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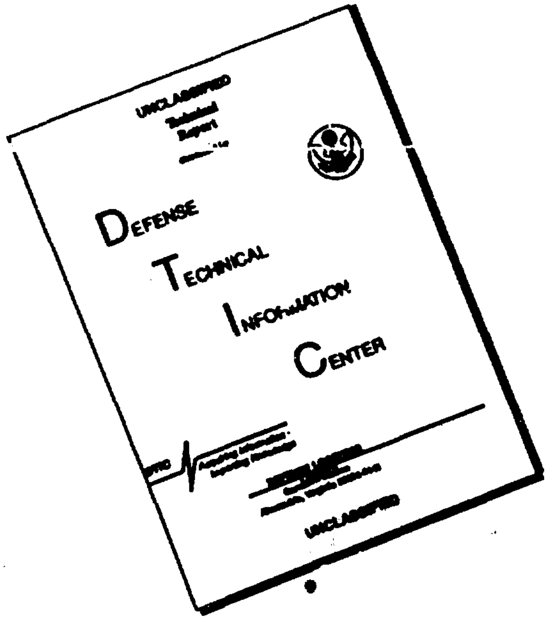

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FINAL REPORT ON GRANT NO. AFOSR-69-1647

(Inference in Stochastic Processes,

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1. **Introduction**. As originally set out in this proposal, the main purpose of this investigation is to prepare a monograph on certain mathematical aspects of the inference theory of stochastic processes. The latter includes the problems of hypothesis testing and estimation. Here the work involves a careful application of several non-trivial aspects of conditional expectations and distributions, as well as the basic existence theory of stochastic processes under various conditions. During the academic year 1968-69, while collecting the material for the monograph, it was found that a comprehensive account of these topics, which are needed for the inference theory, is not available in the books in a sufficient generality in which they can be used. For this purpose it was found necessary to include this theory in the monograph, as it is also of considerable independent interest.

After collecting the material during 1968-69, it was noted that, in preparing the monograph for publication, several gaps existed in the theory for a unified account. However, this is not unexpected, though the work needed was much more than anticipated. During that period as well as the following time, several of these gaps have been filled and the work is outlined in the following sections. Some further work is needed in certain parts, and it is expected that the book will be ready to the printer by the next academic

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year. Preliminary versions of parts of the work have been prepared, and the full account in the final form is expected to be completed during the Principal Investigator's stay at the Institute for Advanced Study in Princeton during 1970-71. It is hoped that this added work will produce a better book than otherwise. A brief summary of the work will now be given in the following six paragraphs.

2. Foundations. A basic problem in most questions of inference theory is the existence of a stochastic process, if a 'compatible' family of distribution (or probability) functions is prescribed. If the stochastic process to be determined is real-valued, an affirmative answer is provided by a fundamental theorem of Kolmogorov and if the process may be vector valued, then the answer may be negative. Since both cases occur in inference problems in stochastic processes, it is necessary to treat this question is some detail.

The general problem involved here goes under the name of 'projective limits of probability spaces', and a comprehensive account of it is included because no standard text on the subject in probability has treated it. Then the problems of separability and measurability of the processes, in the same general context, are considered. Further results on the regularity properties of sample functions of stochastic processes, and some applications conclude this discussion. Problems and complements are included at the end of each chapter both to illustrate the possibilities of the text and also to complement the former.

3. Conditional expectations and distributions. It is well known that conditional probability distributions and expections occupy a central position in the analysis of essentially all the problems of inference -- the estimation of parameters, the tests of hypotheses about parameters and prediction theory. The problem is, unfortunately, quite involved and can be considered simple in only really elementary cases. However, the problem was mostly bypassed in the literature by assuming that conditional distributions exist, and without analyzing the various possibilities. But in the context of the inference theory of stochastic processes (with more than finitely many random variables), this cannot be so assumed. For this reason, a detailed analysis of the problem on conditional probabilities and distributions is presented here. Unless the underlying probability space is very simple (with a finite or countable set of points) the results on the regularity of conditional probabilities have to be treated, using some general theory and ideas of Functional Analysis. This has been done here. As a consequence, for instance, the most general conditions for the following formula (when densities exist) are obtained:

$$f_n(x_1,\ldots,x_n|\theta) = f_1(x_1|\theta)f_2(x_2|x_1,\theta)\ldots f_n(x_n|x_1,\ldots,x_{n-1},\theta)$$

where θ is a parameter, which may be the value of a random variable, and various quantities on the right are conditional densities. For this work, the projective limit theory is needed crucially!

It is also possible to consider the theory of conditional probability directly through a new axiomatic set up. Such a point of view was proposed by A. Renyi, and a discussion of it is given here for comparison. This new theory is still in its infancy however. Some difficulties are illustrated, and various applications, including some on conditional hypothesis testing, and complements, are given in concluding the chapter.

4. Martingales. The basic theory of martingales is available in several books. Consequently here some new or simplified proofs of the standard theory together with a demonstration of the equivalence of the martingale convergence and the Andersen-Jessen theory are presented. It is useful to look at this subject from different angles since it plays an important role not only in the inference theory, but it's of considerable interest in many parts of analysis. The decomposition theory of submartingales (Doob-Meyer), and other related decompositions (Riesz and Krickeberg) together with an application to likelihood ratios and differentiations are given.

There is also a close relation between the projective limit theory of the second paragraph above, and the martingale theory. This is explored here. A discussion of the martingale formulation of ergodic theory and of the 'quasi-martingales' are included. Again many of these considerations have independent interest in other areas, both for theory as well as applications.

5. Stochastic equations. The subject of stochastic difference and differential equations is of considerable interest in both the physical and social sciences, and the topic is growing. The theory depends (in the case of differential equations) on a development of the integral and differential calculus for stochastic processes.

This can be done along two lines: (i) The Itô integral, (ii) the vector integral. Both these ideas are considered here, and particularly in (i), martingale theory plays an important role. Most of the work, in the existing texts, refers to the first order differential equations. However, the second order equations also are of importance, for example, in the theory of the simple harmonic oscillator. This chapter contains, therefore, both the first and second order equations. The latter forming two theses at Carnegie-Mellon, prepared under the supervision of the Principal Investigator, appears in book form for the first time.

The solution processes of stochastic equations are also of interest for the inference theory, particularly in the problems of testing hypothesis, and filtering theory. Some of these are included here. On this problem, there has been considerable amount of work by the Russian school. Some of this is also considered.

6. Gaussian Processes. It is well known that Gaussian processes play a fundamental part in the inference theory as well as in other places where explicit formulas are of interest. Consequently, the basic theory, including the processes with triangular covariance kernels, and Wiener process in particular are considered. The key result of the dichotomy that two Gaussian measures are either equivalent or singular is proved. This enables a calculation of the likelihood ratios of two Gaussian measures when they are equivalent -- a result which forms a central part of the hypothesis testing problem. This problem is presented for various classes of kernels, because of its usefulness. The reproducing kernel

Hilbert spaces play an important role here and an account of these spaces, due to Aronszajn, is included.

Some characterizations of Wiener processes, and a discussion of the abstract Wiener processes, are also included to clarify the structure. Certain other results are considered in the complements and problems section of the chapter.

7. Inference theory. (a) Hypothesis testing. After stating the general principles of hypothesis testing and desirable properties of a test, these are illustrated on classes of Gaussian and certain other 'point' processes. Some applications of martingale theory are included in this context. Wald's theory of decision functions is also discussed in this chapter.

(b) Parametric estimation. Asymptotic properties of maximum-likelihood estimators of parameters in the finite dimensional distributions of processes governed by difference and differential equations form the subject of this chapter. Further special results on estimation of parameters in Markov processes are discussed. The so-called Bayes estimation is also briefly considered here.

(c) Prediction. Theory of non-linear prediction has hardly been included in text books thus far. After a brief discussion of the linear prediction, which is found in several books (generally translations of the Russian works), the theory of non-linear prediction relative to a p^{th} power $(1 < p < \infty)$, as the optimality criterion is treated. The subject is, however, in growth at present, and no complete treatment will be possible. Its connection with Bayes estimation is close and this is discussed. Applications to particular processes are included.

Filtering problems are related to hypothesis testing, and this is also briefly discussed for processes of various kinds, including those that are solutions of stochastic equations.

In conclusion, it may be mentioned that, for a mathematically rigorous treatment of the subject (which is the case in this book), the general considerations of the preceding five chapters becomes essential. In obtaining a unified treatment, several non-trivial gaps need be filled, and this is generally delaying a formal preparation from the rough draft describing the preceding results. The material has all been collected, during 1968-69, and the sketch is prepared later. At the moment the completion of the material, in the form for the book, is being reworked. It is hoped that a monograph will be ready for publication by the end of the 1970-71 academic year. The final copies will be submitted to the AFOSR office in accordance with the terms of the grant.

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13. ABSTRACT

This investigation was designed to prepare a monograph on certain mathematical aspects of the inference theory of stochastic processes, the principal components of which are substantially completed. These include substantive treatments of the foundations of inference theory, i.e., the projective limits of probability spaces, of conditional probability distributions and expectations, which occupy a central position in the analysis of essentially all the problems of inference, some new or simplified proofs of the standard theory of martingales together with a demonstration of the equivalence of the martingale convergence and the Andersen-Jessen theory, of stochastic difference and differential equations in both the physical and social sciences, of Gaussian processes, and of hypothesis testing, parametric estimation, and prediction, as the latter three topics relate to inference theory.

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