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ARPA ORDER NO. 189-1 a/



MEMORANDUM
ARM-6055-ARPA
APRIL 1970

PROPAGATION OF A FINITE OPTICAL BEAM IN AN INHOMOGENEOUS MEDIUM

R. F. Lutomirski

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This research is supported by the Advanced Research Projects Agency under Contract No. DAHC15 67 C 0141. Views or conclusions contained in this study should not be interpreted as representing the official opinion or policy of Rand or of ARPA.

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PREFACE

This Memorandum, prepared for the Advanced Research Projects Agency, is part of a study of the effect of atmospheric turbulence on optical and infrared reconnaissance and guidance equipment.

A quantitative understanding of the manner in which an initially coherent beam of finite cross section propagates is required for the prediction of the performance of various devices employing lasers for target acquisition or guidance in tactical missions, optical communication systems, and other devices. This Memorandum calculates the mean intensity distribution for an arbitrary amplitude and phase distribution in a finite aperture in both the near and far field and examines in detail the case of a uniform distribution across a circular aperture.

These results should be of use to those interested in propagation theory and its applications to laser range finders, laser line scanners, communication systems, and various guidance and other systems employing an illuminating beam.

SUMMARY

The first part of this Memorandum is devoted to extending the Huygens-Fresnel principle to a medium which exhibits a spatial (but not temporal) variation in index of refraction. With the proof of a reciprocity theorem for a monochromatic disturbance in a weakly inhomogeneous medium, it is shown that the secondary wavefront will be determined by the envelope of spherical wavelets from the primary wavefront, as in the vacuum problem, but that each wavelet is now determined by the propagation of a spherical wave in the refractive medium.

In the second part, the above development is applied to the case in which the index of refraction is a random variable; a further application of the reciprocity theorem results in a formula for the mean intensity distribution from a finite aperture in terms of the complex disturbance in the aperture and the modulation transfer function (MTF) for a spherical wave in the medium. The results are applicable in both the Fresnel and Fraunhofer regions of the aperture. Using a Kolmogorov spectrum for the index of refraction fluctuations and a second-order expression for the MTF, the formula is used to calculate the mean intensity distribution for a plane wave diffracting from a circular aperture and to give approximate expressions for the beam spreading at various ranges. It is argued that the spherical-wave MTF is the basic quantity to be measured for computing the atmospheric degradation of an intensity pattern from an arbitrary disturbance in a finite aperture.

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I. INTRODUCTION

The problem of computing the mean diffraction pattern from an optical wave in a finite aperture in the presence of a turbulent atmosphere is both of considerable importance for a number of practical applications (e.g., laser radar and laser line scanners) and a useful tool for evaluating our understanding of turbulence theory.

To solve the problem, it is first shown in Section II that the Huygens-Fresnel principle may be extended to a refractive medium, i.e., the field due to some disturbance specified over an aperture can be computed by superimposing spherical wavelets radiating from all elements of the aperture. The principle follows directly from Green's theorem and the Kirchhoff approximation applied to the scalar wave equation, together with a field reciprocity theorem (proven in the Appendix) between an observation point and a source point of spherical waves in an inhomogeneous medium. To apply the extended principle to the atmosphere, we assume that the refractive index fluctuation is a random function of coordinates and does not depend on time, i.e., the refractive index n does not change appreciably over periods of the order $\Delta t = \lambda/c$ ($\Delta t = 3 \times 10^{-15}$ sec for a wavelength $\lambda = 2\pi/k$ of 1μ), where c is the vacuum speed of light. The time changes in n are considered as changes in the different realizations of the random field $n(\underline{r})$. On the basis of this principle, one can separate the geometry of the problem, i.e., the disturbance in the aperture, from the propagation problem, which is determined by the manner in which a spherical wave propagates through the medium.

The mean intensity, calculated in Section III, is found by first computing the intensity at a point from an arbitrary pair of elements in the aperture. Applying the reciprocity relationship and averaging reveal that the above quantity is essentially the modulation transfer function (MTF) for a spherical wave in the medium. The integration over the aperture is performed as a final step, which results in a simple formula for the mean intensity pattern in both the Fresnel and Fraunhofer regions for an arbitrary complex disturbance in the aperture. The properties of the medium (e.g., the turbulence parameters) appear only in the MTF. The MTF describes the reduction in lateral coherence between different elements of the aperture, effectively transforming the aperture into an equivalent partially coherent radiator, with the degree of coherence decreasing with increasing distance from the aperture. As a practical example, the mean intensity patterns are computed over horizontal ranges of interest (of the order of kilometers) for both a 2-cm and 6-cm aperture using a range of turbulence parameters characteristic of paths of the order of a few meters above the ground. Over certain limited path lengths, approximate formulas exist for the MTF for a Kolmogorov spectrum. Introducing these expressions into the above-mentioned formula yields approximate expressions for the beam spreading at these ranges. It is shown that at distances sufficiently large so that the coherence length of the MTF is less than the size of the smallest inhomogeneities, the intensity pattern is dominated by multiple scattering from these smallest scatterers and is contained in an angle which is independent of both wavelength and aperture size.

It is also shown that (1) for homogeneous isotropic turbulence an optical measurement of the spherical-wave MTF can be inverted to give the turbulence spectrum, and (2) the intensity distribution can be inverted to give the spherical-wave MTF, but only for spatial separations smaller than the diameter of the transmitting aperture.

Finally, it is argued that the spherical-wave MTF is the basic quantity to be measured for computing the degradation of an intensity pattern from an arbitrary disturbance in a finite aperture.

II. AN EXTENSION OF THE HUYGENS-FRESNEL PRINCIPLE
TO A REFRACTIVE MEDIUM

The problem considered in this section is the calculation of the complex field at an arbitrary point in a turbulent medium, given a complex monochromatic disturbance specified over a finite surface (i.e., the aperture).

We assume a scalar wave equation for the propagation of an electromagnetic disturbance in the optical wavelength region through a nonabsorbing refracting medium:

$$(\nabla^2 + k^2 n^2) U = 0 \quad (1)$$

where U is one component of the electric or magnetic field vector and $n(\underline{r}) = 1 + n_1(\underline{r})$ is the refractive index of the medium at the point \underline{r} , where $\langle n \rangle = 1$ and n_1 is the small fluctuating part of the refractive index field. (Angular brackets are used to denote ensemble averages.)

The method to be used is analogous to the integral theorem of Helmholtz and Kirchhoff⁽¹⁾ used in vacuum diffraction theory, and a similar notation will be employed.

Let the turbulent medium occupy a volume V and be bounded by a closed surface S , and assume that U possesses continuous first- and second-order partial derivatives within and on S . Within V , let U' be the field at \underline{r} due to a point source at \underline{P} . Then U' satisfies the equation

$$(\nabla^2 + k^2 n^2) U'(\underline{r}, \underline{P}) = -4\pi \delta(|\underline{r} - \underline{P}|) \quad (2)$$

where δ is the Dirac delta function. Multiplying Eq. (1) by U' , Eq. (2) by U , subtracting, integrating over V , and applying Green's theorem yields

$$U(\underline{P}) = \frac{1}{4\pi} \int_S [U\nabla U' - U'\nabla U] \cdot d\underline{A} \quad (3)$$

where $d\underline{A}$ denotes the surface element with normal directed into V .

To proceed further, we apply the Kirchhoff approximation. A monochromatic disturbance is assumed to propagate through an opening in a plane opaque screen which is large compared to the wavelength but small compared to the distance between P and the screen. The disturbance at P is computed by taking Kirchhoff's integral over a surface S formed by (1) the opening A , (2) a portion B of the non-illuminated side of the screen, and (3) a portion C of a large sphere of radius R , centered at P , which together with A and B forms a closed surface. The contribution to the field at P from C can be argued to vanish by assuming that the radiation field does not exist for all times, but begins to radiate at $t = t_0$ (which actually implies an unimportant departure from monochromacy). Then if for all times of interest R is taken to be $> c(t - t_0)$, the contributions from the surface C could not have reached the point P , and the integral will vanish.* Finally, invoking the Kirchhoff approximation, we assume that, everywhere on B , U and $\frac{\partial U}{\partial n}$ will be approximately zero and that, on A , U and $\frac{\partial U}{\partial n}$ will not differ appreciably from the values obtained in the absence of

*Born⁽²⁾ has shown that this assumption is not essential, but it shortens the discussion.

the screen. Hence the surface integral in Eq. (3) reduces to an integral over the aperture area, A, and from this assumption one can develop an extension to the Huygens-Fresnel principle for vacuum propagation to propagation in a weakly inhomogeneous medium. The principle as applied to propagation in a vacuum states that every point of a wavefront may be considered as a center of a secondary disturbance which gives rise to spherical wavelets, and the wavefront at a later instant may be regarded as the envelope of these wavelets. It will be shown that, for a refractive medium, the "extended" principle is that the secondary wavefront will again be determined by the envelope of spherical wavelets from the primary wavefront, but each wavelet will now be determined by the propagation of a spherical wave through the refractive medium. In either case, the theorem is an approximation valid only when the scattering angles are sufficiently small.

We first assume the aperture to lie in a plane normal to the z direction, and we note that when the radius of curvature at each point of the aperture is large compared to the wavelength, $\frac{\partial U}{\partial z} \approx ikU(\underline{e}_n \cdot \underline{e}_z)$, where \underline{e}_n is the unit vector normal to the wavefront in the aperture. Then, writing $U' = \frac{e^{iks + \psi}}{s}$, where $s = |\underline{r} - \underline{P}|$ is the geometric distance between the point \underline{P} and the elemental area in the aperture, and assuming $ks \gg 1$ yield

$$U(\underline{P}) = \frac{ik}{4\pi} \int_A U(\underline{r}) \frac{e^{iks + \psi}}{s} \left[\underline{e}_s \cdot \underline{e}_z - \underline{e}_n \cdot \underline{e}_z + \frac{\nabla\psi}{ik} \cdot \underline{e}_z \right] d^2\underline{r} \quad (4)$$

where $d^2\underline{r}$ is an element of area at the point \underline{r} in the aperture and

\underline{e}_s is a unit vector in the direction from \underline{P} to \underline{r} .

For $\underline{r} \neq \underline{P}$, substituting $U' = \frac{e^{iks + \psi}}{s}$ into Eq. (2) yields the differential equation for ψ :

$$\nabla^2 \psi + (\nabla \psi)^2 + 2\nabla \psi \cdot \nabla \psi_0 + k^2(2n_1 + n_1^2) = 0 \quad (5)$$

where

$$\psi_0 = \frac{e^{iks}}{s}$$

Expanding $\psi = \sum_{i=1}^{\infty} \psi_i$, where $\psi_i \sim n_1^i$, in Eq. (5) generates the hierarchy of Rytov solutions⁽³⁾ to the field at \underline{r} from a unit spherical wave emitted at \underline{P} in an inhomogeneous medium. In the geometric optics regime, $\psi \sim ik \int n_1 ds$, $|\frac{\nabla \psi}{ik} \cdot \underline{e}_z| \approx |n_1(\underline{e}_s \cdot \underline{e}_z)|$, and in the atmosphere at optical frequencies, $|n_1| \sim 10^{-6}$. In general, due to the smallness of both the scattering "potential" (proportional to n_1) and the ratio of the wavelength to the size of the smallest inhomogeneity (of dimension l_0), from a photon picture the scattered wave from the point source will vary slowly over a wavelength for $\approx (kl_0)^2$ scatterings. The scattering length can be estimated from the decay rate of the average field for a single-scale model to be $\approx (\langle n_1^2 \rangle k^2 l_0)^{-1}$, which implies that the $\nabla \psi/k$ term can be neglected compared to unity for all distances of interest (i.e., $l_0 / \langle n_1^2 \rangle \approx 10^6$ km).

It might be noted that at a range where $|\nabla \psi/k| \sim 1$ one would measure a local beam divergence of the order of a radian due to the scattering, which has never been observed.

Then, noting that in our development the area A could have been

replaced by any other open surface whose rim coincides with the edge of the aperture, we choose instead of A a portion of the incident wavefront which approximately fills the aperture and neglect any small errors near the rim. Over the new region of integration, A', the derivative of U in the direction normal to the surface is $\approx ikU$, and defining $\chi = \pi - \cos^{-1} (\underline{e}_n \cdot \underline{e}_s)$, we obtain

$$U(\underline{P}) = \int_{A'} K(\chi) U'(\underline{r}, \underline{P}) U(A') dA' \quad (6)$$

where $K(\chi) = (-i/2\lambda)(1 + \cos \chi)$.

In the Appendix we show that $U'(\underline{r}, \underline{P}) = U'(\underline{P}, \underline{r})$, i.e., reciprocity holds in that the field and source points may be interchanged in Eq. (2). Equation (6) is thus the extension of the Huygens principle, where the contribution from the element dA' of the wavefront to the field at P is $K(\chi) U(A') \frac{e^{iks + \psi}}{s} dA'$.

When the distance from the observation point to the aperture is large compared with the linear dimensions of the aperture and the wavefront does not vary appreciably over the aperture, $K(\chi)$ may be replaced by $K(\chi_0)$, where χ_0 is the angle between the normal to the screen and a line from the center of the aperture to the observation point. In the same approximation, the factor $1/s$ may be replaced by the reciprocal of the distance from the center of the screen to the point P, $1/s_0$. Then Eq. (6) reduces to

$$U(\underline{P}) = \frac{K(\chi_0)}{s_0} \int e^{iks + \psi} U(\underline{r}) d^2\underline{r} \quad (7)$$

Equation (6) or (7) reveals that if we know how a spherical wave propagates in a given medium, we can determine the response to an arbitrary disturbance in an aperture. In the next section we will give an example of the utility of this formulation by applying it to a problem of practical importance.

III. THE MEAN INTENSITY FROM A DIFFRACTING APERTURE

From Eq. (7), the intensity at the point P is

$$I(P) = |U(P)|^2 = \left(\frac{1 + \cos \chi_0}{2\lambda s_0} \right)^2 \iint e^{ik(s_1 - s_2)} \times e^{\psi(\underline{r}_1) + \psi^*(\underline{r}_2)} U(\underline{r}_1) U^*(\underline{r}_2) d^2 \underline{r}_1 d^2 \underline{r}_2 \quad (8)$$

where s_1, s_2 are the geometric distances between the point P and the points $\underline{r}_1, \underline{r}_2$ in the aperture, respectively, and $\psi(\underline{r}_1), \psi(\underline{r}_2)$ are the perturbations in the field at P due to unit spherical waves emitted at $\underline{r}_1, \underline{r}_2$. For the case in which the index of refraction fluctuations, n_1 , is a random variable, the mean intensity from a finite diffracting aperture can be computed if $\langle \exp [\psi(\underline{r}_1) + \psi^*(\underline{r}_2)] \rangle$ is known for all $\underline{r}_1, \underline{r}_2$. Similarly, higher moments of $I(P)$ can be computed from a knowledge of terms of the form $\langle \exp [\psi(\underline{r}_1) + \psi^*(\underline{r}_2) + \psi(\underline{r}_3) + \psi^*(\underline{r}_4)] \rangle$, etc. For the mean intensity problem, it follows directly from the reciprocity relationship (in the Appendix) that $\exp [\psi(\underline{r}_1) + \psi^*(\underline{r}_2)]$ is the field at \underline{r}_1 multiplied by the complex conjugate of the field at \underline{r}_2 (normalized to the vacuum field) due to a spherical wave at \underline{P} . In particular, the quantity

$$\frac{e^{ik(s_1 - s_2)}}{s_1 s_2} \langle \exp [\psi(\underline{r}_1) + \psi^*(\underline{r}_2)] \rangle$$

is equal to the cross-correlation of the complex fields at

the points $\underline{r}_1, \underline{r}_2$ due to a unit point source at \underline{P} . The function $\langle \exp [\psi(\underline{r}_1) + \psi^*(\underline{r}_2)] \rangle$ is, by definition, the MTF for a spherical wave, $M_s(\underline{r}_1, \underline{r}_2, z)$.

When one observes the intensity as a function of time, one sees the effects of beam "wander" and "breakup," both of which contribute to the intensity scintillation at a point. If the average is taken over times long compared with the periods over which n_1 varies, this mean intensity distribution would include all of the above effects. If averages over an ensemble corresponding to a shorter time period could be obtained, a time history of the intensity distribution corresponding to observations over this shorter time interval could be obtained. For plane waves, the long-term MTF, correct to second order in n_1 , is given by⁽⁴⁾

$$M_p(\rho, z) = \exp \left\{ \frac{2z}{z_c} \left[1 - \frac{\int_0^\infty J_0(K\rho) \phi_n(K) K dK}{\int_0^\infty \phi_n(K) K dK} \right] \right\} \quad (9)$$

where $\rho = |\underline{r}_1 - \underline{r}_2|$, $\phi_n(K)$ is the spectrum of the index of refraction fluctuations, and

$$z_c = [2\pi^2 k^2 \int_0^\infty \phi_n(K) K dK]^{-1} \quad (10)$$

is the propagation distance in which the mean field of a plane wave or spherical wave is reduced by e^{-1} from its vacuum value. The quantity z_c is also the distance over which the perturbations in the field due to the medium become comparable to the field in the absence of the medium, and can be computed by setting $1/2 \langle |\psi_1|^2 \rangle = 1$. It should be noted that, independent of the form of the spectrum,

$z_c \propto \lambda^2$. Consistent with the approximation that the angles χ of interest remain small is the assumption that the points $\underline{r}_1, \underline{r}_2$ may be considered to be lying on a large sphere centered at \underline{P} . In this case, the MTF given by Eq. (9) is modified for spherical waves by substituting for the Bessel function $J_0(K\rho)$ the quantity

$$\int_0^1 J_0(K\rho u) du.$$

Using Eq. (9) for the case of homogeneous isotropic turbulence, one can determine the turbulence spectrum from optical measurements of the spherical wave MTF (e.g., by using a sufficiently small aperture).

First, the distance z_c is determined by considering a sufficiently short path length, z , and measuring the limiting value of $M_s(\rho, z)$ for spatial separations much greater than the largest turbulence scale, L_0 . From Eq. (9), this limiting value is e^{-2z/z_c} .

Then, from the relation between the spherical and plane MTFs, one can construct the quantity $\ln[M_p(\rho, z)]$, which is given by

$$\ln[M_p(\rho, z)] = \frac{\partial}{\partial \rho} [\rho \ln M_s(\rho, z)] \quad (11)$$

With z_c and the logarithm of the plane-wave MTF known, one can invert Eq. (9)⁽⁵⁾ to yield

$$\Phi_n(K) = \frac{1}{2\pi^2 k^2 z_c} \int_0^\infty J_0(K\rho) \left[1 + \frac{z_c}{2z} \ln M_p(\rho, z) \right] \rho d\rho \quad (12)$$

for the turbulence spectrum.

Yura and Lutomirski⁽⁴⁾ have used the spectral density

$$\Phi_n(K) = \frac{0.033 C_n^2 e^{-(K\ell_0/5.92)^2}}{(K^2 + \ell_0^{-2})^{11/6}} \quad (13)$$

an extrapolation of the Kolmogorov spectrum. In Eq. (13), C_n is the index structure constant and $\ell_0 (= 2\pi\ell_0)$ and $L_0 (= 2\pi L_0)$ are the inner and outer scales of turbulence, respectively. In Figs. 1a and 1b we plot z_c versus λ for three values of C_n roughly corresponding to weak ($3 \times 10^{-16} \text{ cm}^{-2/3}$), medium ($3 \times 10^{-15} \text{ cm}^{-2/3}$), and strong ($3 \times 10^{-14} \text{ cm}^{-2/3}$) turbulence. For horizontal propagation near the ground, we have used the nominal values of $\ell_0 = 0.1 \text{ cm}$ and $L_0 = 10 \text{ cm}$ and 100 cm . For $\ell_0 \ll L_0$, one can approximate the integral in Eq. (10) to yield

$$z_c \approx (0.4 k^2 C_n^2 L_0^{5/3})^{-1} \quad (14)$$

The coherence length z_c , and hence the MTF, will depend strongly on the outer scale of turbulence.

Returning to Eq. (8), we thus obtain

$$\langle I \rangle = A^2 \iint e^{ik(s_1 - s_2)} M_s(\underline{r}_1, \underline{r}_2, z) U(\underline{r}_1) U^*(\underline{r}_2) d^2 \underline{r}_1 d^2 \underline{r}_2 \quad (15)$$

where $A^2 = [(1 + \cos \chi_0)/2s_0\lambda]^2$ and M_s is the MTF for spherical waves.

For the case of a plane wave in a circular aperture of diameter D , we take the origin of the coordinate system at the center of the aperture, let

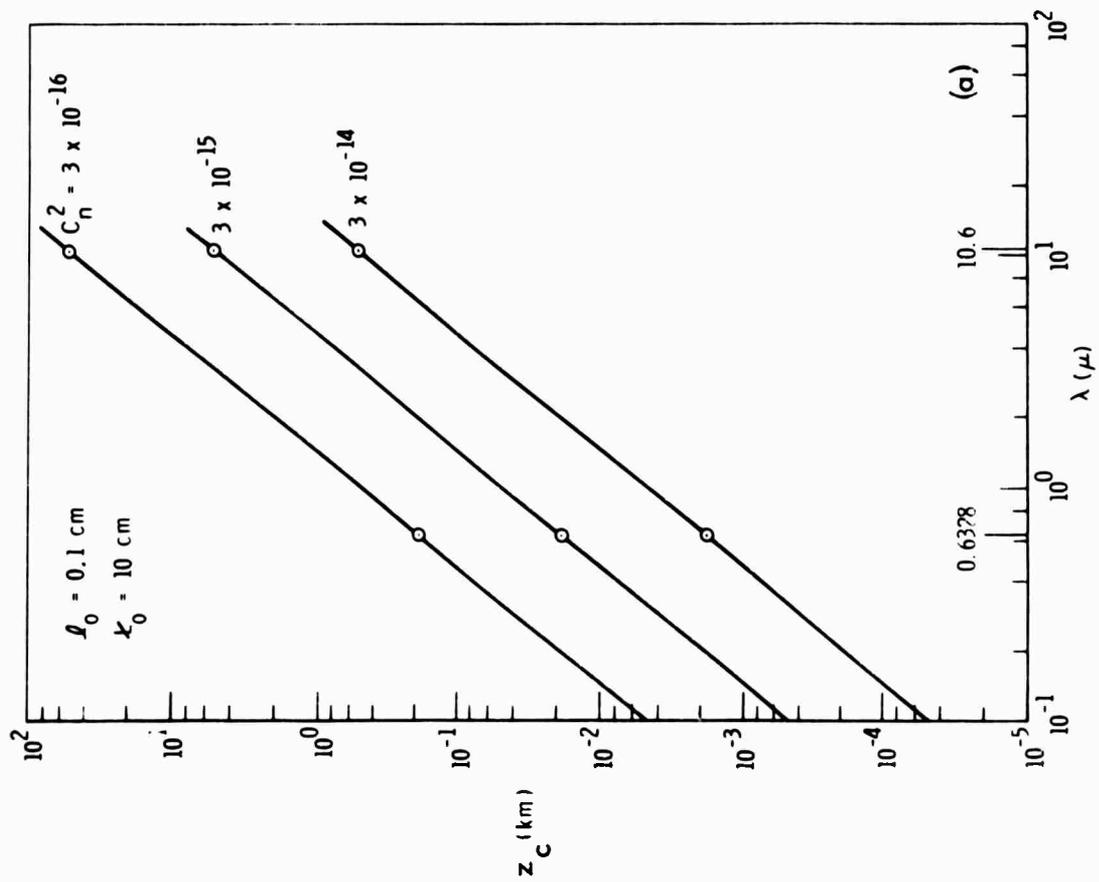
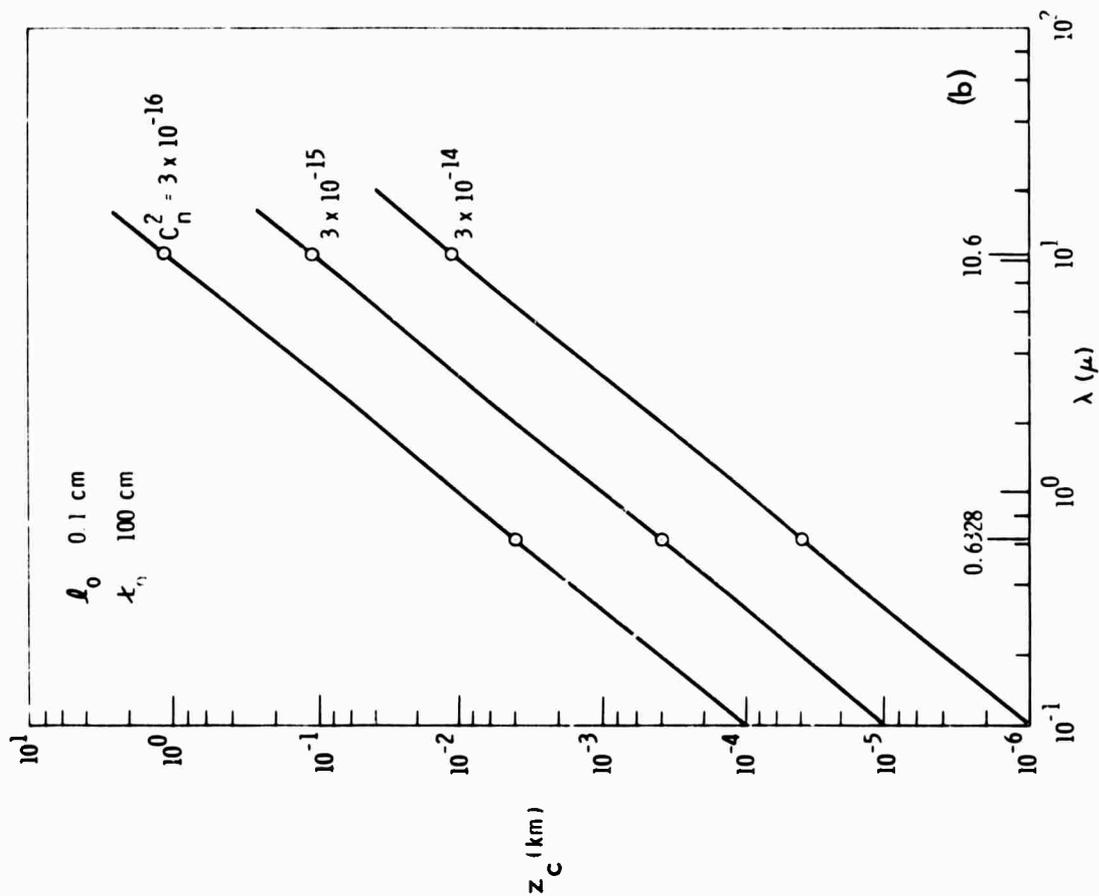


Fig. 1 — The critical length z_c

$$\begin{aligned}
 U(\underline{r}_{1,2}) &= 1, \quad |\underline{r}_{1,2}| \leq D/2 \\
 &= 0, \quad |\underline{r}_{1,2}| > D/2
 \end{aligned}
 \tag{16}$$

and integrate over the entire plane of the aperture. If we define the vector \underline{p} as the normal from the (z) axis of symmetry to the observation point \underline{P} , then

$$s_{1,2}^2 = z^2 + (\underline{p} - \underline{r}_{1,2})^2$$

and, in the small-angle approximation, $A^2 \approx \left(\frac{1}{\lambda z}\right)^2$ and

$$s_1 - s_2 = \frac{-2\underline{p} \cdot (\underline{r}_1 - \underline{r}_2) + r_1^2 - r_2^2}{2z} \tag{17}$$

Changing variables to $\underline{\rho} = \underline{r}_1 - \underline{r}_2$, $\underline{r} = \frac{1}{2}(\underline{r}_1 + \underline{r}_2)$, then for the case of homogeneous isotropic turbulence

$$\begin{aligned}
 \langle I \rangle &= \left(\frac{1}{z\lambda}\right)^2 \int d^2\underline{\rho} M_s(\underline{\rho}, z) e^{-(ik/z)\underline{\rho} \cdot \underline{\rho}} \int U(\underline{r} + \frac{1}{2}\underline{\rho}) \\
 &\quad \times U^*(\underline{r} - \frac{1}{2}\underline{\rho}) e^{(ik/z)\underline{\rho} \cdot \underline{r}} d^2\underline{r}
 \end{aligned}
 \tag{18}$$

For our problem U is real, and inspection of Eq. (18) reveals that the inner integral is the integration of the function $e^{(ik/z)\underline{\rho} \cdot \underline{r}}$ over the area of overlap of two circles each of diameter D , with centers located a distance ρ apart. The integration is straightforward and yields for the inner integral $D^2 \Gamma_\beta(x)$, where

$$\Gamma_{\beta}(x) = \int_0^{\cos^{-1}(x)} \left\{ \frac{\sin [2\beta x (\cos \theta - x)]}{(2\beta x \cos \theta)} - \frac{[1 - \cos [2\beta x (\cos \theta - x)]]}{(2\beta x \cos \theta)^2} \right\} d\theta$$

$$x \leq 1$$

$$= 0, x > 1 \quad (19)$$

where $x = \rho/D$ and $\beta = kD^2/4z$. Then, using polar coordinates for the ρ integration, performing the angular integral, and changing variables from ρ to $x = \rho/D$ yield

$$\langle I \rangle = \frac{8}{\pi} \beta^2 \int_0^1 x J_0(2\alpha x) M_g(Dx, z) \Gamma_{\beta}(x) dx \quad (20)$$

where J_0 is the zero-order Bessel function and $\alpha = \frac{kD\rho}{2z} = \frac{kD}{2} \tan \theta$, where $\theta (= \chi_0)$ is the angle which the direction to \underline{P} makes with the central direction. Normalizing the intensity at a particular range (z) to the on-axis value ($\alpha = 0$) in the absence of turbulence ($M = 1$), we obtain

$$\langle I_N \rangle = \frac{\int_0^1 x J_0(2\alpha x) M_g(Dx, z) \Gamma_{\beta}(x) dx}{\int_0^1 x \Gamma_{\beta}(x) dx} \quad (21)$$

In the Fraunhofer region of the aperture $\beta = 0$, and $D^2\Gamma_0(x)$ reduces to the overlap area of the two circles. From Eq. (19)

$$\Gamma_0(x) = \frac{1}{2} \left[\cos^{-1}(x) - x \sqrt{1 - x^2} \right] \quad (22)$$

and substituting in Eq. (20) with $M = 1$ and integrating by parts yields the vacuum Airy pattern:

$$I_o(\alpha) = \frac{1}{4} \beta^2 \left[\frac{2J_1(\alpha)}{\alpha} \right]^2$$

For arbitrary β , Eq. (21) gives the normalized mean intensity distribution in the Fresnel region as well as in the Fraunhofer region of the aperture. Numerical calculations show that Eq. (22) is a good approximation to $\Gamma_\beta(x)$ for $\beta \lesssim 2$. (See Fig. 2.)

We have numerically integrated Eq. (21) for a 2-cm and 6-cm aperture, using the spectrum of Eq. (13) for propagation paths of 1 and 5 km for $\lambda = 0.6328\mu$ and 5 and 10 km for $\lambda = 10.6\mu$. The results are shown in Figs. 3 through 6 for turbulence parameters characteristic of the strengths and scales found from zero to several meters above the ground. The dashed curve on each graph is the vacuum intensity pattern at the range indicated, normalized to its value at $\theta = 0$. For a given range the 6-cm aperture has less effective coherence than the 2-cm aperture, which results in a correspondingly greater degradation of the distribution in the absence of turbulence. Further, for given range, aperture size, and turbulence conditions, the longer wavelength 10.6μ pattern is the more coherent due to the larger coherence length, z_c .

In Fig. 7a-c we compare the mean intensity distributions for the above wavelengths and aperture diameters. As explained in the next section, the angle for which the intensity is reduced to one-half its axial value becomes independent of both λ and D at large ranges.

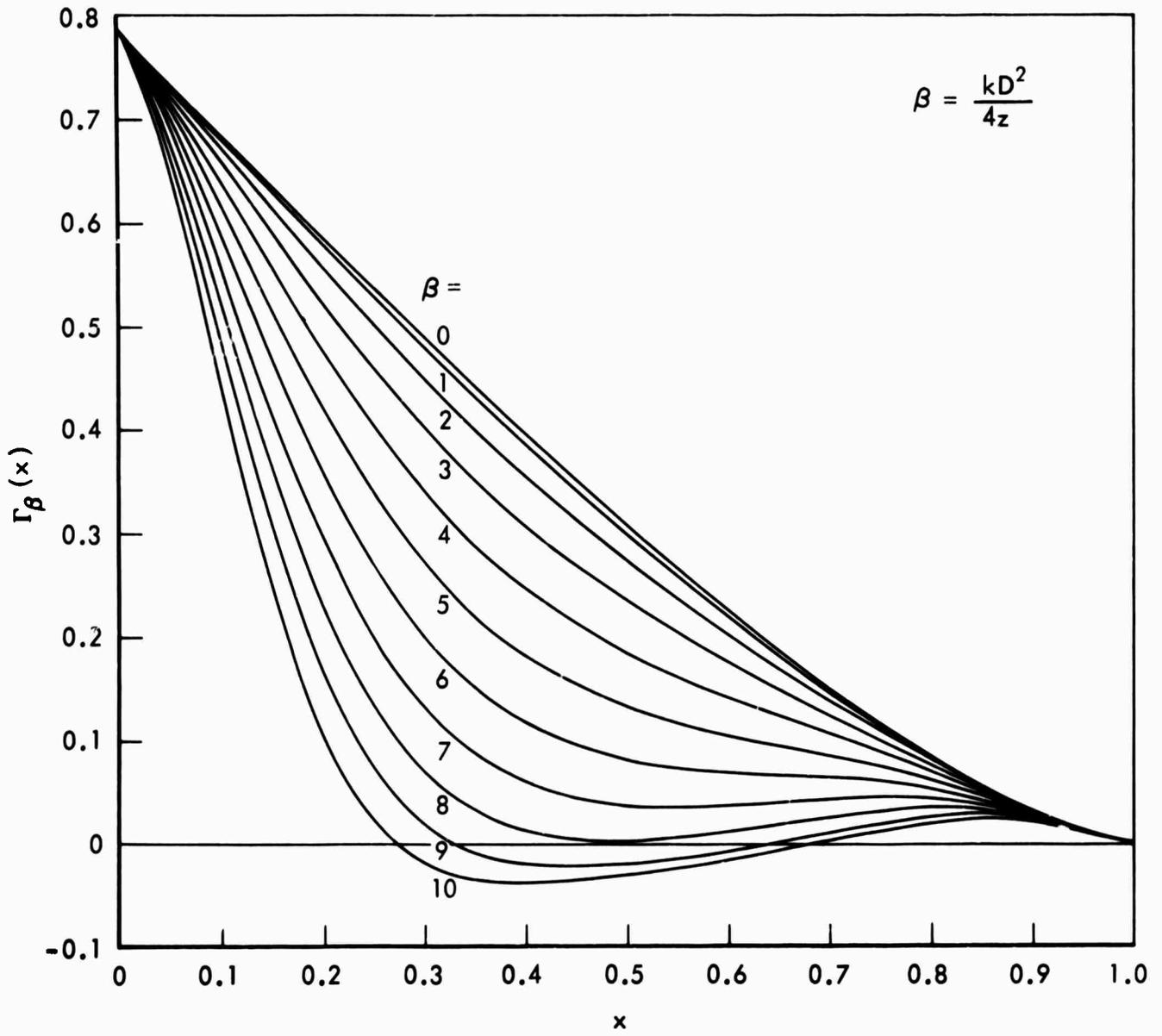


Fig. 2 — The overlap integral defined by Eq. (19)

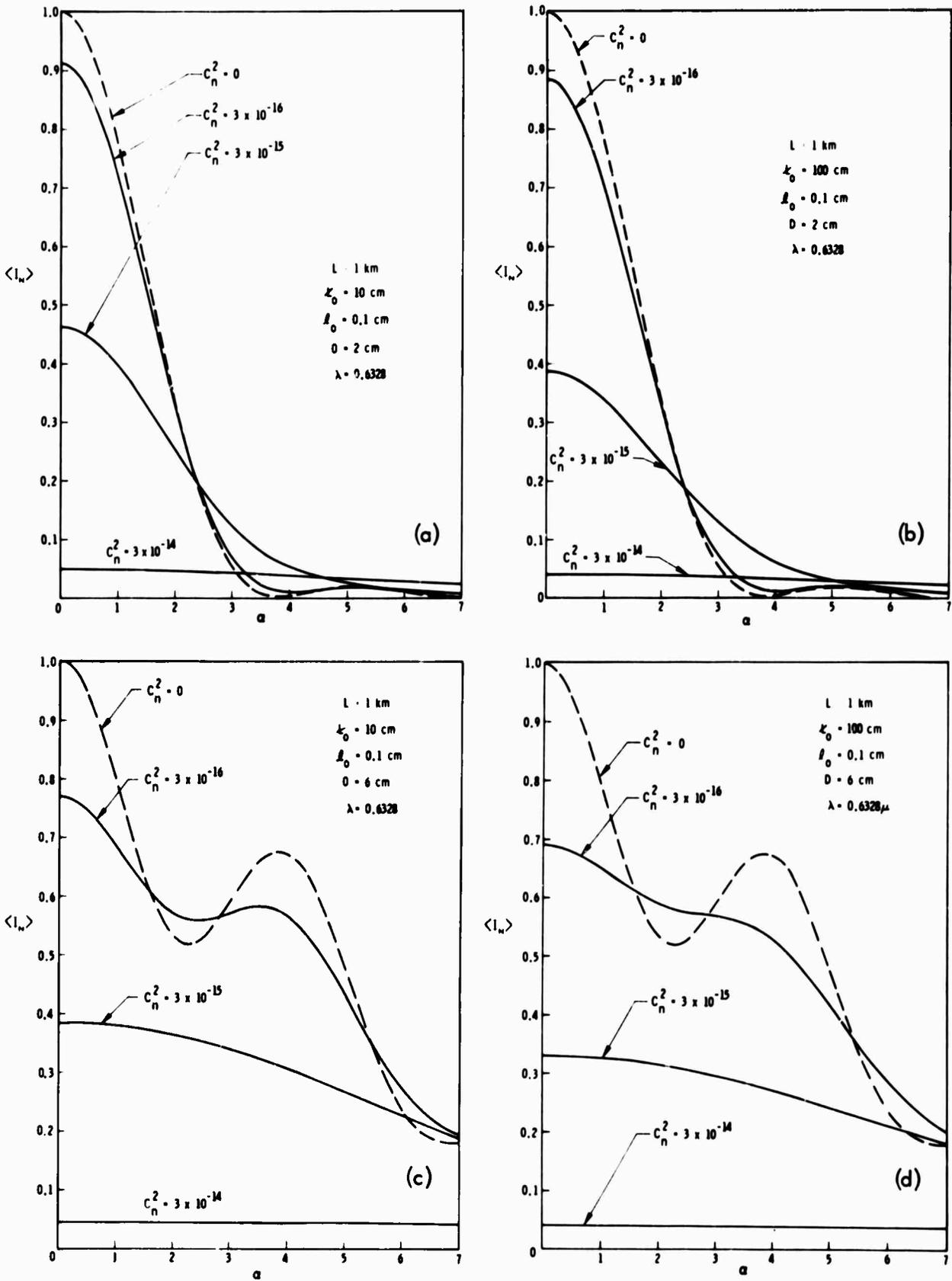


Fig. 3 — Comparison of beam patterns at 1 km for 2-cm and 6-cm apertures with $\lambda = 0.6328 \mu$

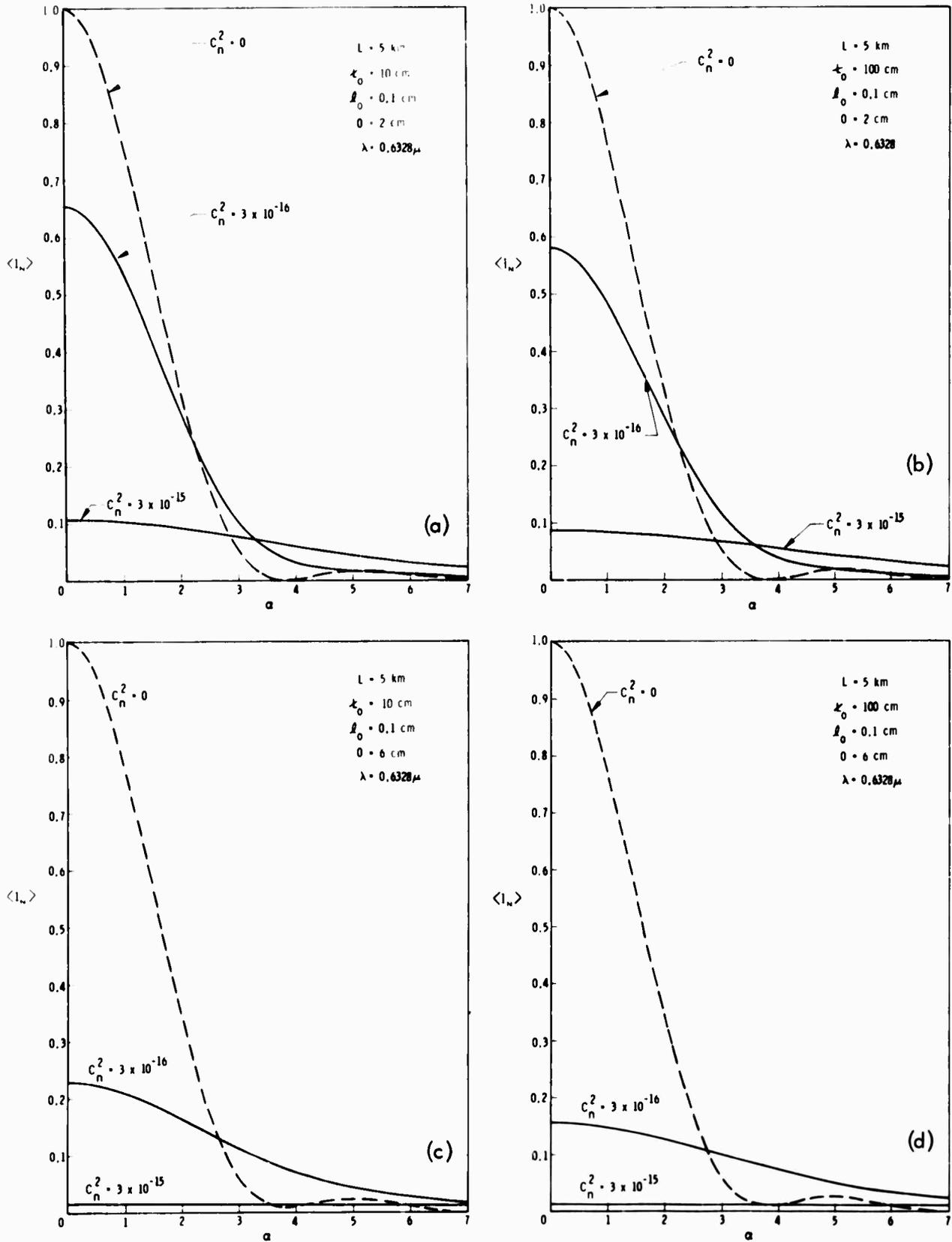


Fig. 4 — Comparison of beam patterns at 5 km for 2-cm and 6-cm apertures with $\lambda = 0.6328 \mu$

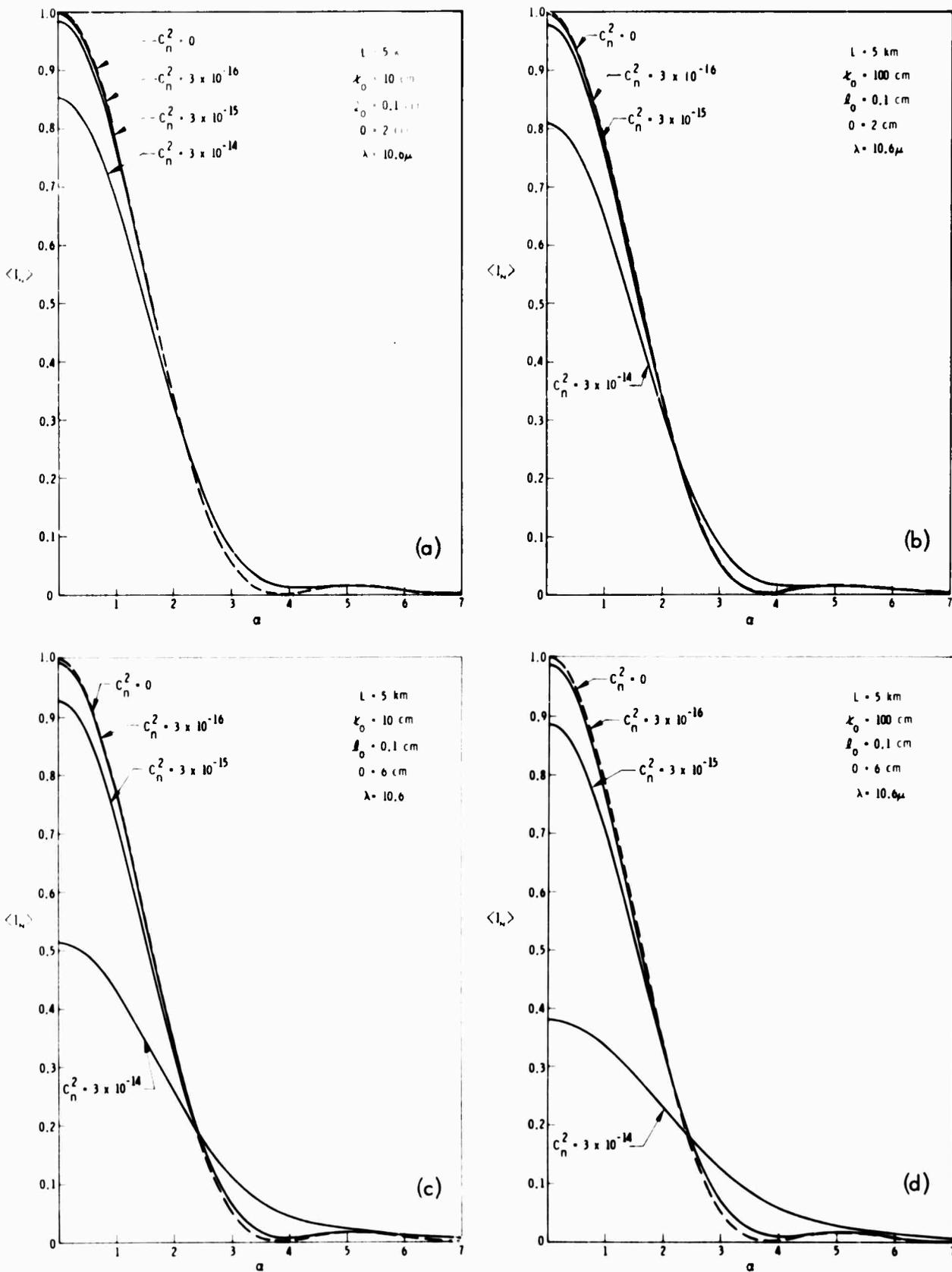


Fig. 5 — Comparison of beam patterns at 5 km for 2-cm and 6-cm apertures with $\lambda = 10.6 \mu$

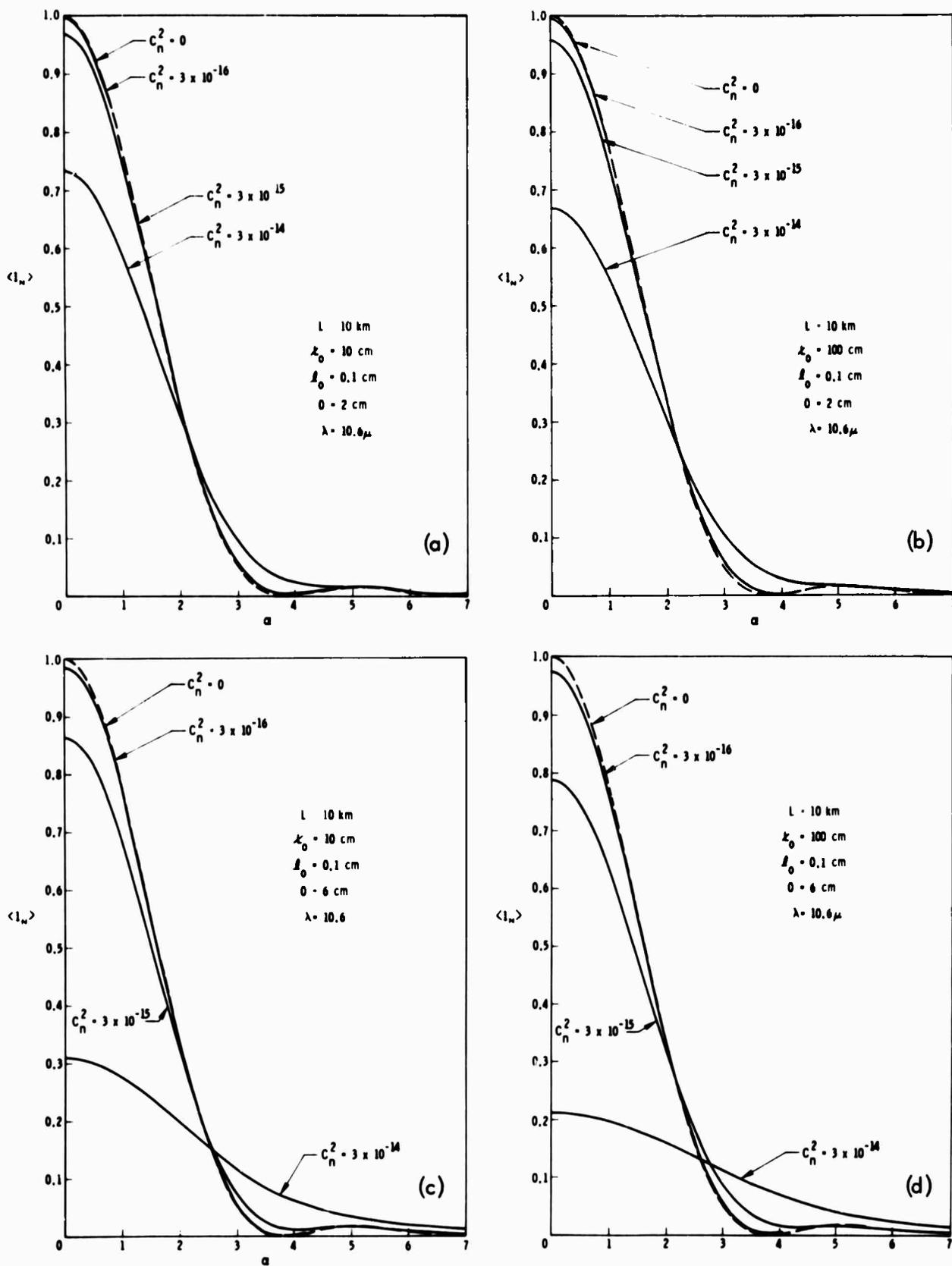


Fig. 6 — Comparison of beam patterns at 10 km for 2-cm and 6-cm apertures with $\lambda = 10.6 \mu\text{m}$

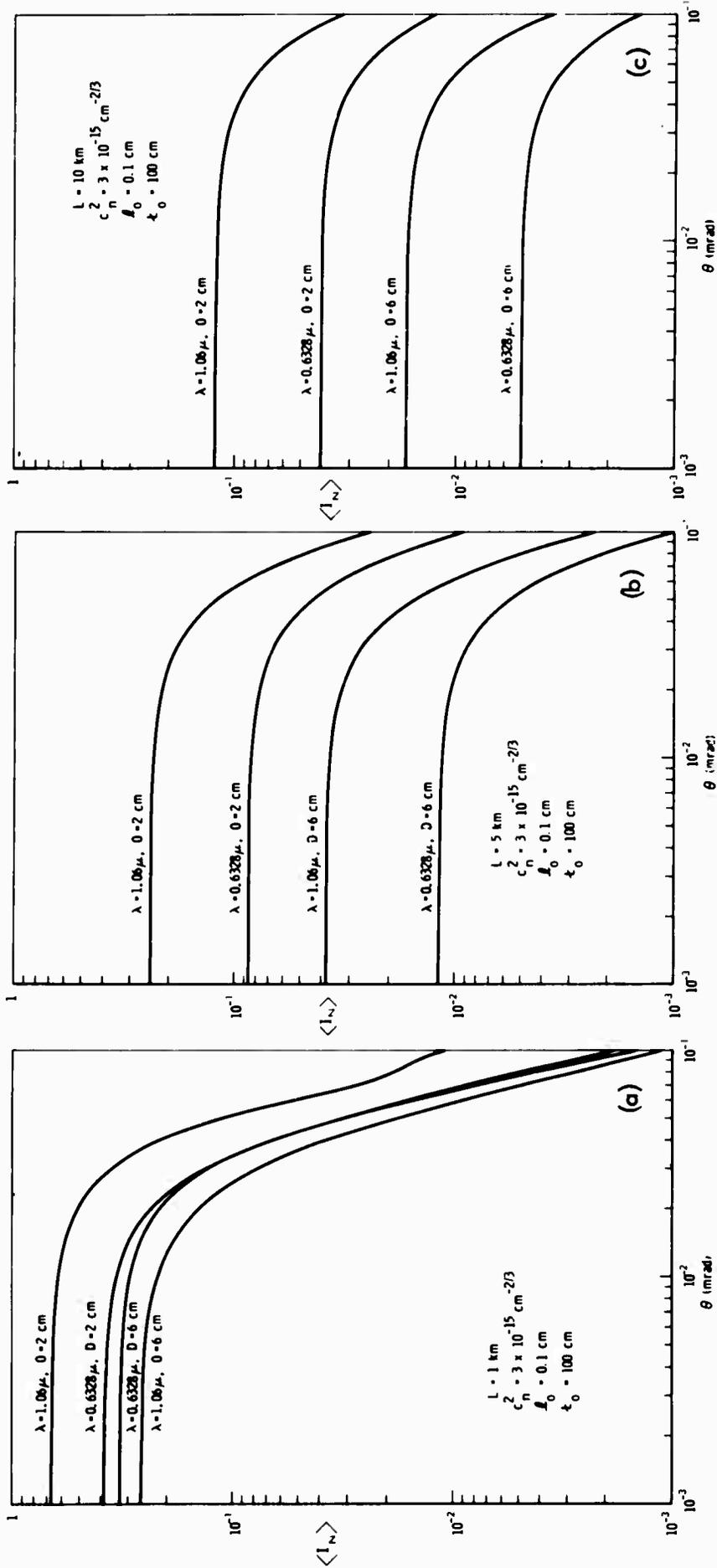


Fig. 7 — Comparison of intensity distributions for different wavelengths and aperture diameters for paths of 1 km, 5 km, and 10 km

IV. APPROXIMATE FORMULAS

The integrals involved in Eq. (20) for computing the mean pattern can be economically performed on a computer. However, for the purpose of relating our work to others and to show precisely what conditions are required for their use, formulas for the beam spread over certain limited ranges are obtained.

If ρ_0 is the transverse distance for which $M_s(\rho_0, z) = e^{-1}$, then for the modified Kolmogorov spectrum of Eq. (13), approximate formulas exist for the MTF when $\rho_0 \ll \ell_0$ and $\ell_0 \ll \rho_0 \ll L_0$. (It should be observed that the condition $|\nabla\psi/k| \ll 1$ implies a lower limit of $\rho_0 \sim \lambda$.) Introducing these MTFs into Eq. (20) results in approximate formulas for the mean intensity, provided that the correlation length ρ_0 in each case is small compared with the size of the aperture.

CASE A: $\rho_0 \ll \ell_0$

When sufficient scattering has occurred so that the mutual coherence between the field from different elements of the aperture is small compared with the smallest scale of turbulence, ℓ_0 , the Bessel function in Eq. (9) can be expanded in powers of (ρ/ℓ_0) , and the MTF for a spherical wave can be approximated by

$$M_s(\rho, z) = \exp \left\{ - \frac{z\rho^2}{6z_c} \frac{\int_0^\infty \phi_n(K) K^3 dK}{\int_0^\infty \phi_n(K) K dK} \right\}$$

$$= \exp \left\{ - k^2 z \rho^2 q \right\} \tag{23}$$

where $q = \pi^2/3 \int_0^\infty \phi_n(K) K^3 dK$. Equation (22) will be valid for all ρ 's

of interest if and only if

$$z \gg z_1 = (k^2 \ell_o^2 q)^{-1} \quad (24)$$

For the spectrum given by Eq. (13), q can be estimated by computing

$$\begin{aligned} \int_0^\infty \frac{k^3 e^{-(k\ell_o/5.92)^2}}{(k^2 + \ell_o^{-2})^{11/6}} dk &= \frac{1}{2} \left(\frac{5.92}{\ell_o}\right)^{1/3} \int_0^\infty \frac{ye^{-y}}{[y + (\ell_o/5.92 \ell_o)^2]^{11/6}} dy \\ &= \frac{1}{2} \left(\frac{5.92}{\ell_o}\right)^{1/3} \Gamma(1/6) + o\left(\frac{\ell_o}{L_o}\right)^2 \end{aligned}$$

where Γ is the gamma function and, for $\ell_o \ll L_o$,

$$q \approx 0.5 C_n^2 \ell_o^{-1/3} \quad (25)$$

Defining $\gamma^2 = k^2 D^2 z q$, it follows from Eq. (24) that for $D \gtrsim \ell_o$, $\gamma^2 \gg 1$. When $M_s(Dx, z)$ is substituted into Eq. (20), $\Gamma_\beta(x)$ can then be approximated by $\Gamma_\beta(0) = \pi/4$, yielding

$$\langle I_a \rangle \approx \frac{2\beta^2}{\gamma} \int_0^\gamma J_0\left(\frac{2\alpha}{\gamma} y\right) e^{-y^2} y dy, \quad z \gg z_1$$

where $z_1 = (0.5 k^2 C_n^2 \ell_o^{5/3})^{-1}$. Taking the upper limit as infinity, the integral can then be evaluated⁽⁶⁾ to yield

$$\langle I_a \rangle = \frac{D^2}{8z^2 \theta_a^2} e^{-\theta^2/2\theta_a^2} = \frac{D^2}{16qz^3} e^{-p^2/4qz^3} \quad (26)$$

where $p = \theta z$ and

$$\theta_a = (2qz)^{1/2} \approx 1.1 C_n \ell_0^{-1/6} z^{1/2} \quad (27)$$

is the angle corresponding to the standard deviation of the gaussian beam pattern. Using Eq. (27) and the inequality Eq. (24) yields

$$\theta_a \gg \frac{1}{k\ell_0}$$

Hence the beam spreading in this region is dominated by many scatterings from the smallest eddies, and for $z = 100$ km, $\theta_a \approx 0.2$ mrad for typical turbulence conditions.

It should be noted that conservation of energy requires

$$2\pi \int_0^\infty \langle I(p,z) \rangle p dp = \pi \frac{D^2}{4} I_{\text{aperture}} \quad (28)$$

which is seen to be satisfied by Eq. (26).

CASE B: $\ell_0 \ll \rho_0 \ll L_0$

Substituting the spectrum of Eq. (13) into Eq. (9), expanding the integrand for $\ell_0 \ll \rho \ll L_0$, and integrating yields

$$M_{pw}(\rho, z) = \exp \left\{ -1.46 k^2 C_n^2 z \rho^{5/3} \left[1 - 0.80(\rho/L_0)^{1/3} + O(\rho/L_0)^2 \right] \right\} \quad (29)$$

The second term in the exponent proves to be an important correction to the wave structure function given by Tatarski,⁽³⁾ whose analysis is based on a spectrum which continues to increase as $K^{-11/3}$ with

decreasing K , even for $K \lesssim \lambda_0^{-1}$. A more complete discussion of this point is given in Ref. 7.

The modification of Eq. (29) for spherical waves can be written

$$M_s(\rho, z) = \exp \left\{ -0.55 k^2 C_n^2 z \rho^{5/3} [1 - 0.71(\rho/L_0)^{1/3}] \right\} \quad (30)$$

For Eq. (30) to describe correctly the mutual coherence between different elements of the aperture for all ρ 's of interest, it is necessary that $M_s(\ell_0, z) \approx 1$ and $M_s(L_0, z) \ll 1$, which is essentially the condition

$$z_c \ll z \ll z_i \quad (31)$$

Defining $\Omega = (0.55 k^2 C_n^2 z)^{3/5} D$, it follows that for $D \gtrsim \lambda_0$, $\Omega \gg 1$. Then in Eq. (20), $\Gamma_\beta(x) \approx \pi/4$, and substituting $u = \Omega x$ results in

$$\langle I_b \rangle = \frac{2\beta^2}{\Omega^2} \int_0^{\approx(z/z_c)^{3/5}} u J_0\left(\frac{2\alpha}{\Omega} u\right) \exp \left\{ -u^{5/3} \left[1 - 0.67 \left(\frac{z_c}{z}\right)^{1/5} u^{1/3} \right] \right\} du \quad (32)$$

where, within the range indicated by Eq. (31), the integral will be insensitive to the precise value of the upper limit. For $D \lesssim \lambda_0$, requiring $M_s(D, z) \ll 1$ leads to the condition

$$z \gg \frac{z_D}{1 - 0.71(D/\lambda_0)^{1/3}} \quad (33)$$

where we have defined $z_D = (0.4k^2 C_n^2 D^{5/3})^{-1}$.

Substituting for β , Ω , and α , one obtains

$$\langle I_b \rangle = \frac{0.18 D^2}{k^{2/5} C_n^{12/5} z^{16/5}} F\left(\frac{z}{z_c}\right) G\left(\frac{z}{z_c}, \frac{\theta}{\theta_b}\right) \quad (34)$$

where we have defined

$$F\left(\frac{z}{z_c}\right) = \int u \exp \left\{ - u^{5/3} \left[1 - 0.67 \left(\frac{z_c}{z}\right)^{1/5} u^{1/3} \right] \right\} du$$

$$G\left(\frac{z}{z_c}, \frac{\theta}{\theta_b}\right) = \frac{\int u J_0\left(\frac{\theta}{\theta_b} u\right) \exp \left\{ - u^{5/3} \left[1 - 0.67 \left(\frac{z_c}{z}\right)^{1/5} u^{1/3} \right] \right\} du}{F\left(\frac{z}{z_c}\right)}$$

and

$$\theta_b = 0.69 k^{1/5} C_n^{6/5} z^{3/5} \quad (35)$$

The function F is a slowly varying correction to the on-axis intensity, while the function G determines the angular distribution (which is "almost" gaussian). $F(z/z_c)$ is plotted in Fig. 8a, while in Fig. 8b we plot the result of a numerical inversion of $G(z/z_c, \theta/\theta_b) = 1/2$, the latter giving the dependence of the half-intensity angle on range. It follows from Fig. 8 that for $15 \lesssim z/z_c \lesssim 10^4$, one can use the formula $\theta_{1/2} = 0.7 k^{1/5} C_n^{6/5} z^{3/5}$ within an error of $\approx \pm 25$ percent.

The inequality (31), together with the condition $z \gg z_D$, implies that

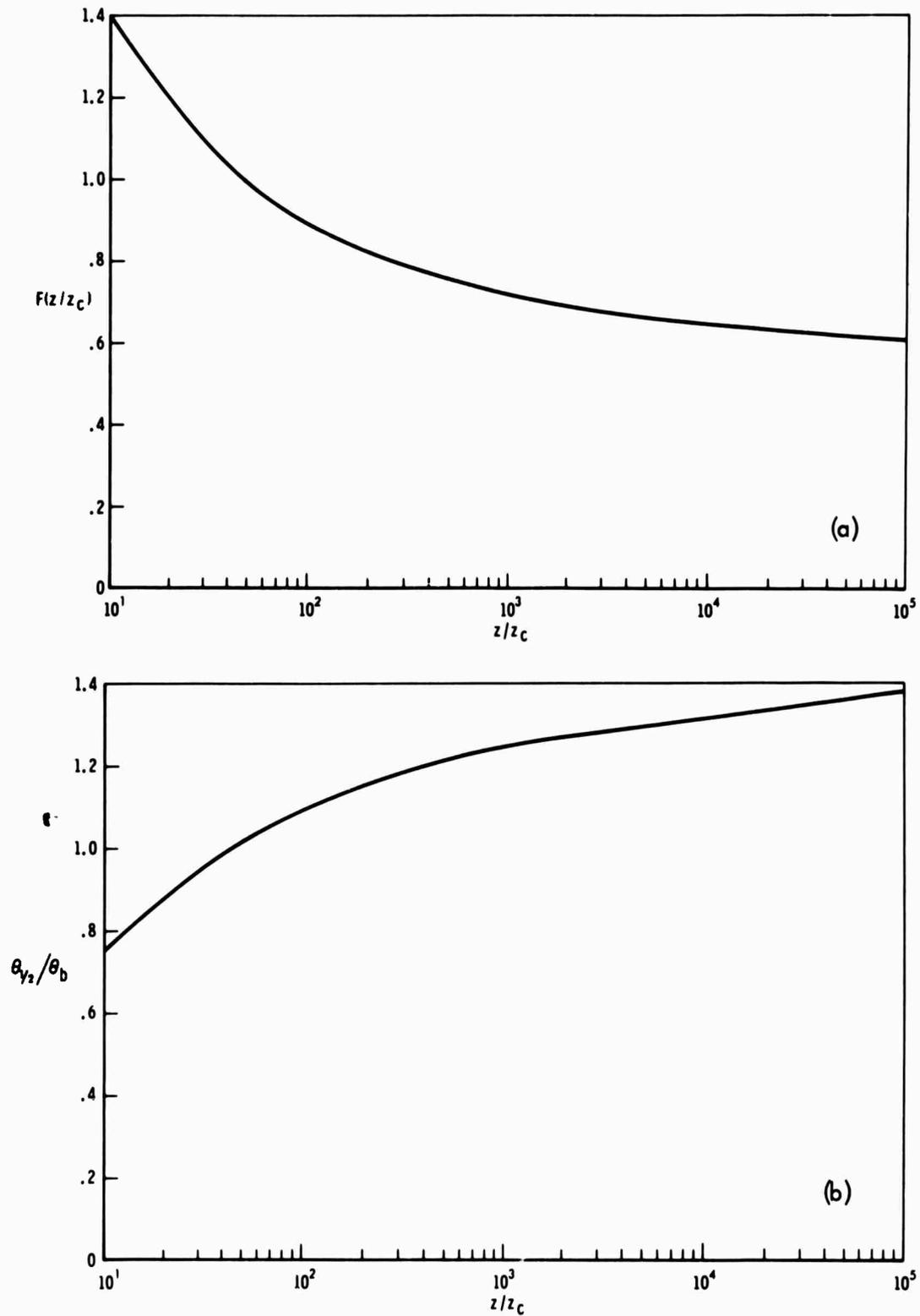


Fig. 8 — The correction to the on-axis intensity, $F(z/z_c)$, and the half-power angle, found by inverting $G(\theta_{1/2}/\theta_D, z/z_c) = 1/2$, as functions of normalized range z/z_c

$$\frac{1}{kD}, \frac{1}{k\lambda_0} \ll \theta_b \ll \frac{1}{k\lambda_0}$$

The scattering in the region where Eq. (34) applies is then dominated by the presence of the largest inhomogeneities and is valid only when θ_b is much greater than the vacuum diffraction angle.

The results of this section are summarized in the table on the following page, where the conditions required for the use of the approximate formulas are indicated.

SUMMARY OF MTFs AND BEAM SPREADS

Range, z	MTF (Spherical Waves)	Intensity Distribution	Half-Power Distance from Axis
$z \ll z_c$	1	Vacuum pattern	$\approx \frac{\pi z}{kD}$
$z \gg z_c, z_D / [1 - 0.71(D/l_o)^{1/3}]$ $z \ll z_i$	$\exp \left\{ -0.55k^2 C_n^2 z \rho^{5/3} [1 - 0.71(\rho/l_o)^{1/3}] \right\}$	Almost gaussian	$\approx 0.7k^{1/5} C_n^{6/5} z^{8/5} \text{ }^a$
$z \gg z_i$ ($D > l_o$)	$\exp \left(-0.55k^2 C_n^2 l_o^{1/3} z \rho^2 \right)$	Gaussian	$1.2C_{n_o}^{-1} l_o^{5/3} / z$

^a See discussion in text for estimate of error.

$$z_c = \left[0.4k^2 C_n^2 l_o^{5/3} \right]^{-1}; z_D = \left[0.4k^2 C_n^2 D^{5/3} \right]^{-1}; z_i = \left[0.5k^2 C_n^2 l_o^{5/3} \right]^{-1}$$

V. GENERAL DISCUSSION AND CONCLUSIONS

From Eq. (18) one can calculate the mean intensity distribution for an arbitrary disturbance in an aperture if the MTF is known. For example, for a focused beam without amplitude perturbations (i.e., $U(r) = e^{-ikr^2/2f}$), the mean intensity is readily shown to be given by Eq. (20), where now in the integrand (only) $\beta = \frac{kd^2}{4} \left| \frac{1}{z} - \frac{1}{f} \right|$ and f is the focal length.

The MTF has previously been shown to determine the limiting resolution obtainable looking along an atmospheric path, as well as the signal-to-noise ratio of a heterodyne receiver.⁽⁸⁾ We would like to emphasize here that in order to predict the beam pattern from an arbitrary wavefront in an aperture for (say) design considerations, it is again the MTF which should be measured rather than specific beam patterns.

To clarify this point, we first observe that Eq. (20) is merely the Fourier-Bessel transform of the quantity $(8/\pi) \beta^2 M_s(D, z) \Gamma_\beta(x)$, which can be inverted to yield

$$D^2 \Gamma_\beta(\rho/D) M_s(\rho, z) = 2\pi \int_0^\infty J_0\left(\frac{k\rho}{z} p\right) \langle I(p, z) \rangle p dp \quad (36)$$

where $\langle I(p, z) \rangle$ is the mean intensity at a distance p from the axis at range z . For an arbitrary disturbance in the aperture, the function $\Gamma_\beta(x)$ given by Eq. (19) would be replaced by the appropriate overlap integral of Eq. (18). Equation (36) thus provides a possible method for determining "part of" the MTF from measurements of the beam pattern.

However, because $\Gamma_\beta(\rho/D) = 0$ for $\rho \geq D$, inverting the intensity distribution can give no information regarding $M_s(\rho, z)$ for spatial

separations larger than the size of the aperture. In particular, in order to determine the distance z_c from a beam pattern measurement, it would be necessary to have an aperture diameter greater than the largest scale of turbulence, L_0 , which might be of the order of meters. Hence, if one can determine the spherical wave MTF for all spatial separations at a given range, one can infer the intensity distribution from an arbitrary aperture distribution at that range, while the reverse is not true unless apertures greater than the coherence length at that range can be constructed. Even if the beam pattern were measured, one would first have to construct $M_g(\rho, z)$ from Eq. (36) from the given measurement to determine the general response.

Finally, although there are many conditions under which the spectrum of Eq. (13) is not expected to apply, the intensity pattern can still be determined from Eq. (15) if the MTF is known. For field applications employing coherent optical devices, measurements of the turbulence spectrum are useful only to the extent to which they yield knowledge of the MTF. It is suggested that MTF measurements be made and correlated, not with detailed spectrum measurements, but with the gross meteorological and topographical measurements of quantities which might be observed to determine the degradation of the higher spatial separations of the MTF, as perhaps wind speed, temperature, and properties of the terrain.

Appendix

PROOF OF RECIPROCITY

Let $[U_1(\underline{r}, \underline{r}_1), U_2(\underline{r}, \underline{r}_2)]$ be the field at point \underline{r} due to a spherical wave source at $(\underline{r}_1, \underline{r}_2)$, respectively. Then

$$(\nabla^2 + k^2 n^2) U_1(\underline{r}, \underline{r}_1) = -4\pi\delta(\underline{r} - \underline{r}_1) \quad (\text{A-1})$$

$$(\nabla^2 + k^2 n^2) U_2(\underline{r}, \underline{r}_2) = -4\pi\delta(\underline{r} - \underline{r}_2) \quad (\text{A-2})$$

Multiplying Eq. (A-1) by $U_2(\underline{r}, \underline{r}_1)$, Eq. (A-2) by $U_1(\underline{r}, \underline{r}_1)$, subtracting, integrating, and using Green's theorem, we have

$$\begin{aligned} \int_S [U_1(\underline{r}, \underline{r}_1) \nabla U_2(\underline{r}, \underline{r}_2) - U_2(\underline{r}, \underline{r}_2) \nabla U_1(\underline{r}, \underline{r}_1)] \cdot d\mathbf{A} \\ = -4\pi [U_2(\underline{r}_1, \underline{r}_2) - U_1(\underline{r}_2, \underline{r}_1)] \end{aligned} \quad (\text{A-3})$$

If the surrounding surface S is taken as a sphere of radius R , then the surface integral can be written

$$\int [U_1 \left(\frac{\partial U_2}{\partial R} - iknU_2 \right) - U_2 \left(\frac{\partial U_1}{\partial R} - iknU_1 \right)] R^2 d\Omega \quad (\text{A-4})$$

If we take the spherical surface outside of the region where $n \neq 1$, then the Sommerfield radiation condition⁽⁹⁾

$$\lim_{R \rightarrow \infty} R \left(\frac{\partial U}{\partial R} - ikU \right) = 0$$

which ensures that at great distances from the source the field represents an outgoing wave, results in the vanishing of the surface term, and establishes the reciprocity theorem.

REFERENCES

1. Born, M., and E. Wolf, *Principles of Optics*, Chapter 8, Pergamon Press, London, 1965.
2. Born, M., *Optik*, Springer-Verlag, Berlin, 1933, p. 149.
3. Tatarski, V. I., *Wave Propagation in a Turbulent Medium*, Chapter 7, McGraw-Hill Book Company, Inc., New York, 1961.
4. Lutomirski, R. F., and H. T. Yura, "Modulation-Transfer Function and Phase-Structure Function of an Optical Wave in a Turbulent Medium," *J. Opt. Soc. Amer.*, Vol. 59, No. 8, August 1969, pp. 999-1000.
5. Morse, P. M., and H. Feshbach, *Methods of Theoretical Physics*, Vol. II, McGraw-Hill Book Company, Inc., New York, 1953, p. 1325.
6. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, London, 1962, p. 393.
7. Yura, H. T., and R. F. Lutomirski, *On the Mutual Coherence Function of an Optical Wave in a Turbulent Atmosphere*, The Rand Corporation (forthcoming).
8. Fried, D. L., "Optical Heterodyne Detection of an Atmospherically Distorted Signal Wave Front," *Proc. IEEE*, Vol. 55, No. 1, January 1967, pp. 57-67.
9. Stratton, J. A., *Electromagnetic Theory*, McGraw-Hill Book Company, Inc., New York, 1941, p. 485.