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DETERMINING THE APPROPRIATE SAMPLE SIZE FOR CONFIDENCE LIMITS FOR A PROPORTION

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ABSTRACT

Earlier papers [1], [2] by the present authors presented formulas for approximating with at least .999 relative accuracy the binomial confidence limits $\overline{p}$ and $\underline{p}$ based on a sample of size n with c "defectives" drawn randomly from an infinite population with probability of p of a defective. The present article, in complementary fashion, presents substantially accurate procedures for determining appropriate sample size based on specifications as to $\hat{p}$ the maximum anticipated value of c/n , $\hat{e}$ the maximum desired value of the error margin $\overline{e}$, which is the interval between c/n and $\overline{p}$, and confidence level γ . The criterion of appropriate sample size is that if c/n proves equal to $\hat{p}$, then $\overline{e}$ will equal $\hat{e}$ -- as nearly as integers for c and n permit.

To maximize accuracy, separate procedures, called the Poisson Procedure and the Modified Normal Procedure, are respectively given for $\hat{p} \leq .25$ and $\hat{p} > .25$ (but not over .50). The results, using the criterion of appropriate sample size, are much more accurate overall than those of the frequently encountered formula $n = Z^2\hat{p}\hat{q}/\hat{e}^2$, where Z is the number of standard deviations for a given confidence level based on two tails of the normal distribution, and $\hat{q} = 1 - \hat{p}$.

Procedures are given for applying the finite population correction if the sample is to be drawn without replacement from a finite population of size N .

1. Introduction

Suppose that a sample of size n is to be drawn randomly from an infinite population in which the probability of an event (item having a specified characteristic) is p. Based on the number of such events c in the sample, the statistician will use c/n as an estimate of p and will calculate upper and lower binomial confidence limits $\bar{p}$ and $\underline{p}$ at confidence level γ for the parameter p. Simple, highly accurate formulas in [1] and [2] permit these confidence limits to be readily calculated for $n \geq 20$ and $c/n \leq \frac{1}{2}$. For $n < 20$, exact confidence limits are available, as in [3] and [4]. If $c/n > \frac{1}{2}$, confidence limits for p are the complements of those for $1 - p$ based on r/n, where $r = n - c$. The following discussion proceeds on the basis that $n \geq 20$ and $c/n \leq \frac{1}{2}$.

Suppose that the statistician wants to determine n in advance of sampling in order to obtain a confidence interval with maximum "error margin" $\hat{e}$ at a given confidence level γ. The length of a confidence interval is $\bar{p} - \underline{p}$, which is the sum of two error margins: $\bar{e} = \bar{p} - c/n$ and $\underline{e} = c/n - \underline{p}$. In the usual symmetrical approach to a two-sided confidence interval (with equal confidence levels for $\bar{e}$ and $\underline{e}$), $\bar{e}$ is the larger error margin when the binomial distribution is used and $c/n < \frac{1}{2}$. We shall assume that the statistician in specifying a maximum error margin $\hat{e}$ has in mind the larger error margin $\bar{e}$.

The binomial error margin $\bar{e}$ depends on the observed sample proportion c/n. For given n and γ, $\bar{e}$ generally increases with

c/n , in most cases until $c/n = \frac{1}{2}$. As n grows large, it becomes increasingly true that $\overline{e}$ reaches a maximum at $c/n = \frac{1}{2}$; this is nearly true once n reaches about 100. Therefore, for practical purposes a reasonably conservative approach is to determine n so that if $c/n = \frac{1}{2}$ -- or $c/(n - 1) = \frac{1}{2}$ when n is odd -- the desired error margin $\hat{e}$ is not exceeded.

Often, however, there is information about the proportion of events that may be anticipated in the sample. This information may be provided by knowledge about the population, previous experience with similar populations, or a pilot sample. For example, when the item is an account, the specified characteristic is the existence of an error in the account and the population is a firm's set of accounts for a given year, an auditor may draw on his earlier experience with this firm for an upper bound to the proportion of errors that may be anticipated in a sample. If the anticipated proportion is appreciably below $\frac{1}{2}$, then for specified $\hat{e}$ and γ the sample size may be reduced; alternatively the specified $\hat{e}$ may be reduced. We denote the largest anticipated sample proportion by $\hat{p}$ (which may be $\frac{1}{2}$).

Based on the normal distribution as an approximation of the binomial distribution, a formula frequently given for determining sample size is

$$(1) \qquad n = \frac{Z^2 \hat{p}\hat{q}}{\hat{e}^2} , \qquad \hat{p} > 0 ,$$

where Z is the number of standard deviations for a given confidence level based on two tails of the normal distribution, and $\hat{q} = 1 - \hat{p}$.

Let us introduce the following criterion of appropriate sample size: If c/n proves equal to $\hat{p}$, then $\bar{e}$ will equal $\hat{e}$ -- as nearly as integers for c and n permit.

The formula (1) fails to meet this criterion when confidence levels are correctly calculated on the basis of the binomial distribution, instead of on the basis of the normal distribution as an approximation of the binomial. The appropriate sample size, which we denote by n_a, is understated by (1), often seriously. To illustrate, assume $\hat{p} = .20$, $\hat{e} = .05$, and $\gamma = .950$ so that $Z = 1.96$. Then (1) indicates a sample size of $(3.84 \times .20 \times .80)/.0025 = 246$. However, n_a is 300 rather than 246; that is, if $n = 300$ and c/n proves to be $.20$, the error margin $\bar{e}$ will be .05 -- as nearly as use of integers for c and n allows. Understatement of n_a may grow more serious as $\hat{p}$ diminishes. For example, if $\hat{p} = .04$, $\hat{e} = .02$, and $\gamma = .950$, (1) indicates $n = 369$, whereas $n_a = 550$.

2. Proposed Procedures

As alternatives to (1), we present two sets of procedures that overall more accurately meet the criterion of appropriate sample size and at the same time are reasonably simple: a "Poisson Procedure" for $\hat{p} \leq .25$, and a "Modified Normal Procedure" for $\hat{p} > .25$. When desired, a refined degree of accuracy can be introduced through a very simple correction factor. Furthermore, adjustment can be made for a finite population when sampling without replacement.

Poisson Procedure

For $p \leq .25$, a procedure for determining the appropriate sample size (based on the Poisson distribution as an approximation of the binomial distribution, described in [1] and [2]) is as follows. First, calculate

$$(2) \qquad \hat{Q} = \frac{\hat{\bar{p}}(2 - \hat{p})}{\hat{p}(2 - \hat{\bar{p}})} \quad ,$$

where $\hat{\bar{p}}$ is defined as $\hat{p} + \hat{e}$. Second, in Table 1 for the desired γ , find Q nearest to $\hat{Q}$, and find the corresponding c , which we denote by $\hat{c}$ (anticipated value of c). If necessary, interpolate linearly between bounding values of Q to find $\hat{c}$ corresponding to $\hat{Q}$; $\hat{c}$ is an integer. Third, for Q within the scope of Table 1,

$$(3) \qquad n = \frac{\hat{c}}{\hat{p}} \quad , \qquad \hat{p} > 0 \ .$$

In Table 1, $Q = \bar{m}/c$, where $\bar{m}$ is the upper confidence limit for the parameter m of a Poisson distribution based on an observed number of events c . Therefore the use of Formulas (2) and (3) to determine sample size may be called the Poisson Procedure, in contrast with the "Normal Procedure" of (1).

Because c is an integer, Q in Table 1 has a limited number of values. In going from $\hat{Q}$ to the nearest tabular value of Q , the effect of rounding upward must be taken into account. The result of using Q higher than $\hat{Q}$ is to reduce $\hat{c}$ and thereby reduce sample size, with the possibility that $\bar{e} > \hat{e}$ and the criterion of appropriate sample size is therefore not met. Hence the user may find it desirable to round <u>down</u> from $\hat{Q}$ to the nearest lower value of Q, even though a higher value of Q is closer to $\hat{Q}$. This situation is particularly apt to occur when $\hat{c}$ is small.

In the area of Table 1 where interpolation may have to be used to find $\hat{c}$, and $\hat{c}$ is adjusted to the nearest integer, it must similarly be recognized that sample size may become too small as the result of adjusting $\hat{c}$ downward. However, $\hat{c}$ is now relatively large, and rounding $\hat{c}$ has relatively small effect on sample size.

The Poisson Procedure provides good accuracy overall. Furthermore, the relative error -- $(n - n_a)/n_a$ -- is in the conservative direction; that is, n obtained by (2) and (3) overstates the sample size. Empirical analysis indicates that for γ from .990 to .600 and for n_a between 100 and 100,000, the maximum relative error is about 4% (at $n_a = 100$, $\hat{p} = .25$, and $\gamma = .990$); and that for n_a between 20 and 100 it is about 10% (at $n_a = 20$, $\hat{p} = .25$, and $\gamma = .990$). The relative error approaches zero as n_a increases, as $\hat{p}$ decreases, and as γ decreases.

Computation of n may be checked and the relative error minimized by two additional steps in the Poisson Procedure. The fourth step is to assume $c = \hat{c}$ and calculate $\bar{p}$ based on c, n, and γ, using (1) or (2) in [1]; calculate $\bar{e} = \bar{p} - c/n$; and compare $\bar{e}$ with $\hat{e}$. If $\bar{e}$ and $\hat{e}$ are not deemed sufficiently close, the fifth and final step is to calculate an adjusted sample size:

$$(4) \qquad n' = n \left(\frac{\bar{e}}{\hat{e}}\right)^2 ;$$

n is an integer.

To illustrate the entire Poisson Procedure, assume $\hat{p} = .25$, $\hat{e} = .10$, and $\gamma = .990$. Thus $\hat{\bar{p}} = .25 + .10 = .35$, and $\hat{Q} = (.35 \times 1.75)/(.25 \times 1.65) = 1.485$. For $\gamma = .990$, the nearest Q in Table 1 is 1.484 , and the corresponding c or $\hat{c}$ is 40 ; $n = 40/.25 = 160$. Formula (2) in [1] yields $\bar{p} = .348$; therefore $\bar{e} = .348 - .25 = .098$, and $n' = 160(.098/.100)^2 = 154$. To check whether 154 is the appropriate sample size, assume $c = .25 \times 154 = 39$ (to the nearest integer). Then $c/n = 39/154 = .253$; $\bar{p}$ (by (2) in [1]) $= .353$, and $\bar{e} = .100$. Hence 154 is the appropriate sample size as nearly as integers allow.

As preceding discussion indicates, the fact that c and n are integers may prevent the sample from exactly meeting the specifications for $\hat{p}$ and/or $\hat{e}$. Inability to meet specifications tends to increase as n and c become small, for rounding to an integer then has a relatively larger effect. If desired, one may adjust sample size so that c/n can exactly meet the specification $\hat{p}$; thus in the above example n could be raised to 156, allowing c/n to be exactly .25. At the same time this affects $\bar{e}$; raising n would reduce $\bar{e}$. On occasion a compromise course may be desirable, permitting both c/n and $\bar{e}$ to be about equally close to specifications.

Instead of from Table 1, $\hat{c}$ can be obtained from

$$(5) \qquad \hat{c} = \left[\frac{Z + \sqrt{Z^2 + 4(\hat{Q} - 1)(\hat{Q} + B - 1)}}{2(\hat{Q} - 1)}\right]^2 - 1 , \qquad \hat{p} > 0 ,$$

where $B = (Z^2 + 2)/3$; $\hat{c}$ is rounded to the nearest integer.

Formulas (1), (2), (3), (4), and (5) presume that $\hat{p} > 0$. If $\hat{p} = 0$, they cannot be used; also, only the upper confidence limit is in point. Sample size can then be estimated very accurately by

$$(6) \qquad n = \frac{\overline{m}}{\hat{e}} - \frac{\overline{m}}{2} , \qquad \hat{p} = 0 .$$

Values of $\overline{m}$ for the upper confidence limit are 5.298 at the .995 confidence level; 4.605 at .990; 3.689 at .975; 2.996 at .950; 2.303 at .900; and 1.609 at .800.

If the population is finite and the sample size is an appreciable fraction of the population, say 5% or more, a finite population correction may be usefully applied when the sample is to be drawn without replacement:

$$(7) \qquad n_{FPC} = \frac{n \times N}{n + N - 1} ,$$

where n_{FPC} is the sample size corrected for finite population. Of course n' may be used in place of n in (7).

To check whether $n_{FPC} = n_a$, find $\overline{e}$ as previously described; based on (4) in [1], calculate

$$(8) \qquad \overline{e}_{FPC} = \overline{e}\sqrt{\frac{N - n_{FPC}}{n - 1}} ;$$

and compare $\overline{e}_{FPC}$ with $\hat{e}$.

Formula (7) tends to understate sample size, as tested by (8) with $\overline{e}$ derived from the binomial distribution. Understatement increases as $\hat{p}$, $\hat{c}$, and n_{FPC}/n decrease. In general, understatement becomes appreciable only when $\hat{p}$ is quite small and the FPC adjustment quite large. Correction for understatement of sample size can be made by calculating

$$n'_{FPC} = \frac{n_{FPC} \times N}{n_{FPC} + (N - n_{FPC})(\hat{e}/\bar{e}_{FPC})^2} \quad (9)$$

To illustrate the use of (7), (8), and (9), assume $\hat{p} = .05$, $\hat{e} - .03$, $\gamma = .950$, and $N = 500$. Then $\hat{\bar{p}} = .05 + .03 = .08$, and $\hat{Q} = (.08 \times 1.95)/(.05 \times 1.92) = 1.625$. For $\gamma = .950$, the nearest Q in Table 1 is 1.624, and the corresponding c or $\hat{c}$ is 16; $n = 16/.05 = 320$. Applying (7), $n_{FPC} = (320 \times 500)/819 = 195$; $\hat{c}_{FPC} = n_{FPC} \times \hat{p} = 195 \times .05 = 10$ (to the nearest integer). Formula (1) in [1] yields $\bar{p} = .09226$ at the .950 confidence level for $n = 195$ and $c = 10$; $\bar{e} = \bar{p} - c/n = .09226 - .05128 = .04098$. Formula (8) yields $\bar{e}_{FPC} = .04098 \sqrt{305/499} = .03196$, which exceeds $\hat{e}$. Hence $n_{FPC} = 195$ is below appropriate sample size. Applying (9) yields $n'_{FPC} = (195 \times 500)/\left[195 + 305(.03/.03196)^2\right] = 210$; $\hat{c}'_{FPC} = 210 \times .05 = 11$ (to the nearest integer). To check whether 210 is the appropriate sample size, we caluclate $\bar{p} = .09177$ at the .950 confidence level for $n = 210$ and $c = 11$; $\bar{e} = .09177 - .03939$; and $\bar{e}_{FPC} = .03939 \sqrt{290/499} = .03000$. Thus 210 is indicated as the appropriate sample size. If n'_{FPC} does not result in $\bar{e}_{FPC}$ sufficiently close to $\hat{e}$, (9) can be used iteratively to obtain a closer approach (so far as integers permit).

Modified Normal Procedure

For $\hat{p} > .25$, an alternative procedure for determining the appropriate sample size (based on the normal distribution as an approximation of the binomial distribution) is

$$n = \frac{Z^2\hat{p}\hat{q} + \hat{e}}{\hat{e}^2} , \tag{10}$$

where $\hat{q} = 1 - \hat{p}$.

Formula (10) has good accuracy overall (for $\hat{p} > .25$). The relative error -- $(n - n_a)/n_a$ -- may be positive or negative, respectively representing overstatement or understatement of sample size. Error generally tends to ge negative as sample size increases and as $\hat{p}$ increases. Systematic empirical analysis indicates that for γ from .990 to .600 and for n_a between 100 and 100,000 the relative error is between about +5% and -3%; and that for n_a between 20 and 100 it is between about +5% and -11%.

Accuracy of n determined by (10) may be checked and improved in essentially the same manner as described for the Poission Procedure. Assuming $c/n = \hat{p}$, calculate c (or $\hat{c}$) = np ; calculate $\overline{p}$ based on c , n , and γ , using (2) in [1]; calculate $\overline{e} = \overline{p} - c/n$; and compare $\overline{e}$ with $\hat{e}$. If $\overline{e}$ and $\hat{e}$ are not deemed sufficiently close, apply (4) to find n' .

The FPC procedures employing (7), (8), and (9) also apply here. So do the comments previously made on the effect of integers.

The procedures proposed in this paper depend on a largest anticipated sample proportion $\hat{p}$. If the statistician calculates the appropriate sample size on the basis of a value of $\hat{p}$ and then obtains a sample with a larger proportion he will nevertheless be able to state a confidence interval with desired confidence coefficient. However, the error margin will usually exceed the desired error margin $\hat{e}$.

3. Calculation of the Formulas

Poisson Procedure

Formulas (1) and (5) in [1] suggest that a fairly good approximation of $\overline{p}$ is

(11) $$\overline{p} \simeq \frac{\overline{m}}{n + (\overline{m} - c)/2} \quad .$$

Rearrangement of terms leads to

(12) $$\frac{\overline{m}}{c} \simeq \frac{\overline{p}(2 - c/n)}{(c/n)(2 - \overline{p})} \quad .$$

For $c/n = \hat{p}$ and $\overline{p} = \overset{\triangle}{p}$ (12) becomes

(13) $$\frac{\overline{m}}{c} \simeq \frac{\overset{\triangle}{p}(2 - \hat{p})}{\hat{p}(2 - \overset{\triangle}{p})} \quad .$$

We define

(14) $$Q = \frac{\overline{m}}{c} \quad ,$$

(2) $$\hat{Q} = \frac{\overset{\triangle}{p}(2 - \hat{p})}{\hat{p}(2 - \overset{\triangle}{p})} \quad .$$

In sum: Specified $\hat{p} + \hat{e} = \overset{\triangle}{p}$; $\overset{\triangle}{p}$ and $\hat{p}$ through Formula (2) yield $\hat{Q}$, which leads to approximately equivalent Q and corresponding c in Table 1; denoting c as $\hat{c}$ (anticipated value of c) , and assuming $\hat{c}/n = \hat{p}$, we obtain

(3) $$n = \frac{\hat{c}}{\hat{p}} \quad .$$

Formula (3) in [1] gives as ~~an approximation~~

(15) $$\overline{m} = c + Z\sqrt{c + 1} + B \quad ,$$

where Z is the number of standard deviations for a given confidence level based on two tails of the normal distribution, and $B = (Z^2+2)/3$. (In [1], A is written in place of Z, with A representing one tail of the normal distribution for a single confidence limit.) Thus we may write as an approximation $\bar{m}/c = Q = (c + Z\sqrt{c+1} + B)/c$, which transforms into $(Q - 1)(c + 1) - Z\sqrt{c+1} - (Q + B - 1) = 0$. Quadratic solution for c, and substitution of $\hat{Q}$ for Q and $\hat{c}$ for c (assuming $c/n = \hat{p}$ and $\bar{p} = \hat{p}$), yields

$$(5) \quad \hat{c} = \left[\frac{Z + \sqrt{Z^2 + 4(\hat{Q} - 1)(\hat{Q} + B - 1)}}{2(\hat{Q} - 1)}\right]^2 - 1, \qquad \hat{p} > 0 .$$

For $c/n = \hat{p} = 0$ and $\bar{p} = \hat{e}$, rearrangement of terms in (11) leads to

$$(6) \qquad n = \frac{\bar{m}}{\hat{e}} - \frac{\bar{m}}{2}, \qquad \hat{p} = 0 .$$

<u>Adjusted Sample Size (n')</u>

As indicated by Formula (1), normal distribution theory suggests that sample size varies inversely with the square of the error margin. Given the error margin $\bar{e}$ for n, and given the specified error margin $\hat{e}$ for an adjusted sample size n', we may write $n'/n = \bar{e}^2/\hat{e}^2$, or

$$(4) \qquad n' = n\left(\frac{\bar{e}}{\hat{e}}\right)^2 .$$

<u>Finite Population Correction</u>

For a sample drawn without replacement from a finite population N, a finite population correction can be applied to the binomial confidence limit to give a good approximation to the exact confidence limit as in Formula (4) in [1]. The FPC is applied to the error margin, and the corrected error margin is then added to c/n. Therefore we may write

$$\bar{e}_{FPC} = \bar{e}\sqrt{\frac{N - n_{FPC}}{N - 1}} \quad . \tag{8}$$

If n (without the FPC) results in an error margin equal to $\bar{e}_{FPC}$, Formula (4) indicates that $\bar{e}_{FPC}/\bar{e} = \sqrt{n_{FPC}/n}$. Thus Formula (8) becomes $\sqrt{n_{FPC}/n} = \sqrt{(N - n_{FPC})/(N - 1)}$. Squaring both sides and rearranging terms, we obtain the conventional FPC formula

$$n_{FPC} = \frac{n \times N}{N + n - 1} \quad . \tag{7}$$

If n_{FPC} in Formula (8) does not result in $\bar{e}_{FPC} = \hat{e}$, we may state that n'_{FPC} results in $\hat{e}$; that is, $\hat{e} = \bar{e}' \sqrt{(N - n'_{FPC})/(N - 1)}$. Based on Formula (4), $\bar{e}/\bar{e}' = \sqrt{n'_{FPC}/n_{FPC}}$. Thus $\bar{e}_{FPC}/\hat{e} = \sqrt{n'_{FPC}/n_{FPC}} \sqrt{(N - n_{FPC})/(N - n'_{FPC})}$. Squaring both sides and rearranging terms leads to

$$n'_{FPC} = \frac{n_{FPC} \times N}{n_{FPC} + (N - n_{FPC})(\hat{e}/\bar{e}_{FPC})^2} \quad . \tag{9}$$

Modified Normal Procedure

If we presume that the value of the parameter p is $\bar{p}$, we can write as an approximation, based on the normal distribution with a continuity correction,

$$\bar{p} = \frac{c}{n} + Z\sqrt{\frac{\bar{p}\bar{q}}{n}} + \frac{1}{2n} \quad , \tag{16}$$

where $\bar{q} = 1 - \bar{p}$. Letting $\bar{e} = \bar{p} - c/n$, Formula (16) becomes

(17) $$\bar{e} = Z\sqrt{\frac{\bar{p}\bar{q}}{n}} + \frac{1}{2n} \quad .$$

Transferring the term $1/2n$ to the left side of Formula (17), squaring both sides, and rearranging terms leads to

(18) $$n = \frac{Z^2\bar{p}\bar{q} + \bar{e}}{\bar{e}^2} - \frac{1}{4n\bar{e}^2} \quad .$$

The relative effect of $1/4n\bar{e}^2$ on sample size is measured by $(1/4n\bar{e}^2)/n = 1/4n^2\bar{e}^2$. Inspection indicates that $1/4n^2\bar{e}^2$ is largest when γ , c/n , and n are minimal. Within the intended scope of the Modified Normal Procedure, the minimal value is .600 for γ , and .25 for c/n ; and we may assume a practical lower limit of 20 for n . Accordingly in Formula (16) we insert .84162 for Z , .25 for c/n , and 20 for n , yielding $1.03541\bar{p}^2 - .58542\bar{p} + .07563 = 0$. Quadratic solution yields $\bar{p} = .36563$; thus $\bar{e} = .36563 - .25 = .11563$. With $n = 20$ and $\bar{e} = .11563$, $1/4n\bar{e}^2$ is about 1, and $1/4n^2\bar{e}^2$ is about 5%. Solution for other values of n (including n below 20) shows then $1/4n\bar{e}^2$ ranges from a low of about .5 to a maximum of about 2 for $\gamma = .600$ and $c/n = .25$. Hence $1/4n^2\bar{e}^2$ diminishes almost inversely in proportion to n ; by the time n is 100 , $1/4n^2\bar{e}^2$ is only about 1% for $\gamma = .600$ and $c/n = .25$. At higher confidence levels and higher values of c/n , $1/4n^2\bar{e}^2$ is still smaller than indicated.

In view of the small effect of $1/4n\bar{e}^2$ on sample size, this term may be dropped from Formula (18). If at the same time we substitute the anticipated values $\hat{p}$, $\hat{q}$, and $\hat{e}$ for $\bar{p}$, $\bar{q}$, and $\bar{e}$, Formula (18) becomes

$$(10) \qquad n = \frac{Z^2\hat{p}\hat{q} + \hat{e}}{\hat{e}^2} .$$

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TABLE 1

VALUES OF Q (= m/c) AT SELECTED CONFIDENCE LEVELS γ
FOR TWO-SIDED CONFIDENCE LIMITS

c \ γ	99%	98%	95%	90%	80%	60%
1	7.430	6.638	5.572	4.744	3.890	2.994
2	4.637	4.203	3.612	3.148	2.661	2.140
3	3.659	3.348	2.922	2.585	2.227	1.838
4	3.149	2.901	2.560	2.288	1.998	1.680
5	2.830	2.622	2.334	2.103	1.855	1.581
6	2.610	2.428	2.177	1.974	1.755	1.513
7	2.448	2.285	2.060	1.878	1.682	1.462
8	2.322	2.175	1.970	1.804	1.624	1.422
9	2.222	2.087	1.898	1.745	1.578	1.391
10	2.140	2.014	1.839	1.696	1.541	1.365
11	2.071	1.954	1.789	1.655	1.509	1.343
12	2.012	1.902	1.747	1.620	1.482	1.325
13	1.961	1.857	1.710	1.590	1.458	1.309
14	1.917	1.818	1.678	1.563	1.438	1.295
15	1.878	1.783	1.649	1.540	1.419	1.282
16	1.843	1.752	1.624	1.519	1.403	1.271
17	1.811	1.724	1.601	1.500	1.389	1.261
18	1.783	1.699	1.580	1.483	1.375	1.252
19	1.757	1.676	1.562	1.467	1.363	1.244
20	1.733	1.655	1.544	1.453	1.352	1.236
21	1.712	1.636	1.529	1.440	1.342	1.229
22	1.692	1.618	1.514	1.428	1.333	1.223
23	1.673	1.602	1.500	1.417	1.324	1.217
24	1.656	1.587	1.488	1.406	1.316	1.212
25	1.640	1.572	1.476	1.398	1.308	1.207
26	1.625	1.559	1.465	1.388	1.301	1.202
27	1.611	1.547	1.455	1.379	1.295	1.197
28	1.598	1.535	1.445	1.371	1.289	1.193
29	1.585	1.524	1.436	1.363	1.283	1.189
30	1.574	1.513	1.428	1.356	1.277	1.185
31	1.563	1.503	1.419	1.350	1.272	1.182
32	1.552	1.494	1.412	1.343	1.267	1.179
33	1.542	1.485	1.404	1.337	1.262	1.175
34	1.533	1.477	1.397	1.331	1.258	1.172
35	1.524	1.469	1.391	1.326	1.253	1.169
36	1.515	1.461	1.384	1.321	1.249	1.167
37	1.507	1.454	1.378	1.316	1.245	1.164
38	1.499	1.447	1.373	1.311	1.242	1.161
39	1.491	1.440	1.367	1.306	1.238	1.159
40	1.484	1.434	1.362	1.302	1.235	1.157

TABLE 1 (CONTINUED)

c \ γ	99%	98%	95%	90%	80%	60%
41	1.477	1.428	1.357	1.298	1.231	1.155
42	1.470	1.422	1.352	1.293	1.228	1.152
43	1.464	1.416	1.347	1.290	1.225	1.150
44	1.458	1.410	1.342	1.286	1.222	1.148
45	1.452	1.405	1.338	1.282	1.220	1.146
46	1.446	1.400	1.334	1.279	1.217	1.145
47	1.441	1.395	1.330	1.275	1.214	1.143
48	1.435	1.390	1.326	1.272	1.212	1.141
49	1.430	1.386	1.322	1.269	1.209	1.139
50	1.425	1.381	1.318	1.266	1.207	1.138
55	1.403	1.361	1.302	1.252	1.196	1.131
60	1.383	1.344	1.287	1.240	1.187	1.124
65	1.366	1.329	1.275	1.229	1.178	1.119
70	1.351	1.315	1.263	1.220	1.171	1.114
75	1.338	1.303	1.254	1.212	1.165	1.110
80	1.326	1.292	1.245	1.204	1.159	1.106
90	1.305	1.274	1.229	1.192	1.149	1.0991
100	1.288	1.258	1.216	1.181	1.141	1.0935
110	1.273	1.246	1.205	1.172	1.134	1.0887
120	1.260	1.234	1.196	1.164	1.127	1.0846
130	1.249	1.224	1.187	1.157	1.122	1.0809
150	1.230	1.207	1.173	1.145	1.113	1.0749
170	1.215	1.193	1.162	1.136	1.106	1.0700
200	1.197	1.177	1.149	1.124	1.0969	1.0642
250	1.175	1.157	1.132	1.111	1.0861	1.0569
300	1.159	1.143	1.120	1.100	1.0782	1.0517
350	1.146	1.132	1.110	1.0925	1.0721	1.0476
400	1.136	1.123	1.103	1.0863	1.0672	1.0444
450	1.128	1.115	1.0968	1.0811	1.0632	1.0417
500	1.121	1.109	1.0916	1.0768	1.0598	1.0395
600	1.110	1.0992	1.0833	1.0698	1.0544	1.0359
700	1.102	1.0915	1.0769	1.0645	1.0502	1.0331
800	1.0947	1.0854	1.0718	1.0602	1.0469	1.0309
900	1.0891	1.0803	1.0675	1.0566	1.0441	1.0291
1000	1.0844	1.0761	1.0640	1.0536	1.0418	1.0275

The reader can extend this table to higher values of c by using (15) to calculate $\bar{m}$ and computing $Q = \bar{m}/c$. The table can be extended to additional confidence levels by deriving $\bar{m}$ from [4] for $c \leq 50$ and using (15) for $c > 50$.

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Earlier papers [1], [2] by the present authors presented formulas for approximating with at least .999 relative accuracy the binomial confidence limits $\overline{p}$ and $\underline{p}$ based on a sample of size n with c "defectives" drawn randomly from an infinite population with probability p of a defective. The present article, in complementary fashion, presents substantially accurate procedures for determining appropriate sample size based on specifications as to $\hat{p}$ the maximum anticipated calue of c/n, $\hat{e}$ the maximum desired value of the error margin $\overline{e}$, which is the interval between c/n and $\overline{p}$, and confidence level γ. The criterion of appropriate sample size is that if c/n proves equal to $\hat{p}$, then $\overline{e}$ will equal $\hat{e}$ -- as nearly as integers for c and n permit. To maximize accuracy, separate procedures, called the Poisson Procedure and the Modified Normal Procedure, are respectively given for $\hat{p} \leq .25$ and $\hat{p} > .25$ (but not over .50). The results, using the criterion of appropriate sample size, are much more accurate overall than those of the frequently encountered formula $n = Z^2\hat{p}\hat{q}/\hat{e}^2$, where Z is the number of standard deviations for a given confidence level based on two tails of the normal distribution, and $\hat{q} = 1 - \hat{p}$. Procedures are given for applying the finite population correction if the sample is to be drawn without replacement from a finite population of size N.

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3. Sample size						
4. Proportion						
5. Normal approximation						
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