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RM-6103-SA
DECEMBER 1969

AN INTRODUCTION TO EQUIPMENT COST ESTIMATING

C. A. Batchelder, H. E. Boren, Jr., H. G. Campbell, J. A. Del Rossi and J. P. Large

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OFFICE OF THE ASSISTANT SECRETARY OF DEFENSE
(SYSTEMS ANALYSIS)

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PREFACE

RAND WAS COMMISSIONED by the Office of the Assistant Secretary of Defense (Systems Analysis) to prepare a book on the subject of military equipment cost-estimating procedures. This memorandum deals with fundamentals of cost analysis and constitutes the introductory portion of such a book. In addition to the material presented here, the complete book will deal with uncertainty, methods and techniques for estimating costs of military equipment such as aircraft, and cost models. Emphasis is placed on cost-estimating techniques that are applicable across a broad spectrum of major military equipment. Consequently, it is hoped that this memorandum, which represents a selection of the more general areas covered in the book, will be useful throughout the Department of Defense and the aerospace industry.

SUMMARY

THIS MEMORANDUM is a compilation of topics related to equipment cost estimating. These topics are treated in five separate sections: (1) cost-estimating methods, (2) data collection and adjustment, (3) statistical methods in development of estimating relationships, (4) use of cost-estimating relationships, and (5) the learning curve.

There are three basic methods used for cost estimation--the industrial engineering, analogy, and statistical approaches. The industrial engineering approach respresents an examination of separate segments of work at a low level of detail and a synthesis of the many detailed estimates into a total. The method of analogy is based on direct comparisons with historical information on like components of existing systems. In the statistical approach, as defined in this memorandum, estimating relationships with parametric explanatory variables, such as weight, speed, power, frequency, and thrust, are used to predict cost. This is usually applied at a higher level of detail than the industrial engineering approach.

Of the three approaches to cost estimating, statistical methods are considered to be the most useful for government analysts in a wide range of application, whether the purpose is long-range planning or contract negotiation. Any estimating method, however, is basically a projection from past experience, and to make this projection it is necessary to have a reliable data base. This must include information on the cost, physical and performance characteristics, and on the development and production history of previous hardware programs. In addition, because the data must be comparable to be useful, adjustments must be made for definitional differences, production quantity differences, and yearly price changes.

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In the discussion on statistical methods, a hypothetical example is used to demonstrate the procedures and techniques of this method. First, attention is given to a simple linear regression, with a single explanatory variable. Next, a logarithmic transformation of this relationship is treated. Finally, multiple regressions are performed in various pairwise combinations of three explanatory variables. These multiple regressions are performed for both linear and nonlinear (logarithmic) relationships.

The limitations of estimating relationships stem from two sources: first, the uncertainty inherent in any application of statistics; and, second, the uncertainty that an estimating relationship is applicable to a particular situation. Important considerations that can be easily overlooked during a purely formal statistical analysis include (1) the reasonableness and structural soundness of the estimating relationship, (2) the importance of the analyst's familiarity with the actual hardware, and (3) systematic bias by the analyst. Although the value of statistical estimating relationships should not be discounted (their widespread use and general applicability attest to their worth), caution is recommended in applying these relationships outside the data base from which they were derived.

The last section covers the subject of learning curves, which are used to predict reductions in cost as the number of items produced increases. The learning process prevails in many industries, and its existence has been verified by empirical data. The factors that account for this learning trend are generally attributed to such items as job familiarization, development of more efficient tools, and improvement in overall management. The basis of learning-curve theory is that each time the total quantity of items produced doubles, the cost per item is reduced to a constant percentage of its previous cost. Such a relationship (log-linear) may be expressed in terms of unit cost or cumulative average cost. In practice, the unit cost is most frequently considered to be linear, but there are sufficient exceptions to suggest that the choice must be based on experience.

When learning curves are displayed graphically, the problem arises of how to plot the average cost for a lot or a complete contract, since,

typically, man-hours or costs are not recorded by each unit. For the cumulative average curve, the plot point is simply the endpoint of each lot, since this is the point where the cumulative average figure is applicable. For the unit curve, calculating the plot point is more complex, and approximations are widely used. The plotting of representative unit costs for contract lots is of importance, especially the early points whose misplacements could lead to improper conclusions about the cost-quantity relationship.

In the application of learning curves to problems associated with cost estimating, the analyst must be cognizant of the wide variations possible and the reasons for such variations. A thorough knowledge of the learning-curve phenomenon is indispensable to persons involved in cost analysis.

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I. COST-ESTIMATING METHODS

A COST ESTIMATE is a judgment or opinion regarding the cost of an object, commodity, or service. This judgment or opinion may be arrived at formally or informally by a variety of methods, all of which are based on the assumption that experience is a reliable guide to the future. In some cases the guidance is clear and unequivocal; e.g., bananas cost 15¢ per pound last week; it is estimated that they will cost about 15¢ per pound next week, barring unforeseen circumstances such as a freeze in Guatemala. At a more sophisticated level, average costs are calculated and used as factors to estimate the cost to excavate a cubic yard of earth, to fly an airplane for an hour, or to drive an automobile a mile. Much, perhaps most, estimating is of this general type, i.e., where the relationship between past experience and future application is fairly direct and obvious.

The more interesting problems, however, are those in which the relationship is unclear, because the proposed item differs in some significant way from its predecessors. The challenge to cost analysts concerned with military hardware is to project from the known to the unknown, to use experience on existing equipment to predict the cost of next-generation missiles, aircraft, and space vehicles. The challenge is not only in new equipment designs; new materials, new production processes, and new contracting procedures also add to uncertainty. These innovations are sometimes accompanied by expectations of cost increases or of cost reductions that must be carefully evaluated.

The techniques used for estimating hardware cost range from intuition at one extreme to a detailed application of labor and material

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cost standards at the other. One of the military services' manuals on cost estimating lists five basic methods--industrial engineering standards; rates, factors, and catalog prices; estimating relationships; specific analogies; and expert opinion. Other sources put the number at two (synthesis and analysis), three (round-table estimating, estimating by comparison, and detailed estimating), or four (analytical appraisal, comparative analysis, statistical analysis, and use of standards). In this section, the discussion will be limited to three techniques--the industrial engineering approach, analogy, and the statistical approach--and it is the latter that will be of primary concern throughout the remainder of the memorandum.

Estimating by industrial engineering procedures can be broadly defined as an examination of separate segments of work at a low level of detail and a synthesis of the many detailed estimates into a total. Statistical estimating is sometimes defined as a statistical extrapolation to produce an estimate-at-completion after progress has been made on a job and costs or commitments have been experienced, but this is not the sense in which the term is used in this study. In the statistical approach, estimating relationships that use explanatory variables such as weight, speed, power, frequency, and thrust are relied on to predict cost at a higher level of aggregation. Figure 1 illustrates this difference in level of detail. At the lowest level of detail, the estimator begins with a set of drawings and specifies each engineering task, tool requirement, or production operation, including the labor and material required. This is sometimes referred to as "grass-roots" estimating.

Table 1 illustrates the detail required at the lowest level of estimating; in this case a labor cost estimate for forming a steel center bracket. The name and number of the operations and the machines that will be used are given with estimates of setup and operating time and labor cost. When they exist, standard setup and operating costs are used in making estimates, but if standards have not been established (which is frequently the case in the aerospace industry), a detailed study is made to determine the most efficient method of performing each operation. A standard may be a "pure" standard or an

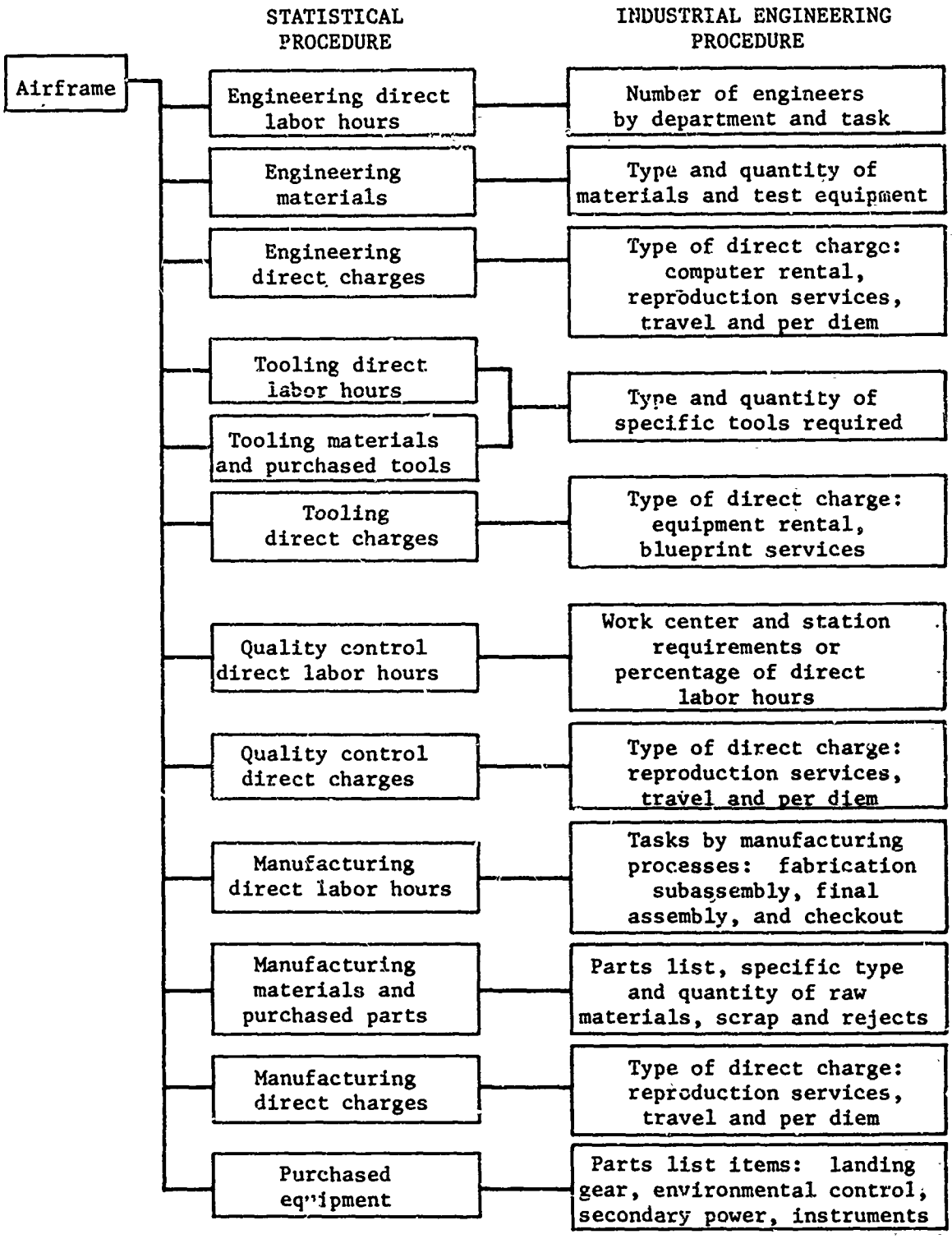

Fig. 1--Levels of aggregation for estimating purposes

Table 1

DETAILED LABOR COST ESTIMATE FOR FORMING A STEEL CENTER BRACKET

Department	Operation Number	Operation	Machine	Setup Charges: Hours	Setup Charges: Rate ($)	Setup Charges: Cost ($)	Operating Labor: Output per Hour	Operating Labor: Rate ($)	Operating Labor: Cost per 1000 ($)
20	16241	Setup	Niagara 462	½	3.40	1.70			
20	16081	Shear to length	Niagara 462				3000	3.40	1.13
20	16242	Setup	No. 4 Bliss press	⅔	3.40	2.27			
20	11571	Perforate and blank	No. 4 Bliss press				1100	3.30	3.00
20	16243	Setup	No. 4 Bliss press	1	3.40	3.40			
20	12951	Form	No. 4 Bliss press				450	3.30	7.26
18	14351	Tap	Tap wheel				900	3.15	3.47
18	16244	Setup	Plain mill	½	3.40	1.70			
18	14661	Mill slots	Plain mill				100	3.40	34.00
18	15541	File burrs	Hand file				95	3.40	35.70
07	16245	Setup	Speed lathe	¼	3.40	.85			
07	11542	Burr 6 slots (and mill)	Speed lathe				600	3.15	5.26
07	16246	Setup	Multiple drill	½	3.40	1.70			
07	11941	Countersink 2 holes	Multiple drill				900	3.15	3.47
07	16247	Setup	Tapping machine	½	3.40	1.70			
07	16561	Tap 2 holes	Tapping machine				400	2.75	6.88
19	15151	Dull nickel plate					1000	3.50	3.50
						13.32			103.67

NOTE: This table is adapted from a detailed labor cost estimate published in L. P. Alford and John R. Bangs (eds.), *Production Handbook*, The Ronald Press Company, New York, 1953, p. 1045.

"attainable" standard, but for a specified condition, it is essentially the minimum time required to complete a given operation and theoretically should be approached asymptotically when the planned production rate is attained.

Standards are not widely used in the aerospace industry for estimating costs, although they are used extensively for other purposes, such as control of shop performance. Standards are best applied when a long, stable production run of identical items is envisaged; in the aerospace industry, however, emphasis is often placed on development rather than on production. The Gemini program provides an extreme example: Twelve spacecraft of varying configurations were developed and produced at a cost of $700 million. Other examples would be less dramatic, but it is true that compared with industry in general, production runs of advanced military and space hardware tend to be short, and both design configurations and production processes may continue to evolve even after several hundred units have been completed. This means that standards are continually changing--one standard applies at unit 50, another at other production quantities. Because changes are unpredictable, it is difficult to establish standards that will be applicable at some specified production quantity in advance of production experience.

Industrial engineering estimating procedures require considerably more personnel and data than are likely to be available to government agencies under any foreseeable conditions. One of the largest aerospace firms judges that the use of this approach in estimating the cost of an airframe requires about 4500 estimates; for this reason, the firm avoids making industrial engineering estimates whenever possible. They take too much time and are costly to both contractor and government during a period of limited funds. Moreover, for many purposes they have been found to be less accurate than estimates made statistically. One reason is simply that the whole often turns out to be greater than the sum of 4500 parts. The detail estimator works under the same disadvantages as do all other estimators before an item has been produced. He works from sketches, blueprints, or word descriptions of some item that has not been completely designed, and he

can assign costs only to work that he knows about. (An attempt is sometimes made to estimate the completeness of the work statement and this estimate becomes a factor to apply to the detail estimate; e.g., if the work statement is judged to be 50 percent complete, the detail estimate is multiplied by two.) The effect of a low estimate is compounded because detail estimating is normally attempted only on a portion of production labor hours. A number of production labor elements, such as rework, planning time, and coordination effort, are usually factored in as percentages of the detail estimate. Then, other cost elements, such as sustaining effort, tool maintenance, quality control, and manufacturing research, are factored in as percentages of production labor. Thus, small errors in the detail estimate can result in large errors in the total.

A second reason for considering industrial engineering standards less accurate than estimates made statistically has already been suggested. Significant variability in the fabrication and assembly of successive production units is, and will continue to be, characteristic of the industry. Production runs of like models tend to be of limited length and are characterized by numerous design changes. In the case of military aircraft, production rates have tended to vary frequently and at times unexpectedly. The proportion of new components in equipment is probably higher in the aerospace industry than in any other. The effect of these factors can be represented statistically by the learning or progress curve so characteristic of this industry. One set of fabrication and assembly modes is succeeded by more efficient production functions, which lower the total labor requirement. The introduction of engineering changes causes discontinuities in this process but does not interfere with the general trend. If new manufacturing processes and techniques are introduced, these may cause changes in past relationships. History, however, seems to show that changes in manufacturing and management techniques, although they may have dramatic impacts in circumscribed areas, tend to result in only gradual changes over the entire process.

Because a private concern generally has information only on its own products, much of the estimating in industry is based on analogy,

particularly when a firm is venturing into a new area. For example, in the 1950s, aircraft companies bidding on ballistic missile programs drew analogies between aircraft and missiles to develop estimates for the latter. Douglas Aircraft Company (now McDonnell-Douglas) made a good estimate on the Thor intermediate range ballistic missile by comparing Thor with the DC-4 transport airplane. This company later based its estimates of the Saturn S-IV stage on its Thor experience. Even with appropriate adjustments for differences in size, the number of engines, higher performance, and insulation problems (the need to cope with liquid hydrogen as well as liquid oxygen), this attempt was not as successful as the first.

At all levels of aggregation, much estimating is performed by this type of analogy: System *A* required 100,000 hours; given the likenesses and differences in design and in performance of proposed System *B*, the requirement for *B* is estimated at, say, 120,000 hours. Or, at a different level, engineers and shop foremen may rely on analogies when making a grass-roots estimate; in this event, analogy becomes part of the industrial engineering approach. The major drawback to estimating by analogy is that it is essentially a judgment process and, as a consequence, requires considerable experience and expertise to be done successfully. For the government cost analyst, analogy can be useful for a rough check of an estimate; however, when making estimates, analogy based on a sample of 1 adjusted by some complexity factor should be avoided. This caveat rests on the contention that first, it is poor statistics; second, it is nonreproducible; and third, it cannot be evaluated by the user of the estimate.

Although statistical procedures are preferable in most situations, there are circumstances when analogy or industrial engineering techniques are required because the data do not provide a systematic historical basis for estimating cost behavior. It may be that a new item is to be constructed of some unfamiliar material, or that a design consideration is so radically different that statistical procedures are inadequate. The use of new structural material for aircraft often requires the development of special cutting and forming techniques with manufacturing labor requirements that differ significantly from

those based on a sample of primarily aluminum airframes. Faced with this problem when titanium was first considered for use in airframe manufacture, airframe companies developed standard-hour values for titanium fabrication on the basis of shop experience in fabricating test parts and sections. Ratios of these values to those for comparable operations on aluminum aircraft were prepared, and these ratios were then used in existing statistical estimating relationships. Thus, while industrial engineering procedures were used to provide input data, the approach remained statistical.

A similar situation occurs in the case of industrial facilities. Requirements for these cannot be estimated without knowing the contractor's identity and the extent and availability of his existing plant. Consequently, the cost of facilities must be estimated from information available for each specific case.

There will always be situations in which analogy or industrial engineering techniques are required, but in general the statistical approach is useful in a wide range of contexts, whether the purpose is long-range planning or contract negotiation. In the former, a more highly aggregated procedure may be used because it ensures comparability when little detailed knowledge about the equipment is available. Total hardware cost may be estimated as a function of one or more explanatory variables; e.g., engine cost as a function of thrust, or transmitter cost as a function of power output and frequency. However, this approach is often a matter of necessity, not choice. Even for long-range planning, it is sometimes desirable to estimate in some detail.

To say that statistical techniques can be used in a variety of situations does not imply that the techniques are the same for all situations. They will vary according to the purpose of the study and the information available. In a conceptual study, it is necessary to have a procedure for estimating the total expected costs of a program, and this must include an allowance for the contingencies and unforeseen changes that seem to be an inherent part of most development and production programs.

Similarly, a long-range planning study will use industry-wide

labor and burden rates and an estimated learning-curve slope; later in the acquisition cycle, data that are specific for a particular contractor in a particular location can be used. In effect, this procedure merely asserts the obvious: As more is known, fewer assumptions are required. When enough is known, and this means when a product is well into production, accounting information and data can be taken directly from records of account and used with a minimum of statistical manipulation. This technique is useful only in those cases when the future product or activity under consideration is essentially the same (both in terms of configuration and scale of production or operation) as that for the past or current period.

In any situation the estimating procedure to be used should be determined by the data available, the purpose of the estimate, and, to an extent, by such other factors as the time available to make an estimate. The essential idea to be conveyed in this section is that, when properly applied, statistical procedures are varied and flexible enough to be useful in most situations that aerospace equipment cost analysts are likely to encounter. Although no specified set of procedures can guarantee accuracy, decisions must be made; it is essential that they be based on the best possible information. The analyst must seek the approaches that will provide the best possible answers, given the basic information that is available.

Although the content of this memorandum is limited to methods of estimating equipment cost, any decision to undertake a new program typically takes into consideration far more than the outlays needed to develop and produce the equipment. For example, there may be a need for complementary hardware, such as launchers or test equipment; possibly additional construction will be needed, such as lengthened runways or hardened shelters. Other investment items may include the cost of personnel training, computer programming services, and development of technical data. However, a number of items that contribute to system operating cost (particularly spares) are usually estimated as a function of total equipment cost.

In addition to the initial investment that is needed to establish a new capability, there are costs of operating and maintaining

equipment that continue as long as it is in the active inventory. These recurring costs include

- Replacement of common (or organizational) equipment.
- Replenishment of spare parts and supplies.
- Fuels, lubricants, and propellants.
- Training ordnance and other expendables.
- Personnel costs.
- Facilities maintenance.
- Training of replacements.
- Maintenance and other logistics support by separate organizations.

These operating costs are far more important in the lifetime total cost computation than their annual figure might suggest. In fact, since the life of a modern weapon system may run ten years (or longer), the investment needed to establish a new system may be dwarfed by the costs required to operate and to maintain it. The practical consequence of this observation is that when the overall study is constrained by time and personnel limitations, as is often the case, the estimation of equipment costs can be accorded only a reasonable share of the time and personnel available for the whole study.

II. DATA COLLECTION AND ADJUSTMENT

THE GOVERNMENT has been collecting cost and program data on weapon and support systems for many years--sometimes in detail, sometimes in highly aggregated form. Consequently, it is surprising that the right data seldom seem to be available when an estimating job is required. It appears that the needs of the cost analyst have not always been considered in designing the many information systems that have been used by the Army, Navy, and Air Force. Data have been collected for program control, for program management, and for program audit, but this information has never been systematically processed and stored. Instead, after a few years it has generally been discarded or placed in not readily accessible warehouses. Moreover, the data were often inconsistent since they were gathered according to the requirements of each military service and each program manager. To obtain the data to develop estimating relationships, the analyst has had to use contractor records.

Data Collection

The Cost Information Report (CIR) was established in 1966 to alleviate the problem of data collection. This reporting system was designed to collect costs and related data on major contracts for aircraft and missile and space programs to assist industry and government in estimating and analyzing the costs of these programs. Information from other sources (contract records, management records, and the like) can be processed to complement the CIR and thus make complete program

histories available. (Subsequent sections of this study describe the methods of analysis that this information was designed to serve.) As data accumulate over a period of years, the need for ad hoc collection efforts should diminish. These efforts will never disappear completely, however, as information systems cannot be designed to satisfy every data requirement. Under ideal conditions, the analyst would have data with which to develop estimating techniques responsive to any demand, but even the largest contractors are reluctant to allocate the resources required to put estimators in such a favorable position, and the cost to the Department of Defense (DOD) for such data--much of which would seldom be used--would be prohibitive. However, a government analyst or estimator has one great advantage over his counterpart in industry: He has a much broader data base to draw on.

A minimum data requirement exists for any given job, but before data collection begins the analyst must consider the scope of his problem, define generally what he wants to do, and decide how to do it. The data required to estimate equipment costs for a long-range planning study can be substantially less than those needed to prepare an independent cost estimate for contract negotiation. In the former, total equipment costs may suffice; in the latter, costs must be collected at the level of detail in which the contract is to be negotiated. For major items, this means a functional breakout, e.g., direct labor, materials, engineering, and tooling. One could postulate problems requiring even a greater amount of detail. Suppose, for example, that two similar hardware items had substantially different costs. Only by examining the cost detail could this difference be explained.

In performing this initial appraisal of the job, the analyst will be aided by a thorough knowledge of the kind of equipment with which he will be dealing--its characteristics, the state of its technology, and the available sample. With this knowledge he can determine the kinds of data that are required and that are available for what he wants to do, where the data are located, and the kinds of adjustments that may be required to make the collected data base consistent and comparable. Only after the problem has been given this general consideration should the task of data collection begin. All too often

large amounts of data are collected with little thought about use. The result is that some portion may be unnecessary, unusable, or not completely understood. Data collection is generally the most troublesome and time-consuming part of cost analysis. Consequently, careful planning in this phase of the overall effort is well worthwhile.

Historical Data

To develop a cost-estimating procedure, at least three different types of historical data are required. First, there are the resource data, usually in the form of expenditures and labor hours. It is customary to apply the word *cost* to both, and that practice is followed throughout this text. A second type of data describes the possible cost-explanatory elements; for hardware such as aircraft and missiles this means performance and physical characteristics. The third type is program data, i.e., information related to the development and production history of past hardware programs.

Resource Data

Resource data are generally classified under end-item categories or functional categories. An example of the former in various possible levels of detail are system, subsystem, component, and part. The functional cost categories, such as engineering, tooling, manufacturing, quality control, purchased equipment, are usually broken down into cost elements--labor, material, overhead, and other direct charges. The data source is the contractor's plant. Generally, the accounting systems will vary from one company to another, and the amount of detail is immense. A typical airframe company, for example, sets up the production process on the basis of a number of different jobs or stations, each identified by a number or symbol. All manufacturing direct labor and material (depending on the type of cost-accounting system) expended on a given job is recorded on a job order or, as is becoming increasingly more common, fed directly into a computer. When such a system is used, the actual hours incurred for every operation are available to management; and these costs can be aggregated as they are needed.

Manufacturing costs of this type can be attributed to a lot or often to a single unit. (Some categories of cost are not identifiable by lot or unit, e.g., tooling and engineering.) But since contractors organize their work differently, different job orders will be used. This means that data at more detailed levels may vary from contractor to contractor and may not be comparable. Also, detailed information of this kind is unnecessary for most government analysis and should rarely be sought.

If there were a need to estimate in more detail, the data required would increase by at least an order of magnitude, and data processing equipment would become a necessity. When to incorporate automatic data processing techniques into the data collection effort is determined primarily by the volume of data to be handled. The trend in the aerospace industry is to rely more and more on computers for internal data needs, and for some purposes data have been provided to the government on punched cards or magnetic tape. Thus, there are no technical reasons why cost data could not be obtained in this form should it be more convenient to the cost analyst but, as mentioned earlier, there are good reasons not to use excessive detail even if it is readily available: Expense increases and accuracy is unlikely to improve.

Theoretical considerations aside, estimating techniques must be based on whatever resource data the analyst can find, and in the past the availability of data has varied from one kind of equipment to another. To illustrate, aircraft airframe estimating procedures tend to be different from those developed for other types of equipment. An airframe model may contain all of the following categories:

- Initial and sustaining engineering.
- Flight test operations.
- Initial and sustaining tooling.
- Manufacturing labor.
- Manufacturing material.
- Quality control.

Such a list of cost categories is desirable for all hardware estimating, but because of data limitations, present procedures for engines often cover only two phases of the procurement cost, development and

production, and avionics procedures only one, procurement cost to the government. The CIR should expand these possibilities in the future.

Physical and Performance Characteristics

Information about the physical and performance characteristics of aircraft and missile and space systems is just as important as resource data. Data collection in this area can be time-consuming, particularly since it is not often clear in advance what data will be required. The goal, of course, is to obtain a list of those characteristics that best explain differences in cost. Weight is a commonly used explanatory variable, but weight alone is seldom enough; speed is almost always included as a second explanatory variable for aircraft airframes. One estimating procedure for aircraft uses all of the following:*

- Maximum speed at optimal altitude.
- Maximum speed at sea level.
- Year of first delivery.
- Total airframe weight.
- Increase in airframe weight from unit 1 to unit n.
- Weight of installed equipment.
- Engine weight.
- Electronics complexity factor.

In addition, the following characteristics were considered for inclusion as part of the estimating procedure, although they were not used:

- Maximum rate of climb.
- Maximum wing loading.
- Empty weight.
- Maximum altitude.
- Design load factor.
- Maximum range.
- Maximum payload.

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*Planning Research Corporation, *Methods of Estimating Fixed-wing Airframe Costs*, Vol. I (Revised), PRC R547A, Los Angeles, April 1967.

At the outset of a study undertaken to develop an estimating relationship for aircraft cost, the cost analyst would not know which of all these characteristics would provide the best explanation of variations among the cost of different aircraft; he would of necessity try to be as comprehensive as possible. An analyst who is familiar with the type of hardware under study will have some idea of the most likely candidates, but he will generally consider more characteristics than will eventually be used.

Program Data

A third type of essential data is drawn from the development and production history of hardware items. The acceptance date of the item, the significant milestones in the development program, the production rates, and the occurrence of major and minor modifications in production--all such information can contribute to the development of cost-estimating relationships. The list of explanatory variables discussed in the previous section includes year of first delivery and increase in airframe weight from unit 1 to unit n, information that would be included in the category *program data*.

An airframe typically changes in weight during both development and production as a result of engineering changes. For example, the weight of the F-4D varied as follows:

Cumulative Plane Number	*Airframe Unit Weight (lb)*
1- 11	8456
12-186	8941
187-241	8541
242-419	9193

Since labor hours are commonly associated with weight to obtain hours-per-pound factors, it is important to obtain weights applicable to each production lot if airframe weights by unit are not available.

The need for other kinds of program data will be clarified under the discussion on data adjustment. To cite one example here, the year in which expenditures occur must be known to adjust cost data for price

level changes. (This is the reason for at least one CIR submission annually.) A certain amount of program data cannot be specified with this degree of precision nor can the use of these data be foretold, but the information is important nonetheless. It is what might be called background information--data on other activities in the contractor's plant at the time a particular hardware item is being built; unusual problems the contractor may be encountering; attempts to compress or stretch out the program; and inefficiencies that are noted. This information may be useful in explaining those factors that appear to be aberrations when the resource data are compared with those from other development and production programs. In addition, a history of a contractor's overhead, general and administrative costs, and labor rates is useful for analyzing and predicting costs.

Data Adjustment

To be useful to the cost analyst, data must be consistent and comparable, and in most cases the data as collected are neither. Hence, before estimating procedures can be derived, an adjustment must be made for definitional differences, production quantity differences, yearly price changes, and so on. The more common adjustments are examined in this section. It is by no means an exhaustive treatment of the subject: The list of possible adjustments is long and many of them will apply only in a very small number of cases. Also, evidence on certain types of adjustments (for contractor efficiency, for contract type, for program stretch-out) consists largely of opinion rather than hard data. While the cost analyst may allude to such adjustments, the research necessary to treat them in some definitive way has not yet been done.

Definitional Differences

Different contractor accounting practices and make or buy arrangements are primary reasons why adjustment of the basic cost data is generally necessary. Companies record their costs in different ways. Often they are required to report costs to the government by categories

that differ from those used internally. Also, government reporting categories change from time to time. Because of these definitional differences, one of the first steps in cost analysis is to state the definition that is being used and to adjust all data to this definition. With the inception of the CIR, a standard set of definitions for airframes has been established for use throughout the DOD. A primary purpose of the CIR is to overcome the problem of definitional differences in hardware cost data. For the next few years, however, most data will antedate the CIR and some adjustment will be required.

As an example of what may be expected, a cost analyst may be examining data from a sample of ten hardware items and discover that the cost category Quality Control is missing for some of the earlier items. He may conclude that no quality control was exercised in the 1950s or that this function is included in another cost element. The latter assumption is correct. Traditionally, Quality Control was carried in the burden account, and it was only in the late 1950s that it began to appear (at the request of the DOD) as a separate element. Hence, to use cost data on equipment built prior to this change requires converting a portion of overhead cost to Quality Control.

A more current example involves Planning, which in the CIR definition is included in Tooling. Planning consists of two components--tool planning and production planning. A company may put the first in Tooling and the second in Manufacturing. Other practices are to include tool planning in Engineering, to put all planning in Manufacturing, or to include a portion in Overhead.

Table 1 illustrates this problem more concretely. A slightly abbreviated version of the CIR list of cost elements appears on the left; on the right, the cost elements used by a large aerospace company and the nonrecurring costs of a proposed airframe. The lists are different and, as shown by Table 2, a simple rearrangement of the contractor cost elements does not solve the adjustment problem. Four of the contractor cost elements remain: Developmental Material ($2.6 million), Outside Production ($70 thousand), Other Direct Charges ($2.7 million), and Manufacturing Overhead ($28.94 million). These are not trivial adjustments: These four elements can amount to well over half the total

cost of a large production contract. Developmental Material presumably would be split between Engineering Material and Manufacturing Material; Other Direct Charges would have to be allocated among Engineering, Tooling, Quality Control, and Manufacturing; and part of Manufacturing Overhead would be apportioned to Tooling Overhead and Quality Control Overhead. In each of these instances, the contractor who furnished the CIR information would be able to make the necessary adjustments from his own

Table 1

ILLUSTRATIVE COMPARISON OF CIR AND AIRFRAME CONTRACTOR COST ELEMENTS

CIR Cost Element	*Airframe Contractor* Cost Element	*Nonrecurring costs ($ thousands)*
Engineering	Engineering	8,600
Direct labor		
Overhead	Manufacturing	
Material	Developmental direct labor	2,500
Other direct charges	Tooling direct labor	11,600
Tooling	Production direct labor	850
Direct labor	Developmental material	2,600
Overhead	Tooling material	2,600
Materials and purchased tools	Production material	500
Other direct charges	Purchased equipment	5
	Outside production	70
Quality Control		
Direct labor		
Overhead	Inspection	620
Other direct charges	Other Direct Charges	2,700
Manufacturing	Overhead	
Direct labor	Engineering	10,200
Overhead	Manufacturing	28,940
Materials and purchased parts		
Other direct charges		
Purchased Equipment		
Material Overhead		

accounting records. Outside Production costs, although small in this example, may constitute 30 to 40 percent of the total cost of an airframe in some cases. When this happens, the labor hours and material costs incurred by the prime contractor fall far short of the total required to build an airplane; a method of arriving at a total must be

Table 2

AIRFRAME CONTRACTOR COST ELEMENTS ARRANGED IN CIR FORMAT

CIR Cost Element	*Airframe Contractor Cost Element*	*Nonrecurring Costs ($ thousands)*
Engineering		
Direct labor	Engineering	8,600
Overhead	Engineering overhead	10,200
Material		--
Other direct charges		--
Tooling		
Direct labor	Tooling direct labor	11,600
Overhead		--
Materials and purchased tools	Tooling material	2,600
Other direct charges		--
Quality control		
Direct labor	Inspection	620
Overhead		--
Other direct charges		--
Manufacturing		
Direct labor	Developmental direct labor	2,500
	Production direct labor	850
Overhead		--
Materials and purchased parts	Production material	500
Other direct charges		--
Purchased equipment	Purchased equipment	5
Material overhead		--

devised to permit the data to be analyzed on a comparable basis, i.e., on an equivalent 100-percent inplant basis. Ordinarily, the contractor would have a detailed breakout of costs only for subcontractors on cost-reimbursable contracts, and other Outside Production costs would have to be allocated to the specified categories. Production labor hours incurred out of plant, for example, are often estimated on the basis of the weight of that portion of the airframe being built out of plant. In using historical data, the analyst may be in a similar position: When the amounts involved are large, he should be guided by whatever information the contractor can provide.

Physical and Performance Considerations

A problem that resembles the one discussed above is the need for consistency in definitions of physical and performance characteristics. For example, *speed* can be defined in many ways--maximum speed at optimal altitude, true speed, equivalent speed, indicated speed. All of these defining terms differ in exact meaning and value. The weight of an aircraft or missile depends on what is included. Gross weight, empty weight, and airframe unit weight apply to aircraft, but each of these terms also differs in exact meaning and value. Some agencies include sweep volume in their definition of the physical volume of an aircraft fire control system; others exclude it. Differences such as these can lead an analyst unfamiliar with the equipment to use inconsistent or varying values inadvertently. When data are being collected from a variety of sources, an understanding of the terms used to describe physical and performance characteristics is at least as important as an understanding of the content of the various cost elements.

Nonrecurring and Recurring Costs

Another problem that involves questions of definition concerns nonrecurring and recurring costs. Recurring costs are a function of the number of items produced; nonrecurring costs are not. Thus, for estimating purposes it is useful to distinguish between the two, and the CIR provides for this distinction. Unfortunately, historical cost

data frequently show such cost elements as nonrecurring and recurring engineering hours as an accumulated item in the initial contract. Various analytical techniques have been developed for dividing the total into its two components synthetically, but it is not clear at this time whether the nonrecurring costs that are obtained by ex post facto methods will be comparable with those reported in the CIR. The CIR instructions state:

> it is preferable to identify the point of segregation between nonrecurring and recurring engineering costs as a specific event or point in time. Ideally, the event used would be the point at which "design freeze" takes place as a result of a formal test or inspection, and after which formal Engineering Change Proposal (ECP) procedures must be followed to change design. If no reasonable event can be specified for this purpose, then all engineering costs incurred up to the date of 90 percent engineering drawing release may be used.*

Although it would be premature to consider the kinds of adjustments needed before a body of CIR data exists, splicing historical data to CIR data may also involve adjustments.

A more subtle problem arises when nonrecurring costs on one product are combined with recurring costs on another, i.e., when the contractor is allowed to fund development work on new products by charging it off as an operating expense against current production. This practice is especially prevalent in the aircraft engine industry. Separation of the nonrecurring and recurring costs means an adjustment of the production costs shown in contract or audit documents to exclude any amortization of development. The nonrecurring expense that has been amortized can then be attributed to the item for which it was incurred. Such an adjustment can only be accomplished in cooperation with the accounting department of the companies that are involved. It would not be necessary, of course, for equipment on which CIR data are available.

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*U.S. Department of Defense, *Cost Information Report (CIR) for Aircraft, Missile, and Space Systems*, Budget Bureau No. 22-R260, Washington, D.C., April 21, 1966, p. 43.

Price-level Changes

Figure 1 shows the change in average hourly earnings of production workers on manufacturing payrolls from 1920 to 1965. Although these earnings declined slightly during the early 1920s and again during the Depression, the trend has been steadily upward since 1934. The hourly wage rate has increased by a factor of 4.75 over a 45-year period; in other words, a manufacturer paid $4.75 for labor in 1965 that would have cost him $1.00 in 1920. The implication for equipment cost is clear. If the labor component of an automobile cost $500 in 1920, the cost for the same car today would be something over $2000; however, the hours required in 1965 would be less because of increased productivity.

The relevance of these observations to the subject of data adjustment is that the manufacturing date of the different hardware items in a sample are normally spread over a period perhaps as long as ten to fifteen years. To compare a missile built in 1955 when labor cost about $2.35 per hour with a missile built ten years later when the labor rate

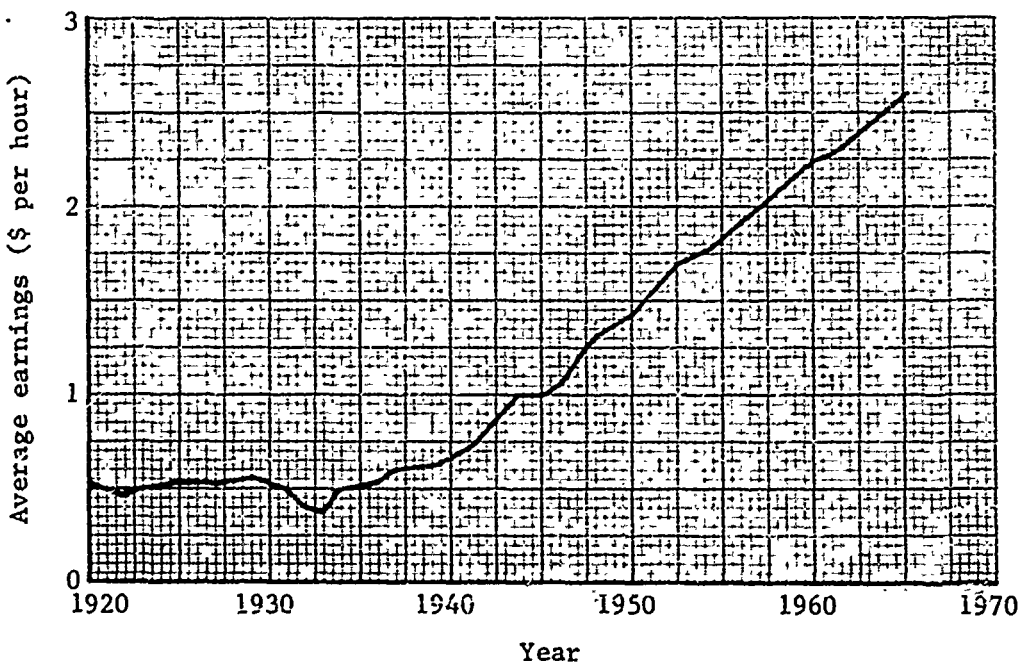

Fig. 1--Change in hourly earnings

had increased to over $3.35 per hour, requires that the labor cost of both be adjusted to a common base. (This problem is obviated by dealing in hours rather than dollars, but an adjustment would still be needed for raw material and purchased parts.) Adjustments are made by means of a price index constructed from a time-series of data in which one year is selected as the base and the value for that year expressed as 100. The other years are then expressed as percentages of this base. The hourly earnings from 1950 to 1960 for production workers could be converted to an index using any of the years as the base; in Table 3, 1950 and 1960 have both been used as base years.

Table 3

AVERAGE HOURLY EARNINGS INDEX

Year	*Average Hourly Earnings ($)*	*Index with 1950 as Base Year*	*Index with 1960 as Base Year*
1950	1.44	100	64
1951	1.56	108	69
1952	1.65	115	73
1953	1.74	121	77
1954	1.78	124	79
1955	1.86	129	82
1956	1.95	135	86
1957	2.05	142	91
1958	2.11	147	93
1959	2.19	152	97
1960	2.26	157	100

SOURCE: U.S. Department of Labor, *Employment and Earnings Statistics for the United States, 1909-66*, Bulletin No. 11312-4, Washington, D.C., October 1966.

The information needed to construct a labor index is available in the Bureau of Labor Statistics (BLS) monthly publication *Employment and Earnings*, and Table 4 presents indexes based on this source. Changes in materials costs are available in another BLS monthly publication, *Wholesale Prices and Price Indexes*. These indexes can be used to develop a materials price index for a given type of equipment by selecting from

Table 4

LABOR PRICE INDEXES

Year	*Aircraft*	*Aircraft Engines and Engine Parts*	*Other Aircraft Parts and Equipment*	*Motor Vehicles and Equipment*	*Electrical Equipment and Supplies*	*Ship and Boat Building*
1952	.59	.61	na[a]	.61	.64	.62
1953	.63	.63	na	.64	.67	.67
1954	.66	.65	na	.66	.69	.68
1955	.69	.67	na	.69	.71	.70
1956	.72	.71	na	.70	.76	.74
1957	.75	.74	na	.74	.79	.79
1958	.80	.79	.79	.76	.82	.82
1959	.84	.83	.83	.81	.85	.85
1960	.86	.86	.86	.84	.88	.88
1961	.88	.89	.88	.86	.91	.93
1962	.91	.92	.91	.90	.93	.95
1963	.94	.94	.94	.93	.95	.99
1964	.95	.97	.97	.96	.97	1.00
1965	1.00	1.00	1.00	1.00	1.00	1.00

[a]Not available for years prior to 1958. For the years 1952-1957, the labor price index for aircraft should be used.

tl ... dity groups in the Wholesale Price Index a list of materials repre... tative of those used in constructing the equipment; these materials are then weighted according to estimates of the value of each in fabricating the equipment. A composite aircraft raw-materials index might be based on the following materials and weights:

Finished steel	.02
Stainless steel sheet	.04
Titanium sponge	.07
Aluminum sheet	.29
Aluminum rod	.11
Aluminum extrusions	.20
Wire and cable	.12
Rivets, nuts, bolts	.15

For any given year a price index for each of these is obtained and a composite index constructed by summing the individual index numbers multiplied by the weightings as shown in Table 5. Weights in an index

Table 5

AIRCRAFT RAW-MATERIALS INDEX

Commodity	*1967 Index Number*[a]	*Weight*	*Index Number × Weight*
Finished steel	105.8	.02	2.12
Stainless steel sheet	108.0	.04	4.32
Titanium sponge	60.3	.07	4.22
Aluminum sheet	99.8	.29	28.94
Aluminum rod	110.4	.11	12.14
Aluminum extrusions	75.6	.20	15.12
Wire and cable	126.0	.12	15.12
Rivets, nuts, bolts	133.2	.15	19.98
Composite index number			101.96

[a]1957-1959 = 100.

need to be updated from time to time to reflect changing technology; it may be that those shown in Table 5 are applicable only to current aircraft. Table 5 merely illustrates the principle of deriving a composite index; the reader who wishes to pursue the matter will find index numbers discussed in textbooks on economic statistics.* Another type of composite index is used in those instances in which labor and material costs cannot be separated and the price-level adjustment has to be made to the total cost of an engine, airframe, or missile. Such an index can be derived in the manner illustrated in Table 4 with the labor and material elements weighted according to the pattern that has been found to exist in the past (e.g., labor, 80 percent; materials, 20 percent). Overhead, which is a mixture of indirect labor, materials, and items such as rent, utilities, taxes, and fringe benefits, is adjusted in most cases by the same percentage as direct labor. To decide whether a different adjustment factor should be used, it would be necessary to examine each of these components.

*See, for example, W. A. Spurr, L. S. Kellogg, and J. H. Smith, *Business and Economic Statistics,* rev. ed., Richard D. Irwin, Inc., Homewood, Illinois, 1961. It is important to recognize the differences in indexes that may result from weighting by base year or a given year, i.e., Laspeyres' or Paasche's index. These are also discussed in textbooks on economic statistics.

The adjustment of costs for yearly price changes is not always as straightforward as the foregoing discussion may imply. One problem is that price indexes are inherently inexact and their use, while necessary, can introduce errors into the data. The average hourly earnings for all aircraft production workers may increase by $.05 in a given year, but at any particular company they will increase more or less than that amount. Use of the average number to adjust the data for a given company may bias the data up or down. Also, for many specialized items of equipment, a good published price index does not exist. In fact, the usual indexes are oriented toward the civilian economy and may be misleading, i.e., they may understate the change experienced in defense and space industries. The United States, with many other countries, furnishes the Office of Economic Cooperation and Development in Paris with an index applicable to government defense expenditures in general. This index, shown in Table 6 for 1952-1964, is a useful reference when detailed index numbers seem questionable or are nonexistent.

Table 6

DEFENSE EXPENDITURES INDEX, 1952-1964

Year	*Index Number*	*Year*	*Index Number*
1952	84	1959	102
1953	83	1960	104
1954	84	1961	105
1955	88	1962	106
1956	93	1963	108
1957	97	1964	113
1958	100		

Another problem is that of identifying the years in which expenditures occur when the only data available show total contract cost. Production and cash flow may have been spread out over a period of several years, and in principle the costs should be adjusted for each year separately. Although the CIR will provide the information needed to do this in the future, this information may be unavailable today and some reasonable approximation of the expenditure pattern must suffice.

One method of obtaining this approximation is to use a percent-of-cost versus percent-of-time curve of the type illustrated in Fig. 2. These curves are developed from historical data on a number of programs involving the same kind of hardware--large ballistic missiles in this case--and can be used to break total research and development or total production cost into annual expenditures. For example, to determine the annual expenditures in a five-year R&D program amounting to a total of $50 million the following percentages would be obtained from the R&D curve of Fig. 2:

Time	*Expenditures*
20	5.5
40	23.0
60	65.0
80	92.0
100	100.0

These percentages are cumulative, of course, so the annual percentages and the amount they represent would be:

	Expenditures	
Year	*Percent*	*$ Millions*
1	5.5	2.75
2	17.5	8.75
3	42.0	21.00
4	27.0	13.50
5	8.0	4.00

In the production phase, a technique that can be used is to develop lag factors by examining delivery schedules and production lead times. Costs are then lagged behind delivery dates by some reasonable factor.

A more fundamental question than any of those raised above is whether yearly price changes should be made at all. It is sometimes argued that the upward trend in wage rates has been accompanied by a parallel trend in the output per employee or productivity rate. This argument implies that there has been little change in the real costs of aerospace equipment because increases in wages and materials cost have been offset by a decrease in the number of employees required per dollar of output. However, the real dollar output per man is difficult to measure in an industry in which continual change rather than

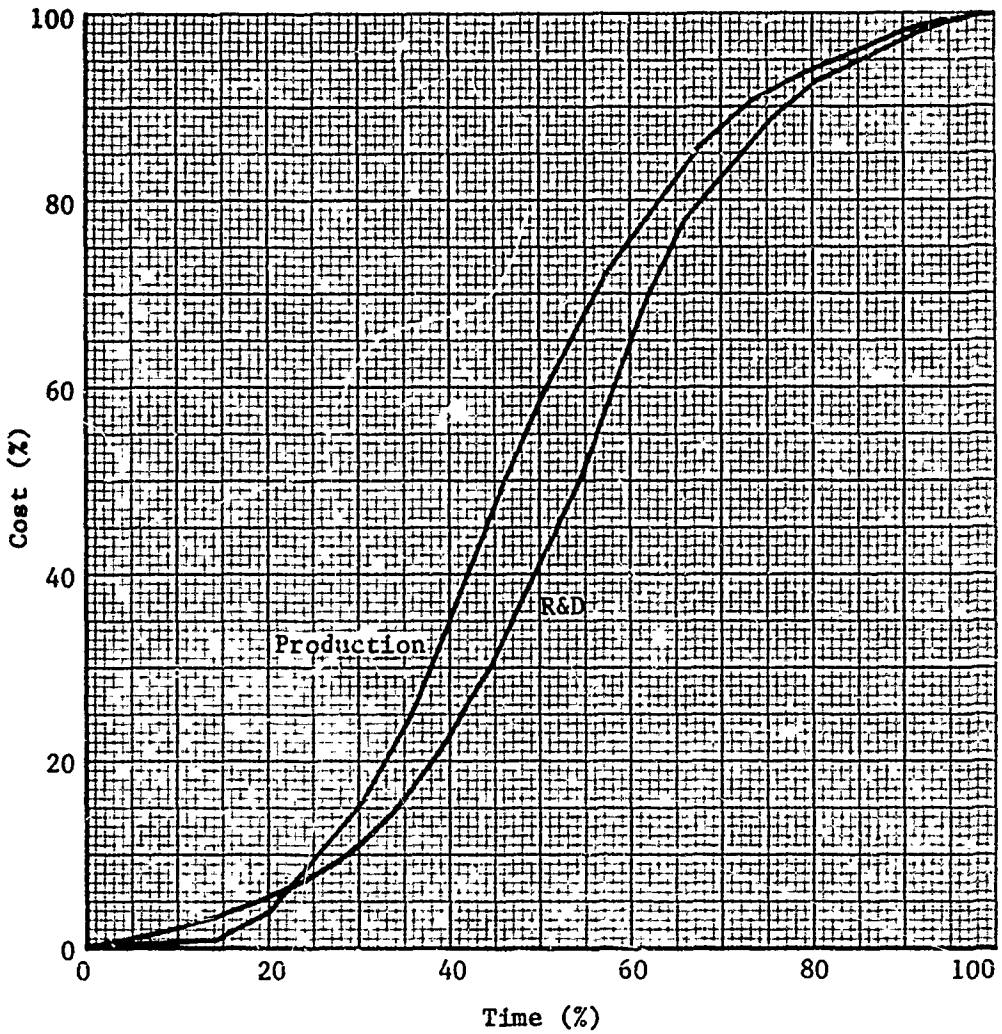

Fig. 2--Percent-of-cost versus percent-of-time curves

standardization is the rule. Certainly the growth in productivity is not uniform for aircraft, missiles, ships, and tanks, and to develop a productivity index for each would be a difficult and contentious task. Present practice, therefore, is to apply the price-level adjustment factors to obtain constant dollars and, at the same time, to remain alert to inequities that may be introduced by following this procedure. As an illustration of the significance of price-level adjustments, Fig. 3 shows the effect of adjusting production costs incurred over the period 1959-1965 (open circles) to 1962 dollars (closed circles). Both

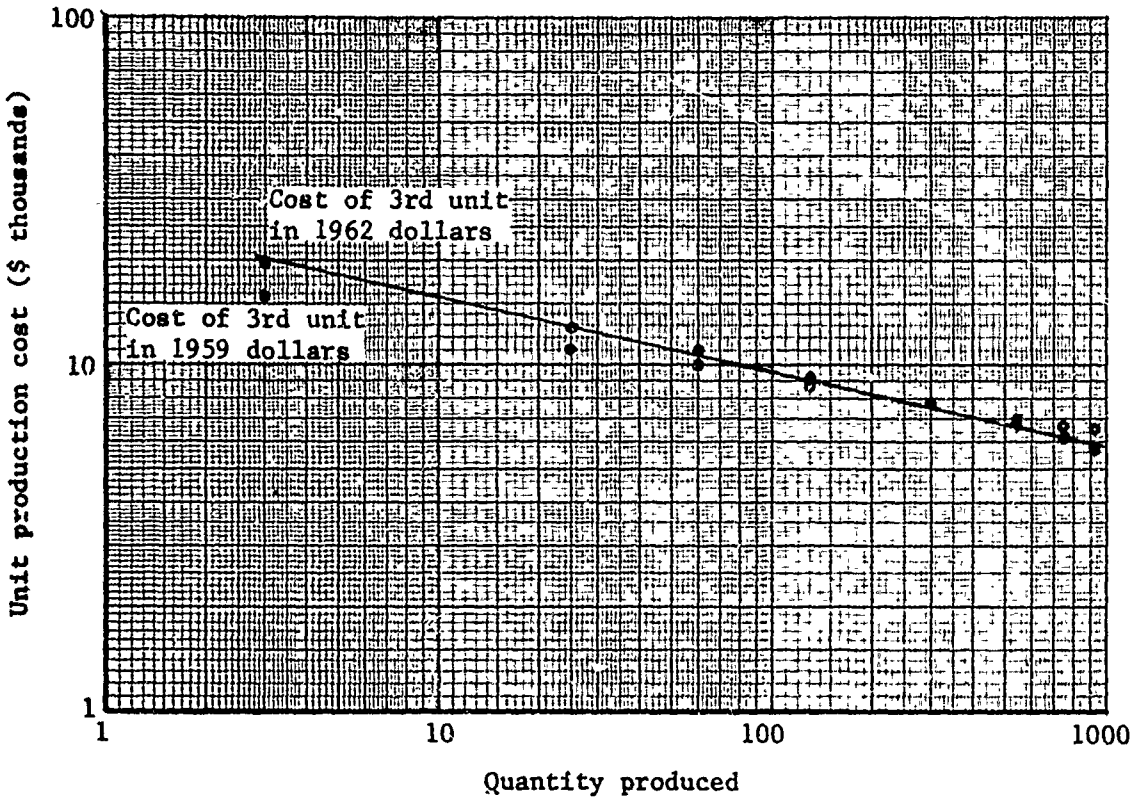

Fig. 3--Effect of adjustment for price-level changes

the level of cost and the slope of the curve change as a result of the price-level adjustment. (In this example a crossover occurs because the year 1962 has been selected as a base for adjustment.)

Cost-quantity Adjustments

The cost-quantity relationship, discussed at length in Sec. V, is usually known in the aerospace industry as the learning curve. The cost-quantity relationship may be defined in brief as follows: Each time that the total quantity of items produced doubles, the cost per item is reduced to some constant percentage of its previous value. Whether or not this particular formulation is accepted, the fact remains that, for most production processes, costs are invariably a function of quantity: As the number of items produced increases, cost normally decreases. Thus, in speaking of cost, it is essential that a given quantity be associated with that cost. An equipment item can be

said to cost $100,000, $80,000, $64,000, or $51,200, and all of these numbers will be correct.

Which cost should be used by the cost analyst? The answer will depend on a number of factors; if his purpose is to compare one missile with another, the cumulative quantity must be the same for both missiles. The adjustment to a specific quantity is a simple matter if the slope of the learning curve is known or if it can be inferred from the data. Take, for example, the costs for three missiles:

Missile	*Unit Number*	*Cost/Unit ($)*
A	50	1000
B	100	1000
C	200	1000

Although the cost is the same for each, the number of units is different. Thus, for a cost comparison, the units must be adjusted to a common quantity. If 100 is chosen and an 80-percent learning curve assumed for all three missiles, the adjusted costs will be as follows:

Missile	*Unit Number*	*Cost/Unit ($)*
A	100	800
B	100	1000
C	100	1250

To project labor requirements for the 100th unit when only 50 units have been produced is somewhat uncertain, but to ignore the cost-quantity relationship will in most instances result in greater error than such a projection introduces. (The learning curve is most frequently depicted as a straight line on logarithmic scales as shown in Fig. 3.)

Other Possible Cost Adjustments

The lack of a way to adjust cost data for productivity changes over time is illustrative of the current situation in which more kinds of cost adjustments have been theorized than have been quantified. For example, it has been suggested that adjustment may be required because of differences in contract type (fixed-price, fixed-price-incentive, cost-plus-fixed-fee contracts) or differences in the type of procurement (competitive bidding or sole source). The hypothesis is

that the type of contract or procurement procedure will bias costs up or down, but this hypothesis is difficult to substantiate.

Another question concerns manufacturing techniques. What are the effects of varying amounts of capital investment or capital improvement and of changes in manufacturing state of the art? A related question concerns the efficiency of the contractor. It may be surmised that Contractor *A* has been a lower cost producer than Contractor *B* on similar items, but this is extremely difficult to prove. A low-cost producer may be one who, because of his geographical location, pays lower labor rates. Contractors in Fort Worth, Texas, and in Atlanta, Georgia, may have a considerable advantage in this regard over their competitors in Los Angeles and San Francisco, California, and in Seattle, Washington. Table 7 does not give a fair picture of comparative rates because differences among industries in the various cities tend to be more important than differences in location. But, for two cities as close together as Los Angeles and San Francisco, labor rates differ by 10 percent. Thus, although it might not be possible to adjust cost data on the basis of contractor efficiency, adjustments can be made for differences in location by using the specific area labor rates.

Table 7

AVERAGE HOURLY EARNINGS OF PRODUCTION WORKERS ON MANUFACTURING PAYROLLS, NOVEMBER 1965
(in dollars)

Atlanta	2.69
Boston	2.69
Chicago	2.91
Detroit	3.45
Los Angeles	3.04
New Orleans	2.72
New York	2.63
Philadelphia	2.79
St. Louis	2.96
San Francisco	3.35
Seattle	3.25

SOURCE: U.S. Department of Labor, Bureau of Labor Statistics, *Employment and Earnings*, Washington, D.C., January 1966.

III. STATISTICAL METHODS IN DEVELOPMENT OF ESTIMATING RELATIONSHIPS

MANY ESTIMATING RELATIONSHIPS are simple statements that indicate that the cost of a commodity is directly proportional to the weight, area, volume, or other physical characteristic of that commodity. These estimating relationships are simple averages; they are useful in a variety of situations and, because of their simplicity, they require little explanation. In this section, the statistical considerations involved in developing cost-estimating relationships for advanced equipment are examined. The emphasis is on the derivation of more complex relationships, i.e., equations that are able to reflect the influence on cost of more than one variable. The intent is to illustrate a general approach to the development of such relationships and to introduce basic concepts of statistical analysis. The emphasis is not on statistics per se; the basic statistical theory as well as the computational aspects involved in developing these relationships are included only to clarify practical considerations. Statistical analysis can help provide an understanding of factors that influence cost, but estimating relationships are no substitute for understanding; regression analysis, which will be discussed in this study, does not offer a quick and easy solution to all the problems of estimating cost.

The outstanding characteristic of a cost factor is that the relationship between cost and the explanatory variable is direct and obvious; thus, cost per pound is widely used because of the generally satisfying thesis that as a ship, tank, or airplane increases in weight it becomes more costly. Weight changes alone do not always adequately

explain cost changes, however, and additional explanatory variables are often needed. The problem is to find these variables and their relationship to cost. The procedure is to decide what variables are logically or theoretically related to cost and then to look for patterns in the data that suggest a relationship between cost and the variables. Table 1 contains a set of data on cost and selected variables that can be analyzed for such patterns. The costs of ten airborne radio communication sets are given with the weight, power output, and frequency of each. It is to be expected that cost would increase with weight or with power output. Frequency is also included because in the past higher and higher frequencies have been sought to increase communication capacity and, for a given power output, higher frequency sets have been more costly.

A graphic analysis of the data in Table 1 shows that cost is not a simple linear function of any of the three explanatory variables. Cost tends to increase with weight, but there are notable exceptions to the trend, as illustrated by the scatter diagram of Fig. 1. Cost plotted against power output as shown in Fig 2 is even less promising, partly because the arithmetic scale does not enable an observer to distinguish among the points between .5 and 30 watts. The change from an arithmetic to a logarithmic scale shown in Fig. 3 spreads the points in the low-power range and indicates that a trend may exist, but with a very wide scatter.

Table 1

TEN AIRBORNE RADIO COMMUNICATION SETS

Cost ($)	*Weight (lb)*	*Power Output (w)*	*Frequency (MHz)*
22,200	90	20	400
17,300	161	400	30
11,800	40	30	400
9,600	108	10	400
8,800	82	10	400
7,600	135	100	25
6,800	59	6	400
3,200	68	8	156
1,700	25	8	42
1,600	24	0.5	258

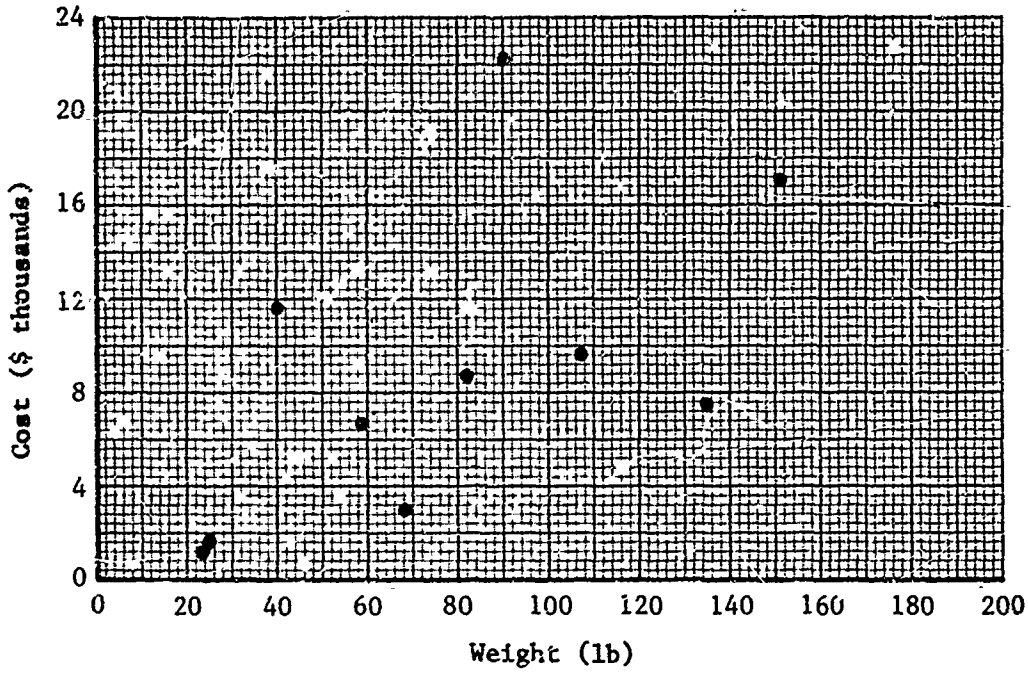

Fig. 1--Scatter diagram of cost versus weight for sample data

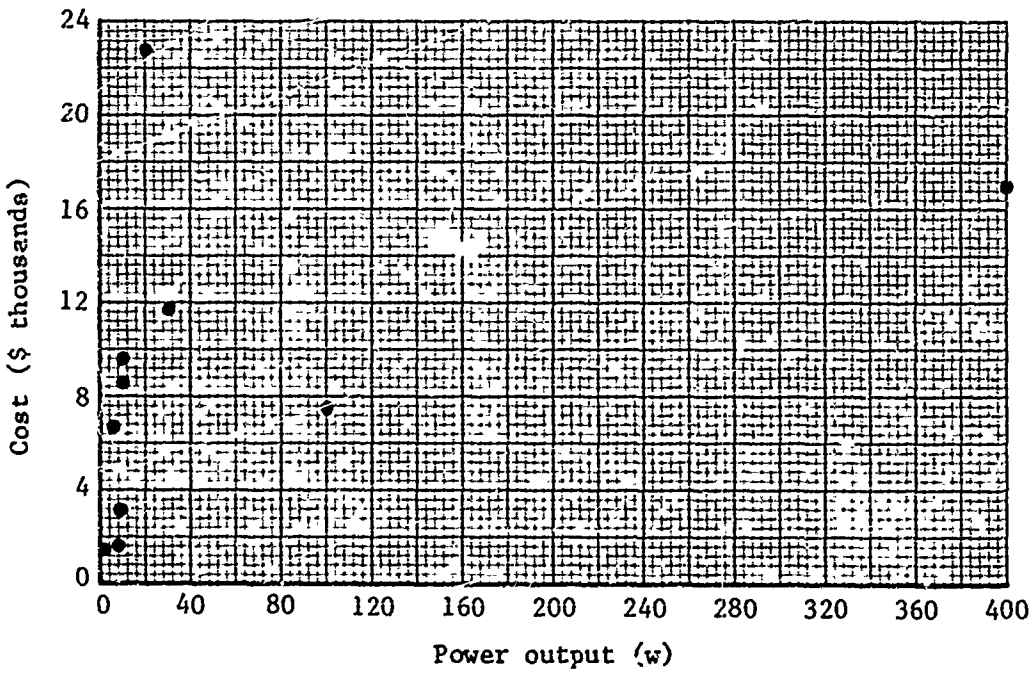

Fig. 2--Scatter diagram of cost versus power output for sample data

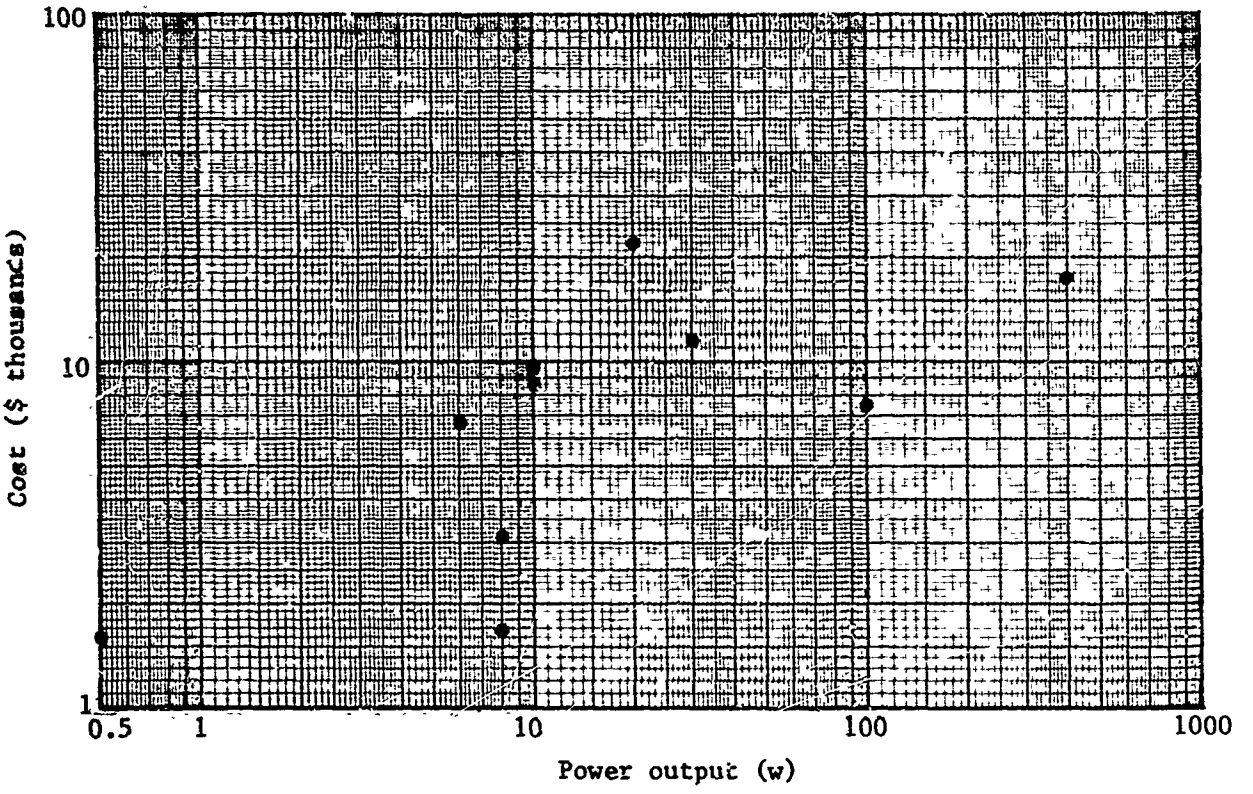

Fig. 3--Scatter diagram on logarithmic grid of cost versus power output for sample data

The wide scatter in Fig. 3 is explained in part by recognizing the effect of frequency. In Fig. 4, each point is identified by frequency class: High Frequency (HF), up to 30 MHz; Very High Frequency (VHF), 30 to 300 MHz; and Ultra High Frequency (UHF), above 300 MHz. A clearer relationship exists between cost and power output within each frequency class than exists for the whole sample scattered without regard to frequency. This suggests that the sample is not homogeneous. Each frequency band may constitute a separate sample, or possibly HF and VHF costs are on one level and UHF costs are on another.

At this point, it is not clear if any of the explanatory variables, either singly or in combination, will yield a useful estimating relationship, or if a single relationship can serve for all frequencies. To illustrate techniques that are commonly employed in deriving estimating relationships, assume that cost can be related to a single predictive

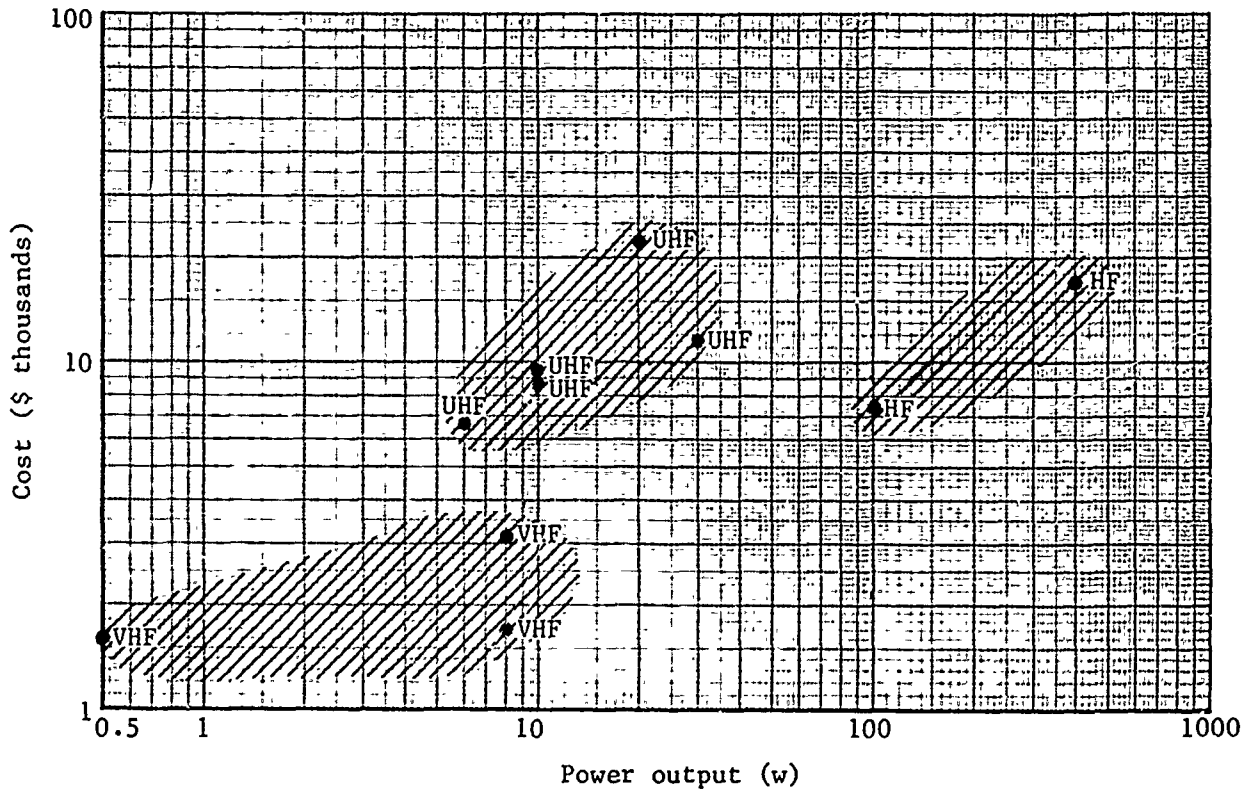

Fig. 4--Identification by frequency class

variable--that of weight. The results of a linear normal simple regression model will then be examined. Later, several variables in a multiple regression analysis will be considered, and the problem of the apparent nonhomogeneous character of the sample illustrated in Fig. 4 will be reexamined.

Regression has become a widely accepted tool for cost analysis, and it is frequently used to develop estimating relationships. The technique of regression analysis can be thought of as consisting of two distinct stages. The first is that of estimating the constant and coefficients of the equation, and the second is that of inferring the reliability and significance of the results of the estimate on the basis of assumed (and to a degree verifiable) properties possessed by the data and the results. Regression analysis as a technique is applicable only to the two stages performed together. Estimating coefficients or

curve fitting is simply a mathematical exercise. Only when these estimating procedures are used as a basis for making statistical inferences can they be viewed as part of a regression analysis.

Simple Linear Regression

The form of the relationships between cost and the explanatory variable(s) depends on the problem. It may reflect either an underlying physical law or a structural relationship. When no particular functional form is suspected, a simple (two-variable) linear model is frequently used to describe the relationship between two variables. In this case, the equation of the model is

$$y = a + bx, \tag{1}$$

where y is the dependent variable and x is the explanatory variable. The symbols a and b are the constant and coefficient, respectively, of the equation estimated from the data. Here y could represent the cost of a radio communication set and x could represent the weight. If it is assumed that b is greater than zero, the model indicates that heavier equipment will cost more than lighter equipment. When the values of a and b are known, it is possible to compute y (cost) for any given value of x (weight).

Least-squares Estimating

Given Eq. (1), the basic problem in the first phase of the regression analysis is to derive estimates of the parameters a and b. The standard procedure is the method of least-squares. The values of a and b are determined by the requirement that the sum of the squares of the deviations of the sample observations from the estimated line will be at a minimum. Symbolically, this minimum is expressed as

$$\min \sum_{i=1}^{n} (y_i - \hat{y}_i)^2, \tag{2}$$

where y_i is the ith observation and $\hat{y}_i$ is the value of y_i estimated from the equation

$$\hat{y}_i = \hat{a} + \hat{b}x_i. \tag{3}$$

The carets over $\hat{a}$ and $\hat{b}$ indicate that $\hat{a}$ and $\hat{b}$ are least-squares estimates of the true but unknown values of a and b. Thus $\hat{y}_i$ is the least-squares estimate of y_i and the term $(y_i - \hat{y}_i)$ indicates the difference between each observed y_i and between each corresponding estimated value $\hat{y}_i$. This is illustrated in Fig. 5, which shows the actual (y) and estimated $(\hat{y})$ value of the dependent variable that corresponds

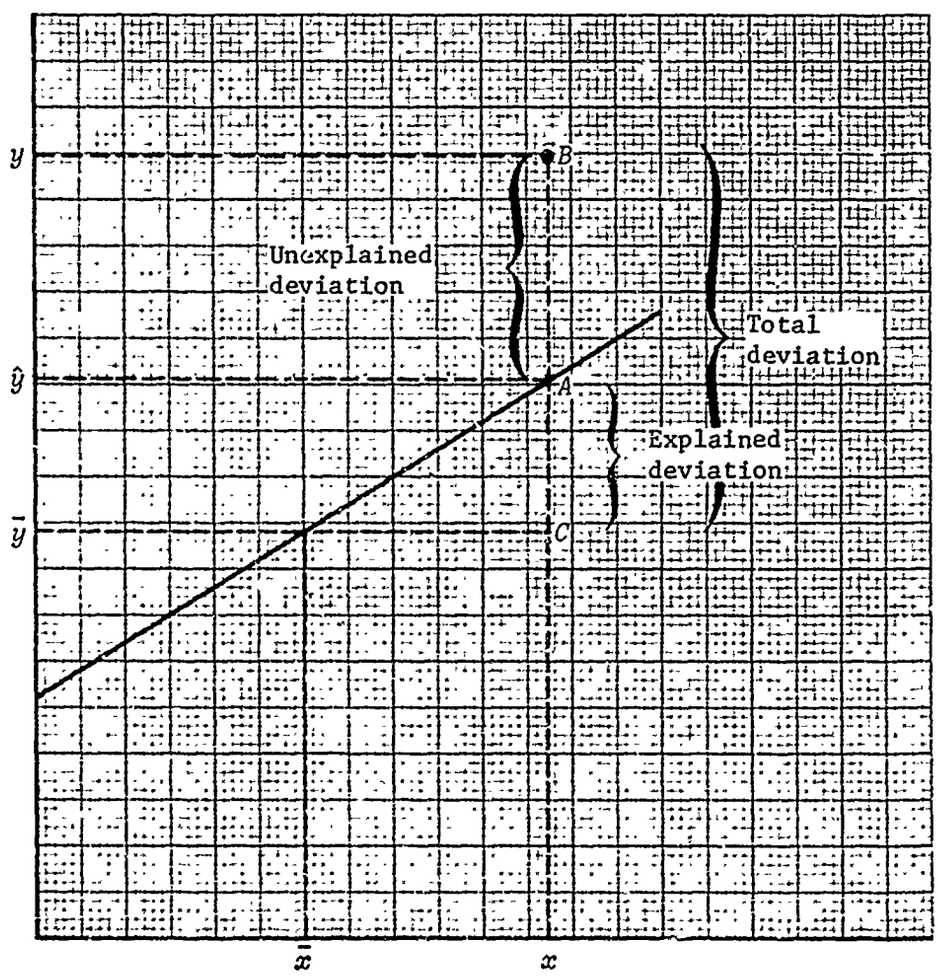

Fig. 5--Deviation of actual value from estimated value and sample mean

to a specific value of the explanatory variable x. The line shown in Fig. 5 is the line that represents Eq. (3). All of the estimated values of $\hat{y}_i$ fall on this line. The vertical distance from point A to point B is the difference between the actual value (y) and the estimated value ($\hat{y}$). The summation of all such differences that are squared (as illustrated in Eq. (2)) is the quantity to be minimized in estimating the line.

The minimum value for this sum is satisfied by substituting Eq. (3) in Eq. (2), taking the partial derivatives of Eq. (2) with respect to $\hat{a}$ and $\hat{b}$, and setting the results equal to zero. This process yields two equations that are called normal equations and that can be solved for $\hat{a}$ and $\hat{b}$:

$$\sum y = n\hat{a} + \hat{b} \sum x,$$

$$\sum xy = \hat{a} \sum x + \hat{b} \sum x^2,$$

where y = cost of airborne radio equipment in thousands of dollars,

x = weight of airborne radio equipment in pounds,

n = number of items in the sample,

Σ = summation (e.g., $\Sigma\, y$ = the sum of all y's).

Table 2 contains the numerical values and totals required to solve the

Table 2

DATA FOR REGRESSION ANALYSIS OF COST AND WEIGHT

x	y	x^2	xy
90	22.2	8,100	1,998.0
161	17.3	25,921	2,785.3
40	11.8	1,600	472.0
108	9.6	11,664	1,036.8
82	8.8	6,724	721.6
135	7.6	18,225	1,026.0
59	6.8	3,481	401.2
68	3.2	4,624	217.6
25	1.7	625	42.5
24	1.6	576	38.4
792	90.6	81,540	8,739.4

normal equations when data from Table 1 are used. The costs are expressed in thousands of dollars. When the values from Table 2 are substituted in the normal equations, the following expressions are obtained for the sample data points ($n = 10$):

$$90.6 = 10a + 792b,$$

$$8739.4 = 792a + 81{,}540b.$$

Solved simultaneously, these equations give*

$$\hat{a} = 2.477,$$

$$\hat{b} = .083,$$

and thus from Eq. (3)

$$\hat{y} = 2.477 + .083x. \tag{4}$$

The line represented by this equation is shown in Fig. 6 as the solid line with the actual observations plotted as dots. The extent of the dispersion of the observations relates inversely to the usefulness of the line as a tool for estimating the values of y from the values of x. The greater the dispersion of observed values of y about the line, the less accurate the estimates that are based on the line are likely to be. The measure of the dispersion about the regression line is called the standard error of estimate (SE) of the equation and is shown by the dashed lines.

One measure of dispersion in a collection of data points is called the variance. The variance is defined as the sum of the squared distances to each of the data points from a central reference point divided by the degrees of freedom (df), which equal the number of independent bits of information contained in the sample. (In analyzing the data

*Slight variations may exist in the last significant figure in the examples throughout this section because of rounding and logarithmic transformations.

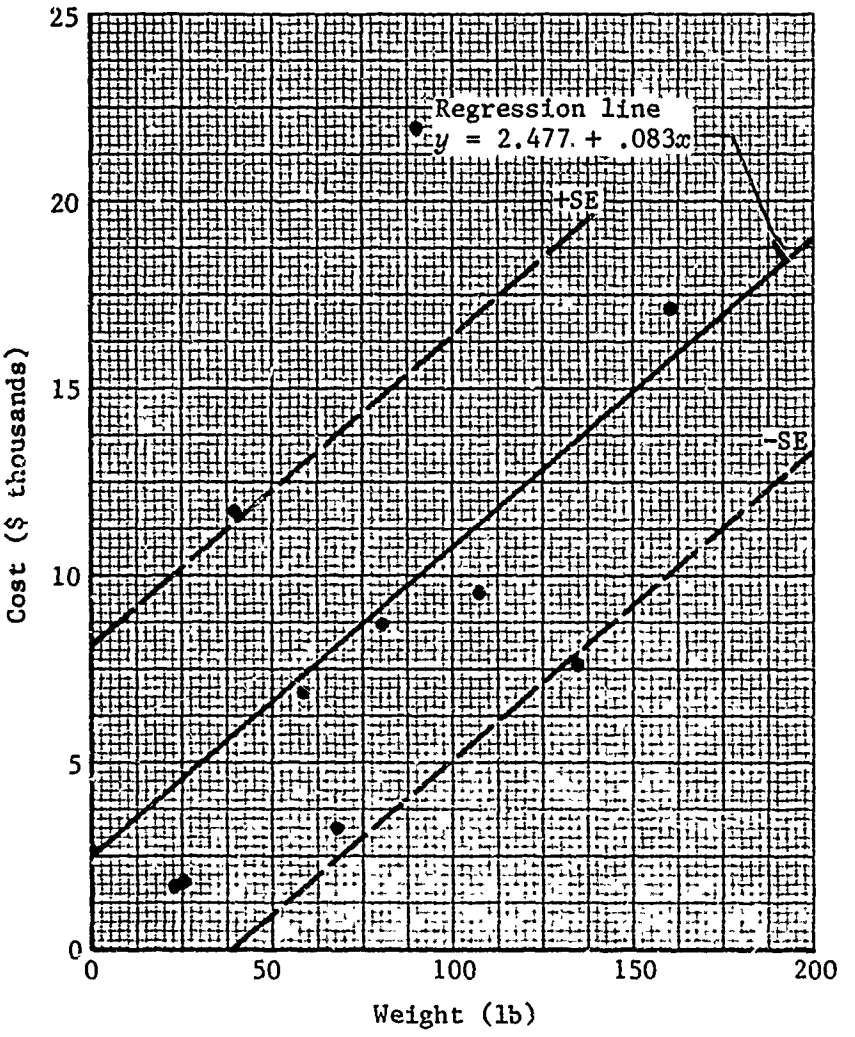

Fig. 6--Regression line and standard error of estimate

that are given in Table 1, the degrees of freedom equal $(n - 2)$; i.e., the number of observations n less the number of constraints, 1 each for a and b.)

In least-squares procedures, the central point of reference for calculating the variance of each variable is its sample mean, which causes the least-squares line to have the property of passing through the means of the variables used to estimate the line. This characteristic is shown in Fig. 5; it can be verified by dividing both sides of the first normal equation by n, since the sample mean of any variable y is

$$\frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}. \tag{5}$$

By referring to Fig. 5, it can be seen that the total distance from y_i to $\bar{y}$ for any observation on y is the distance from C to B. The sum of all such distances squared and divided by the degrees of freedom is called the total variance of y:

$$\text{Total variance of } y = \sum \frac{(y_i - \bar{y})^2}{n - 1}. \tag{6}$$

The distance from C to A indicates the amount of the total deviation of y from $\bar{y}$ which is explained by the estimating relationship. Consequently, the sum of the distances from $\bar{y}$ to the line, squared and divided by the degrees of freedom, is called the explained variance:

$$\text{Explained variance of } y = \sum \frac{(\hat{y}_i - \bar{y})^2}{n - 2}. \tag{7}$$

The remaining distance from A to B is the residual or unexplained deviation from y_i to $\bar{y}$, or the unexplained variance:

$$\text{Unexplained variance of } y = \sum \frac{(y_i - \hat{y}_i)^2}{n - 2}. \tag{8}$$

The standard error of estimate is defined as the square root of the unexplained variance of the y's:

$$\text{SE} = \sqrt{\frac{\sum (y_i - \hat{y}_i)^2}{n - 2}}. \tag{9}$$

For the equation $y = 2.477 + .083x$, the standard error of estimate is \$5,808. This value has been plotted above and below the regression line in Fig. 6. The interpretation and significance of these results will be discussed in connection with the use of prediction intervals.

In comparing one SE with another, it is useful to compute a relative

standard error of estimate. One such measure is the coefficient of variation (CV), which relates the SE to the mean of the sample y's:

$$CV = \frac{SE}{\bar{y}}. \tag{10}$$

Continuing the analysis of the data in Table 1, the mean of the y's is \$9,060. Therefore, the value of CV is

$$\frac{\$5,808}{\$9,060} = .641.$$

This value is high. Although the question of reliability of an estimating equation is relative to the context in which the equation is to be used, a value at least as small as 10 to 20 percent for the coefficient of variation is desirable.

The standard error of estimate gives a measure of the magnitude of the unexplained variance. Another related measure of dispersion is given by the coefficient of determination that shows the proportion of total variance accounted for by the estimating relationship:

$$r^2 = \text{Coefficient of determination} = \frac{\text{Explained variance}}{\text{Total variance}}$$
$$= 1 - \frac{\text{Unexplained variance}}{\text{Total variance}}. \tag{11}$$

When all the observed points in the sample are on the least-squares line, the coefficient of determination equals 1 and there is no unexplained or residual variance. As the proportion of total variance that remains unexplained increases, the coefficient of determination approaches zero. The square root of the coefficient of determination is called the correlation coefficient.* Correlation has no substantive

*Since total variance, Eq. (6), and the standard error, Eq. (9), have been adjusted for degrees of freedom, the resulting correlation coefficient, the square root of Eq. (11), is also adjusted. Some computer programs do not adjust; the variance figures are then biased downward and the correlation coefficient will appear larger than in the

meaning unless both the dependent and explanatory variables are assumed to be normal random variables. The ordinary assumption in using regression analysis for developing estimating relationships is that only the dependent variable is random. Consequently, it is not considered good practice for the correlation coefficient to be used in documenting the results in this particular application of regression analysis. The inclusion of the correlation coefficient, however, causes no serious problem since it is simply the square root of the coefficient of determination. When analysts review the results, they can easily calculate the latter from the former. Since the coefficient of determination is always in the range between zero and one, its square root will always be larger, except at the boundary points of zero and one.

The coefficient of determination for Eq. (4) is .325, which is relatively low and further substantiates the evidence that weight alone is not a good predictor of the cost of airborne radio communication equipment.

Statistical Inference

The standard error of estimate, the coefficient of variation, and the coefficient of determination indicate the degree of accuracy with which the estimating equation describes the sample observations. However, the analyst is primarily interested in using the estimating equation to predict costs among the population of items that the sample represents; the standard error of estimate and the coefficient of determination do not furnish a good measure of the reliability of the estimating equation for predictive purposes.

The problem of reliability raises other considerations. First, the question arises whether x and y are actually related in the manner

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unadjusted case. The practical implications of these adjustments is minimal except in extremely small sample cases. However, to fully understand the results, the analyst should know whether the total variance, standard error, and correlation coefficient are adjusted in any particular program or set of results. A discussion of adjustments for degrees of freedom is given in M. J. B. Ezekiel and K. A. Fox, *Methods of Correlation and Regression Analysis*, 3d ed., John Wiley & Sons, Inc., New York, 1959, pp. 300-305.

indicated by the regression equation. A particular sample could show such a relationship out of pure chance when, in fact, none exists. Second, the regression equation obtained from the sample is one of a family that could be obtained from different samples within the same population. Finally, when the equation is used to estimate a value for y based on an x that is outside the range of the sample, the reliability of the estimate of y may be suspect because the estimated relationship may not hold beyond the sample range or because the x is a point from a different population rather than an extrapolation from the sample. An example of an extrapolation for which the relationship might not hold is that of an aircraft that is much larger than any in the sample. The problem of moving to a new population appears in a case in which an aircraft is to be constructed of titanium when the sample contains only aluminum aircraft. In the latter case, if a substitution of titanium for aluminum is expected to increase the cost, the estimating relationship developed from the aluminum sample may be used by an experienced analyst as an approximate indicator of the lower bound; however, adjustments based on such personal judgments are not a part of statistical theory.

Statistical inference may be used to answer the two questions that arise in connection with the problem of reliability. To decide whether x and y are actually related, test for statistical significance; to evaluate predictions, establish a prediction interval for the regression line. However, certain assumptions and conditions must be met before standard techniques of statistical inference and testing can be validly applied to least-squares results; namely, the data are assumed to be a sample taken from a larger population, which meet the following conditions:

1. The x values are nonrandom (fixed) variables.
2. The residual deviations are independent random variables with normal distributions.
3. The expected value of the distribution of each of these random variables is zero, and the unknown variance is the same for all values of x.

Under these assumptions, the hypothesized relationship between y and x becomes

$$y_i = a + bx_i + u_i, \tag{12}$$

where $i = (1, \ldots, n)$,

u_i = the normally distributed random error terms with zero expected value and a common and unknown variance.

Further, under these assumptions, the least-squares method produces unbiased maximum likelihood estimators. Standard statistical techniques can be applied to the least-squares results to test for significance and to make inferences about reliability and accuracy in a probabilistic sense.* A graphic illustration of these assumptions as they relate to the simple (two-variable) regression case is shown in Fig. 7.

Although the subject of statistical testing is too complex to treat comprehensively here, the method of testing the significance of the relationship between x and y in the simple regression of Fig. 6 will be examined briefly. Basically, the procedure involves establishing the null hypothesis that x and y are not related (i.e., that $b = 0$), and testing to determine whether the hypothesis should be rejected. The test that is commonly used for this purpose is known as the t-test because it uses the t-ratio, or ratio of a coefficient to its standard error. For this simple regression, the ratio is expressed as

$$t_b = \frac{\hat{b}}{s_b}, \tag{13}$$

where $\hat{b}$ = the estimated regression coefficient (from the equation $\hat{y} = \hat{a} + \hat{b}x$),

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*A more comprehensive statement of these assumptions and considerations is given in W. A. Spurr and C. P. Bonini, *Statistical Analysis for Business Decisions*, Richard D. Irwin, Inc., Homewood, Illinois, 1967, pp. 564-565; A. M. Mood, *Introduction to the Theory of Statistics*, McGraw-Hill Book Company, Inc., New York, 1950, pp. 152-154; and John Johnson, *Econometric Methods*, McGraw-Hill Book Company, Inc., New York, 1963, pp. 3-9.

s_b = the standard error of b,

$$= \frac{SE}{\sqrt{\Sigma\ (x_i - \bar{x})^2}},$$

SE = the standard error of estimate as defined in Eq. (9).

The value of t_b for Eq. (4) is 1.96.

A standard table of t-ratios is required to use Eq. (13) to test the null hypothesis.* The relevant row is shown in Table 3. If the calculated value t_b falls below the appropriate value of t selected from this table, the null hypothesis that $b = 0$ would be accepted, and it would be concluded that b is, in fact, not significantly different from zero. The level of significance above each of the t-values indicates the probability that the calculated value could be as high strictly by chance as the values that are shown in the table. In other words, these levels of significance indicate the probability that the null hypothesis will be rejected when it is true.

Table 3

VALUES OF t-RATIOS FOR 8 DEGREES OF FREEDOM
(One-sided Test)

Degrees of Freedom	*Level of Significance*					
	.20	.15	.10	.05	.025	.01
.	*t-Ratio*					
.						
.						
8	.889	1.108	1.397	1.860	2.306	2.896

If there were evidence to justify the assumption that the sign of the coefficient could be only positive (or only negative) if it were different from zero, the level of significance associated with each t could be read directly from Table 3. However, the common practice in

*All of the references in the Bibliography to this section contain t-tables.

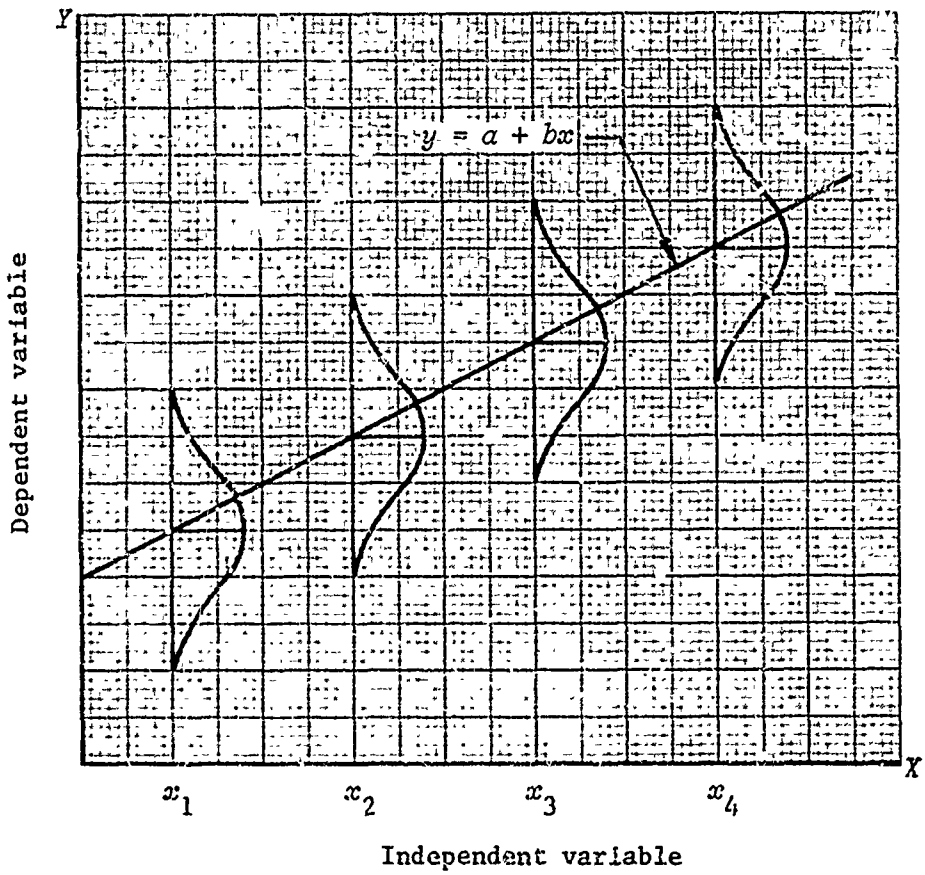

Fig. 7--Simple linear population regression model

regression analysis is not to make this assumption, but to test as though the value of t (if it were different from zero) could be either positive or negative. Because of the symmetry of the distribution of the t-ratios, the level of significance for the two-sided test is twice the level of significance for the one-sided test. Thus, the levels of significance of the t-values shown in the table are only half the actual levels for the two-sided test. For example, the value 1.86 has a level of significance of .05. For the two-sided test, double this amount and read the level of significance as .10. In the two-sided test, the probability is 10 percent that the absolute value of t_b is as large as 1.86 when b is actually equal to zero. Since in the example $t_b = 1.96$, if the required level of probability for rejecting the null hypothesis when it is true is as high as 10 percent but no higher, the hypothesis

that $b = 0$ is rejected, and the relationship is considered significant. On the other hand, if a .05 level of significance ($\bar{t} = 2.306$) seems appropriate, the hypothesis must be accepted. In this case, the coefficient of x, and therefore the equation, is considered as *not* significant.*

The question at this point is, What should the level of significance be for rejecting the hypothesis? Unfortunately, no simple answer is possible. The values of .10, .05, and .01 are those that are most commonly used, but the analyst must make a decision based on the risk that is assumed when a true hypothesis is rejected.** For the purpose of this discussion, we will accept a value of .10 in testing significance and in establishing a prediction interval for the regression line.

Prediction Intervals

The procedure for calculating the prediction interval for a simple regression is as follows. For a given value of the explanatory variable, say x, the estimating equation is used to obtain a predicted value of the dependent variable:

$$\hat{y} = \hat{a} + \hat{b}x. \tag{14}$$

The prediction interval puts a boundary around $\hat{y}$:

$$\hat{y} \pm A_{\varepsilon/2}. \tag{15}$$

There is a certain level of confidence $(1 - \varepsilon)$ that the cost of a set weighing x will be in that interval.

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*A more comprehensive discussion of the use of statistical tests is given in W. A. Wallis and H. V. Roberts, *Statistics*, The Free Press, New York, 1963, pp. 399-402, 413-426.

**For further discussion, see W. A. Spurr, L. S. Kellogg, and J. H. Smith, *Business and Economic Statistics*, Richard D. Irwin, Inc., Homewood, Illinois, 1961, pp. 251-255.

Values for $\varepsilon/2$ rather than ε are used since $\hat{y}$ is to be bounded on both sides. The values of ε can be divided by two since under the assumptions, the probability distribution about $\hat{y}$ is normal and therefore is symmetrical. In statistical terminology, a two-tailed t distribution for constructing the intervals is used.

In the case of simple regression, a $100(1 - \varepsilon)$-percent prediction interval for an estimated value of the dependent variable can be constructed as follows:

$$\hat{y} \pm A_{\varepsilon/2}, \tag{16}$$

where

$$A_{\varepsilon/2} = (\mathrm{SE})\, t_{\varepsilon/2} \sqrt{\frac{n+1}{n} + \frac{(x - \bar{x})^2}{\Sigma\,(x_i - \bar{x})^2}}, \tag{17}$$

and where SE = the standard error of the estimating equation from which $\hat{y}$ was obtained,

$t_{\varepsilon/2}$ = the value obtained from a table of t-values for the $\varepsilon/2$ significance level,

n = the size of the sample,

x = the specified value of the explanatory variable used as a basis for obtaining $\hat{y}$,

$\bar{x}$ = the mean of the x's in the sample,

$\sum (x_i - \bar{x})^2$ = the sum of the squared deviations of the sample x's from their sample mean.

When the estimating equation derived previously is used, the cost of a communications set weighing 100 lb is estimated at $10,777. To establish around this value a 90-percent prediction interval (i.e., one with a 10-percent level of significance), the necessary data are

$$\mathrm{SE} = 5.808,$$
$$\varepsilon = 0.1,$$
$$\varepsilon/2 = 0.05,$$
$$t = 1.86,$$
$$n = 10,$$

$$df = 8,$$
$$x = 100 \text{ lb},$$
$$\bar{x} = 79.2 \text{ lb},$$
$$\sum (x - \bar{x})^2 = 18{,}813.6 \text{ lb}.$$

By substituting these data in Eq. (17), solving for $A_{\varepsilon/2}$, and multiplying by 1000, we obtain

$$A_{\varepsilon/2} = \$11{,}447.$$

Therefore, for $x = 100$ lb, the 90-percent prediction intervals in dollars are

$$\hat{y} \pm A_{\varepsilon/2} = \$10{,}777 \pm \$11{,}447.$$

The percentage $100(1 - \varepsilon)$ is the confidence level of the prediction intervals, which means that if repeated observations on the cost of communications sets that weigh 100 lb were taken, $100(1 - \varepsilon)$ percent of the time these observations would lie within the range set by the $100(1 - \varepsilon)$ prediction intervals. This is the only sense in which a level of confidence can be associated with prediction intervals. It is erroneous to infer that there is a $100(1 - \varepsilon)$-percent probability that the actual value for any particular case will lie within the interval.

Further, prediction intervals are valid outside the range encompassed by the sample data that are used to generate the estimating relationship and the interval *only if* the estimating relationship is itself valid outside that range. For example, if there were occasion for the line to curve up or down or if a discontinuity in the form of a discrete jump in cost occurred for weights outside the sample range, this fact would not be reflected in the prediction interval. Thus, it must be clearly indicated when the intervals are used for estimates based on values outside the sample range.

This prediction interval procedure can be repeated for other values of x and the results plotted to obtain a 90-percent prediction

interval band around the regression line, as shown in Fig. 8. In this case, the 90-percent confidence region is fairly wide because of the relatively large standard error of this equation. The formula for the prediction interval is such that the width of the interval is sensitive to the size of the standard error; large standard errors indicate that much of the cost variation in the observed data is unexplained by the equation.

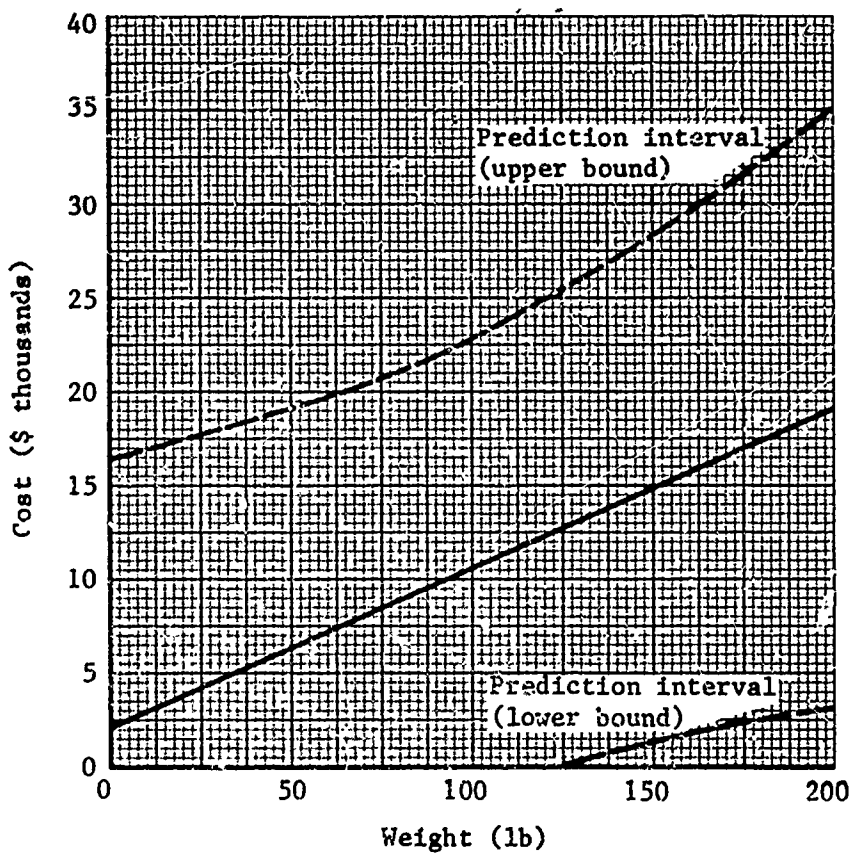

Fig. 8--The 90-percent prediction interval band for estimated costs based on sample data

The prediction interval becomes wider as values of x that are farther from the mean of the sample are selected. From Eq. (4), the prediction interval (multiplied by 1000) for the mean 79.2 lb is \$9,051 ± \$11,329; for x = 200 lb, the prediction interval is \$19,077 ± \$14,794.

In the latter case, the width of the interval is about 1.3 times the width for the mean weight. This change in the size of the prediction interval occurs because the formulas are derived to allow for the possibility that the estimated values of $\hat{a}$ and $\hat{b}$ differ from the true values of a and b. Such a situation can occur when the sample data contain chance fluctuations that prevent the data from reflecting the true relationship that exists in the total population or when there are not sufficient data in the sample.

Figure 9 illustrates the way in which errors in the estimates of $\hat{a}$ and $\hat{b}$ affect the accuracy of estimates. The solid line represents the true relation between x and y. The dashed line represents an equation in which the estimated values of $\hat{a}$ and $\hat{b}$ differ from the true

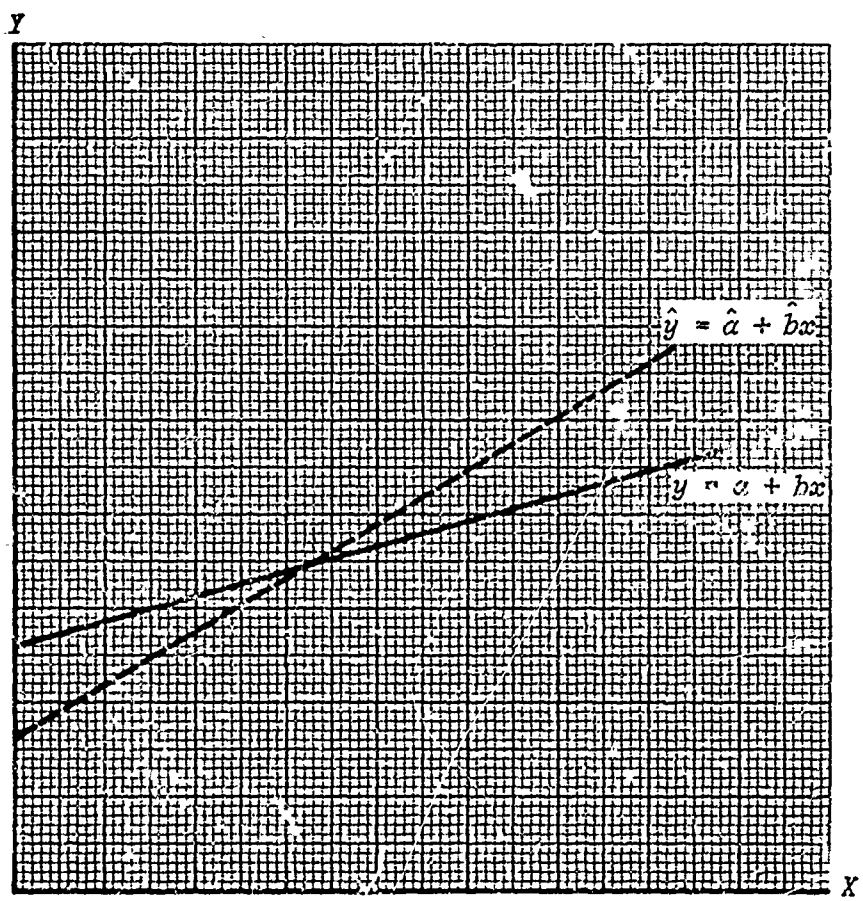

Fig. 9--Effects of estimating errors on accuracy of predictions

values. The figure shows that the effect of these errors increases with movement toward the extreme ranges of x.

The width of the prediction interval is also sensitive to the level of confidence that is specified and to the number of degrees of freedom. That level was set at 90 percent (i.e., $\varepsilon/2 = 0.05$). Suppose that only a 70-percent level of confidence is required ($\varepsilon/2 = 0.15$). The only change in the inputs used in the previous calculations is the value of t. With a 90-percent level of confidence, $t_{.05} = 1.86$; with a 70-percent level, $t_{.15} = 1.11$. This change will make a difference in the width of the prediction interval. Since the level of confidence is lower, the prediction interval is narrower; for lower levels of confidence, the band will be even more narrow. For $\varepsilon = .10$ and the degrees of freedom = 8, the value of $t_{\varepsilon/2}$ is 1.86. If the degrees of freedom were 16, $t_{\varepsilon/2}$ would be 1.746. Thus, if there are twice as many degrees of freedom for an equation with the same standard error, the prediction interval for $\varepsilon = .10$ is smaller. However, the difference in prediction interval size because of differences in degrees of freedom is more significant for small samples than for large samples; the value of t for any given level of significance becomes almost constant for degrees of freedom over 30. For example, the smallest value of $t_{\varepsilon/2}$ for $\varepsilon = .10$ is 1.645.

Before concluding this section, there are two additional points to be made. First, even when the coefficient of determination r^2 is high, it is possible for the standard error of estimate to be large. This is explained by the fact that r^2 is based on a proportion and the standard error is based on an absolute quantity:

$$r^2 = \frac{\text{Explained variance}}{\text{Total variance}},$$

$$\text{SE} = \sqrt{\text{Unexplained variance.}}$$

Thus, even if the explained variance represents a high fraction of the total variance, it is possible for the unexplained variance to be large relative to the estimated cost. This outcome would be indicated by the coefficient of variation.

Second, the statistical significance of regression relationships does not necessarily imply existence of a causal relationship. The following excerpt from an Institute of Defense Analyses (IDA) memorandum illustrates the importance of this distinction in cost analysis:*

> Frequently during cost effectiveness studies, the distinction between a "causation" cost model and a "correlation" cost model is overlooked. A simple example will be used to illustrate the distinction between the two types of cost models and show how a sensitivity analysis performed with a correlation cost model, rather than a causation model, can lead to erroneous conclusions.
>
> Example: Estimate the cost of assembling a piece of hardware. The assembly consists merely of bolting various elements together The overwhelming majority of the cost of the assembly process is the salary paid to the men who do the bolting. Careful analysis of all the available cost data might yield a correlation cost model given by Equation 1.
>
> $$C = a \times w \qquad (1)$$
>
> where w is the total weight of all the bolts that go into the assembly,
> C is the cost of the assembly,
> a is a regression coefficient.
>
> By all of the various statistical measures of goodness of fit, Model 1 is a valid prediction equation.
>
> The causation cost model is given by Equation 2.
>
> $$C = k \times h \times n \qquad (2)$$
>
> where k is the hourly wages of the assemblers,
> h is the number of hours it takes to fasten and bolt,
> n is the number of bolts used in the final assembly,
> C is the cost of the assembly.
>
> It should be noted that the correlation cost model and the causation cost model are interrelated by Equation 3.
>
> $$w = F \times N \qquad (3)$$

---

*Morris Zusman, "Use of Cost Models in Sensitivity Analysis and as a Design Aid," Institute of Defense Analyses, N-587(R), September 1968. In this discussion, the term *correlation* is used figuratively in the sense that it is statistically significant in explaining the amount of variance rather than in the sense that both the dependent and independent variables are random.

where B is the weight of a single bolt,
w is the total weight of all of the bolts that go into the assembly.

Thus any design or sensitivity analysis performed on Equation 1, the correlation cost model, will lead to the correct results if Equation 3 is not violated. For example, an analyst would be correct in predicting that a cost reduction would occur if he reduced the weight of the fasteners used by using less fasteners. He would be incorrect if he predicted a cost reduction would occur if he reduced the weight of the fasteners by substituting aluminum for steel bolts while keeping the number of bolts constant. The reason that a substitution of aluminum for steel bolts would not reduce the cost, is because the underlying relationship between the number of bolts and the weight of the fasteners (Equation 3), which is the reason for the good cost weight relationship of the correlation model, has been violated.

In mathematical terms both a causation and a correlation cost model have the following properties.

$$\text{Cost} = f\ (\text{characteristics}) \tag{4}$$

But <u>only</u> a causation model can be manipulated as Equation 5.

$$\text{Characteristics} = f^{-1}\ (\text{cost}) \tag{5}$$

The problem of determining whether a cost model is a correlation or a causation model is, except for the trivially simple type of problem illustrated here, very difficult since all causation models can be transformed into correlation models. There exist no statistical tests to determine whether a model is a causation model or a correlation model.

The types of explanatory variables used in the cost model generally will give a good guide as to whether a model is a correlation model or a causation model. For example, weight as an explanatory variable in a cost model where the material cost did not dominate, would be a good indication that the cost model was a correlation model.

If the model is a correlation model and the analyst performs a sensitivity analysis, he runs the risk of violating the unknown underlying relationships between the correlation and causation models. If these underlying relationships are violated the sensitivity analysis will be erroneous.

This example illustrates that regression analysis is an aid to, and not a substitute for, experience and understanding.

Curvilinear Analysis

Until this point the analysis has been confined to a simple (one explanatory variable) linear regression. Although a cursory examination of the scatter diagram of cost versus weight illustrated in Fig. 1 indicates that a linear relationship may be adequate, it cannot be concluded definitely that a curvilinear relationship might not be preferable. These relationships can be examined by transforming the data to permit the relationships to be estimated using linear estimating techniques. The equation

$$y = a + bx^2 \tag{18}$$

can be estimated using the least-squares method by substituting x^2 for each x and solving the normal equations as before.

Another type of nonlinear relationship that is frequently used and that will be examined in discussing cost-quantity relationships in Sec. V is of the form

$$y = ax^b. \tag{19}$$

For this form, a logarithmic transformation of both variables is made to obtain an equation that is linear in the logarithms of the original variables:

$$\log y = \log a + b(\log x). \tag{20}$$

The regression analysis is then conducted in terms of the logarithms of the variables rather than in terms of the variables themselves. (Throughout this section, logarithms to the base 10 will be used.*)

*It is possible to estimate relationships such as those represented by Eq. (19) directly. For example, see C. A. Graver and H. E. Boren, Jr., *Multivariate Logarithmic and Exponential Regression Models*, The Rand Corporation, RM-4879-PR, July 1967. Although direct nonlinear estimating techniques have some desirable properties, they are much less widely used in cost analysis than the linear methods.

However, to permit the standard techniques of statistical inference based on linear least-squares regression to be used, it is assumed that the dependent variable $\log y_i$ is linearly related to the independent variable $\log x_i$ and to the normally distributed random variable u_i by the equation

$$\log y_i = \log a + b \log x_i + u_i \qquad (i = 1, \ldots, n). \tag{21}$$

When antilogarithms are used, Eq. (21) is implicitly of the form

$$y = ax_i^b 10^{u_i}. \tag{22}$$

Because of this difference in form, statistics derived for Eq. (22) are not directly comparable with those derived for Eq. (12). Similarly, statistics on predictions made by the two models will not be easily comparable because in the one case error is additive and in the logarithmic case error is exponential and multiplicative.

The first step in estimating the coefficients for Eq. (20) is to convert to logarithms the data for cost (in thousands of dollars) and for weight shown in Table 1. The next step is to calculate the least-squares estimates of b and $\log a$. The results of these calculations are

$$\begin{aligned} \log y &= -1.0425 + 1.0241(\log x), \\ r^2 &= .560, \\ \mathrm{SE}_{\log} &= .2763, \\ t_b &= 3.19, \\ \mathrm{df} &= 8. \end{aligned} \tag{23}$$

The antilogarithms of both sides of Eq. (23) give

$$y = (.09067)x^{1.0241}, \tag{24}$$

where y = cost in thousands of dollars,

x = weight in pounds.

Based on the coefficient of determination (r^2) and the calculated t-value (t_b), these results appear to be slightly better than those obtained with the linear case. However, care must be exercised in comparing the logarithmic with the linear form and in evaluating the logarithmic form itself. There are significant differences between the two forms. A hint of these differences is given by the fact that the standard error for the logarithmic case (SE_{log}) is the standard error of the logarithms of the original numbers and not the standard error of the numbers themselves. For this reason, the standard error for the logarithmic case (SE_{log} = .2763) is about 20 times smaller than the standard error for the arithmetic or linear case (SE = 5.808). Thus, the relative sizes of these standard errors do not give a direct indication of the equation that has the smaller standard error in terms of the original numbers, which are the numbers of interest in cost analysis.

A review of the manner in which least-squares estimators are calculated will help to clarify this difference and to explain how these results can be compared. The technique is to find $\hat{a}$ and $\hat{b}$ such that

$$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \tag{25}$$

is minimized. In the logarithms of the numbers, however, this is equivalent to finding the minimum value of

$$\sum_{i=1}^{n} \left[\log \left(\frac{y_i}{\hat{y}_i} \right) \right]^2, \tag{26}$$

since $(\log y_i - \log \hat{y}_i) = \log y_i/\hat{y}_i$. Thus, by transforming the variables to logarithms, the sum of the squares of the logarithms of the ratios rather than the sum of the squares of the differences between the observed and actual values of y are minimized.

The full impact of this change can be best illustrated by an examination of the way in which the difference affects the calculation of

prediction intervals. To obtain prediction intervals for cost estimates when a logarithmic equation is used, the intervals are first calculated directly with the logarithmic data and they are then converted to natural numbers. Thus, the end points of the interval in logarithmic form are

$$\log \hat{y} - A_{\varepsilon/2} \quad \text{and} \quad \log \hat{y} + A_{\varepsilon/2}, \tag{27}$$

where

$$A_{\varepsilon/2} = (\mathrm{SE}_{\log}) t_{\varepsilon/2} \sqrt{\frac{n+1}{n} + \frac{(\log x - \log \bar{x})^2}{\Sigma\,(\log x_i - \log \bar{x})^2}}.$$

For the case where $x = \bar{x}$, these end points become

$$\log \hat{y} - (.2763) t_{\varepsilon/2}(1.049) \quad \text{and} \quad \log \hat{y} + (.2763) t_{\varepsilon/2}(1.049). \tag{28}$$

When antilogarithms of these numbers are used, the following prediction interval end points for the ε level of significance are obtained:

$$(\hat{y}) 10^{-.2898 t_{\varepsilon/2}} \quad \text{and} \quad (\hat{y}) 10^{.2898 t_{\varepsilon/2}}, \tag{29}$$

which are equivalent to

$$\frac{\hat{y}}{10^{.2898 t_{\varepsilon/2}}} \quad \text{and} \quad (\hat{y}) 10^{.2898 t_{\varepsilon/2}}. \tag{30}$$

These results show that the prediction interval band for the original numbers, based on a logarithmic regression analysis, is both nonsymmetrical and proportional to the predicted values. Further, the standard error for the logarithmic case ($\mathrm{SE}_{\log}$) is more comparable with the coefficient of variation (CV) for the arithmetic case than it is with the standard error (SE) for the arithmetic case, because the standard error for the logarithmic case (like the coefficient of variation for the linear case) is a proportion.

The band for the standard error is delineated by the following locus of points for the various values of y:

$$\frac{y}{10^{.2763}} \quad \text{and} \quad (y)10^{.2763} \tag{31}$$

Thus, the upper and lower bounds of the standard error band at the sample mean value of y (9.06) based on the logarithmic regression analysis is given by the following numbers:

$$\frac{9.06}{10^{.2763}} \quad \text{and} \quad (9.06)10^{.2763},$$

which equals 4.80 and 17.12, respectively. When these numbers are expressed as differences around the mean, 8.06 is obtained for the upper half of the interval and 4.26 for the lower half.

Figure 10 shows a graph of the values of the standard error for other values of y and the band for the 90-percent prediction intervals plotted above and below the regression line. These bands about the regression line illustrate both the nonsymmetry and the proportionality of these measures for the logarithmic case: nonsymmetry in that the distance between the regression line and the upper bounds is greater than that for the lower bounds; and proportionality in that the bounds become wider as y becomes larger. Because the standard error for the logarithmic case is a constant percentage of y, the absolute value of the bounds change as the value of y changes.

In Fig. 11, an interval of plus and minus $5,808 (the amount of the standard error in the arithmetic case) and the standard error as shown in Fig. 10 have been plotted about the regression line that was obtained with the logarithmic transformation. Figure 11 illustrates the way in which the standard error based on the logarithmic regression analysis compares with the results that were obtained from the arithmetic equation. The interval of plus $5,808 intersects the upper bound of the standard error at the point where $x = 65$ lb. The interval of

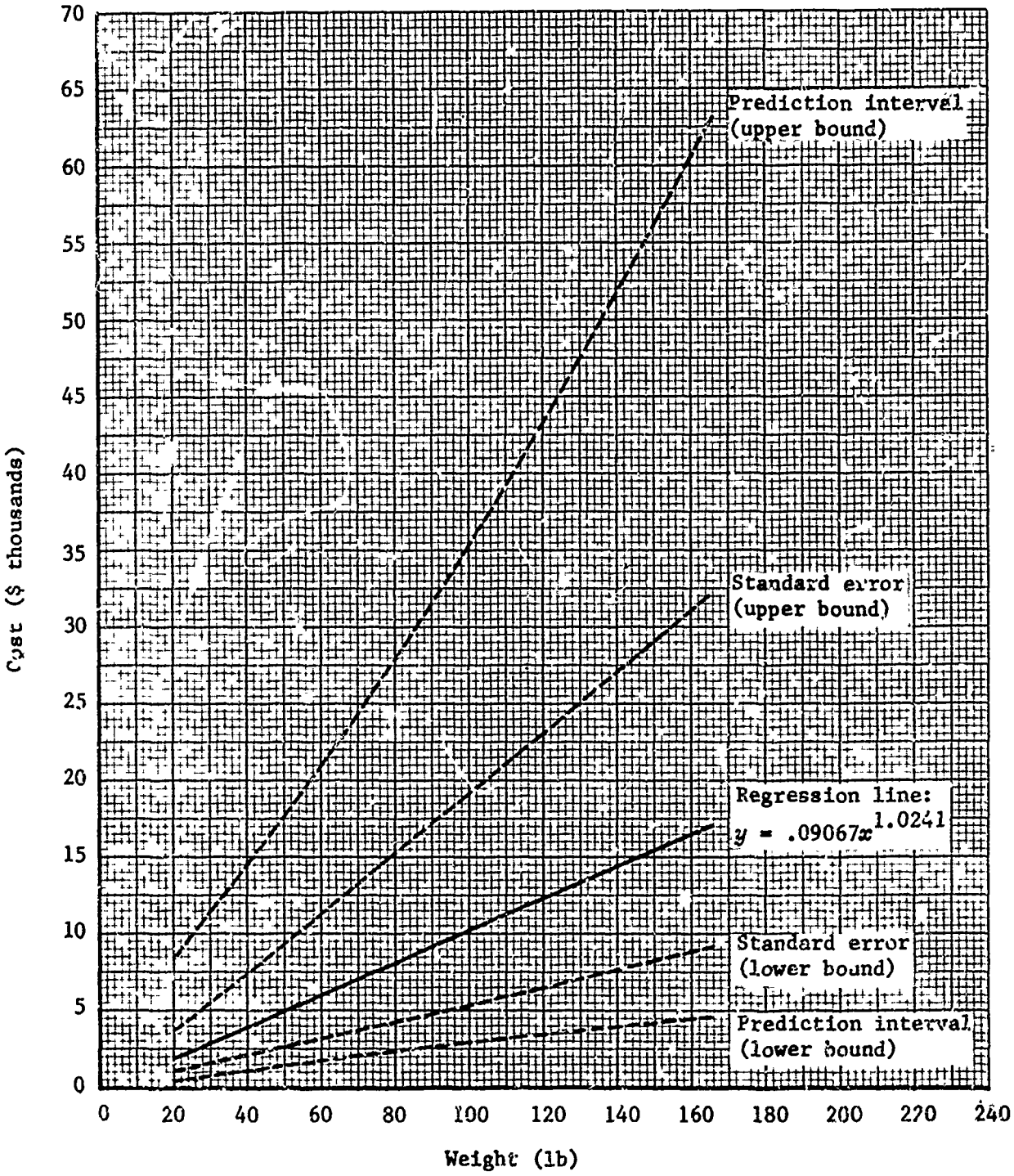

Fig. 10--Logarithmic equation with standard error and 90-percent prediction intervals

minus \$5,808 intersects the lower bound at x = 121 lb. Thus, for all estimated values of y greater than \$12,300, the interval based on the value of the standard error of the arithmetic case is less than the lower bound of the standard error calculated from the logarithmic analysis. Similarly, for all estimated values of y greater than \$6,500, this interval is less than the upper bound (logarithmic case).

On the basis of these considerations, it can be seen that the comparisons of the logarithmic results and the arithmetic results are difficult and can often be misleading. Higher coefficients of determination for the logarithmic case do not necessarily imply that this case is better from the viewpoint of explaining cost variance in the original numbers. Comparisons of the standard errors for these two cases is usually not possible without a full examination of the differences as illustrated in Figs. 10 and 11.

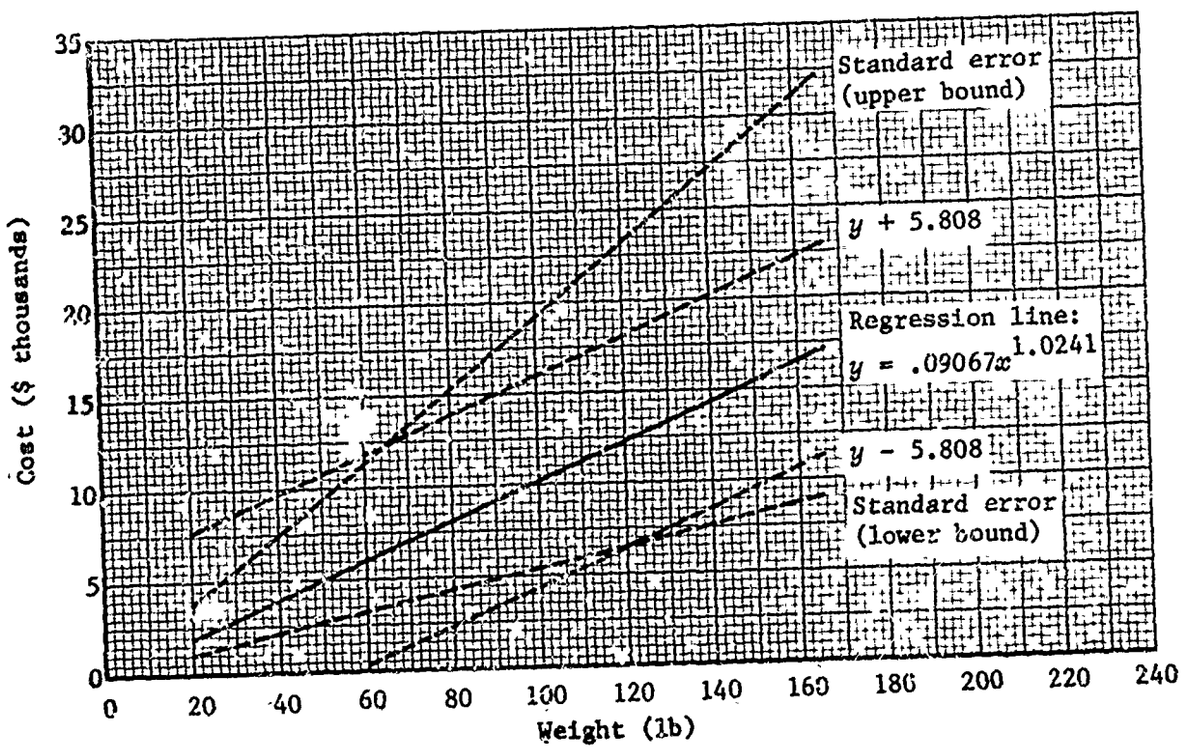

Fig. 11--Comparison of standard error for logarithmic equation with interval based on standard error from arithmetic equation

However, on the positive side, some relationships are, in fact, nonlinear, and logarithmic transformations provide a practical means for estimating nonlinear exponential relationships with linear estimating techniques. Although there are techniques for estimating exponential forms directly with nonlinear estimating techniques,* there are also some difficulties in comparing and evaluating these results. Because the direct estimating techniques for exponential forms are nonlinear, they do not possess all the properties that are required to permit the direct application of standard regression analysis.

Another useful application of logarithmic regression analysis arises in cases in which empirical evidence or experience indicates that the assumption of proportional variance, rather than constant variance, seems more appropriate. Frequently, a simple scatter diagram such as that shown in Fig. 6 is sufficient to indicate whether proportional or constant variance is more appropriate. Alternatively, the sample could be divided into two or more groups, and tests could be performed on the means of the absolute values of the residuals in the linear case in each group. If the higher values of the dependent variables have residuals that are greater in value, the assumption of proportional variance would be indicated. The use of a logarithmic transformation is a convenient way to transform the data to conform to the requirement of proportional variance. If constant variance is assumed in the logarithms of the numbers, standard regression analysis can be performed in the logarithms. However, the assumption of constant variance in the logarithms implies proportional variance in the original numbers.

Multiple Regression Analysis

To this point, simple (one explanatory variable) regression analysis has been used to examine both the linear and the nonlinear relationship between cost and weight. With the array of data shown in

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*See, for example, Graver and Boren.

Table 1 and the logarithmic transformations of these data, multiple (more than one explanatory variable) regression analysis will now be examined. This section covers the multiple linear and the multiple nonlinear (exponential) case; for the latter, logarithmic transformations will be used. Because the sample documented in Table 1 contains only ten observations, the examination will be limited to various combinations of two rather than three explanatory variables. If additional observations were included in the sample, three explanatory variables might be considered under certain circumstances; however, this number of variables used with ten observations would detract from the credibility of the results. In any event, there is no great loss in limiting the number of variables to two; the essential differences between simple and multiple regression can be illustrated with the two-explanatory variable case.

In the linear case, the estimating equation is of general form

$$y = a + bx + cz. \tag{32}$$

The results for each of the possible combinations of two from the set of three explanatory variables are as follows:

$$C = -3.752 + \underset{(2.61)}{.104(W)} + \underset{(1.72)}{.018(F)}, \tag{33a}$$

$$C = 2.930 + \underset{(1.12)}{.074(W)} + \underset{(0.19)}{.0047(P)}, \tag{33b}$$

$$C = -0.526 + \underset{(2.82)}{.045(P)} + \underset{(2.38)}{.027(F)}, \tag{33c}$$

where C = cost in thousands of dollars,

W = weight in pounds,

F = frequency in megahertz,

P = power in watts.

The number in parentheses below each of the estimated coefficients is the value of the t-ratios for each of these coefficients. However,

since an additional variable has been added, the degrees of freedom for these equations is 7 rather than 8, as it was for the simple case. Thus, the appropriate value of t in testing the null hypothesis for each of the coefficients is 1.895 rather than 1.860.

To understand the use of t-ratios in multiple regression equations, the meaning of the multiple regression coefficients must be understood. In each case, the multiple regression coefficients shows the *net* effect of an explanatory variable. For example, Eq. (33a) can be interpreted as follows: For a given frequency, a 1-lb increase in weight will cause a $104 increase in cost. Alternatively, for a given weight, a 1-MHz change will cause the expected cost to change by $18. As the independence between the explanatory variables decreases, the validity of this interpretation and the use of multiple variables diminish. For example, if weight and frequency are related in such a way that a change in weight cannot be assumed with frequency constant, the use of both variables in a single multiple regression equation can produce spurious results (e.g., the wrong sign on a coefficient, such as a negative sign for the weight coefficient).

Fortunately, there are quantitative indicators that are useful in evaluating empirically the significance of such interdependencies on regression results. Allowance for interdependence is built into the formula for calculating the standard error of each coefficient in multiple regression equations. Thus, the t-ratios in a multiple regression not only serve to indicate the significance (or nonsignificance) of each of the explanatory variables but also indicate when there is an unacceptably strong relationship between these variables.

From Eq. (33b), it can be seen that the inclusion of power with weight causes weight to become nonsignificant at the 10-percent level of significance. Weight was, however, significant at this level in the simple regression case. The coefficient of determination between weight and power is .553, which indicates that over 50 percent of the total variance in weight could be explained by a regression of weight on power. Thus, the adverse effect on the significance of weight that results from the inclusion of power can be attributed to the existence of interdependence between these two variables.

As the degree of interdependence increases, regression results become less stable and more indeterminant. As a consequence, the t-ratio should not be the sole test for assessing the amount of interdependence present.* Further, it is not possible to give a precise cutoff point at which explanatory variables must always be considered too interdependent. A coefficient of .9 or more will almost certainly cause problems; one of .3 or less usually will not. The array of correlations and coefficients of determination among the explanatory variables should always be examined in the early stages of analysis, and, to the extent possible, the use of interdependent explanatory variables should be avoided.

It is also possible for variables to be nonsignificant in multiple regression equations, even when there is no high level of interdependence. For example, in Eq. (33a) the coefficient of frequency is nonsignificant at the 10-percent level although the coefficient of determination between frequency and weight is only .091. Frequency in conjunction with weight is simply not a useful explanatory variable. Regardless of the reason, nonsignificant variables should not ordinarily be retained in regression equations used for cost estimating. Only one of the three multiple regression equations shown above produces an acceptable result: This is Eq. (33c), in which frequency and power are used as explanatory variables, and both are statistically significant.

The question arises, For cost-estimating purposes, is the multiple regression with power and frequency preferable to the simple regression with weight as the explanatory variable? To find an answer, the other measures by which the regression equations are judged must be compared: the standard error of estimate, the coefficient of variation, and the coefficient of determination. These are shown in Table 4 for each of the multiple regressions for comparison with the results obtained from

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*In the limiting case of the two explanatory variable regressions in which one variable is an exact linear function of another, the regression results become completely indeterminant since the attempt is then to fit a plane in two dimensions, and there are an infinite number of planes intersecting each line in the two-dimensional space. An excellent discussion of this point is found in John Johnson, *Econometric Methods*, McGraw-Hill Book Company, Inc., New York, 1963, pp. 201-207.

the simple regression. The primary concern in this comparison is between the multiple regression with frequency and power and the simple regression with weight, since the power and frequency equation is the only one in which both the explanatory variables are significant. For completeness, however, the results for all three of the linear multiple regressions are shown and will be discussed.

Table 4

COMPARISON OF MULTIPLE-LINEAR WITH SIMPLE-LINEAR REGRESSION RESULTS

	Explanatory Variables			
Statistical Measures	*Weight*	*Weight and Frequency*	*Weight and Power*	*Frequency and Power*
Standard error	5.808	5.204	6.192	4.999
Coefficient of variation	0.641	0.574	0.683	0.552
Coefficient of determination	0.325	0.526	0.329	0.563
Degrees of freedom	8	7	7	7

Equation (33a), in which weight and frequency are used, appears to give slightly better results in a comparison with the other measures. However, the coefficient of the frequency variable is not significant at the 10-percent level. As a consequence, the improvement is not a statistically significant one. The generalized test to determine whether the incremental improvement associated with the addition of a variable is significant uses an F-statistic.* The test performed with this statistic is similar to the t-test. In this case, the null hypothesis is that the increment is not significant. The statistic used to test this null hypothesis is

$$F = \frac{\text{Increment of explained variance} \div \text{degrees of freedom}}{\text{Remaining unexplained variance} \div \text{degrees of freedom}}.$$

*See F. E. Croxton, D. J. Cowden, and S. Klein, *Applied General Statistics*, 3d ed., Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1960, p. 627.

This can be rewritten as

$$F = \frac{(R^2 - r^2)/1}{(1 - R^2)/7}, \tag{34}$$

where R^2 = the coefficient of determination of the equation that includes weight and frequency,

r^2 = the coefficient of determination of the equation with weight alone.

Equation (34) shows only 1 degree of freedom involved in the numerator, which is the incremental degree of freedom lost by adding another coefficient. The degrees of freedom in the denominator equal the number of observations in the sample less the number of coefficients estimated.

Substituting the appropriate coefficients of determination in the formula for the F-statistic, we obtain

$$F = \frac{(.526 - .325)}{(1 - .526)/7} = \frac{(.201)(7)}{.474} = 2.97. \tag{35}$$

This value falls short of the critical value of F, which equals 3.95 at the 10-percent level of significance. Thus, the null hypothesis is accepted, and we conclude that the net increment in explained variance associated with the addition of frequency to the equation containing weight is insufficient to establish that the improvement is not due to chance.

In Eq. (33b), in which weight and power are used as explanatory variables, it can be seen that the loss of the degree of freedom associated with adding another variable more than offsets the slight increase in the proportion of explained variance (R^2). As a result, the standard error in this case is greater than it is for the case where weight is used alone (6.192 versus 5.808). Thus, not only are the variables not significant, but the equation would also produce slightly less satisfactory (larger) prediction intervals than simple regression, although the coefficient of determination is slightly larger.

Equation (33c), in which power and frequency are used as explanatory variables, compares favorably with the simple regression in which

weight is used, and thus far appears to be the best estimating equation derived. However, to complete the analysis, the nonlinear equations should be examined. These equations, expressed in the logarithms of the original numbers, have the general form

$$\log y = \log a + b(\log x) + c(\log z). \tag{36}$$

The results for each of the possible different combinations of two that can be developed from the set of three explanatory variables are as follows:

$$\log C = -1.8576 + \underset{(3.78)}{1.1385}(\log W) + \underset{(1.62)}{.2743}(\log F), \tag{37a}$$

$$\log C = -0.6582 + \underset{(1.46)}{.7145}(\log W) + \underset{(.842)}{.1542}(\log P), \tag{37b}$$

$$\log C = -1.1933 + \underset{(8.44)}{.5756}(\log P) + \underset{(5.91)}{.6085}(\log F), \tag{37c}$$

where C = cost in thousands of dollars,

W = weight in pounds,

F = frequency in megahertz,

P = power in watts.

The other measures required to complete the comparisons between the various equations are shown in Table 5.

The major patterns in the nonlinear multiple regression equations compared with the nonlinear simple case are similar to those for the linear equations. The use of both frequency and weight produces slightly better results, but the coefficient of the frequency variable is not statistically significant at the 10-percent level. The use of power with weight again produces a larger standard error than the simple case although the coefficient of determination is slightly larger. In all respects, the best nonlinear equation is the equation that uses power and frequency as explanatory variables. In addition, this nonlinear equation has a significantly larger coefficient of determination

than the best linear equation. The best linear equation also uses power and frequency and has a coefficient of determination of .563. The nonlinear form has a coefficient of determination of .913.

Table 5

COMPARISON OF MULTIPLE-NONLINEAR WITH SIMPLE-NONLINEAR REGRESSION RESULTS

	Explanatory Variables			
Statistical Measures	*Log Weight*	*Log Weight and Log Frequency*	*Log Weight and Log Power*	*Log Frequency and Log Power*
Standard error	0.2763	0.2518	0.2814	0.1312
Coefficient of determination	0.560	0.680	0.600	0.913
Degrees of freedom	8	7	7	7

The remaining question is whether the nonlinear results are sufficiently superior to the linear results to conclude that the nonlinear equation should be used in preference to the linear one. The standard error for each in the original numbers at the mean and as a percentage of the mean should be compared. If the results show that the standard error for the nonlinear case is smaller, this evidence, and the fact that the coefficient of determination for the nonlinear case is much larger, can be used as a basis to judge in favor of the nonlinear form.

When the formulas shown in Eq. (31) are used, the end points that delineate the standard error at the mean for the nonlinear equation are

$$\frac{9.060}{10^{.1312}} \quad \text{and} \quad (9.060)10^{.1312}$$

When the end points are simplified, the following values are obtained:

$$6.698 \quad \text{and} \quad 12.255.$$

These results, expressed as differences from the mean, give values of 2.362 below and 3.195 above the mean. Thus, the lower band of the

standard error for the nonlinear case is 26 percent of the mean and the upper bound is 35 percent. This compares favorably with the coefficient of variation from the linear case, which is about 55 percent. Thus, given the inherent limitations of the small sample size of 10, the use of the nonlinear form improves the results significantly. The preferred equation is

$$\log C = -1.1933 + .5756(\log P) + .6085(\log F), \tag{38}$$

or

$$C = (.0641)P^{.5756}F^{.6085},$$

where C = cost in thousands of dollars,
P = power in watts,
F = frequency in megahertz,
log = logarithm base 10.

This equation is also acceptable on logical grounds since the estimated relationships between cost and power and cost and frequency are positive.

Documentation

Once an estimating relationship has been developed, a report that documents the data, assumptions, and analytical results is indispensable. The following guidelines for preparing the report are suggested:

1. Describe the scope and coverage of the study and of the equations that have been developed.
2. Assuming that the study has provided for a survey of work already performed in the area of interest (a desirable part of any cost-research study), prepare a summary of the survey results.
3. Describe the major input data used in the study. The raw and adjusted data, which includes data for both the dependent and explanatory (independent) variables, should be documented to

the extent that is feasible. Include data not only for those cost categories and characteristics used in the final estimating equations, but also for those characteristics that were considered but were eliminated in the process of analysis. Describe and explain fully any adjustments to the raw data; indicate limitations and accuracy. Because one of the outputs of a cost-research study is the data base itself, documentation should be such that the data base will be useful in future studies.

4. Identify sources and dates of the data.
5. Define each dependent and explanatory variable considered in the study. (Unambiguous definitions of weapon system characteristics and cost elements are usually more involved than appears at first glance.)
6. Provide the major dependent- versus single-explanatory-variable scatter diagrams used in the study. The diagrams should be labeled to identify each data point.
7. Document the final equations as well as the other major equation forms examined in the study; include such statistics as the standard error of estimate, coefficient of determination, coefficient of variation, and prediction intervals to the extent that they are derived for each equation. Other criteria that are considered appropriate for indicating the goodness of fit and prediction capabilities of the equations should be described.
8. For the major final equations, prepare a table such as Table 6 to show the observed values of the dependent variables, the estimated values, the deviations, and the percent deviation from the observed values. In addition, prepare a scatter diagram, such as that illustrated in Fig. 12, on which the observed values versus the estimated values are plotted. The points on the diagram should be labeled to identify each item. (Figure 12 shows that the apparent problem of stratification illustrated in Fig. 4 has been eliminated by including frequency as an explanatory variable.)

9. Describe the alternative equations that were considered and why they were rejected. The report should convey a sense of the improvement that results from a high degree of selectivity in choosing the final forms. The alternative equations could show
   a. The use of different explanatory variables;
   b. Different forms of the equations, e.g., linear, multiplicative (linear in the logarithms), or other nonlinear forms;
   c. The use of different forms of the dependent variables, e.g., cost per pound or cost per item;
   d. The use of stratified dependent variables grouped into subcategories that are determined by such factors as ship or missile type, weight, frequency, or speed regime.
10. Describe any special methodology in an appendix if only of special interest (e.g., a sophisticated mathematical approach).
11. Describe the cost-estimating methods fully and clearly. It should be possible to reconstruct the results of the study from the data base as it is given in the report. The major assumptions, statistical and otherwise, used in the derivation of the equations should be explicitly stated.

Table 6

ACTUAL AND ESTIMATED COSTS OF AIRBORNE COMMUNICATION EQUIPMENT

Actual Cost ($)	*Estimated Cost ($)*	*Deviation (Actual less estimate) ($)*	*Percent Deviation*
22,200	13,768	+8,432	+38
17,300	16,970	+1,330	+8
11,800	17,388	-5,588	-47
9,600	9,238	+362	+4
8,800	9,238	-438	-5
7,600	6,435	+1,165	+15
6,800	6,885	-85	-1
3,200	4,581	-1,381	-43
1,700	2,062	-362	-21
1,600	1,261	+339	+21

Average of absolute value of percent deviations = 20

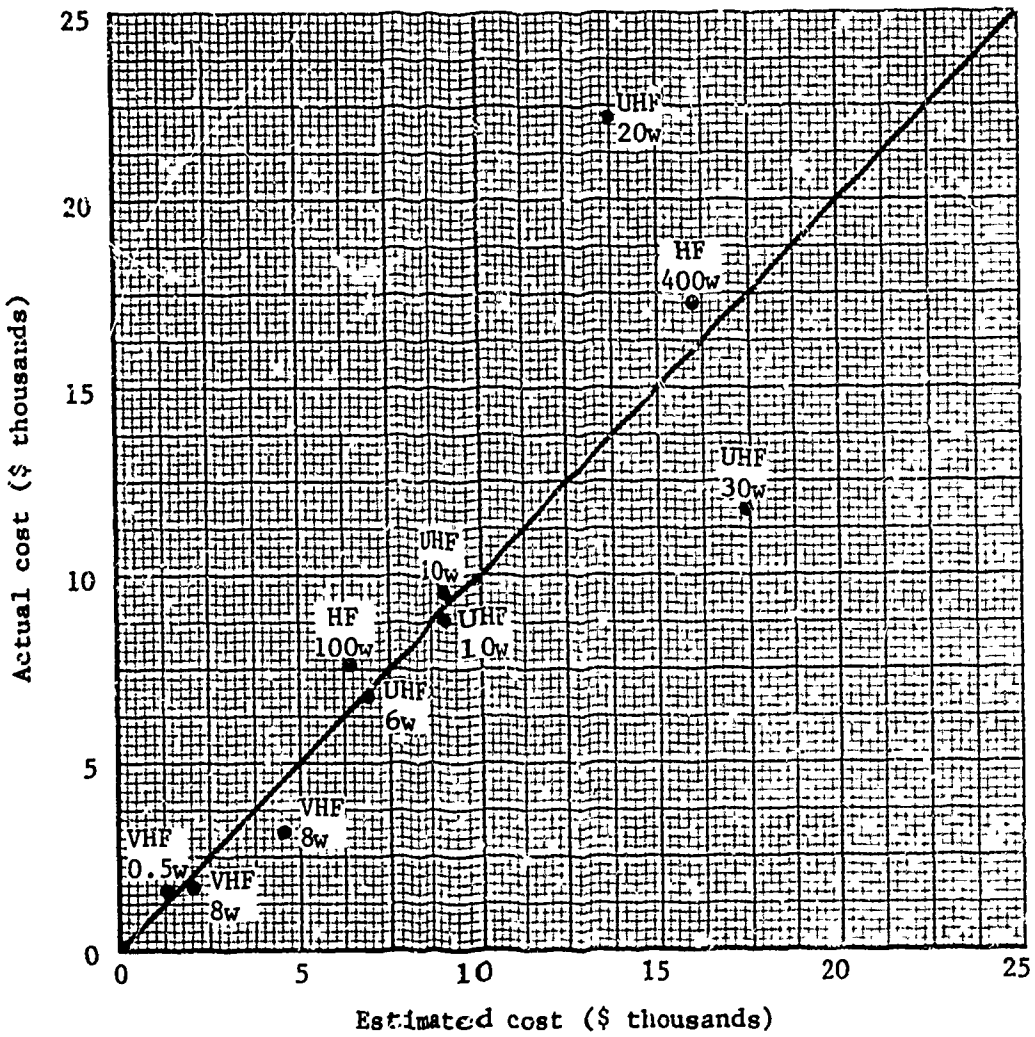

Fig. 12--Actual cost versus estimated cost

12. Provide an example to illustrate the procedure for using the final cost-estimating relationship.
13. Describe the limitations of the final equations as specifically as possible. State the range of characteristics over which the estimating procedure applies and any other restrictions on the population covered by the equations.

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IV. USE OF COST-ESTIMATING RELATIONSHIPS

THE WIDESPREAD USE of estimating relationships in the form of simple cost factors, equations, curves, nomograms, and rules of thumb attests to their value and to the variety of situations in which they can be helpful. But an estimating relationship can only be derived from information on past occurrences, and the past is not always a reliable guide to the future. As all horseplayers know, the favorite runs out of the money often enough to prove that an estimate based on past performance is very likely to be wrong. Admittedly, there may be other factors at work in a horserace, but the problem remains the same as that encountered in any attempt to predict the course of future events, i.e., how much confidence can be put in the prediction? This question dominates all other considerations in any discussion of the use of estimating relationships.

These remarks are not intended to depreciate the value of estimating relationships. They are an important tool in an estimator's kit and, in many cases, the only tool. Thus, it is essential that their limitations be understood to preclude their improper use. The limitations of estimating relationships stem from two sources: first, the uncertainty inherent in any application of statistics and second, the uncertainty that an estimating relationship is applicable to a particular article. The first pertains primarily to articles well within the bounds of the sample on which the relationship is based; even here, uncertainty may be found. The second source refers to those cases in which the article has characteristics somewhat different from those of the sample. Although extrapolation beyond the sample is universally

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deplored by statisticians, it is universally practiced by cost analysts in dealing with advanced hardware because, in most instances, it is precisely those systems outside the range of the sample that are of interest. The question is whether the equation is relevant to the case under investigation, although good statistical practice would question the validity of such an approach.

Characteristics of the Estimating Relationship

The degree of emphasis placed on statistical treatment of data can cause two fundamental points to be overlooked: first, that an estimating relationship must be reasonable and second, that it must have predictive value.

Reasonableness can be tested in various ways--by inspection, by simple plots, and by complicated techniques that involve an examination of each variable over a range of possible values. Inspection will often suffice to indicate that an estimating relationship is not structurally sound. For example, the following equation is the result of an exercise at the Air Force Institute of Technology in which students were asked to develop cost-estimating relationships for small missiles:

$$C = 8347.5 + 150.6W - 1149.1R, \qquad (1)$$

where C = cost of airframe + guidance and control,

W = weight in pounds,

R = range in miles.

This equation fits the data very well, but it states that as range increases, the cost decreases; such an assumption appears to be in error. If cost is a function of range, the relationship should be direct rather than inverse. To investigate further, choose two hypothetical but reasonable values for W and R within the range of the sample data: 38.5 - 157 lb for W, 5.0 - 14.8 mi for R. Table 1 shows that Missile B, although heavier and with greater range than Missile A, is estimated as the cheaper of the two, which is contrary to experience. A reexamination of the sample data and the equation is in order.

Table 1

SAMPLE COST COMPARISON OF TWO MISSILES

Hypothetical Missile	*Airframe Weight + Guidance and Control (lb)*	*Range (mi)*	*Estimated Airframe Cost + Guidance and Control ($)*
A	50	5	10,132
B	75	10	8,152

When an estimating relationship is developed to make a particular estimate, it may have little predictive value outside a narrow range. As an example, consider the following equation for estimating the cost of solid-propellant motors for small missiles:

$$\text{Cost} = 1195.6 + .000003I^2, \tag{2}$$

where I = total impulse.

The equation fits the sample data very well:

Missile Motor	*Observed Cost ($)*	*Estimated Cost ($)*
A	2600	2660
B	1700	1693
C	1250	1265
D	1750	1781

If it were appropriate to use statistical measures for a sample of 4, Eq. (2) explains over 99 percent of the total variance. But, note that the constant 1195.6 accounts for 94 percent of the cost of Motor *C* and that the cost of all motors smaller than Motor *C* will be about $1200. Because of the I^2 term, the influence of total impulse is likely to be too pronounced for motors larger than those in the sample.

A common method of examining the implications of an estimating relationship for values outside the range of the sample is to plot a scaling curve as shown in Fig. 1. Scaling curves may be plotted on either arithmetic or logarithmic graph paper as Fig. 1 illustrates; cost analysts usually prefer the log-linear representation. The theory on which a scaling curve is based is as follows: As an item increases in weight (or another dimension), the incremental cost of each additional pound

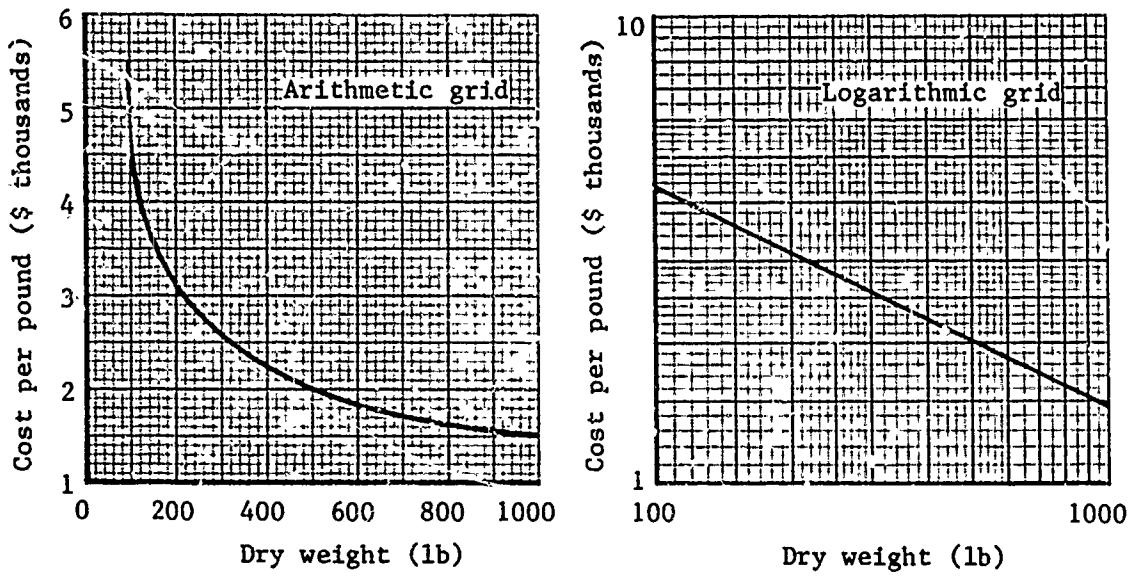


Fig. 1--Scaling curve: cost per pound versus dry weight

(or square foot, watt, horsepower) will decrease or increase in a predictable way. Thus, in Fig. 1 the cost per pound of an electrical power subsystem in a manned spacecraft decreases from about $4200 to $1400 as the total weight increases from 100 to 1000 lb. The slope of the curve is fairly steep; if the curve were extended to the right, it might be expected to flatten. Eventually, the curve might become completely flat at the point at which no more economies of scale can be realized, but it is unlikely that the slope would ever become positive.

Now examine Fig. 2 in which total impulse is plotted against cost per pound-second based on values obtained from an estimating relationship. Two differences are immediately seen. First, the lefthand portion of the curve is unusually steep. Second, the slope becomes positive when total impulse exceeds about 22,000 lb-sec. In some instances, fabrication problems increase with the size of the object being fabricated and a positive slope may result. No such problems are encountered in the manufacture of small, solid-propellant rocket motors, however, and continued economies of scale are to be expected.

Figure 2 illustrates another point: A more useful estimating relationship could have been obtained by drawing a trend line rather than

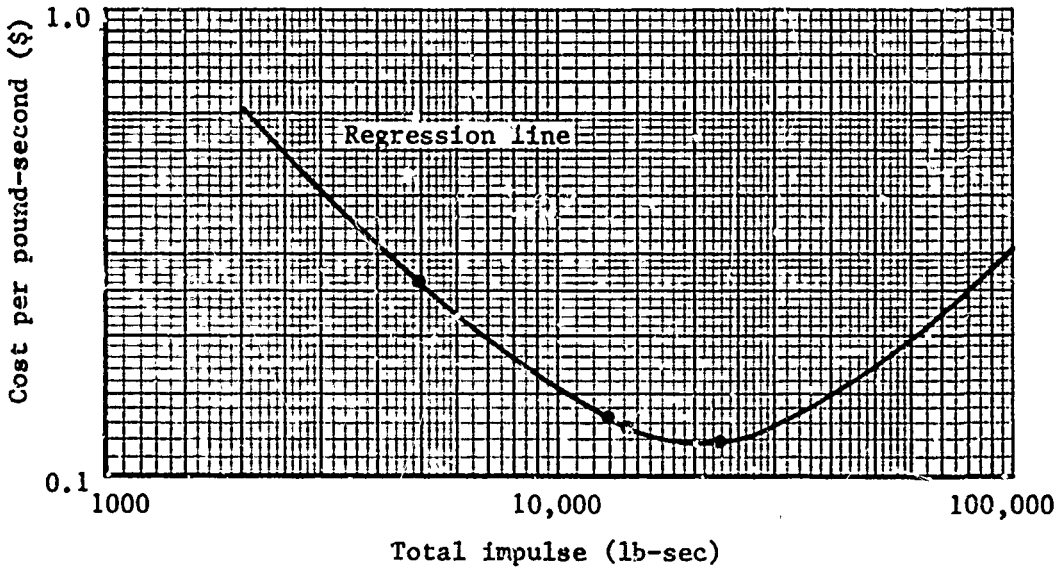


Fig. 2--Cost per pound-second versus total impulse

by fitting a curve to the four data points. With a small sample, it is often possible to write an equation that fits the data perfectly, but the equation is useless outside the range of the sample. Statistical manipulation of a sample this size rarely produces satisfactory results.

A final example of the kind of error that undue reliance on statistical measures of fit may bring about is based on an estimating equation for aircraft airframes. Initially, the equation for estimating airframe production labor hours was based on a sample of 44 aircraft. It then seemed that a grouping of the aircraft by type should give better correlation and, in fact, when the bombers, fighters, trainers, and cargo aircraft were considered separately, the average deviation between estimates and actual values was markedly reduced. For example, in the case of trainer aircraft, the average deviation was reduced from 20 to 6 percent, and a more useful estimating relationship was obtained. In the case of fighters, however, although average deviation was reduced from 15 to 11 percent, the estimating equation exhibited the flaw shown in Eq. (3):

$$\text{Manufacturing hours} = 4.28\ (\text{weight})^{1.08}(\text{speed})^{.4}. \qquad (3)$$

The exponent of weight is greater than 1.0, which means that when speed is held constant and weight increased, the man-hours per pound of airframe weight will increase. This can be seen in Fig. 3. The dashed lines show scaling curves derived from the total sample of 44 aircraft. These portray the normal relationship--as weight increases, hours per pound decrease. The regression equation gives the opposite results because the general trend in fighter aircraft has been for increased speed to be accompanied by increased weight, which causes an emphasis on the weight variable. It cannot be assumed, however, that all new fighters will conform to this trend; the equation, if used at all, would have to be used with great care.

The advice is frequently given that an estimating relationship should not be used mechanically. This implies (1) that the function must be thoroughly understood and (2) that the hardware involved must be understood as well. To illustrate the first point, examine an estimating relationship for direct manufacturing hours derived from a

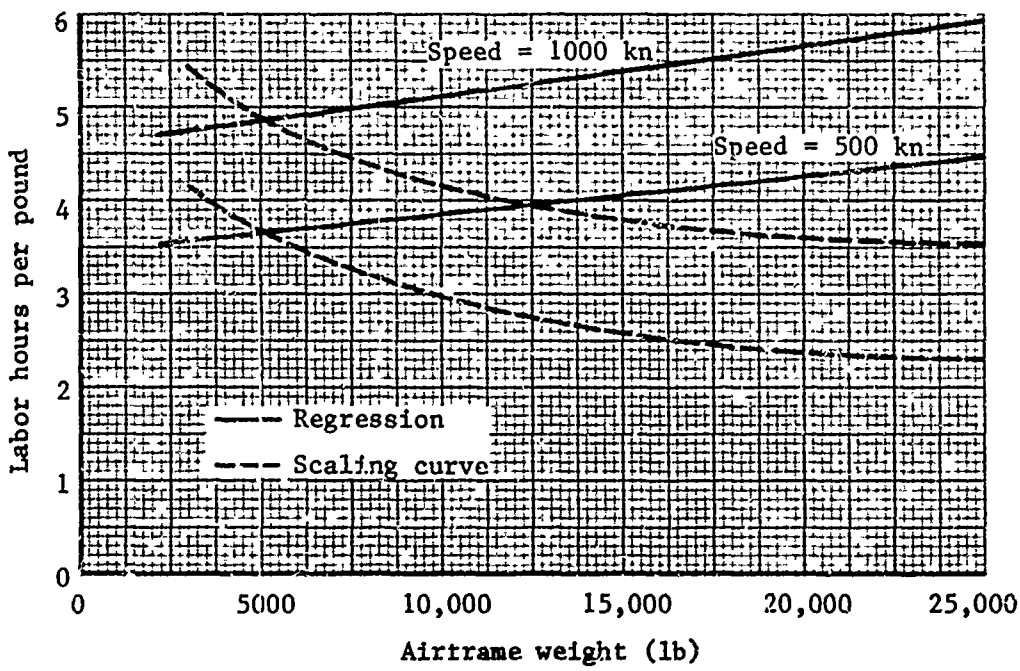


Fig. 3--Comparison of regression lines with scaling curves

sample of Navy and Air Force airframes:

$$H_{100} = 1.45W^{.74}S^{.43}, \tag{4}$$

where H_{100} = manufacturing labor hours required to produce the 100th airframe,

W = gross takeoff weight in pounds,

S = maximum speed in knots.

The multiple correlation coefficient is 0.98 and the coefficient of variation is .016 in logarithmic terms. Despite these satisfactory measures of fit, a comparison of the actual manufacturing hours for each airframe in the sample with those estimated by the equation provides a better understanding of how the relationship relates to the real world. In such a comparison, as shown by Table 2, 33 percent of the estimates differ from the actuals by more than 20 percent, and 7 percent differ by more than 30 percent. These figures imply that an analyst with only the estimating relationship on which to rely may or may not obtain a good estimate. However, if the less acceptable results can be explained in some way, the analyst is then in a much better position to understand the strengths and weaknesses of the equation.

Since this estimating relationship is based on gross takeoff weight and maximum speed, an initial hypothesis to explain the variations might be that the estimates decrease in quality at one end of the weight or speed range or in certain combinations of weight and speed. In this

Table 2

COMPARISON OF ACTUAL AND ESTIMATED MANUFACTURING HOURS

Difference Between Actual Hours and Estimated Hours (%)	*Number of Airframes*	*Percentage of Sample*
10 or less	15	56
11-20	3	11
21-30	7	26
31-40	2	7

case, however, as shown in Fig. 4, the poorer estimates are scattered throughout the sample, which indicates no consistent bias because of the explanatory variables.

A second hypothesis might be that the manufacturing history of the airframes in the sample explains the discrepancies and, in general, this hypothesis is valid. Of the nine airframes in the sample for which estimates differed from actuals by 20 percent or more, several were considered problem airframes, i.e., airframes for which the manufacturer encountered an abnormal number of problems in meeting weight and performance specifications. Interestingly enough, these were not aircraft in which a major state-of-the-art advance was being attempted. Another cause for discrepancy was the interspersion of different models of the same aircraft in a single lot: For example, reconnaissance versions of a bomber were interspersed among bomber airframes. Situations of

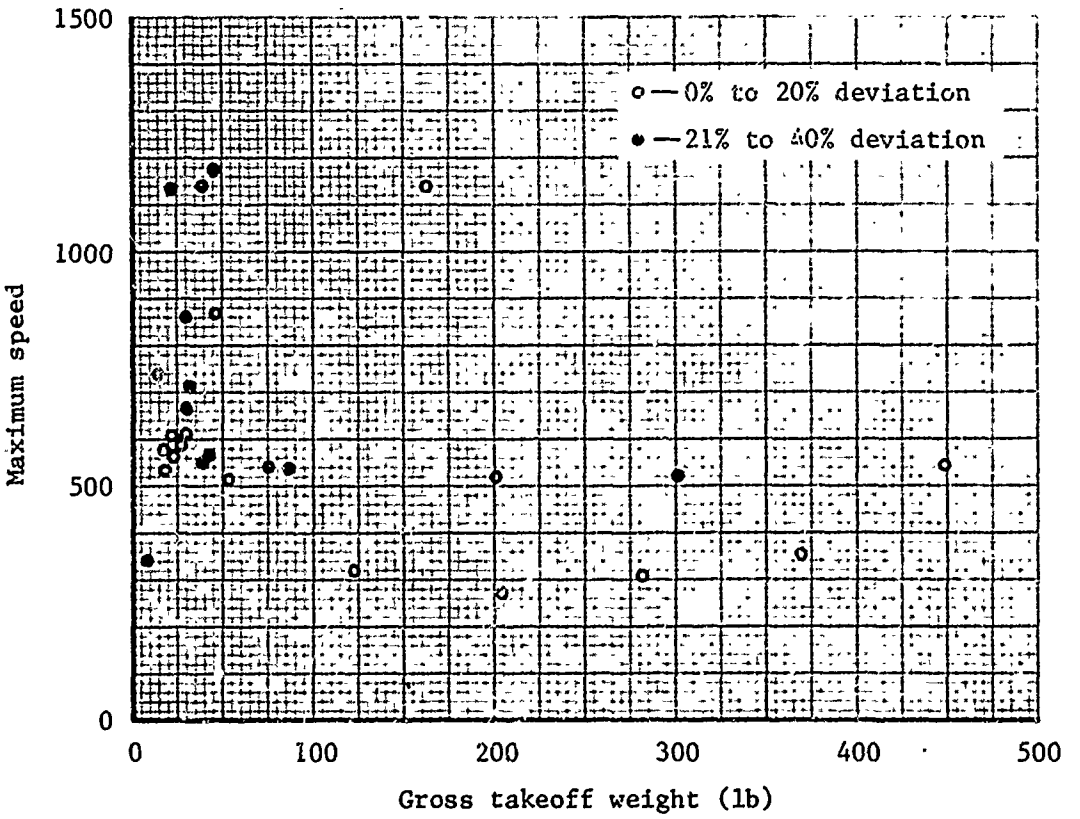


Fig. 4--Plot of sample data

this kind increase direct labor requirements. The two airframes for which the estimates were the poorest and for which almost 40 percent less labor than the equation predicted was required, were vastly different ones--a large transport and a supersonic fighter. Production of one of these airframes benefited from the manufacturer's concurrent experience with a commercial airplane of similar configuration. The other case cannot be explained. The amount of labor involved in producing the airplane was unusually low.

Although it is not possible to resolve all uncertainties with the information available, an estimator can feel reasonably confident that the estimating relationship does not contain a systematic bias, that it should be applicable to normal production programs, and that it provides reasonable estimates throughout the breadth of the sample.

Hardware Considerations

The sample included aircraft having gross takeoff weights of 6100 lb to 450,000 lb and maximum speeds of 300 kn to 1200 kn. Suppose that a proposed new aircraft has a gross weight of 600,000 lb and a maximum speed of 1700 kn. Should Eq. (4) be used as the estimating equation in this case? The same question could arise for an aircraft with weight and speed that are in the sample range, but which is to be fabricated by a new process or out of a new material. Again, the estimator must decide whether the equation is relevant or how it can be modified to be useful. An estimating relationship can be used properly only by a person familiar with the type of equipment whose cost is to be estimated. To say that an analyst who estimates the cost of a destroyer should be familiar with the characteristics of destroyers is a truism; however, an estimator is sometimes far removed from the actual hardware. Further, he may be expected to provide costs for air-to-air missiles one week and for a new antiballistic missile system the next. The tendency in such a situation may be to use the equation that appears most appropriate without taking the required measures to determine whether the equation is applicable.

To illustrate the problem, assume that a new supersonic bomber is proposed having a gross weight of 450,000 lb and a maximum speed of 1700 kn. Equation (4) may be inappropriate because the speed is far beyond the range of the sample. On the other hand, no equation exists for aircraft in that speed range, and an estimate is required. This situation may be regarded as the normal one, and there is no choice but to use what is available. In this example, Eq. (4) gives 542,000 direct labor manufacturing hours.

The next step is to compare the result with other similar systems to see if the estimate appears reasonable. In this instance manufacturing hours versus gross weight are plotted for several other large aircraft as shown in Fig. 5. The supersonic bomber estimate SSB_1 is substantially above the trend as it should be, because a 1700-kn airframe will be more difficult to build than a subsonic airframe of the same size. If other information is lacking, an estimator might accept the figure of 542,000 hr. In this case, however, all the airframes in the sample were fabricated almost entirely of aluminum; an airframe built to withstand the heat generated by sustained flight in the atmosphere at a speed of about Mach 3 will require a metal such as stainless steel or titanium. The question that occurs is whether the speed variable in the equation fully accounts for this change in technology.

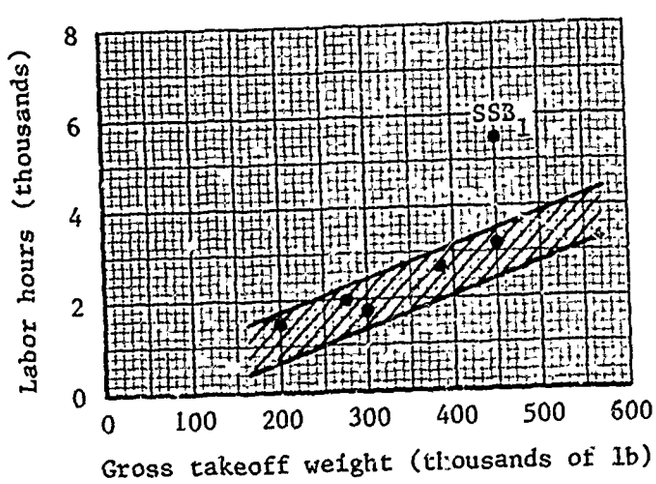


Fig. 5--Trend line for large aircraft

One way to answer this question is to plot a second scatter diagram, with speed as the independent variable. Figure 6 shows labor hours per pound of airframe weight plotted against speed with a calculated line of best fit drawn through the scatter. If an airframe weight of 125,000 lb out of a gross weight of 450,000 lb is assumed, the estimate of 542,000 hr is equal to 4.3 hr-lb of airframe, which not only is below the calculated trend line, but is also below any reasonable trend line that can be drawn through the sample. (This point is shown as SSB_1 in Fig. 6.)

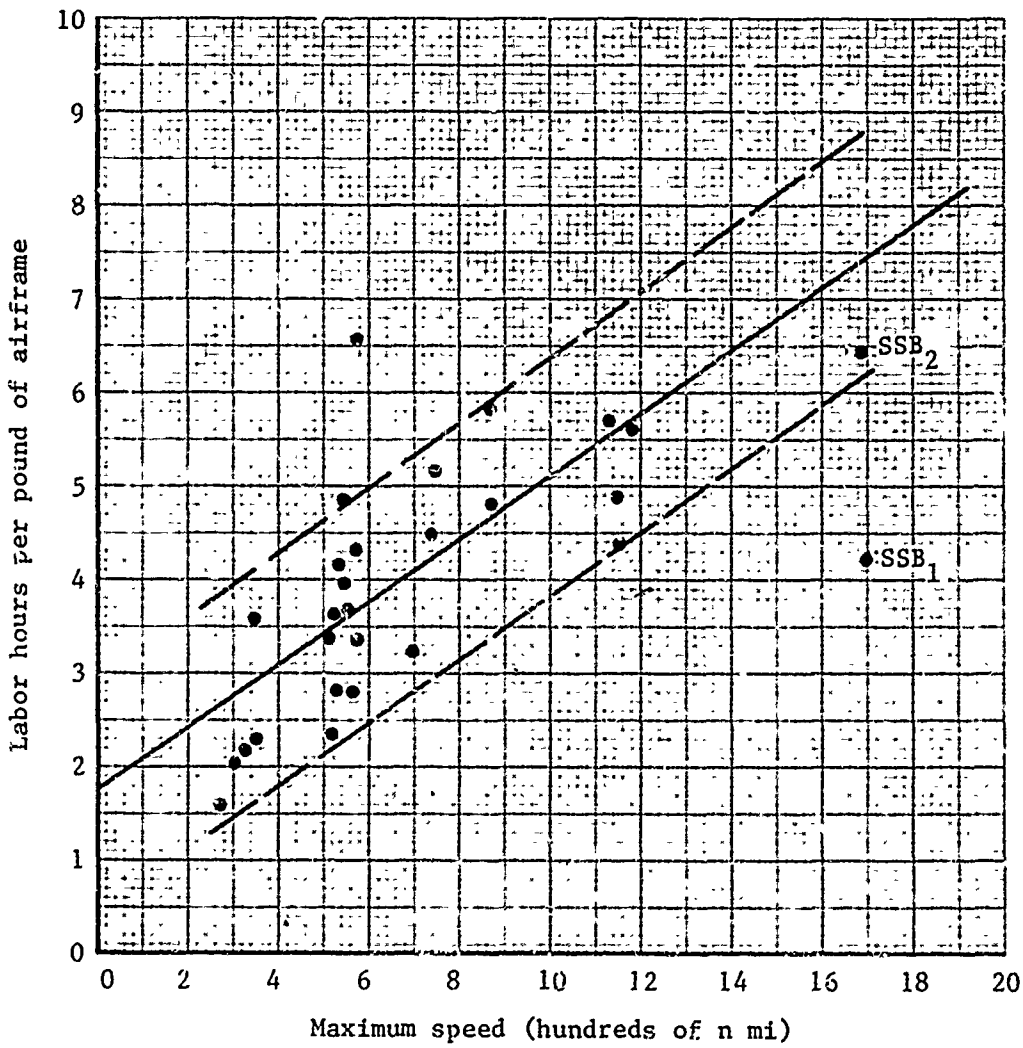


Fig. 6--Labor hours per pound versus maximum speed

Three possible estimates can now be considered: 542,000 hr based on speed and weight; about 300,000 hr based on weight alone as shown by Fig. 5; and about 925,000 hr based on speed alone as shown by the regression line in Fig. 6 (7.4 hr-lb × 125,000 lb = 925,000 hr). More information is needed to narrow the range.

Although data are less than abundant, several experimental and prototype aircraft have been fabricated using stainless steel and titanium. On the basis of prototype experience, one manufacturer maintains that a titanium airframe requires twice the number of hours that an aluminum airframe requires; however, manufacturing hours for an aluminum airframe can vary considerably. A second approach is more precise. An examination of actual data for different airframes with speeds of Mach 3 and above shows that these airframes require about 1.5 times as many hours as the estimating relationship of Eq. (4) indicates, which implies 813,000 hr or 6.5 hr-lb for the supersonic bomber. (This point is shown as SSB_2 in Fig. 6.) On the basis of current knowledge, the estimate appears to be reasonable. Further measures could be taken in the form of another independent estimate that uses a different estimating relationship. An estimator does not have this option for most kinds of hardware, because estimating relationships are not plentiful. However, in the case of airframes, a number of equations have been developed over the years; it is good practice to use one to confirm an estimate made with another.

Judgment in Cost Estimating

The need for judgment is often mentioned in connection with the use of estimating relationships. Although this need may be self-evident, one of the problems in the past has been too much reliance on judgment and too little on estimating relationships. The problem of introducing personal bias with judgment has been studied in other contexts, but the conclusions are relevant to this discussion. In brief, a person's occupation or position seems to influence his forecasts. Thus, a consistent tendency toward low estimates appears among those persons whose interests are served by low estimates, e.g., proponents of a new weapon or

support system whether in industry or in government. Similarly, there are people in industry and in government whose interests are served by caution. As a consequence, their estimates are likely to run higher than would be the case were they free from all external pressures. (In fairness to this latter group, however, overestimates are rare enough to suggest that caution is not a quality to be despised.)

The primary use of judgment should be to decide first, whether an estimating relationship can be used for an advanced system, and second, if so, what adjustments will be necessary to take into account the effect of a technology that is not present in the sample. Judgment is also required to decide whether the results obtained from an estimating relationship are reasonable. This does not mean reasonable according to a preconception of what the cost ought to be, but reasonable in a comparison with the past cost of similar hardware. A typical test for reasonableness is to study a scattergram such as Fig. 7 of costs of analogous equipment at some standard production quantity. The estimate of the article may be outside the trend lines of the scattergram and still be correct, but an initial presumption exists that a discrepancy has been discovered and that this discrepancy must be investigated. An analyst who emerges from his deliberations with an estimate implying that new, higher performance equipment can be procured for less than the cost of existing hardware knows that his task is not finished. If, after research, he is convinced that the estimate is correct, he should then be prepared to explain the new development that is responsible for

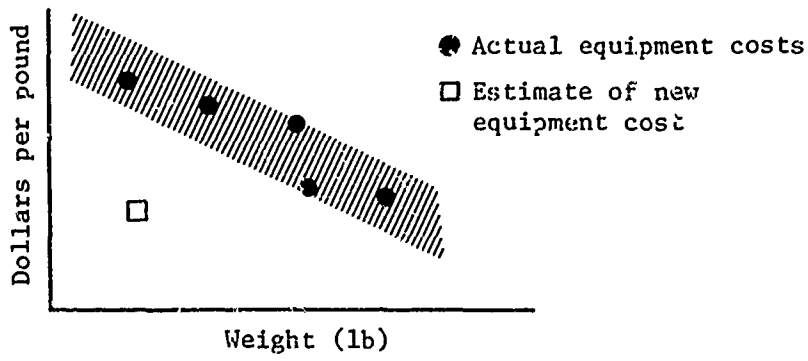


Fig. 7--Cost comparison of analogous equipment

the decrease in cost. He should not raise the cost arbitrarily by a percentage to make the figure appear more acceptable or because he feels that the estimate is too low. (Such adjustments are the province of management and are generally occasioned by reasons somewhat removed from those discussed here.) Judgments must be based on well-defined evidence. The only injunction to be observed is that any change in an estimate be fully documented to ensure that the estimate can be thoroughly understood, and to provide any information that may be needed to reexamine the equations in the light of the new data.

V. THE LEARNING CURVE

FOR MANY YEARS the aerospace industry has made use of what variously have been called "learning," "progress," "improvement," or "experience" curves to predict reductions in cost as the number of items produced increases. The learning process is a phenomenon that prevails in many industries; its existence has been verified by empirical data and controlled tests. Although there are several hypotheses on the exact manner in which the learning or cost reduction can occur, the basis of learning-curve theory is that each time the total quantity of items produced doubles, the cost per item is reduced to a constant percentage of its previous cost. Alternative forms of the theory refer to the incremental (unit) cost of producing an item at a given quantity or to the average cost of producing all items up to a given quantity. For example, if the cost of producing the 200th unit of an item is 80 percent of the cost of producing the 100th item, and if the cost of the 400th unit is 80 percent of the cost of the 200th, and so forth, the production process is said to follow an 80-percent *unit* learning curve. If the average cost of producing all 200 units is 80 percent of the average cost of producing the first 100 units, the process follows an 80-percent *cumulative average* learning curve.*

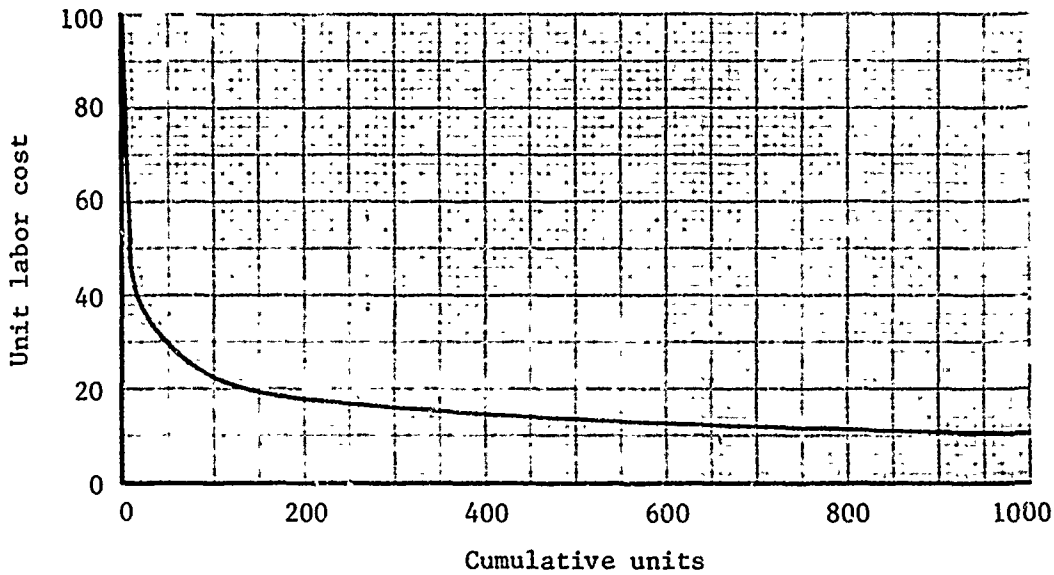
*The quantities mentioned in connection with the learning concept presuppose the inclusion of all items. As concerns the J-79 engine used on the F-4 airplane, one would expect engine costs for the first 100 F-4s to be more than that for the second 100 airplanes. Although this is true, what is important is that the J-79 has been used on several other types of aircraft, and these uses, including full spare engines, must be considered in learning-curve analysis.

Either formulation of the theory results in a power function that is linear on logarithmic grids. Figure 1 shows a unit curve for which the reduction in cost is 20 percent with each doubling of cumulative output, the upper figure showing the curve on arithmetic grids and the lower on logarithmic grids. The arithmetic plot illustrates that the percentage reduction in cost in each unit is very pronounced for the early units. On an 80-percent curve, for example, cost decreases to 28 percent of the original value over the first 50 units. Over the next 50 units, it declines only 5 more percentage points, i.e., down to 23 percent of unit 1 cost. The factors that account for the decline in unit cost as cumulative output increases are numerous and not completely understood. Those most commonly mentioned are

1. Job familiarization by workmen, which results from the repetition of manufacturing operations.
2. General improvement in tool coordination, shop organization, and engineering liaison.
3. Development of more efficiently produced subassemblies.
4. Development of more efficient parts-supply systems.
5. Development of more efficient tools.
6. Substitution of cast or forged components for machined components.
7. Improvement in overall management.

The above list of relevant factors is not complete, and it tends to understate the importance of the item sometimes considered the most important--labor learning. Labor cost, however, cannot decline through experience gained by workmen unless management also becomes more efficient. In other words, it is necessary for management to organize and coordinate more efficiently the work of all manufacturing departments so that parts and assemblies will flow smoothly through the plant.

Labor cost is not the only element of manufacturing that declines as cumulative output increases. A learning curve exists for unit materials cost. The materials category frequently includes much purchased equipment, which in turn includes a substantial number of engineering, tooling, and labor hours. Unit hours decline as production quantities



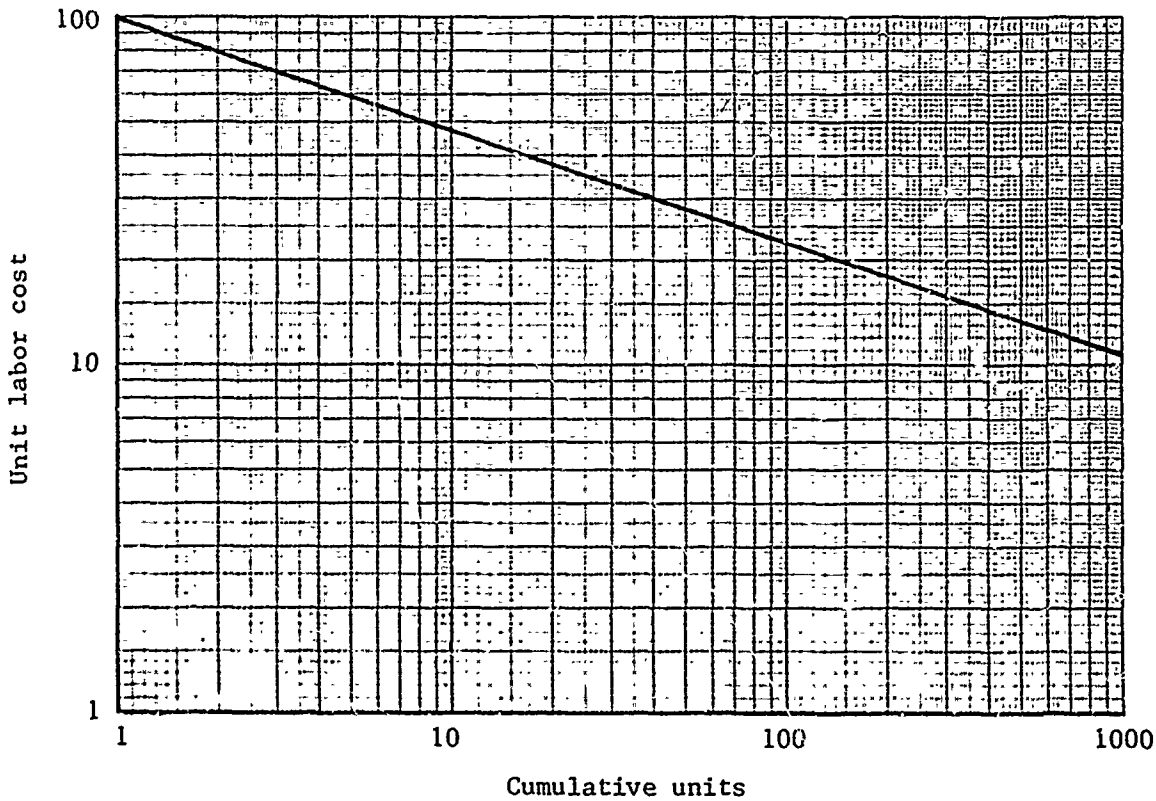


Fig. 1--The 80-percent learning curve on arithmetic and logarithmic grids

increase, and the contractor who buys in successive lots is generally able to negotiate a lower price for each lot. Decreases in raw material costs are generally attributed to two factors as cumulative output increases: The workmen learn to work the raw materials more efficiently, cutting down spoilage and reducing the rejection rate, and management learns to order materials from suppliers in shapes and sizes that reduce the amount of scrap that must be shaved and cut from the pieces of sheet or bar to fabricate the item of equipment. Substitution of forgings for machined parts also reduces the amount of scrap material.

A second factor that is probably responsible to a lesser extent for the decline in materials cost is the pricing policy of the raw material suppliers. These suppliers generally reduce the price per pound for the various kinds of raw materials if an order is sufficiently large. Although the learning curve pertains to cost reductions as materials are applied to successive lots and not to reductions due to volume purchases, segregation of the two effects is imperfect. This may account for differences observed in learning-curve slopes.

A third major component of cost--overhead--also declines with cumulative output, but as a result of the method of allocating overhead and not because of a perceptible relationship between overhead rate and cumulative output. Direct labor hours per unit decline as cumulative output increases, and overhead is distributed to each unit on the basis of direct labor cost or hours. As a consequence, it is inappropriate to discuss a learning curve for this element of cost.

The Log-linear Hypothesis

The relationship between cost and quantity may be represented by a power (log-linear) equation of the form

$$y = ax^b,$$

where x equals the cumulative production quantity. The relationship

corresponds to a unit or a cumulative average learning curve according to whether y is the cost of the xth unit or the average cost of the first x units. The constant a is the cost of the first unit produced. The exponent b, which measures the slope of the learning curve, bears a simple relationship to the constant percentage to which cost is reduced as the quantity is doubled. If S represents the fraction to which cost decreases when quantity doubles, the equation becomes

$$S = \frac{y_{2x}}{y_x} = \frac{a(2x)^b}{ax^b} = 2^b \quad \text{or} \quad b = \frac{\log S}{\log 2}.$$

This equation shows that for a value of S equal to 75 percent, the corresponding value of b is*

$$\frac{\log .75}{\log 2} \quad \text{or} \quad -.415.$$

Log-linear Unit Curve

If a production process follows a unit learning curve of the form $y_u = ax^b$, the cumulative cost T of producing the first n units is

$$T = a \sum_{x=1}^{n} x^b.$$

The cumulative average cost y_c of producing the first n units is then

$$y_c = \frac{T}{n} = \frac{a}{n} \sum_{x=1}^{n} x^b.$$

The relationship between the unit curve and the cumulative average curve is shown by Fig. 2. The function y_c is not log-linear; however, as x becomes larger, y_c approaches asymptotically the value

*In learning-curve literature, the term slope often refers to this percentage reduction; e.g., a 75-percent slope means a curve with a b value of -.415.

$$\frac{a}{b+1} x^b,$$

which differs from the expression for unit cost only by the constant factor $1/(b + 1)$. Consequently, if unit cost has been estimated at a sufficiently large quantity, the cumulative average cost for the same quantity may be approximated by multiplying the unit measure by $1/(b + 1)$.*

Log-linear Cumulative Average Curve

When a production process follows a log-linear cumulative average curve rather than a unit curve, the basic functional form is still $y = ax^b$ but can be written $y_c = ax^b$, where y_c is the average cost of the first x units. The cumulative cost for producing x units is simply $y_c x$, or ax^{b+1}, and the unit cost is obtained from the function

$$a[x^{b+1} - (x - 1)^{b+1}].$$

The relationship between a linear cumulative average curve and the resulting unit curve is illustrated in Fig. 3. The unit curve is not log-linear; however, as x becomes larger, y_u quickly approaches asymptotically the value

$$(b + 1)ax^b,$$

which differs from the cumulative average cost equation only by the constant factor $(b + 1)$.

These equations may appear cumbersome, but in practice much of the work involved in using learning curves has been simplified by the

*Whether a quantity is sufficiently large for the asymptotic method to provide a good approximation depends on the slope of the learning curve. For a 90-percent curve, the asymptotic method produces an error of about 1 percent at quantity 100; for a 75-percent curve, the error at quantity 100 is almost 5 percent and does not decrease to 1 percent until a quantity of almost 2000 has been reached.

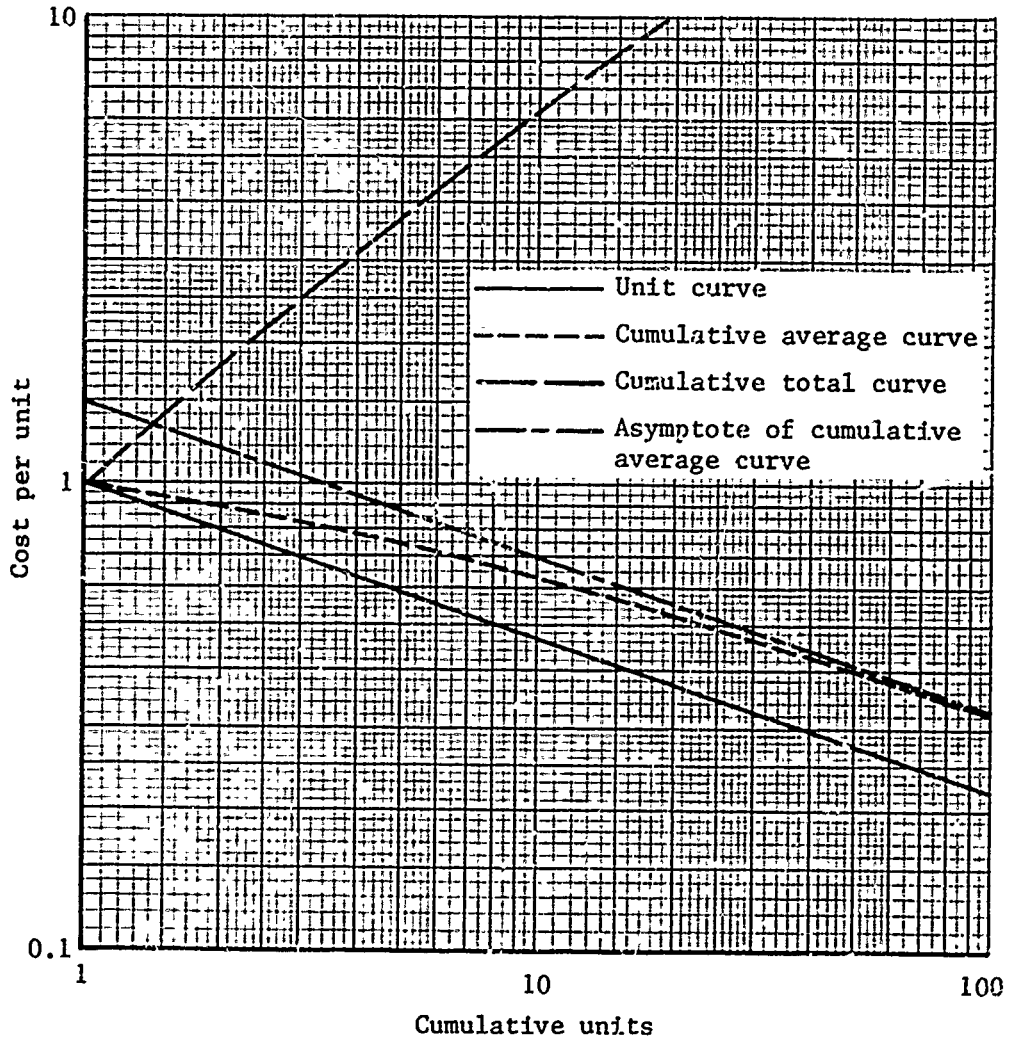


Fig. 2--Log-linear unit curve (80-percent slope)

preparation of tables giving the relationship between cumulative total, cumulative average, and unit cost for a range of slopes and quantities. Table 1 gives values for these relationships for a 70-percent curve when a, the cost of the first unit produced, is equal to 1. To illustrate how such a table is used, assume a log-linear unit curve and a quantity n of 20 units. The total cost of 20 units is approximately 7.4, the cumulative average cost of 20 units is .37, and the cost of the 20th unit is .214, in terms of the cost of the first unit. The unit cost of .214 appears in the dual-headed column, y_u, y_c, since a

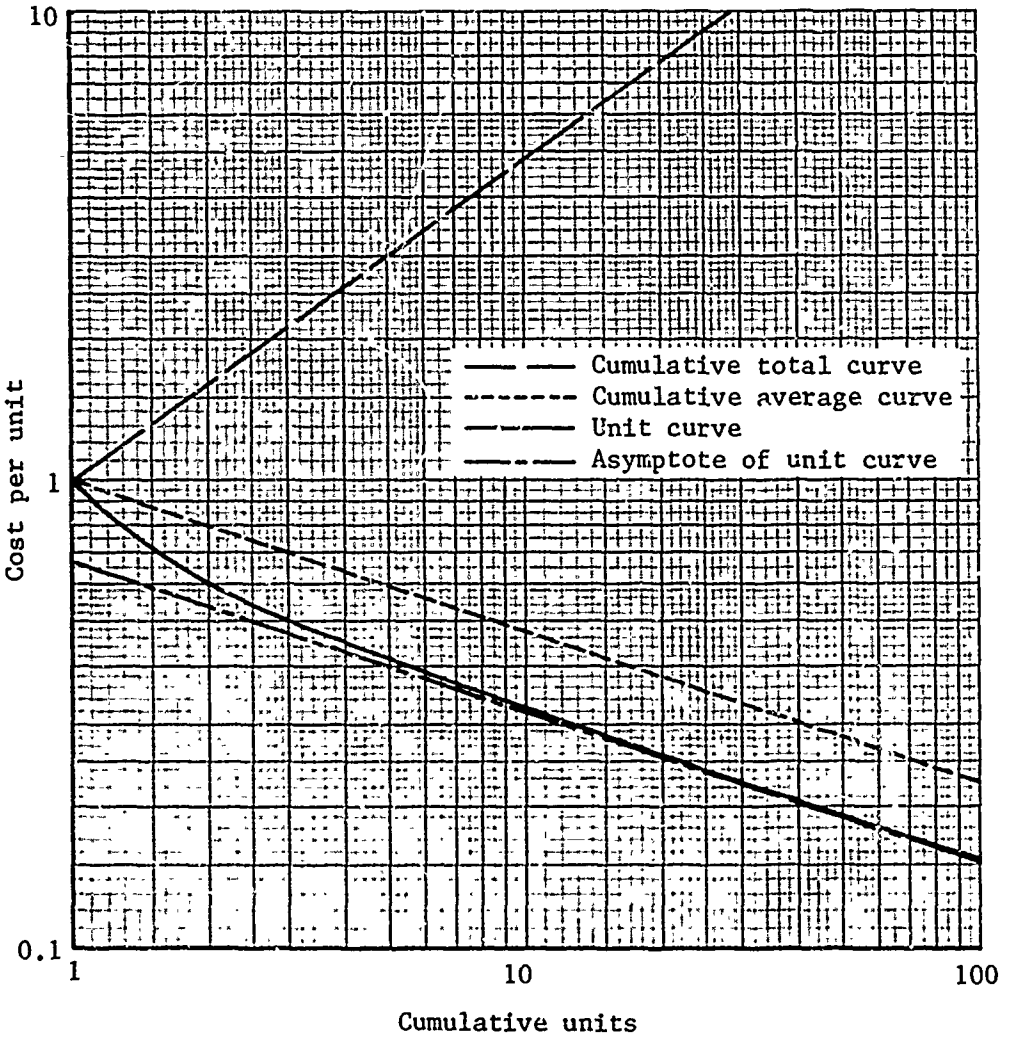


Fig. 3--Log-linear cumulative average curve (80-percent slope)

log-linear unit curve is assumed. If a log-linear cumulative average cost curve is assumed, this column presents the cumulative average cost. One column serves to present both log-linear unit and log-linear cumulative average since the functional form of the equation, $y = ax^b$, is the same in either case.

In practice, the unit cost is most frequently considered to be linear, but there are sufficient exceptions to suggest that the choice must be based on past experience. Once the choice is made, however, it is of the utmost importance to apply the technique consistently.

Table 1

70-PERCENT CURVE DATA

	Log-linear Unit			Log-linear Cumulative Average	
	Cumulative Total	Cumulative Average	Unit	Cumulative Average	Unit	Cumulative Total
n	T	y_c	y_u, y_c		y_u	$y_c n$
1	1.000000	1.000000	1.000000		1.000000	1.000000
2	1.700000	0.850000	0.700000		0.400000	1.400000
3	2.268180	0.756060	0.568180		0.304541	1.704541
4	2.758180	0.689545	0.490000		0.255459	1.960000
5	3.195027	0.639005	0.436846		0.224232	2.184232
6	3.592753	0.598792	0.397726		0.202125	2.386357
7	3.960150	0.565736	0.367397		0.185419	2.571777
8	4.303150	0.537894	0.343000		0.172223	2.744000
9	4.625979	0.513998	0.322829		0.161460	2.905460
10	4.931771	0.493177	0.305792		0.152465	3.057925
11	5.222928	0.474812	0.291157		0.144802	3.202727
12	5.501336	0.458445	0.278408		0.138173	3.340900
13	5.768511	0.443732	0.267174		0.132365	3.473266
14	6.025688	0.430406	0.257178		0.127222	3.600487
15	6.273896	0.418260	0.248208		0.122626	3.723113
16	6.513996	0.407125	0.240100		0.118487	3.841600
17	6.746721	0.396866	0.232726		0.114734	3.956334
18	6.972702	0.387372	0.225980		0.111310	4.067644
19	7.192481	0.378552	0.219780		0.108171	4.175816
20	7.406536	0.370327	0.214055		0.105279	4.281095
21	7.615284	0.362633	0.208748		0.102604	4.383699
22	7.819094	0.355413	0.203810		0.100119	4.483818
23	8.018295	0.348622	0.199201		0.097804	4.581622
24	8.213180	0.342216	0.194886		0.095639	4.677261
25	8.404015	0.336161	0.190835		0.093609	4.770870
26	8.591037	0.330425	0.187022		0.091702	4.862572
27	8.774462	0.324980	0.183425		0.089904	4.952476
28	8.954487	0.319803	0.180024		0.088206	5.040682
29	9.131290	0.314872	0.176803		0.086600	5.127282
30	9.305035	0.310168	0.173745		0.085076	5.212359
31	9.475873	0.305673	0.170838		0.083629	5.295988
32	9.643943	0.301373	0.168070		0.082252	5.378240
33	9.809373	0.297254	0.165430		0.080940	5.459180
34	9.972281	0.293302	0.162908		0.079687	5.538867
35	10.132777	0.289508	0.160496		0.078490	5.617357

As is evident from Table 1, large errors could result if one type of curve was confused with the other.

Nonlinear Hypothesis

Throughout this section it will be assumed that the log-linear hypothesis applies, i.e., that the learning curve is linear when plotted on logarithmic grids. It must be mentioned, however, that this is not the only possible formulation of the learning curve. A number of studies have suggested that the curve is not log-linear. One of the best known of these is the Stanford Research Institute investigation of 20 World War II aircraft. The study proposed

$$y = \frac{a}{\sqrt{x + B}}$$

as a more reliable expression of the relationship between man-hour cost and cumulative output. The decision to find a substitute function was apparently prompted by a visual inspection of several series that seemed to indicate a concavity when viewed from below in the unit learning curve.* This concavity has been recognized independently in other studies.

However, in some cases both the labor and production cost curves develop convexities beyond certain values of cumulative output. In the theory of a linear unit curve, it is implicitly assumed that constituent curves (fabrication, subassembly, and major and final assembly) are parallel to the linear unit curve, implying that the rate of learning on all production jobs in all departments is the same. However, it is to

*In this context, concavity means that when plotted on logarithmic grids the curve declines at an increasingly steep slope as it moves away from the y-axis. In the formulation

$$y = \frac{a}{\sqrt{x + B}},$$

the curve becomes essentially linear as x becomes large relative to B.

be expected that the departmental learning curves could have different slopes from each other (e.g., fabrication, 80 percent; subassembly, 75 percent; and major and final assembly, 70 percent). The sum of these curves (the unit curve) would be convex when viewed from below and approach as a limit the flattest of the departmental curves.

Much literature is available describing the bases for, and hypotheses about, learning curves, and it is beyond the scope of this section to attempt to cover this background material in any detail.* For this discussion, it is stipulated that the learning curve is a useful and accepted estimating tool, particularly in the aerospace industry, that the log-linear curve is the one most commonly used, and that a knowledge of its mechanics is indispensable to persons making or using cost estimates.

Plotting a Curve

In the graphical display of learning curves, the problem is to represent the average cost for a lot or a complete contract, since typically, man-hours or costs are not recorded by unit. See, for example, the following table:

Lot	*Units*	*Manufacturing Hours per Lot*
1	1-10	5,830
2	11-20	4,370
3	21-50	10,550
4	51-100	14,750

*There is one subject that is not discussed in the literature: the effect of production rate on unit cost. Economic theory generally holds that this relationship can be described by a U-shaped function: First, cost declines as production rate increases; next, it is insensitive to rate over some range; and eventually, it begins to rise again. In learning-curve applications, on the other hand, it is assumed implicitly that cost is not affected by rate of output (or that the rate is constant). Empirical evidence of the interaction between the volume and rate effects is scanty. For further discussion, see Lee E. Preston and E. C. Keachie, "Cost Functions and Progress Functions: An Integration," *American Economic Review*, Vol. 54, No. 2, Part I, March 1964, pp. 100-107.

To plot a cumulative average curve from these data, the cumulative average hours are computed at the final unit in each lot:

Plot Point	*Manufacturing Hours per Lot*	*Computation*	*Cumulative Average Hours*
10	5,830	5,830 ÷ 10	583
20	4,370	10,200 ÷ 20	510
50	10,550	20,750 ÷ 50	415
100	14,750	35,500 ÷ 100	355

The cumulative average at the 10th unit is 583 hours; this is the first plot point. Successive plot points are at the end of each lot, since these are the points where the cumulative average hour figures apply.

To plot the unit curve it is first necessary to compute the unit hours and then to establish plot points. The unit hours can be taken as an average for each lot:

Lot	*Computation*	*Unit Hours*
1	5,830 ÷ 10	583
2	4,370 ÷ 10	437
3	10,550 ÷ 30	352
4	14,750 ÷ 50	295

The lots can be represented by these unit hour values. The question is, where should the values be plotted? To plot at the lot arithmetic midpoint is to assume that the learning curve can be approximated by a linear curve on arithmetic grids, but as suggested by Fig. 1 such a method of approximation only becomes reasonable for lots following a large number of previous units. Thus, when dealing with a log-linear function, the arithmetic midpoint plot produces the unequal distribution of the area under the curve, as shown in Fig. 4.

The true midpoint is defined as that unit, x_m, which represents the entire lot and which must also reflect the average unit cost, y_m, of the lot. The total cost (or total hours) of the lot is equal to the product of y_m and the number of units in the lot, n. This product will approximate the area under the curve for n units (see Fig. 5).*

*If n represents only integers, the limits of the area must be modified. (See H Asher, *Cost-quantity Relationships in the Airframe Industry*, The Rand Corporation, R-291, July 1, 1956, pp. 34-38.)

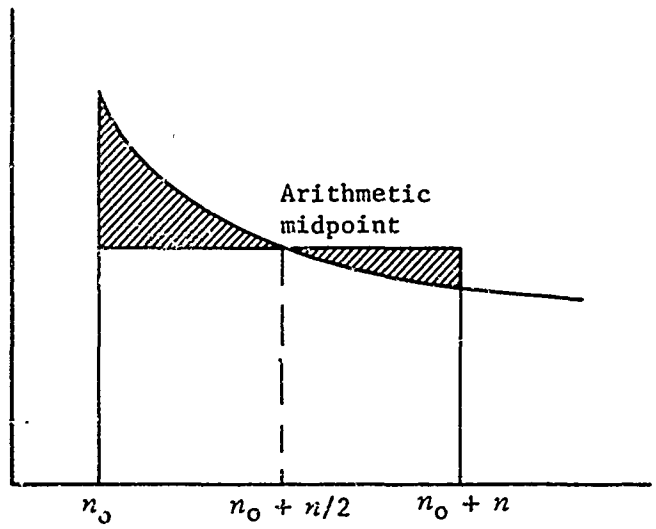


Fig. 4--Learning curve on arithmetic grids

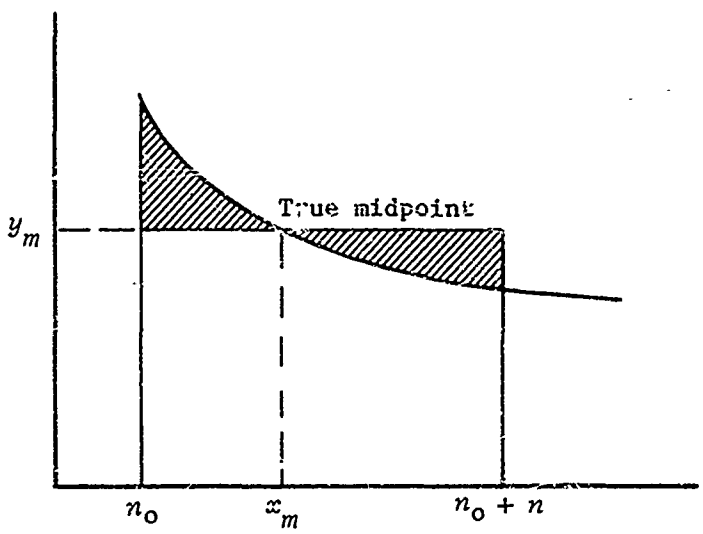


Fig. 5--True lot midpoint on arithmetic grids

Note that if the area under the curve is equal to $y_m n$, the two cross-hatched areas in the figure must be equal. In fact, the exact determination of a true lot plot point for plotting purposes depends on (1) the lot quantity; (2) the type of curve hypothesized, i.e., whether the unit curve or the cumulative average curve is log-linear; and (3)

the *true* value of the slope. Therefore, these values must be known or assumed. The first, the lot quantity, will be known. The second, the nature of the curve, must be assumed. The third, the true value of the slope, is actually never known, and is usually approximated based on prior experience.

It is possible to ascertain the exact lot plot points for each type of curve over a range of slopes and quantities. However, because of the assumptions mentioned above that will usually have to be made regarding both the type of curve and its approximate slope, in most situations there is little need to strive for extreme accuracy. The following discussion provides methods of approximation that do not involve the complicated calculations required to derive the true lot plot point.

As illustrated in Fig. 5, y_m is the average cost for the lot as well as the unit cost of the lot plot point x_m. Therefore, tables similar to Table 1 can be used to derive acceptably accurate plot points. To illustrate, assume a log-linear *unit* curve of 70 percent, a first lot of 10 units, and a first unit cost of 1. Then, the cumulative average cost y_c of the first 10 units is .493. This average cost lies between unit cost values y_u of .568 and .490, i.e., between units 3 and 4 on the unit curve. Arithmetic interpolation yields a value for x_m of slightly less than 4, which is the plot point for this particular lot when a 70-percent log-linear unit curve is assumed. An exact solution to the plot point equation would show the true plot point for a 70-percent curve to be 3.95. Similarly, if the first unit cost is 1 and if a 70-percent log-linear *cumulative average* curve is assumed, data from Table 1 yield a plot-point approximation of slightly less than 3 (the cumulative average cost for 10 units is .306, which lies between unit cost values of .400 and .304, i.e., between units 2 and 3 on the unit curve); the true plot point is 2.98. In this example, the plot points vary because of the assumption that one or the other of the curves is log-linear. This method of approximation produces accurate first-lot plot points for all but very small lot sizes. As a general rule, the steeper the slope and the smaller the lot size, the less accurate this approximation method becomes.

For the successive lots following a preceding quantity, the same

procedure can be used for approximating plot points. To illustrate, again using Table 1, assume that a quantity of 10 units follows the first lot of 10 units. If a 70-percent log-linear *unit* curve and a unit cost of 1 are assumed, the total cost of the second lot may be obtained by subtracting 4.93 (the total cost of the first 10 units) from 7.4 (the total cost of 20 units), or a difference of 2.47. This represents an average cost of .247 for the 10 items in the lot. This value falls between units 15 and 16 on the unit curve, and simple interpolation gives a value of 15.1 for the plot point. If a log-linear *cumulative average* curve is assumed, the approximation value of the plot point is also 15.1. In other words, from Table 1, the difference between the cumulative total for 20 and 10 units, 4.28 and 3.06, respectively, is 1.22, or an average of .122 for the 10 units in the lot. This unit cost lies between .1226 and .1185 or units 15 and 16 on the unit curve.

Tables to permit computation of lot plot points for a range of slopes and lot quantities are available in the literature.* In addition, an easier-to-use, but less accurate, approximation method will be discussed that provides plot points for early lot quantities of less than 100.

Figure 6 presents an approximation of the plot point for the first lot. It illustrates that substantial errors are possible when deriving first-lot plot points. The abscissa represents first-lot quantity and the ordinate the first-lot plot points associated with each quantity. For the upper dashed curve, a 95-percent log-linear unit curve is assumed; for the upper solid line, a 95-percent log-linear cumulative average curve is assumed. Similarly, for the lower lines, 65-percent curves are assumed. Approximation methods suitable for one type of curve cannot be used for another type unless extremely large quantities are dealt with, i.e., well beyond those shown in the figure. Figure 6 also shows the greater sensitivity to slope exhibited by the log-linear cumulative average curve for moderately small first lots.

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*See, for example, H. E. Boren and H. G. Campbell, *Learning Curve Tables*, Vols. 1-3, The Rand Corporation, RM-6191-PR, to be issued.

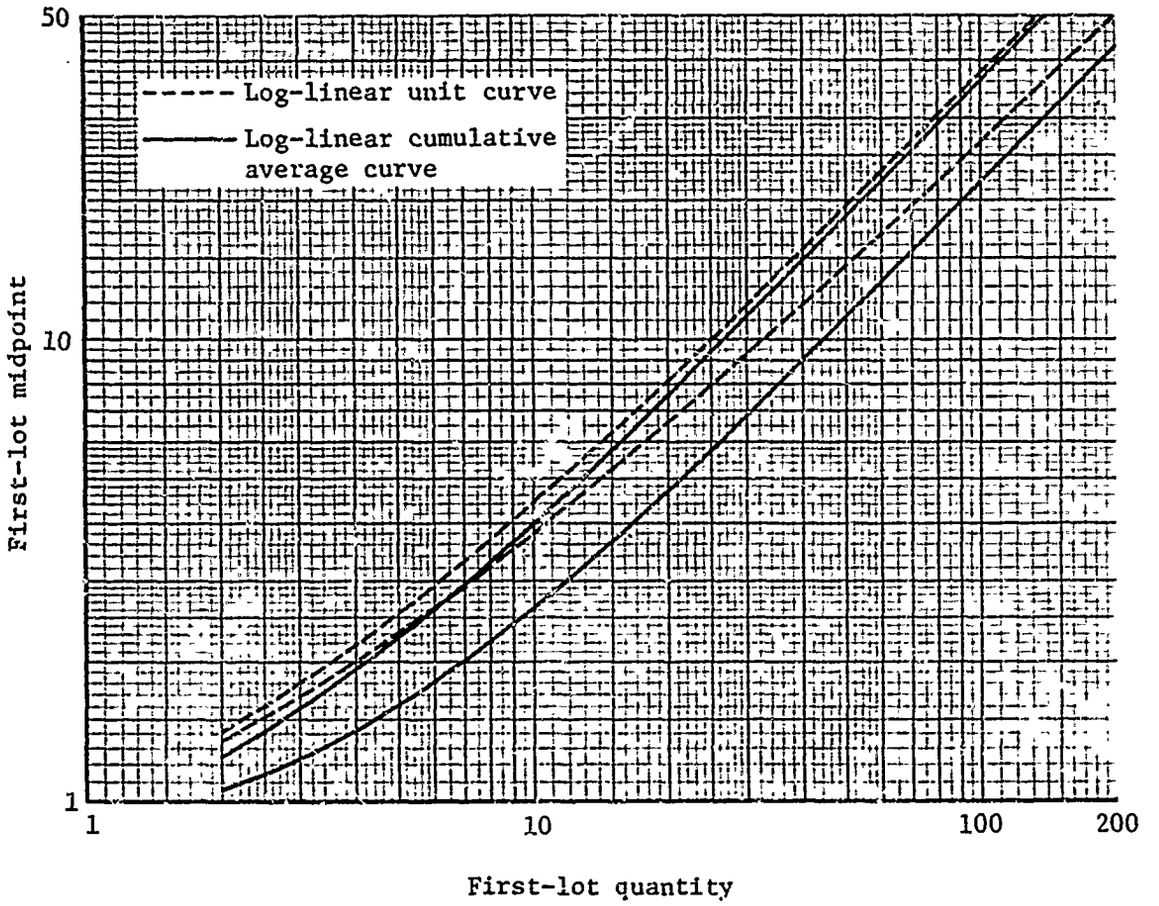

Fig. 6--First-lot midpoints versus first-lot quantities

In addition, it affords an opportunity to approximate quickly the range of error that can be introduced by inappropriate plotting of the cost of the first lot.

Figure 7 gives plot points for follow-on lots. These points represent an average of the range obtained from 65- to 95-percent curves and the range obtained from a log-linear unit or a log-linear cumulative average curve. The graph is used as follows:

1. The first unit of the contract lot is found on the 45-deg line.
2. The curve extending out from this unit is followed to the point on the horizontal axis that represents the last unit of the lot.

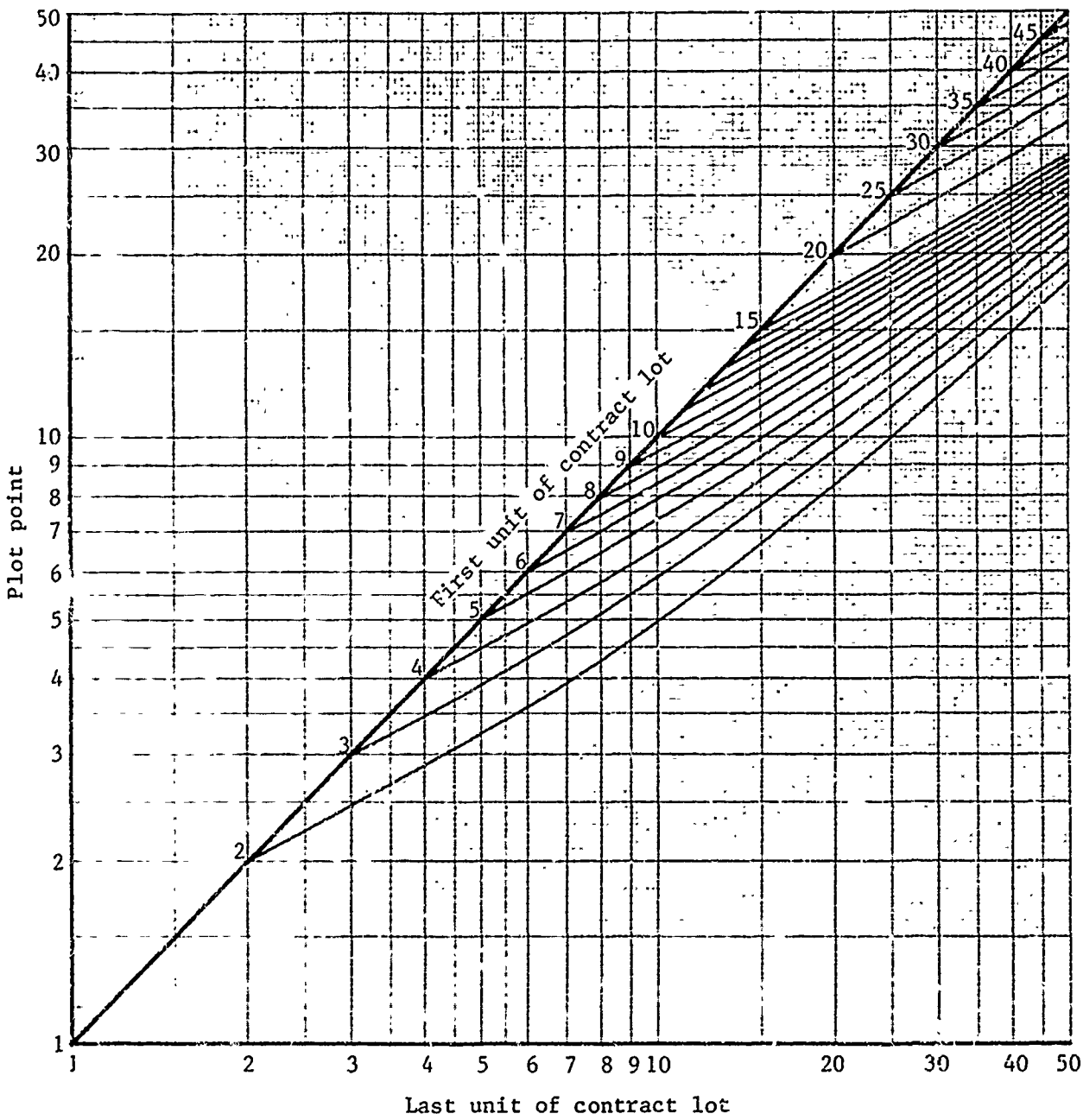

Fig. 7--Plot points for average costs

3. The plot point is read off the vertical axis at that point. Thus, for a lot of 10 units following 10 previous units, the plot point would be slightly over 15.

In practice, plot points for only the first two or three lots, if these comprise more than about 25 units, need be taken from the graph. For succeeding lots, the arithmetic lot midpoint is usually adequate.

As a further illustration, Fig. 8 shows two sets of curves. The lower set of curves was constructed from a series of small contract lots, 10, 29, and 31 units. The upper set of curves was based on two

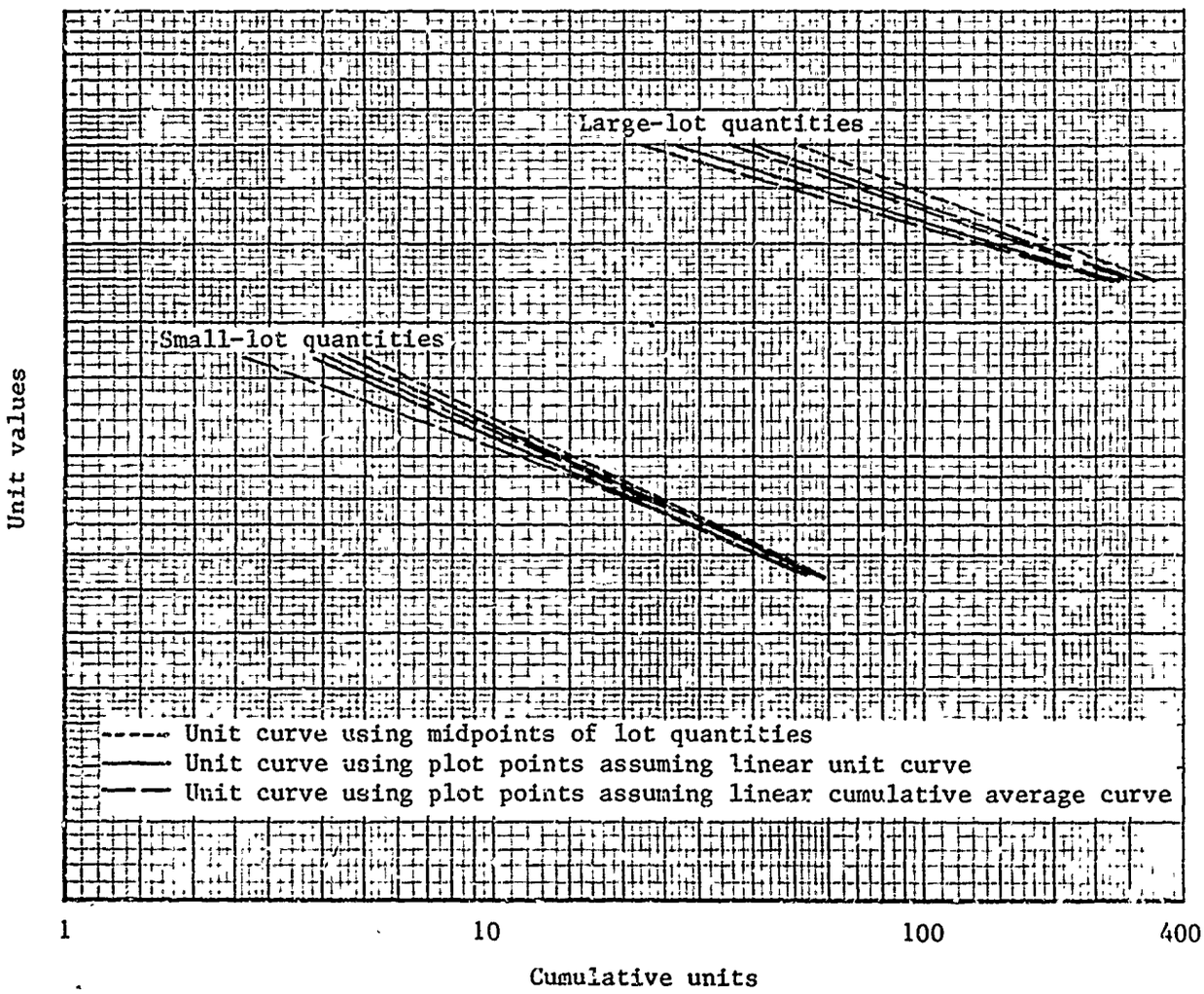

Fig. 8--Unit curves from contract lot averages

large contract lots, 100 and 500 units. With lot average costs, the costs were plotted (1) at lot quantity arithmetic midpoints, (2) at plot points where a log-linear unit curve for 65- and 95-percent slopes was assumed, and (3) at plot points where a log-linear cumulative average curve for 65- and 95-percent slopes was assumed.

From Fig. 8 it can be seen that the distance between the unit curve constructed with the arithmetic midpoint and the unit curve constructed with the true plot points depends on the size of the lot quantity. The larger the lot quantity, the greater the distance between the midpoint line and the other lines. In both sets the unit curves exhibit the widest variation for the first lot. However, for a series of small contract lots the range of plot points is of interest only for the first few lots. The midpoint of even the second-lot quantity may often provide a good approximation of the unit curve.

It is not the purpose of this discussion to recommend any particular technique. Rather, it is to underline that plotting representative unit costs for contract lots is of importance. The gross misplacement of early points could lead to improper conclusions about cost-quantity relationships.

Variations

The examples used earlier tend to suggest that data points generally fall along a straight line, as one would expect from the log-linear hypothesis. The truth is that plots of the type illustrated in Fig. 9 are not unusual and that fitting a curve to these points is more than a matter of understanding the least-squares method of curve fitting. The types of plots in Fig. 9 are common enough to have been given names by the airframe industry. The "scallop" is generally caused by a model change or some other major interruption in the production process. Characteristic of a scallop is the abrupt rise in manufacturing hours, followed by a rapid decline, the basic slope of the curve remaining relatively unchanged. When a model change is sufficiently great, as in the case of the change to the F-106B from the F-106A, the result is not

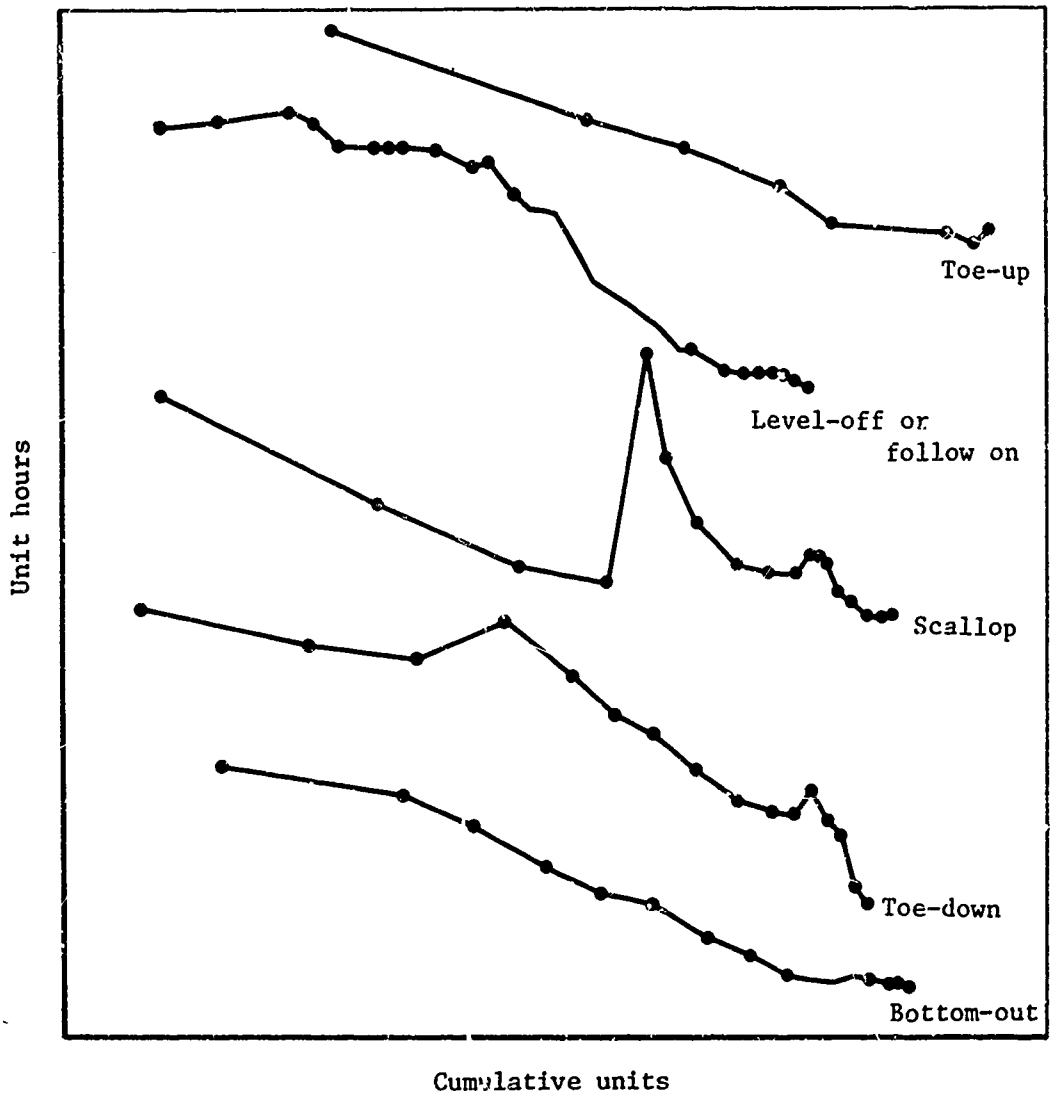

Fig. 9--Illustrative examples of learning-curve slopes

a scallop but a change to a new curve. In this case, a "level-off" or "follow-on" is characteristic of the initial portion of the new curve. This is attributed to learning from a previous model that carries over and flattens the curve during initial production. Such an effect can also occur when production is halted for a long period or when production is transferred to a new facility.

To "bottom-out" is the tendency for a learning curve to flatten at high production quantities. It seems reasonable that at some point

no further learning should occur or that whatever slight learning does occur would be offset by the effect of other factors. In addition, it can be established empirically that bottoming-out has occurred in a number of cases. There are those who argue, however, that learning *can* continue indefinitely, or at least as long as the attempt is made to obtain man-hour reductions. The classic case relates to the assembly of candy boxes, in which operation the learning curve was found to have continued for the preceding 16 years when 16 million boxes were assembled by one person.* The problem for the estimator, of course, is that while bottoming-out may occur in any given case, it is difficult to predict where it will occur. One study found that for the sample of airframes examined it was fairly typical for flattening to begin at the 300th unit,** but in the past this has not been true for many airframes. The B-17 curve maintained a 70-percent slope out to the 6000th unit and then exhibited a toe-up.

"Toe-ups" and "toe-downs" are the names given to the rather sharp rises or falls in hours that sometimes occur at the end of a production series. The upward trend has been explained as resulting from the transfer of experienced workers to other production lines, an increase in the amount of handwork as machines are disassembled, failure to replace or repair worn tooling at the normal rate, tool disassembly, or a production lag at the end of a program to forestall unemployment.*** Toe-downs are thought to be caused by fewer engineering changes at the end of a production run and also by the ability of the manufacturer to salvage certain items fabricated in previous lots.

It is important to realize that such variations in production do occur, and not occasionally but frequently. In the analysis of man-hour or cost data, use of the unit curve reveals these variations, and

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*Glen E. Ghormley, "The Learning Curve," *Western Industry* (now *Western Manufacturing*), September 1952, pp. 31-34.

**Planning Research Corporation, *Methods of Estimating Fixed-wing Airframe Costs*, Vol. I (Revised), PRC R-547A, Los Angeles, April 1967.

***Glenn M. Brewer, *The Learning Curve in the Airframe Industry*, School of Systems & Logistics, Air Force Institute of Technology, Report SLSR-18-65, August 1965.

for this reason the unit curve is generally preferred. The cumulative average curve tends to smooth out aberrations to such an extent that even major changes can be obscured, as shown in Fig. 10. The data points are taken from a fighter aircraft production program that had more than its share of problems. The solid line shows how a cumulative average curve dampens the effect of these problems. The choice between working with the unit or the cumulative average curve depends on the problem. The unit curve better describes the data and is therefore preferred. In addition, its use can aid the cost analyst in determining whether the basic curve is best represented by a log-linear cumulative average or unit function, what slope is most appropriate, and what follow-on projections can be made. The log-linear cumulative average

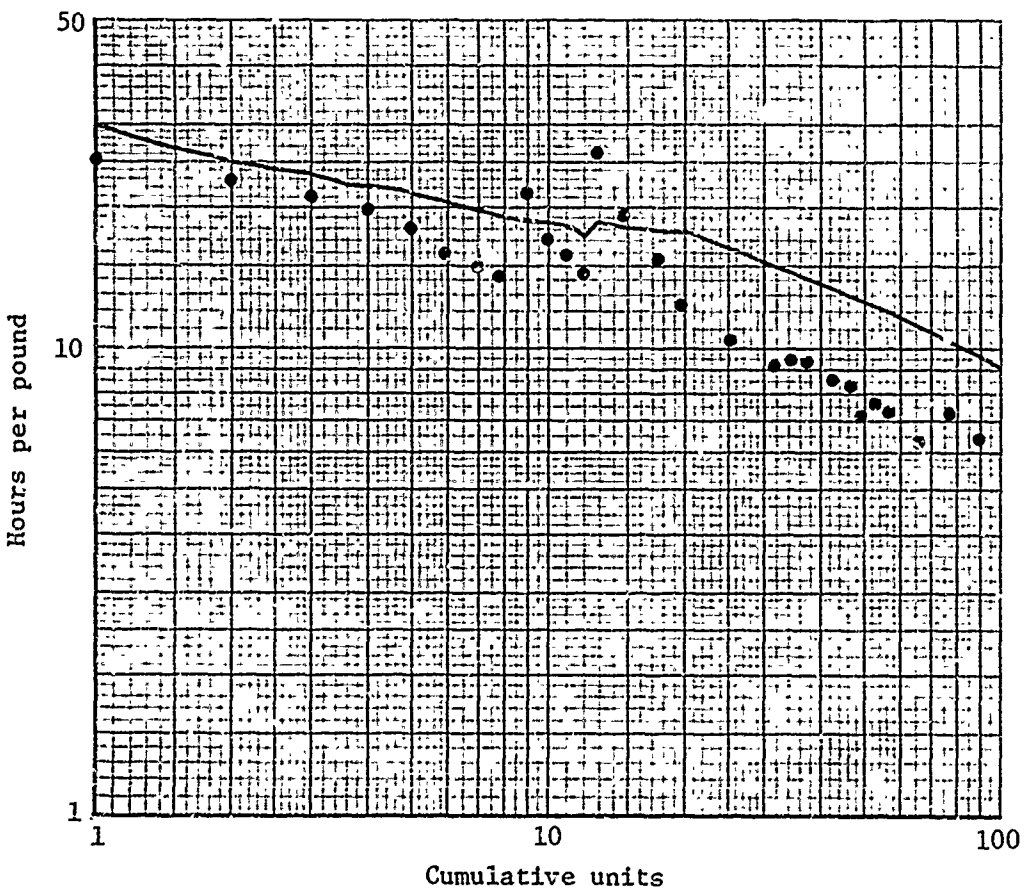

Fig. 10--Smoothing effect of cumulative average curve

curve is widely preferred in predictive models because of its computational simplicity, i.e., the cost of n items is simply the cumulative average cost of the nth item times n. However, it is important to understand all curves well enough to choose intelligently between them.

Applications

The learning curve is used for a variety of purposes and in a variety of contexts; how the curve is drawn will depend on the purpose and the context. In long-range planning studies, for example, the curve must be constructed on the basis of generalized historical data, and the possible error is considerable. Empirical evidence does not support the concept of a single slope for all fighter aircraft, all solid propellant missiles, or all spacecraft. Therefore, the practice of assuming that manufacturing hours on the airframe will follow an 80-percent curve (as was common for many years) or that electronic equipment will follow, say, a 90-percent curve, can lead to very large estimating errors.

In regard to airframes, Table 2 shows the slope of the manufacturing-hour curves for 25 post World War II Air Force and Navy aircraft and indicates that a slope steeper than 80 percent is the rule. Since the learning-curve slopes of the table show important differences, it would be desirable to relate slope to aircraft characteristics. Such a relation is accomplished by a technique suggested by the Planning Research Corporation.* Separate estimating equations based on aircraft characteristics are derived for four different production quantities--10, 30, 100, and 300--and a learning curve is developed from the estimates at these four points. However, on a theoretical level the concern is with aircraft characteristics that influence the rate of learning. It seems reasonable to expect relatively little learning for a model that represents a small modification over a preceding type, because the previous model would have already absorbed a considerable

**Fixed-wing Airframe Costs.*

Table 2

LEARNING CURVES FOR MANUFACTURING
(Labor for Airframe Only)

Aircraft	*Learning Curve* (%)
Fighter	77
Fighter	73
Fighter	74
Fighter	73
Fighter	78
Fighter	71
Fighter	74
Fighter	76
Fighter	77
Fighter	79
Fighter	82
Fighter	76
Fighter	75
Fighter	74
Bomber	76
Bomber	73
Bomber	70
Bomber	71
Bomber	79
Cargo	74
Cargo	78
Cargo	77
Cargo	75
Trainer	74
Trainer	75
Mean	75
Standard Deviation	2.7

SOURCE: G. S. Levenson and S. M. Barro, *Cost-estimating Relationships for Aircraft Airframes*, The Rand Corporation, RM-4845-PR (Abridged), May 1966, p. 56.

learning effect. On the other hand, if an aircraft contains radically new design features, a high initial cost is to be expected, followed by a rapid decline with increased production quantities. In other words, it has been suggested that the "newness" of an aircraft should be a major determinant of learning-curve slope, but explicit techniques for taking newness into account have yet to be developed.

For estimating to be effective, therefore, learning curves must be established on the basis of historical data relevant to the specific problem. Such curves are equally applicable to missiles, electronic equipment, aircraft, ships, and other types of equipment, but the slopes may be different for each of these. A recent study of avionics, for example, showed slopes ranging from 84 to 91 percent with a median value of 88 percent. If a comparison is being made between two weapon systems, one involving aircraft and the other missiles, the learning-curve slope chosen for each could play a significant part in the total system cost comparison. For example, the effect of using a 92-percent rather than a 90-percent cumulative average curve is an increase of 25 percent in the total cost of 1500 items. As one would guess, the situation is worse when steeper slopes are involved. If a slope of 62 percent instead of 60 percent is assumed, there is a 42-percent difference in the cost of 1500 items and a 25-percent difference in the cost of 100 items.* In practice, errors of this type can be minimized by originating the curve at the estimated cost of the 100th unit rather than at the first. Table 3 shows how this reduces the effect of a 2-percent change in slope on total cost.

Once a few data points are available either for developmental or production items, the situation should improve, but, as illustrated by Fig. 11, the first few points may be misleading. Suppose an estimator had been asked to calculate the cost of a large production contract after the fabrication of the first 30 units. By fitting a curve to the existing data he would have projected a learning curve with an 88- or 89-percent slope and at a level considerably higher than that later

*The assumption regarding the type of curve is important. For example, if a log-linear unit curve (rather than a log-linear cumulative average curve) were assumed, these differences would be only 25 and 13 percent, respectively.

Table 3

EFFECT OF VARYING SLOPE ASSUMPTIONS

Change in Slope	*Change in Total Cost of 1500 Units (%)*
From 90% to 92%	
Origin of curve:	
Unit 1	25
Unit 100	9[a]
From 60% to 62%	
Origin of curve:	
Unit 1	42
Unit 100	14

[a]If a log-linear unit curve is assumed, this value would be less than 6 percent.

experienced. In this situation it is important to realize that such a flat learning curve for airframe production is improbable. The estimator should have an idea of what the answer is likely to be and should investigate differences.

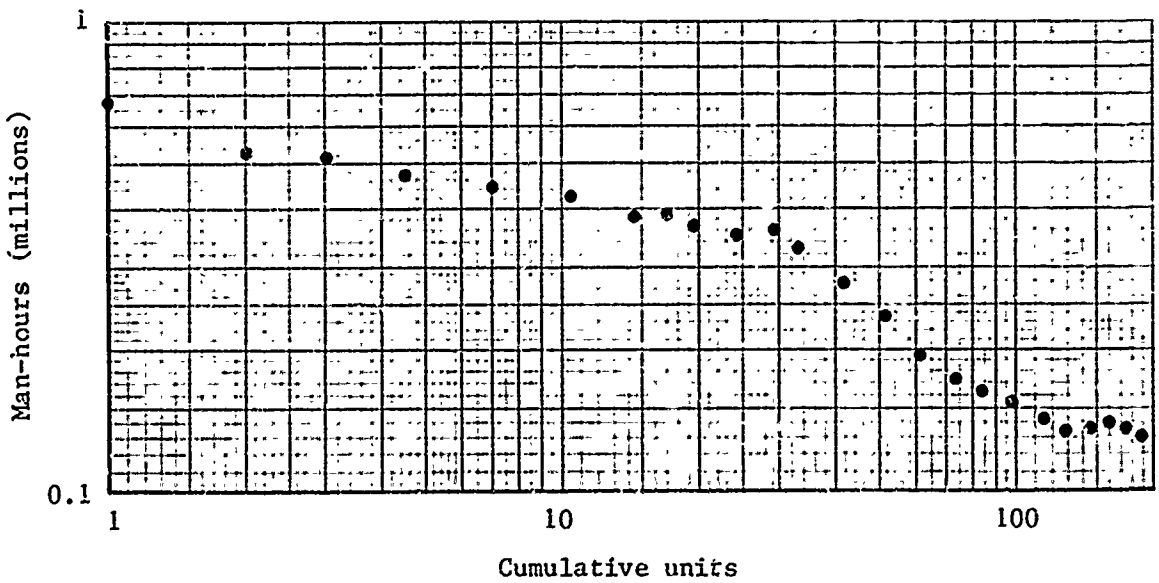

Fig. 11--Direct labor hours for a transport aircraft

With a small sample of data, where a learning curve is fitted to a few points, the correlation may be perfect, i.e., all the points may lie on the fitted line, but the results can still be unreliable. The points used in fitting must be sufficiently numerous and reasonably homogeneous with the points implied by extending the curve to offer a reasonable probability of success in predicting costs.

The manufacturing history of the item to be fabricated is the most valuable information the estimator can have. Variations from the norm may be caused by particular problems, configuration changes, or changes in manufacturing methods. In the curve of Fig. 11, the initially flat portion (out to the 30th airframe) is explained by the manufacturer as being typical of the initial production period. In this manufacturer's experience, the curve begins to steepen when

1. Manpower has stabilized or reached its peak.
2. The engineering configuration has stabilized.
3. The parts flow has stabilized.

Thus, it may be preferable to explain certain points and exclude them rather than to include them and bias the curve in height or slope.*

Whether to include all the points depends, in addition, on the anticipated use of the resulting curve. If a unit cost curve that includes all costs and changes is desired, a line of best fit through the unit plot points may be appropriate. If the curve is to be used in negotiating a follow-on contract, the effect of changes should be eliminated by constructing a curve through the lower portion of the plotted individual unit points, as in Fig. 12. In effect, this assumes that the introduction of changes raises the hours initially but that these decrease again to the approximate level of the original curve.

Whatever the basic technique, it is important to remember that on logarithmic grids the points at the right are usually more important than those at the left. In visually fitting a line, the analyst should avoid the tendency to be unduly influenced by plot points for small early lots. Early units are often incomplete because they are used for

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*It is also possible to have a segmented unit curve, as implied by Fig. 11, and several manufacturers subscribe to this concept.

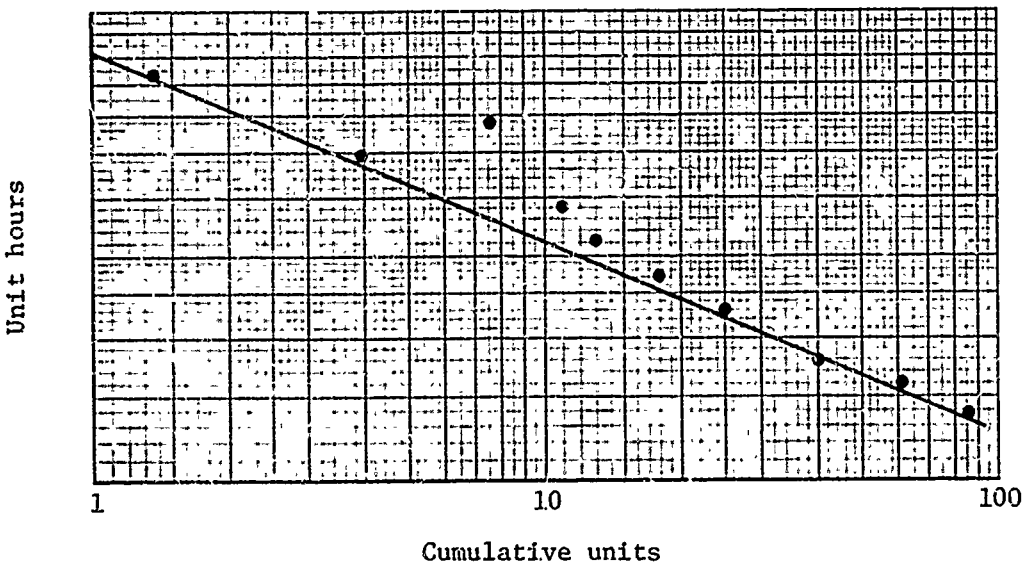

Fig. 12--Effect of changes on the learning curve

test purposes. It is equally possible that early units will include certain nonrecurring problems incident to startup and for this reason may be above the level suggested by later plot points.

Of course, variations in unit cost (or hour) data may happen for reasons other than the introduction of changes. An interruption in production can be an important factor. Interruptions may occur because of production cutbacks, labor disturbances, or funding problems. Whatever the reason, if significant time periods are involved, the learning curve will be affected in much the same way as illustrated in Fig. 12. Those units produced after a significant amount of interruption can be expected to exhibit sharp increases in costs, followed by a recovery to the approximate projected level of the earlier preinterruption period.

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10. ABSTRACT

There are three basic methods used for cost estimation--the industrial engineering, analogy, and statistical approaches. The statistical method is considered the most useful for government cost analysts, whether the purpose is long-range planning or contract negotiation. An estimating procedure must rely on a data base that includes cost, physical and performance descriptions, and a development and production history of previous equipment programs. Cost is predicted by estimating relationships based on parametric explanatory variables such as weight, speed, power, frequency, or thrust. Often these variables are examined through the use of statistical procedures. Experience has proved the value of the learning-curve concept. Its basis is that each time the total quantity of items produced doubles, the cost per item is reduced to a constant percentage of its previous cost. A thorough knowledge of the learning-curve phenomenon is indispensable to persons involved in cost analysis.

11. KEY WORDS

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