CORRELATION BETWEEN TWO HOTELLING'S $T^2$

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Technical Report No. 79
Department of Statistics THEMIS Contract

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1. Introduction:

Let $B$ be a $p \times p$ symmetric matrix having the Wishart distribution

$$W_p(B|I|f)dB = C_p f^{(f-p-1)/2} e^{-1/2 \text{tr}B} dB,$$

where

$$C_p = \frac{2^{fp/2} \pi^{p(p-1)/4}}{\prod_{i=1}^{p}(f+1-2)alt} ,$$

and $dB$ stands for the product of the differentials of the $p(p+1)/2$ distinct elements of $B$. Let $x$ and $y$ be two vector variables of $p$ components, distributed independently of $B$, and also independently of each other, as

$$\frac{1}{(2\pi)^{p/2}} e^{-1/2 x'x} dx ,$$

and

$$\frac{1}{(2\pi)^{p/2}} e^{-1/2 y'y} dy ,$$

respectively. While considering the problem of multivariate statistical outliers, Wilks (1963) used statistics of the type,

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He has remarked that the exact distribution (joint) of \( r \) and \( s \) is complicated and has given the expected values, variances and covariance of \( r \) and \( s \). Unfortunately, his expressions for the variance and covariance are in error. The purpose of this note is to derive the exact joint distribution of \( r \) and \( s \) and to give correct expressions for the moments.

2. Joint distribution:

In the joint distribution of \( B, x \) and \( \gamma \), make the following transformation

\[
A = B + xx' + yy',
\]
\[
u = A^{-1/2} x,
\]
\[
\nu = A^{-1/2} y,
\]

where \( A^{-1/2} \) is any matrix such that \( A^{-1/2} \cdot A^{-1/2} = A^{-1} \). The Jacobian of transformation from \( B \) to \( A \) is 1 and that from \( x \) to \( u \) or \( \gamma \) to \( v^\prime \) is \( |A|^{1/2} \) and hence, the joint distribution of \( A, u \) and \( \nu \) comes out as

\[
f + 2 - p - 1 |A|^{-1/2} e^{-1/2 \text{tr} A} \cdot |I - uu' - vv'|^{2} dA du dv,
\]

as \( |B| = |A - A^{1/2} uu' A^{1/2} - A^{1/2} vv' A^{1/2}| = |A| |I - uu' - vv'| \).

This shows that \( A \) has a Wishart distribution of \( f + 2 \) degrees of freedom and is independent of \( u \) and \( \nu \). Splitting the constant suitably, the joint distribution of \( u \) and \( \nu \) is

\[
\frac{\Gamma(f+1)}{(2\pi)^{f/2} \Gamma(f-p+1)} |I - uu' - vv'|^{f-p-1} du dv.
\]

Observe that the statistics \( r, s \) of Wilks are given by

\[
r = \frac{|B + yy'|}{|B + xx' + yy'|} = \frac{|A - xx'|}{|A|} = |I - uu'| = 1 - u'u,
\]
and
\[(2.5) \quad s = \frac{|B+xx'|}{|B+xx'+yy'|} = 1 - \nu'\nu \] .

Also observe that in (2.3)
\[(2.6) \quad |l - uu' - vv'| = (1 - u'u)(1 - v)v - (u'v)^2 \]
\[= rs - (u'v)^2 \] .

In (2.3), transform from \(v\) to \(w = [w_1, w_2, ..., w_p]'\), by an orthogonal transformation
\[(2.7) \quad w = L\nu , \]
where
\(L\) is a \(p \times p\) orthogonal matrix, whose last row is \(u'/\sqrt{u'u} \). The Jacobian of this transformation is \(|L| = 1\) and \(v'v = w'w = 1-s\) . Also
\[(2.8) \quad u'v = u'L'L\nu = [0 \ldots 0, \sqrt{u'u}]w = \sqrt{1-r} \cdot w_p \] .

The joint distribution of \(u\) and \(w\) is, therefore,
\[(2.9) \quad \frac{\Gamma(f+1)}{(2\pi)^{f/2}\Gamma(f-p+1)} \frac{1}{|rs - (l-r)w^2|^2} \text{dudw} \] .

From \(u\), transform to \(r = 1 - u'u\) and \(p-1\) other variables
\[(2.10) \quad \phi_1, \phi_2', ..., \phi_{p-1} \text{ by} \]
\[u_1 = (1-r)^{1/2} \cos\phi_1 \cos\phi_2 \ldots \cos\phi_{p-1} \]
\[u_j = (1-r)^{1/2} \cos\phi_1 \cos\phi_2 \ldots \cos\phi_{p-j} \sin\phi_{p-j+1} \quad (j=2, 3, ..., p) \]

Similarly, transform from \(w\) to \(s = 1 - w'w\) and \(p-1\) other variables
\[(2.11) \quad \theta_1, \theta_2', ..., \theta_{p-1} \text{ by} \]
\[v_1 = (1-s)^{1/2} \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{p-1} \]
\[v_j = (1-s)^{1/2} \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{p-j} \sin\theta_{p-j+1} \quad (j=2, 3, ..., p) \]
The Jacobian of transformation from $u$ to $r$, $\phi_1$, ..., $\phi_{p-1}$ is

$$\frac{1}{2} \sum_{i=1}^{p-1} \frac{1}{(1-r)^2 \prod \cos^2 \phi_i}$$

and a similar expression in $s$ and $\theta$ for the Jacobian of transformation from $w$ to $s$ and the $\theta$'s. Now $\theta_{p-1}$ and $\phi_{p-1}$ vary from 0 to $2\pi$, the other $\theta$'s and $\phi$'s vary from $-\pi/2$ to $\pi/2$ while $r$ and $s$ vary from 0 to 1. Integrating out all the $\phi$'s and all $\theta$'s except $\theta_1$, we obtain the joint distribution of $r$, $s$ and $\theta_1$ as

$$\Gamma(f+1) \frac{1}{4\pi^p(p-1)\Gamma(f+p+1)} \left\{ rs - (1-r)(1-s)\sin^2 \theta \right\}^2 \cos^{p-2} \theta \, drdsd\theta$$

where $\theta_1$ is replaced by $\theta$.

The joint distribution of $r,s$ alone can now be obtained by integrating out $\theta$ but this does not seem to yield a manageable expression, as the bracket in (2.12) will have to be expanded in a series.

3. **Moments of $r,s$**

Only the product moment of $r$ and $s$ is difficult to obtain. The mean and variance of $r$ (or $s$) can be very easily obtained from the marginal distribution of $r$, which is related to the well-known Hotelling's $T^2$ by $r = \frac{1}{1+\frac{(\bar{z})^2}{f+1}}$. In the joint distribution of $u$ and $\bar{z}$, given by

$$\Gamma(f/2) \frac{1}{\pi^{f/2}(f-p)} \frac{\Gamma(f/2)}{\Gamma(f-p)} \left| I - uu' \right|^2 \, du$$

(3.1) if we transform to $z = \begin{bmatrix} z_1, ..., z_p \end{bmatrix}'$ from $u$ by

$$\bar{z} = (I - uu')^{1/2} \bar{z}$$

we shall find that $u$ and $\bar{z}$ are independently distributed as

(3.2) $K(u|f)du = \frac{f}{\pi^{p/2}(f-p)} \frac{\Gamma(f/2)}{\Gamma(f-p)} \left| I - uu' \right|^{-2} du$

(3.3) and $K(\bar{z}|f-1)d\bar{z}$, respectively.
From \((3.2)\), one can easily show that
\[
E(r^h) = E[(1 - u'u)^h] = E[1 - uu']^h
\]
\[
= \frac{\Gamma(f+2h-p)}{(f-p)(f+2h)} \cdot \frac{\Gamma(f-p)}{\Gamma(f/2)} \cdot \frac{\Gamma(f/2 + h)}{\Gamma(f+2h)} = \frac{\Gamma(f-p)}{\Gamma(f/2)} \cdot \frac{\Gamma(f/2 + h)}{\Gamma(f+2h)}
\]
This will also be the \(h\)th moment of \(s\) by symmetry. This leads to
\[
E(r) = \frac{f-p+2}{f+2}, \quad v(r) = \frac{2p(f-p+2)}{(f+2)(f+4)}
\]
Now
\[
\text{Cov}(r,s) = E((1-u'u)(1-u'u)) - E(r)E(s)
\]
\[
= E((1-u'u)(1-z(I uu'))z) - (E(r))^2 \quad \text{by} \ (3.1)
\]
\[
= E(r) - E(r(z'z-(z'u)^2)) - (E(r))^2
\]
\[
= E(r) - E(r)E(z'z) + E(r(z'u)^2) - (E(r))^2,
\]
as \(z\) and \(r\) are independent. Since \(z\) has the same distribution as \(u\) with \(f\) changed \(f-1\),
\[
E(z'z) = 1 - E(1 - u'u) \quad \text{with} \ f \ \text{replaced by} \ f-1
\]
\[
= \frac{p}{f+1}
\]
Hence \((3.6)\) reduces to
\[
\text{Cov}(z,s) = \frac{-p(f-p+2)}{(f+1)(f+2)^2} + E(r(z'u)^2).
\]
Now
\[
E(r(z'u)^2) = (1-u'u)K(y|f)K(z|f-1)dudz
\]
where the integration is over the range of values of \(u\) and \(z\) such that \(u'u \leq 1, z'z \leq 1\). Transform from \(z\) to \(\xi = [\xi_1, \ldots, \xi_p]^{\top}\) by the transformation
\[
\xi = Lz,
\]
where \(L\) is already defined to be a \(p \times p\) orthogonal matrix, whose last row is \(u'\sqrt{u'u}\). Then,
\[
z'u = z'L'Lu = \xi'Lu = \xi_p \sqrt{u'u} = (1-r)^{1/2}\xi_p.
\]
Hence (3.9) reduces to

$$/r(1-r)K(u|f)du \cdot /l_f K(x|f-1)dx = K(r-r^2) \cdot \frac{1}{p} E(f|x)$$, due to symmetry of the distribution of $x$. Now $x$ has the same distribution as $y$ with $f$ replaced by $f-1$ and hence finally, (3.10) reduces to

$$\frac{p(f-p+2)}{(f+4)(f+2)} \cdot \frac{1}{f+1}$$

The covariance between $x$ and $y$, therefore, is: (from (3.5))

$$\frac{-2p(f-p+2)}{(f+1)(f+2)^2(f+4)}$$

Remarks:

Wilks considers a sample of size $n$ and has a Wishart matrix based on $n-1$ degrees of freedom as deviations are from the sample means. He then removes two observations as outliers and thus his $(n-1)-2$ corresponds to our $f$. His $E(r)$ agrees with our result, with this correspondence but the other moments are in error.

Reference

Correlation between two Hotelling’s $T^2$

If $B$ is a Wishart matrix and $x, y$ are two vectors of $p$ components each having a multinormal distribution and if all these quantities are independently distributed, the joint distribution of the two statistics

$$r = \frac{|B + xy'|}{|B + xx' + yy'|} \quad \text{and} \quad s = \frac{|B + xx'|}{|B + xx' + yy'|}$$

is derived in this paper. The correlation between $r$ and $s$ is also obtained. $r$ and $s$ are related to Hotelling’s $T^2$ and are useful in problems of testing multivariate outliers.