 is given by

$$\alpha_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \alpha_2 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

giving back the general Levi-Civita transformation

$$M = \begin{pmatrix}\omega_1 & -\omega_2\\ \omega_2 & \omega_1\end{pmatrix} \quad .$$

For $n = 3$, there is no solution, while the problem can partially be solved for $n = 4$ or $n = 8$ (Hurwitz, 1933).

In order to study the problem for $n = 4$ we here make use of quaternions as opposed to the above spinor solution of the case $n = 2$.

The four-dimensional case

We follow the results in Hurwitz (1919) and define the imaginary units i,j,k, satisfying the multiplication rules: $i^2 = j^2 = k^2 = -1$, $i\cdot j = -j\cdot i = k$, etc.

The rotation matrix M is then generated by

$$dx_1 + idx_2 + jdx_3 + kdx_4 = 2(f_1 + if_2 + jf_3 + kf_4)\cdot$$
$$\cdot\,(du_1 + idu_2 + jdu_3 + kdu_4)$$

which we write $dx^* = 2f^* \cdot du^*$. We easily find that M is represented by

$$\begin{pmatrix} f_1 & -f_2 & -f_3 & -f_4 \\ f_2 & f_1 & -f_4 & f_3 \\ f_3 & f_4 & f_1 & -f_2 \\ f_4 & -f_3 & f_2 & f_1 \end{pmatrix} \tag{38}$$

where, as was seen, the f_j must be linear functions of the u_j or homogeneous of first degree of the ω_j. The above matrix is orthogonal whatever the f_j are (not all zero). The matrices α_j (j=1,2,3,4) are obtained immediately by setting one of the f_j equal to unity and the others equal to zero, in succession.

By imposing integrability conditions on the first three rows of M we obtain the conditions

$$\begin{aligned} f_1 &= \phi(u_1,u_2,u_3) \\ f_2 &= f_2(u_1,u_2,u_4) \\ f_3 &= f_3(u_1,u_3,u_4) \\ f_4 &= f_4(u_2,u_3,u_4) \end{aligned}$$

where ϕ is linear while f_2, f_3, f_4 must satisfy the relations

$$\left\{ \begin{array}{lll} \partial f_2/\partial u_1 = -\partial\phi/\partial u_2 \ , & \partial f_2/\partial u_2 = \partial\phi/\partial u_1 \ , & \partial f_2/\partial u_4 = -\partial\phi/\partial u_3 \\ \partial f_3/\partial u_1 = -\partial\phi/\partial u_3 \ , & \partial f_3/\partial u_3 = \partial\phi/\partial u_1 \ , & \partial f_3/\partial u_4 = \partial\phi/\partial u_2 \\ \partial f_4/\partial u_2 = -\partial\phi/\partial u_3 \ , & \partial f_4/\partial u_3 = \partial\phi/\partial u_2 \ , & \partial f_4/\partial u_4 = -\partial\phi/\partial u_1 \end{array} \right. \tag{3g}$$

If integrability of the fourth row is imposed, there results $\phi = 0$, so that complete integrability is impossible. In other words, whatever ϕ is ($\neq 0$), the differential form

$f_4du_1 - f_3du_2 + f_2du_3 + f_1du_4$ is not integrable. This problem, of course, does not represent an obstacle but only means that to the holonomic constraint $x_4 = 0$ in the physical space there corresponds the non holonomic constraint $(f^* \cdot du^*)_k = 0$ in the u-space. This problem has been discussed in several works for the particular transformation (KS-Transformation) introduced by Kustaanheimo and Stiefel (1965).

We now obtain the general solution of the problem by setting

$$f_1 = \beta_1\omega_1 + \beta_2\omega_2 + \beta_3\omega_3 \tag{40}$$

so that, applying Equation (39), we easily find

$$\begin{aligned} f_2 &= -\beta_2\omega_1 + \beta_1\omega_2 - \beta_3\omega_4 \\ f_3 &= -\beta_3\omega_1 + \beta_1\omega_3 + \beta_2\omega_4 \\ f_4 &= -\beta_3\omega_2 + \beta_2\omega_3 - \beta_1\omega_4 \end{aligned} \tag{41}$$

where

$$\omega_j = Ku_j + a_j$$

The transformation M depends therefore on seven parameters, while K is just a scaling factor which we can set equal to unity without loosing anything. Not all β's can be zero.

In components form, the transformation can now be written

$$
\begin{aligned}
Kdx_1 = \; & \beta_1 d(\omega_1^2 - \omega_2^2 - \omega_3^2 + \omega_4^2) + \\
& + \beta_2 d(\omega_1\omega_2 - \omega_3\omega_4) + \\
& + \beta_3 d(\omega_1\omega_3 + \omega_2\omega_4)
\end{aligned}
$$

$$
\begin{aligned}
Kdx_2 = \; & \beta_1 d(\omega_1\omega_2 + \omega_3\omega_4) + \\
& + \beta_2 d(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2) + \\
& + \beta_3 d(-\omega_1\omega_4 + \omega_2\omega_3)
\end{aligned}
$$

$$
\begin{aligned}
Kdx_3 = \; & \beta_1 d(\omega_1\omega_3 - \omega_2\omega_4) + \\
& + \beta_2 d(\omega_1\omega_4 + \omega_2\omega_3) + \\
& + \beta_3 d(-\omega_1^2 - \omega_2^2 + \omega_3^2 + \omega_4^2)
\end{aligned}
$$

$$
\begin{aligned}
Kdx_4 = \; & \beta_1(-\omega_4 d\omega_1 - \omega_3 d\omega_2 + \omega_2 d\omega_3 + \omega_1 d\omega_4) + \\
& + \beta_2(\omega_3 d\omega_1 - \omega_4 d\omega_2 - \omega_1 d\omega_3 + \omega_2 d\omega_4) + \\
& + \beta_3(-\omega_2 d\omega_1 + \omega_1 d\omega_2 - \omega_4 d\omega_3 + \omega_3 d\omega_4)
\end{aligned}
\tag{42}
$$

and it is clear that dx_4 is not an exact differential. Setting $K = 1$, $a_j = 0$ $(j=1,2,3,4)$, $\beta_2 = \beta_3 = 0$, we obtain essentially the KS-Transformation.

Other than transformation (42) one could consider rotations, translations and inversions which are analytic and can be used to make the transformation satisfy other conditions than the ones required in this work. For example, they were used by Waldvogel (1967) to generalize Birkhoff's transformation for the 3-dimensional restricted problem of three bodies.

In that work, the author gives an excellent discussion of the general problem.

Generalized forms of generators for regularization were given by Szebehely (1967a), Giacaglia (1967a), Giacaglia (1966), Giacaglia (1967b), and they are totally discussed by Szebehely (1967b).

The results given by (42) show that it is not possible to fully generalize Equation (24) into a four-dimensional complex analogy (quaternion formalism) which would look like

$$dx^* = 2Q^*(\beta) \cdot (Ku^* + a^*) \cdot du^*$$

with $Q^*(\beta)$ a rotation quaternion. It is easy to verify that the quaternion product $u^* \cdot du^*$ is not integrable. The KS-Transformation corresponds to

$$\begin{aligned} dx^* &= 2(u_1 + iu_2 + ju_3 - ku_4)(du_1 + idu_2 + jdu_3 + kdu_4) \\ &= 2u_4^* \cdot du^* \quad , \end{aligned} \tag{43}$$

where $u_4^* = u_1 + iu_2 + ju_3 - ku_4$,

while (42) can be written as

$$Kdx^* = 2(\beta_1 u_4^* \cdot du^* - \beta_2 u_3^* \cdot du^* - \beta_3 u_2^* \cdot du^*) \tag{44}$$

with the same notation of Equation (43), that is,

$$u_3^* = u_1 + iu_2 - ju_3 + ku_4$$

$$u_2^* = u_1 - iu_2 + ju_3 + ku_4 \quad .$$

All combinations $u_4^* \cdot du^*$, $u_3^* \cdot du^*$, $u_2^* \cdot du^*$ correspond to integrable relations leading to orthogonal transformations. In any event the time transformation is given by Equation (34), that is,

$$\frac{dt}{d\tau} = g(\underline{u}) = \sum_{j=1}^{4} \omega_j^2$$

where $\omega_j = Ku_j + a_j$, an exact generalization of Levi-Civita's general correspondent Equation (25).

It is important to notice that, in quaternion notation, the integrated (3 → 4) KS-Transformation can be easily written as

$$x_1 + ix_2 + jx_3 + k0 = (u_1 + iu_2 + ju_3 + ku_4) \cdot (u_1 + iu_2 + ju_3 - ku_4)$$

or

$$x^* = u^* \cdot u_4^* \quad .$$

In the same manner one can write the integrated form of (44) as

$$Kx^* = \beta_1 u^* \cdot u_4^* - \beta_2 u^* \cdot u_3^* - \beta_3 u^* \cdot u_2^*$$

the k-th component of this being identically zero.

Quaternion notation has been neglected for a long time but just recently, Arenstorf (1969) has shown its usefulness in relation to the transformation introduced by Waldvogel (1967).

Conclusions

The primary reason for this paper was to obtain KS-Transformation via the direct solution of the original problem. As a result we obtained a system of Partial Differential Equations from which arose the equation connecting the time and coordinates transformations. We then obtained a general form for the time transformation and with this we concluded that the coordinates transformation matrix had to be linear in the new variables, or homogeneous of first degree in a well defined linear combination of these. These combined conditions led to a transformation which contains the KS as a special case. It is important to note that orthogonality of the Jacobian matrix for the transformation and absence of velocity dependent terms was imposed as a starting point.

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