APPENDIX III

AN ALGORITHM FOR LEAST-SQUARES RATIONAL APPROXIMATION
OF COMPLEX FUNCTIONS ON THE UNIT CIRCLE

Bo Jansson

Seismological Institute
University of Uppsala
Uppsala, Sweden

Contract No. AF 61(052)-702, Mod. 1-4
July 26, 1965
Prepared for

GEOPHYSICS RESEARCH DIRECTORATE
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

Work sponsored by Advanced Research Projects Agency
PROJECT VELA-UNIFORM

ARPA Order No. 292, Amendment No. 12, dated 20 Dec 1963
Project 0652, Task 065203
Project Code 4810, Program Element 62506015

Reproduced by
NATIONAL TECHNICAL
INFORMATION SERVICE
Springfield, Va 22151
AN ALGORITHM FOR LEAST-SQUARES RATIONAL APPROXIMATION OF COMPLEX FUNCTIONS ON THE UNIT CIRCLE

Bo Jannsen

Abstract. An algorithm for rational approximation on the unit circle will be given. An immediate application is computation of recursive filters for use on a digital computer.

1. Definitions

Let \( f(\omega) \) be a given complex-valued function defined for real values of \( \omega \), and let

\[
R_{n,n}(\omega) = \frac{a_0 + a_1 e^{-i\omega} + \ldots + a_n e^{-in\omega}}{b_0 + b_1 e^{-i\omega} + \ldots + b_n e^{-in\omega}}
\]

be a complex-valued rational approximating function defined for real values of \( \omega \). The sets \( \Lambda = \{a_0, a_1, \ldots, a_n\} \) and \( B = \{b_0, b_1, \ldots, b_n\} \) are real parameters of \( R_{n,n}(\omega) \). In the formulas below \( a_i \) and \( b_i \) are assumed to be equal to zero if they do not belong to \( \Lambda \) or \( B \), i.e. if their subscripts are outside the intervals \( 0 - n \) or \( 0 - n \) respectively. We will always assume that \( m \geq n \).

2. Least-squares approximation

We want to determine the parameters \( a_0, a_1, \ldots, a_n \) and \( b_0, b_1, \ldots, b_n \) such that

\[
L_2(f - R) = \left[ \int_0^{2\pi} \left| f(\omega) - R_{n,n}(\omega) \right|^2 d\omega \right]^{1/2}
\]
attains a minimum (assuming it exists). The "linear" approach of taking the partial derivatives with respect to all the parameters is not feasible, as it will result in a system of non-linear equations which is hard to solve. We will avoid this difficulty by using an algorithm given by the author (Jansson, 1965).

3. The algorithm.

Following the algorithm mentioned in the last paragraph we start by adding a real constant \( h \) to \( f(\omega) \), thus having \( g(\omega) = f(\omega) + h \).
Let \( \alpha^0 = \{1, 0, \ldots, 0, \ldots, 0\} \) and consider \( L_2(g - R) \) as a function of \( a_0, a_1, \ldots, a_n \) only. We can minimize \( L_2(g - R) \) by minimizing \( L_2(g - R)^2 \), and this makes the analysis simpler. It may be shown that a unique minimum exists. Furthermore, it is clear that \( L_2(g - R) \) does not have a maximum value, for we can place \( R(\omega) \) as far away from \( f(x) \) as we please. Since \( L_2(g - R)^2 \) is a differentiable function of \( a_0, a_1, \ldots, a_n \), the minimum must occur where all of the partial derivatives

\[
\frac{\partial L_2(g - R)^2}{\partial a_k} = 0, \quad k = 1, 2, \ldots, n, \tag{3}
\]

are zero.

From the fact that \( L_2(g - R) \) is a strictly convex function of \( a_0, a_1, \ldots, a_n \), it follows that there is only one set of parameters for which all of the partial derivatives are zero.

We have

\[
I = |g(\omega) - R_{n,n}(\omega)|^2 =
\]

\[
= |g(\omega)|^2 + |\chi_n(\omega)|^2 - \text{Re}\{R(\omega) R_{n,n}(\omega)\}, \tag{4}
\]
where

$$|R_{m,n}(\omega)|^2 =$$

$$= \frac{(a_0^2 + a_1^2 + \ldots + a_m^2) + 2(n_0 a_1 + n_1 a_2 + \ldots + n_{m-1} a_m) \cos \omega + \ldots + 2(n_0 a_n) \cos m \omega}{(b_0^2 + b_1^2 + \ldots + b_n^2) + 2(b_0 b_1 + b_1 b_2 + \ldots + b_{n-1} b_n) \cos \omega + \ldots + 2(b_0 b_n) \cos n \omega}$$

and

$$\text{Re}\{g(\omega) R_{m,n}(\omega)\} =$$

$$= \left[\text{Re}\{g(\omega)\} \text{Re}\{P_n(\omega) \bar{Q_n}(\omega)\}\right] - \text{Im}\{g(\omega)\} \text{Im}\{P_n(\omega) \bar{Q_n}(\omega)\} / |Q_n(\omega)|^2.$$

We have that

$$P_n(\omega) \bar{Q_n}(\omega) = a_0 b_n e^{i \omega} + a_0 b_{n-1} e^{i (n-1) \omega} + \ldots +$$

$$+ (a_0 b_0 + a_1 b_1 + \ldots + a_n b_n) + (a_1 b_1 + a_2 b_2 + \ldots + a_{n+1} b_{n+1}) e^{-i \omega} +$$

$$+ \ldots + a_m b_0 e^{-i m \omega}.$$  \hspace{1cm} (5)

It is now easy to compute the partial derivatives

$$\frac{\partial}{\partial a_k} = 2 \sum_{j=0}^{m} (a_{k+j} + a_{k-j}) \cos j \omega -$$

$$- \text{Re}\{g(\omega)\} \sum_{j=-m}^{m} b_{k+j} \cos j \omega + \text{Im}\{g(\omega)\} \sum_{j=-m}^{m} b_{k+j} \sin j \omega \right] / |Q_n(\omega)|^2$$

\hspace{1cm} (6)
Setting all the partial derivatives equal to zero and substituting into (3) we get the matrix equation

\[ M_A \cdot A^0 = G_A \]  \hspace{1cm} (7)

\( M_A \) is a \( m \times n \) Toeplitz matrix, \( A^0 \) is the column vector (\( a_0, a_1, \ldots, a_m \))^T, and \( G_A \) a column vector. Let us write out the elements of \( M_A \) and \( G_A \):

\[ M_{Aij} = \frac{2\pi}{\pi} \frac{e^{-|i-j| \omega}}{|q_n(\omega)|^2} \ d \omega \]

\[ G_{Ai} = \frac{2\pi}{\pi} \left[ \text{Re} \{ g(\omega) \} \sum_{j=-m}^{m} b_{j+i} \cos j \omega - \text{Im} \{ g(\omega) \} \sum_{j=-m}^{m} b_{j+i} \sin j \omega \right] \frac{|q_n(\omega)|^{-2}}{|q_n(\omega)|^2} \ d \omega \]  \hspace{1cm} (8)

To solve for \( A \) we just have to compute

\[ A^0 = M_A^{-1} G_A \]

which is very easy since \( M \) is a Toeplitz matrix that can be inverted in \( cm^2 \) operations. Here \( c \) is a constant and \( m \) is the number of rows and columns. An algorithm for this purpose has been given by Robinson (1963).

We have now done the first step and computed a trial solution \( A^0 \). Let us in the next step compute a trial solution \( B^1 \). We will do this by minimizing
Let us write

\[ I = \left| e^2(\omega) \left( \frac{1}{R_{m,n}(\omega)} - \frac{1}{\varphi(\omega)} \right) \right|^2 = \]

\[ = \left| e^2(\omega) \left( \frac{Q_n(\omega)}{\Gamma_n(\omega)} - \frac{1}{e(\omega)} \right) \right|^2 = \]

\[ = \left| e^2(\omega) \frac{Q_n(\omega)}{\Gamma_n(\omega)} \right|^2 - |e(\omega)|^2 - 2\text{Re}\{e^2(\omega) \frac{Q_n(\omega)}{\Gamma_n(\omega)} - e(\omega)\} \] (10)

The parameters \( B \) enter linearly in the expression above so that we may take the partial derivatives and solve for \( B^1 \) just as we solved for \( A^0 \).

Again we get a matrix equation

\[ M_B \begin{bmatrix} B^1 \end{bmatrix} = \begin{bmatrix} G_B \end{bmatrix} \] (11)

where \( M_B \) is a Toeplitz matrix and \( G_B \) is a column vector. We solve for \( B^1 \) by computing

\[ B^1 = M_B^{-1} G_B \]

The elements of \( M_B \) and \( G_B \) may be written

\[ M_{Bij} = \int_0^{2\pi} e^2(\omega) \cos |i - j| \omega \frac{1}{|\Gamma_m(\omega)|^2} d \omega \]
\[ c_{Bij} = \int_0^{2\pi} \left[ \sin^2(\omega) \sum_{j=-m}^m n_{j+1} \cos j \omega - \cos^2(\omega) \sum_{j=-m}^m n_{j+1} \sin j \omega \right] \cdot |1_\infty(\omega)|^{-2} d\omega \] (12)

We have now completed one iteration of the algorithm and we just have to continue to determine \( A^g \) and \( B^g \), \( A^{g+1} \) and \( B^{g+1} \) etc until we are close enough to the point in the parameter space \( AUB \) where the minimum (if it exists) is located. We have then computed a sufficiently good approximation \( f_{m,n}(\omega) \approx g(\omega) \). Obviously

\[ f_{m,n}(\omega) = f_{m,n}(\omega) - h \frac{p_{m}^{g}(\omega) - h q_{n}^{g}(\omega)}{q_{n}^{g}(\omega)} = \frac{p_{m}^{g}(\omega)}{q_{n}^{g}(\omega)} \] (13)

is an equally good approximation to \( f(\omega) \).

References


Acknowledgment

This research was carried out at the Seismological Institute, Uppsala, with the sponsorship of the Cambridge Research Laboratories of the Office of Aerospace Research, United States Air Force, through its European Office, as part of the Advanced Research Projects Agency's project Vela-Uniform, under contract AF 61(052)-702.