A SEQUENTIAL STOCHASTIC ASSIGNMENT PROBLEM

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CYRUS DERMAN, GERALD J. LIEBERMAN, and SHELDON M. ROSS

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DEPARTMENT OF OPERATIONS RESEARCH
AND
DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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0. Summary

Suppose there are n men available to perform n jobs. The n jobs occur in sequential order with the value of each job being a random variable X. Associated with each man is a probability p. If a "p" man is assigned to an "$X = x$" job, the (expected) reward is assumed to be given by px. After a man is assigned to a job, he is unavailable for future assignments. The paper is concerned with the optimal assignment of the n men to the n jobs so as to maximize the total expected reward. The optimal policy is characterized, and a recursive equation is presented for obtaining the necessary constants of this optimal policy.

In particular, if $p_1 \leq p_2 \leq \cdots \leq p_n$ the optimal choice in the initial stage of an n stage assignment problem is to use p_i if x falls into an i^{th} non-overlapping interval comprising the real line. These intervals depend on n and the CDF of X, but are independent of the p's.

The optimal policy is also presented for the generalized assignment problem, i.e., the assignment problem where the (expected) reward if a "p" man is assigned to an "x" job is given by a function $r(p,x)$.

1. Introduction

The sequential stochastic assignment problem can be described as follows. Suppose there are n men available to perform n jobs.

The n jobs arrive in sequential order, i.e., first job 1 appears, followed by job 2, etc. Associated with the j^{th} ($j = 1,2, \ldots, n$) job is a random variable X_j which takes on the value x_j. It will be assumed that the X's are independent and identically distributed random variables. This j^{th} job is then referred to as a "type x_j" job. If a "perfect" man is assigned to the type x_j job, a reward x_j is obtained (the type job may then be viewed as the maximum potential value of a job). However, none of the n men may be perfect, and whenever the i^{th} man is assigned to any type x_j job, the (expected) reward is given by $p_i x_j$, where $0 \leq p_i \leq 1$,[1/] $i = 1,2,\ldots,n$ are known constants. After a man is assigned to a job, he is unavailable for future assignments. The problem is to assign the n men to the n jobs so as to maximize the total expected reward. An assignment of men is equivalent to a sequential assignment of the p's to the X's. Let a policy be any rule for assigning men to jobs. In particular, if the random variable i_j is defined to be the man (identified by number) assigned to the j^{th} arriving job, then the total expected reward is given by

$$(1) \qquad E\left[\sum_{j=1}^{n} p_{i_j} X_j\right],$$

and the desired policy is the one which maximizes (1). It should be noted that $(i_1, i_2, \ldots, i_n)$ is a random permutation of the integers $1,2, \ldots, n$.

[1/] Actually, the constraint, $0 \leq p_i \leq 1$, is given for clarity of application, and none of the ensuing results are dependent upon it.

There are other interpretations of the above model which may be useful to the reader. Suppose there exists n cards. Associated with the i^{th} card is a probability p_i. A sequence of independent identically distributed random variables $X_1, X_2, \ldots, X_n$ are observed in a sequential manner. When the random variable X_j appears, a card must be chosen and played on that random variable. If the i^{th} card is played when $X_j = x_j$ is observed, then the expected reward is given by $p_i x_j$. An example of this form occurs when x_j is received with probability p_i and zero is received with probability $1 - p_i$. The problem is to choose the n plays of the cards to maximize the total expected reward, i.e., maximize (1).

Finally a special case of this model is a generalization of the "house hunting" problem [1]. Suppose that there are $k \leq n$ identical houses to be sold. Offers arrive in a sequential manner. These offers will be assumed to be a sequence of independent identically distributed random variables $X_1, X_2, \ldots, X_n$. The seller may accept or reject the offers but must dispose of all k houses by no later than the n^{th} offer. In the above "card interpretation" let k of the cards have associated p's equal to 1 and let $(n-k)$ of the cards have associated p's equal to 0. If the seller accepts the j^{th} offer he assigns it a card having an associated p equal to 1 and receives x_j, and that house and card become unavailable. If the seller rejects the j^{th} offer he assigns it a card having an associated p equal to 0 and hence receives nothing. This procedure continues until all the houses (and cards) are disposed of. The problem is to determine which offers to accept in order to maximize the total expected profit (or reward), i.e., maximize (1).

Section 2 characterizes the optimal policy, and presents a recursive equation for obtaining the constants of the optimal policy. In Section 3, it is assumed that the choice of the values of p is available to the decision maker, and results are presented for an optimum allocation. Section 4 contains a detailed example which illustrates the concepts presented in the earlier sections. Finally, Section 5 generalizes the assignment problem to include the case where the (expected) reward if a "p" man is assigned to an "x" job is given by a function $r(p,x)$.

2. Optimal Policy

The key result needed to determine the optimal policy is to show that it is of the following form. If there are n stages to go (n men to assign or n cards to play) and probabilities $p_1 \leqq p_2 \leqq \cdots \leqq p_n$, then the optimal choice in the initial stage is to use p_i (implying using the i^{th} man or the i^{th} card in the appropriate interpretation) if the random variable X falls into an i^{th} non-overlapping interval comprising the real line. Furthermore, these intervals depend on n and the cumulative distribution function of X but are independent of the p's.

In proving the main result, a well known theorem due to Hardy [2] will be used

Lemma (Hardy's Theorem). If $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ and $y_1 \leqq y_2 \leqq \cdots \leqq y_n$ are sequences of numbers, then

$$(2) \qquad \max_{(i_1,i_2,\ldots,i_n)\in P} \sum_{j=1}^{n} x_{i_j} y_j = \sum_{j=1}^{n} x_j y_j ,$$

where P is the set of all permutations of the integers $(1,2, \ldots, n)$. This result implies that the maximum sum is achieved when the smallest of the x's and y's are paired, the next smallest of the x's and y's are paired, and continued until the largest of the x's and y's are paired.

The following notation will now be introduced: Let

$f(p_1,p_2, \ldots, p_n)$ = Total expected reward under an optimal policy when the probabilities are $p_1,p_2, \ldots, p_n$,

$f(p_1,p_2, \ldots, p_n|x)$ = Total conditional expected reward given $X_1 = x$ under an optimal policy when the probabilities are $p_1,p_2, \ldots, p_n$.

That, in fact, optimal policies exist can be shown by induction. Denote by $G_X(z)$ the cumulative distribution function of the random variable X. It is assumed that $X_1,X_2, \ldots, X_n$ are independent identically distributed random variables with CDF $G_X(z)$, and that $\mu = E(X) = \int_{-\infty}^{\infty} z dG_X(z) < \infty$.

The optimal policy is embodied in the following theorem which will be proven by induction.

<u>Theorem 1.</u> For each $n \geq 1$, there exist numbers

$$-\infty = a_{0,n} \leq a_{1,n} \leq a_{2,n} \leq \cdots \leq a_{n,n} = +\infty$$

such that whenever there are n stages to go and probabilities $p_1 \leq p_2 \leq \cdots \leq p_n$ then the optimal choice in the initial stage is to use p_i if the random variable X_1 is contained in the interval

$(a_{i-1,n}, a_{i,n}]$. The $a_{i,n}$ depend on G_X but are independent of the p's. Furthermore $a_{i,n}$ is the expected value, in an (n-1) stage problem, of the quantity to which the i^{th} smallest p is assigned (assuming an optimal policy is being followed), and

$$(3) \quad f(p_1, p_2, \ldots, p_{n-1}) = \sum_{i=1}^{n-1} p_i a_{i,n} \quad \text{for all} \quad p_1 \leqq p_2 \leqq \cdots \leqq p_{n-1} .$$

Proof. A proof by induction is employed. Suppose that there exist numbers $\{a_{j,m}\}_{j=1}^{m-1}$, $m = 1, 2, \ldots, n-1$ such that the optimal policy in an m stage problem is to initially use the i^{th} smallest p if the initial value is contained in the interval $(a_{i-1,m}, a_{i,m}]$, where $a_{0,m} = -\infty$ and $a_{m,m} = \infty$. Then, in the n stage problem where p_k is selected first

$$(4) \quad f(p_1, p_2, \ldots, p_n | x) = \max_k [x p_k + f(p_1, p_2, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n)] .$$

However, by the induction hypothesis, it follows that the optimal policy for an (n-1) stage problem is independent of the (n-1) values of p. Hence defining $a_{i,n}$ as the expected value (under the optimal policy) of the quantity to which the i^{th} smallest p is assigned in the (n-1) stage problem, the total expected reward of that problem is given by

$$(5) \qquad f(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_{n-1}) = \sum_{i=1}^{n-1} \bar{p}_i a_{i,n} ,$$

for every $\bar{p}_1 \leqq \bar{p}_2 \leqq \cdots \leqq \bar{p}_{n-1}$ (the $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_{n-1}$ represent the remaining (n-1) p's of the original n p's after the first, i.e., p_k, is chosen in the n stage problem). Furthermore, since

$a_{i,n}$ is independent of the p's and other policies are obtained by permuting the p's, any sum of the form $\sum_{i=1}^{n-1} p_{j_i} a_{i,n}$ (where $j_1, j_2, \cdots, j_{n-1}$ is a permutation of the integers) can be obtained for the total expected reward of the $(n-1)$ stage problem. Hence, using Hardy's theorem (lemma 1) it follows that

$$(6) \qquad a_{1,n} \leqq a_{2,n} \leqq \cdots \leqq a_{n-1,n} ,$$

since by the induction assumption $f(\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1})$ must be a maximum.

Using the results of (5) and (6), equation (4) can now be expressed as

$$(7) \qquad f(p_1, p_2, \ldots, p_n | x) = \max_k \left[x p_k + \sum_{i=1}^{k-1} p_i a_{i,n} + \sum_{i=k+1}^{n} p_i a_{i-1,n} \right] .$$

Again, using Hardy's theorem (lemma 1),

$$f(p_1, p_2, \cdots, p_n | x) = x p_{k*} + \sum_{i=1}^{k*-1} p_i a_{i,n} + \sum_{i=k*+1}^{n} p_i a_{i-1,n} ,$$

where $k*$ is such that (with $a_{0,n} = -\infty$, $a_{n,n} = +\infty$)

$$a_{k*-1,n} < x \leqq a_{k*,n} .$$

This result follows because the p's and a's are ordered so that if x is greater than or equal to the $(k*-1)$ smallest a, then the corresponding p (i.e., p_{k*}) must be greater than or equal to the $(k*-1)$ smallest p. Hence, the first choice in an n stage problem is to choose p_i if $x \in (a_{i-1,n}, a_{i,n}]$. Noting that the result is trivial for $n = 1$ completes the induction. Equation

(5) follows immediately from equation (5), and the theorem is complete.

Theorem 1 presents the form of the optimal policy, but does not indicate how to obtain the $a_{i,n}$. The constants may be calculated from the results of Corollary 1.

Corollary 1. Define $a_{0,n} = -\infty$, $a_{n,n} = +\infty$. Then

$$(8) \quad a_{i,n+1} = \int_{a_{i-1,n}}^{a_{i,n}} z\,dG_X(z) + a_{i-1,n}G(a_{i-1,n}) + a_{i,n}[1-G(a_{i,n})] ,$$

for $i = 1,2, \ldots , n$, where $-\infty \cdot 0$ and $\infty \cdot 0$ are defined to be 0.

Proof. The result follows by recalling that $a_{i,n+1}$ is the expected value, in an n stage problem, of the quantity to which the i^{th} smallest p is assigned. The result then follows by conditioning on the initial x, and recalling that p_i is used if and only if this value lies in the interval $(a_{i-1,n}, a_{i,n}]$.

The previous results assume that the X's are independent, identically distributed random variables. An alternative set of conditions leads to the following theorem.

Theorem 2. Suppose that the successive values $X_1, X_2, \ldots , X_n$ form a sub-martingale, i.e.,

$$E[X_j | X_1, X_2, \ldots , X_{j-1}] \geqq X_{j-1}, \quad \text{for all } j \geqq 2 ,$$

then the optimal policy is to use p_1, then $p_2, \ldots ,$ and finally p_n, whenever $p_1 \leqq p_2 \leqq \cdots \leqq p_n$.

Proof. Again, a proof by induction is employed. The result is trivial for $n = 1$. Assume it is true for all $m \leqq n-1$. For the n stage problem where p_k is selected equation (4) still holds. However, by the induction hypothesis, the optimal policy is specified for the $(n-1)$ stage problem, and the total expected reward for this optimal policy in the $(n-1)$ stage problem can easily be expressed in terms of the conditional expectations of the ensuing X's given $X_1 = x$. Hence, equation (4) reduces to

$$f(p_1, p_2, \cdots, p_n|x) = \max_k \left[xp_k + \sum_{i=1}^{k-1} p_i E[X_{i+1}|X_1 = x] \right. \tag{9}$$
$$\left. + \sum_{i=k+1}^{n} p_i E[X_i|X_1 = x] \right] .$$

Using the properties of sub-martingales, it follows that $E[X_i|X_1 = x]$ is monotone increasing in i, for $i \geqq 1$. Again, using Hardy's theorem (lemma 1), it follows that

$$f(p_1, p_2, \cdots, p_n|x) = xp_1 + \sum_{i=2}^{n} p_i E[X_i|X_1 = x] , \tag{10}$$

and the induction is complete.

It can be remarked that if the successive values $X_1, X_2, \cdots, X_n$ form a super-martingale, i.e.,

$$E[X_j|X_1, X_2, \cdots, X_{j-1}] \leqq X_{j-1} \quad \text{for all} \quad j \geqq 2 ,$$

then the same reasoning shows that the optimal ordering is $p_n, p_{n-1}, \cdots, p_1$ whenever $p_1 \leqq p_2 \leqq \cdots \leqq p_n$.

3. Allocation of p's

In the previous sections it was assumed that the p's were a fixed set of given numbers. An extension is to allow the p's to be determined in some optimal fashion. In the context of the stochastic sequential assignment problem, a company has the opportunity to hire skilled men, i.e., those having large p's, but at the expense of large salaries. In particular, suppose that $c(p)$ denotes the cost to retain a man having an associated p. Let $a_1, a_2, \cdots, a_n$ denote the expected value of the quantity to which the i^{th} smallest p is assigned (these a's are the $a_{i,n+1}$'s of Section 2 except that the second subscript is surpressed; since only an n stage problem is being considered no confusion should result). Then the appropriate total expected reward for a given allocation $p_1, p_2, \cdots, p_n$, where $p_1 \leqq p_2 \leqq \cdots \leqq p_n$, is given by

$$(11)\quad \varphi(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} a_i p_i - \sum_{i=1}^{n} c(p_i) = \sum_{i=1}^{n} [a_i p_i - c(p_i)] \, .$$

The problem can now be stated as follows:

$$\text{maximize} \quad \varphi(p_1, p_2, \cdots, p_n)$$

subject to

$$(12)\qquad 0 \leqq p_i \leqq 1, \quad i = 1, 2, \ldots, n$$

and

$$(13)\qquad p_1 \leqq p_2 \leqq \cdots \leqq p_n \, ,$$

and solutions are presented for the following five cases.

Case 1:

If $c(p) = c \cdot p$ with $c \geqq 0$, then

$$\varphi(p_1, p_2, \dots, p_n) = \sum_{i=1}^{n} (a_i - c)p_i \; .$$

It is evident that $\varphi(p_1, p_2, \dots, p_n)$ is maximized subject only to (12) if $p_i = 1$ when $a_i - c \geqq 0$, and $p_i = 0$ when $a_i - c < 0$. However, it has already been shown in expression (6) that $a_1 \leqq a_2 \leqq \cdots \leqq a_n$. Hence, choosing i^* so that it is the smallest integer such that $a_i - c \geqq 0$ (if all the $a_i - c < 0$, then i^* may be interpreted as equal to $n+1$) it follows that the optimal values of p_i subject to (12) are

$$p_i = 0, \quad \text{for} \quad i < i^*$$

and

$$p_i = 1, \quad \text{for} \quad i \geqq i^* \; .$$

However, this solution also satisfies (13) so that it is a solution to the problem. Note also that the a_i's, and hence i^*, are calculable by corollary 1.

<u>Case 2</u>:

If $c(p) = cp + bp^2$ where $c \geqq 0$ and $b \geqq 0$, then

$$\varphi(p_1, p_2, \dots, p_n) = \sum_{i=1}^{n} [(a_i - c)p_i - bp_i^2] \; .$$

Each $(a_i - c)p_i - bp_i^2$ is maximized at

$$p_i = \frac{a_i - c}{b}, \quad \text{for} \quad (a_i - c) \geqq 0$$

and

$$p_i = 0 \quad , \quad \text{for} \quad (a_i - c) \leqq 0 \; .$$

Therefore, the optimal values of p_i subject to (12) are

$$p_i = 0 \qquad , \quad \text{for} \quad i < i^*$$

and

$$p_i = \min\left(\frac{a_i - c}{b}, 1\right), \quad \text{for} \quad i \geq i^* ,$$

where i^* is the smallest integer such that $a_i - c \geq 0$. If all the $a_i - c < 0$, then i^* may be interpreted as equal to $n+1$. Note that this solution also satisfies (13) so that it is a solution to the problem.

Case 3:

If $c(p)$ satisfies $c(0) = 0$ and $c(p)$ is non-decreasing and convex, then

$$\varphi(p_1, p_2, \dots, p_n) = \sum_{i=1}^{n} [a_i p_i - c(p_i)] .$$

Using the same argument given for Case 2 and noting that $a_i p_i - c(p_i)$ is concave in p, the optimal solution takes on the form

$$p_i = 0 \qquad , \quad \text{for} \quad i < i^*$$

and

$$p_i = \min (p^*, 1), \quad \text{for} \quad i \geq i^* ,$$

where i^* is the smallest integer such that $a_i - c'(0) \geq 0$ and p^* satisfies $a_i - c'(p^*) = 0$. If all the $a_i - c'(0) < 0$, then i^* may be interpreted as equal to $n+1$.

Case 4:

If $c(p)$ satisfies $c(0) = 0$ and $c(p)$ is non-decreasing and concave, then

$$\varphi(p_1, p_2, \cdots, p_n) = \sum_{i=1}^{n} [a_i p_i - c(p_i)] .$$

Following the same argument as in the two preceding cases and noting that $a_i p_i - c(p_i)$ is convex in p, the optimal solution assumes the form

$$p_i = 0, \quad \text{for } i < i^*$$

and

$$p_i = 1, \quad \text{for } i \geqq i^* ,$$

where i^* is the smallest integer such that $a_i - c(1) \geqq 0$. If all $a_i - c(1) < 0$, then i^* may be interpreted as equal to $n+1$.

Case 5:

This case will be concerned with the allocation of the p's when the p's can take on only a finite set of possible values

$$\pi = \{\pi_1, \pi_2, \cdots, \pi_k\}, \text{ with } \pi_1 < \pi_2 < \cdots < \pi_k .$$

The allocation problem can now be written as follows

$$\text{maximize } \varphi(p_1, p_2, \cdots, p_n)$$

subject to

$$p_1 \leqq p_2 \leqq \cdots \leqq p_n$$

and

(14) $$p_i \in \pi .$$

This is the original allocation problem with expression (12) being replaced with (14). The four cases considered for the original problem will be considered for this new model.

Case 1' - linear cost function:

The arguments are identical to those presented for Case 1 up to and including the expression for determining i^*. However, for $i < i^*$, the p_i should be chosen as small as possible, whereas for $i \geqq i^*$, the p_i should be chosen as large as possible. Therefore, the optimal values of p_i subject to (14) are

$$p_i = \pi_1, \quad \text{for} \quad i < i^*$$

and

$$p_i = \pi_k, \quad \text{for} \quad i \geqq i^* .$$

Again, this solution satisfies (13) so that it is a solution to the problem.

Case 2' - quadratic cost function:

The arguments are similar to those presented for Case 2, with i^* determined as in Case 2. The optimal values of p_i subject to (14) are

$$p_i = \pi_1, \quad \text{for} \quad i < i^*$$

and

$$p_i = \left\{ \begin{array}{l} \pi_1, \quad \text{if} \quad 0 \leqq \dfrac{a_i - c}{b} < \pi_1; \\[2ex] \pi_r, \quad \text{if} \quad \dfrac{a_i - c}{b} = \pi_r, \quad r = 1,2,\ldots,k; \\[2ex] \text{either } \pi_r \text{ or } \pi_{r+1}, \quad \text{if} \\[1ex] \qquad \pi_r < \dfrac{a_i - c}{b} < \pi_{r+1} \quad \text{and} \quad r < k; \\[2ex] \pi_k, \quad \text{if} \quad \pi_k < \dfrac{a_i - c}{b} \; ; \end{array} \right\} \text{for} \quad i \geqq i^* .$$

Again this solution satisfies (13) so that it is a solution to the problem.

Case 3' - convex cost function:

The arguments are again similar to those presented for Case 3, with i^* determined as in Case 3. The optimal values of p_i subject to (14) are

$$p_i = \pi_1, \quad \text{for} \quad i < i^*$$

and

$$p_i = \left\{\begin{array}{l} \pi_1, \text{ if } p_i < \pi_1 \text{ in Case 3;} \\ \pi_r, \text{ if } p_i = \pi_r,\ r = 1,2,\ldots,k, \text{ in Case 3;} \\ \text{either } \pi_r \text{ or } \pi_{r+1}, \text{ if } p_i, \text{ in Case 3,} \\ \quad \text{satisfies } \pi_r < p_i < \pi_{r+1} \text{ with } r < k; \\ \pi_k, \text{ if } p_i \geqq \pi_k \text{ in Case 3;} \end{array}\right\}, \quad \text{for} \quad i \geqq i^* .$$

Again, this solution satisfies (13) so that it is a solution to the problem.

Case 4' - concave cost function:

The arguments are again similar to those presented for Case 4 with i^* determined as in Case 4. The optimal values of p_i subject to (14) are

$$p_i = \pi_1, \quad \text{if} \quad p_i = 0 \quad \text{in Case 4}$$

and

$$p_i = \pi_k, \quad \text{if} \quad p_i = 1 \quad \text{in Case 4} .$$

Again, this solution satisfies (13) so that it is a solution to the problem.

4. Example

In the context of the stochastic sequential assignment problem, suppose there are four men available to perform four jobs occurring in sequential order. Each man has an associated p_i and is labeled so that $p_1 \leqq p_2 \leqq p_3 \leqq p_4$. The type job, X, is assumed to be a uniformly distributed random variable over the range (0,1000), i.e.,

$$G_X(z) = \begin{cases} 0 \quad , & \text{for } z < 0 \\ z/1000, & \text{for } 0 \leqq z \leqq 1000 \\ 1 \quad , & z > 1000 . \end{cases}$$

Using this information, and equation (8), the required $a_{i,n}$ are obtained as follows:

(i) $a_{0,1} = -\infty, \quad a_{11} = +\infty$

(ii) $$\begin{cases} a_{0,2} = -\infty \\ a_{1,2} = \int_{-\infty}^{\infty} [y/1000]dy = 500 \\ a_{2,2} = \infty \end{cases}$$

(iii) $$\begin{cases} a_{0,3} = -\infty \\ a_{1,3} = \int_{-\infty}^{a_{1,2}} [y/1000]dy + a_{1,2}[1 - G_X(a_{1,2})] = 375 \\ a_{2,3} = \int_{a_{1,2}}^{\infty} y/1000 \ dy + a_{1,2}G_X(a_{1,2}) = 625 \\ a_{3,3} = \infty \end{cases}$$

$$
\text{(iv)}\begin{cases}
a_{0,4} = -\infty \\
a_{1,4} = \displaystyle\int_{-\infty}^{a_{1,3}} [y/1000]dy + a_{1,3}[1-G_X(a_{1,3})] = 304.6875 \\
a_{2,4} = \displaystyle\int_{a_{1,3}}^{a_{2,3}} [y/1000]dy + a_{1,3}G_X(a_{1,3}) + a_{2,3}[1-G_X(a_{2,3})] = 500 \\
a_{3,4} = \displaystyle\int_{a_{2,3}}^{\infty} [y/1000]dy + a_{2,3}G_X(a_{2,3}) = 695.3125 \\
a_{4,4} = \infty .
\end{cases}
$$

Suppose that the first job to come in is a \$800 job. The optimal policy calls for assigning the "best" man to this job, i.e., p_4, since it lies in the interval $(695.3125,\infty)$. Suppose that the next job to arrive is a \$450 job. There are now 3 men available and the optimal policy calls for assigning man 2 to this job, i.e., p_2, since it lies in the interval $(375,625]$ Suppose that the next job to arrive is a \$400 job. There are now 2 men available and the optimal policy calls for assigning man 1 to this job, i.e., p_1, since it lies in the interval $(0,500]$. The remaining man, man 3 (associated with p_3) is then available for the last assignment.

It should be noted that the assignment did not depend upon the values of the p's but only on the ordering. Suppose that the choice of p's are available to the decision maker, and the cost to retain a man having an associated p is given by $c(p) = 50p + 300p^2$. The results for Case 2 of Section 3 are then applicable. It is necessary to determine the a_i. Recall that $a_i = a_{i,n+1}$ so that the $a_{i,5}$ are required. These are as follows:

$$a_1 = a_{1,5} = 258.3$$

$$a_2 = a_{2,5} = 421.4$$

$$a_3 = a_{3,5} = 578.6$$

$$a_4 = a_{4,5} = 741.7 .$$

Thus, $a_1 - 50$ is positive so that $i^* = 1$. Therefore,

$$p_i = \min\left(\frac{a_i - 50}{300}, 1\right), \quad \text{i.e.,}$$

$$p_1 = 0.69, \quad \text{and}$$

$$p_2 = p_3 = p_4 = 1 .$$

5. General Assignment Problem

The previous sections were concerned with a very special form of the assignment problem, i.e., if a "p" man is assigned to an "x" job, the expected reward is given by px. The general assignment problem is concerned with an expected reward function of a more arbitrary form. In particular, denote by $r(p,x)$ the expected reward if a "p" man is assigned to an "x" job. The analogous characterization of $f(p_1,p_2,\ldots,p_n|x)$ presented in equation (4) is now given by

$$(15) \quad f(p_1,p_2,\ldots,p_n|x) = \max_k [r(p_k,x)+f(p_1,p_2,\ldots,p_{k-1},p_{k+1},\ldots,p_n)] .$$

Also, it should be noted that

$$(16) \quad f(p_1, p_2, \ldots, p_n) = \int f(p_1, p_2, \ldots, p_n|x)dG_X(x) ,$$

and

(17) $$f(p_1|x) = r(p_1,x) \; .$$

The characterization of the form of the optimal assignment policy is embodied in Theorem 3.

Theorem 3. Assume that $r(p,x)$ is differentiable and

(18) $$\frac{\partial}{\partial x}\frac{\partial}{\partial p} r(p,x) \geqq 0 \; ,$$

then for each $p_1, p_2, \cdots, p_n$ there exist numbers

$$-\infty = a_{0,n} < a_{1,n} \leqq a_{2,n} \leqq \cdots \leqq a_{n-1,n} < a_{n,n} = +\infty \; ,$$

such that whenever there are n stages to go and probabilities $p_1 \leqq p_2 \leqq \cdots \leqq p_n$ then the optimal choice in the initial stage is to use p_i if the random variable X is contained in the interval $(a_{i-1,n}, a_{i,n}]$.

Proof. For any $p_2 > p_1$ and $x_2 > x_1$, expression (18) indicates that

$$\int_{x_1}^{x_2}\int_{p_1}^{p_2} \frac{\partial}{\partial x}\frac{\partial}{\partial p} r(p,x)\,dp\,dx \geqq 0 \; ,$$

or equivalently

(19) $$r(p_2,x_2) - r(p_1,x_2) \geqq r(p_2,x_1) - r(p_1,x_1) \; .$$

Now, let

$$x_1^* = \sup\{x\colon f(p_1, p_2, \cdots, p_n|x) = r(p_1,x) + f(p_2, p_3, \cdots, p_n)\}$$

and let x_1^* be $-\infty$ if the above set is vacuous. (Note that

$f(p_1, p_2, \ldots, p_n)$ may easily be shown to be continuous by induction, and hence, if x_1^* is finite then the supremum is actually a maximum.) Suppose $x < x_1^*$, then let $\bar{x} \in (x, x_1^*]$ be such that $f(p_1, p_2, \ldots, p_n|\bar{x}) = r(p_1, \bar{x}) + f(p_2, p_3, \ldots, p_n)$. Now for any $j > 1$, expression (19) can be written as

$$r(p_j, \bar{x}_1) - r(p_1, \bar{x}_1) \geqq r(p_j, x) - r(p_1, x) ,$$

or alternatively,

$$r(p_1, x) + f(p_2, \ldots, p_n) - r(p_j, x) - f(p_1, p_2, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n)$$

$$\geqq r(p_1, \bar{x}_1) + f(p_2, \ldots, p_n) - r(p_j, \bar{x}_1) - f(p_1, p_2, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n) .$$

However, the right-hand side of the inequality must be greater than or equal to zero since

$$r(p_1, \bar{x}_1) + f(p_2, \ldots , p_n) = f(p_1, p_2, \ldots , p_n|\bar{x}_1) .$$

Hence, for $x < x_1^*$

$$r(p_1, x) + f(p_2, \ldots, p_n) \geqq r(p_j, x) + f(p_1, p_2, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n) .$$

Therefore, it follows that the optimal policy uses p_1, if and only if, $x \leqq x_1^*$. By defining x_2^* as

$$x_2^* = \sup\{x > x_1^* : f(p_1, p_2, \ldots, p_n|x) = r(p_2, x) + f(p_1, p_3, \ldots, p_n)\} ,$$

it follows from the same reasoning that the optimal policy uses p_2, if and only if, $x \in (x_1^*, x_2^*]$. Similar reasoning completes the proof.

Several comments are in order.

i) Although Theorem 3 appears to be similar to Theorem 1, it differs in that the a's are not, in general, independent of the p's, nor are the a's easily calculable.

ii) For the allocation problem in the general assignment model, a result similar to that given for Case 4 may be obtained. In particular, if $c(p)$, the cost to retain a man having an associated p, is concave, and $r(p,x)$ is convex, then the optimal p's are either zeros or ones. This follows by showing that $f(p_1,p_2, \dots , p_n)$ is convex (in the vector) so that the objective function

$$f(p_1, p_2, \dots , p_n) - \sum_{i=1}^{n} c(p_i)$$

is convex. The result that $f(p_1,p_2, \dots , p_n)$ is convex is embodied in Lemma 2.

<u>Lemma 2</u>. If for all x, $r(p,x)$ is convex in p, then $f(p_1,p_2,\dots,p_n)$ is convex (in the p vector).

<u>Proof</u>. A proof by induction on the number of terms in the p vector is employed.

For any fixed x, it will be shown that $f(p_1,p_2, \dots , p_n|x)$ is convex. From (17), $f(p_1|x)$ is convex. Assume that $f(p_1,p_2, \dots , p_m|x)$ is convex for all $m \leq n-1$. It follows that

$$(20) \qquad r(p_k,x) + f(p_1, p_2, \dots , p_{k-1}, p_{k+1}, \dots , p_n)$$

is convex for all $k = 1,2, \dots , n$. Hence, the maximum, over k, of expression (20) is also convex since the maximum of a finite number of convex functions is also convex. Using equation (15),

$f(p_1, p_2, \dots, p_n | x)$ is seen to be convex. The lemma is then proved by employing equation (16). It can be noted that a similar proof can be used to show that $f(p_1, p_2, \dots, p_n)$ is monotone (in the p vector) if $r(p,x)$ is monotone in p.

iii) Throughout this paper it has been assumed that the number of assignments is equal to the number of men. This restriction can be relaxed easily. Let m denote the number of assignmentsand let n denote the number of men. If $n > m$, choose only those m men having the highest p's associated with them (assuming that $r(p,x)$ is non-decreasing). If $n < m$, add $(n-m)$ "pseudo men" having p's equal to zero associated with them (assuming $r(0,x) = 0$).

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13. ABSTRACT

Suppose there are n men available to perform n jobs. The n jobs occur in sequential order with the value of each job being a random variable X. Associated with each man is a probability p. If a "p" man is assigned to an "$X = x$" job, the (expected) reward is assumed to be given by px. After a man is assigned to a job, he is unavailable for future assignments. The paper is concerned with the optimal assignment of the n men to the n jobs so as to maximize the total expected reward. The optimal policy is characterized, and a recursive equation is presented for obtaining the necessary constants of this optimal policy.

In particular, if $p_1 \leq p_2 \leq \cdots \leq p_n$ the optimal choice in the initial stage of an n stage assignment problem is to use p_i if x falls into an i^{th} non-overlapping interval comprising the real line. These intervals depend on n and the CDF of X, but are independent of the p's.

The optimal policy is also presented for the generalized assignment problem, i.e., the assignment problem where the (expected) reward if a "p" man is assigned to an "x" job is given by a function $r(p,x)$.

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