ANALYSIS OF TRANSVERSELY ISOTROPIC LAMINATED CYLINDERS UNDER AXISYMMETRIC MECHANICAL AND THERMAL LOADS

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ABERDEEN PROVING GROUND, MARYLAND
ABSTRACT

A theory for the analysis of stresses in laminated circular cylindrical shells subjected to arbitrary axisymmetric mechanical and thermal loadings has been developed. This theory, specifically for use with pyrolytic graphite type materials, differs from the classical thin shell theory in that it includes the effects of transverse shear deformation and transverse isotropy, as well as thermal expansion through the shell thickness.

Solutions in several forms are developed for the governing equations. The form taken by the solution function is governed by geometric considerations. A range in which the various solution forms occur was determined numerically.

As a sample problem, the slow cooling of pyrolytic graphite deposited onto a commercial graphite mandrel was considered. Investigation of normal and shear stress behavior at the pyrolytic graphite - mandrel interface showed that these stresses decrease in magnitude with increasing E/G_c ratio and increasing deposit to mandrel thickness (h_d/h) ratio. This implies that a thin mandrel and a material weak in shear are desirable to minimize the possibilities of flaking and delamination of the pyrolytic graphite.
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NOTATION

\( a, b \) subscripts indicating upper or lower lamina

\( a_{ij}, b_{ij} \) constants defined by equations (C.2) and (C.3)

\( A, B \) constants defined by (C.8)

\( c \) preferred direction in a transversely isotropic material

\( c_i \) roots of equation (22) defined by (C.11) and (C.13)

\( C_i = \frac{E_i h_i}{1 - \nu_i^2} \quad (i = a, b) \)

\( D_i = \frac{E_i h_i^3}{12(1 - \nu_i^2)} \)

\( D \) \( \frac{d}{dx} \)

\( e \) natural base

\( E, E_c \) Young's Modulus in the plane of isotropy and "c" direction respectively

\( G_c \) shear modulus relating stress and strain across the plane of isotropy \((G_{xz} = G_{yz} = G_{zx} = G_{yz})\)

\( g_i \) constants defined by equation (C.4)

\( h_a, h_b \) individual lamina thicknesses

\( h \) \( h_a + h_b \)

\( i, j \) subscripts

\( k_i \) constants defined by equation (C.6)

\( m, n \) constants defined by equation (C.7)

\( \bar{m} \) constant defined by (11a)

\( M_x, M_\theta \) stress couples
$M_{Tx}, M_{T0}$ thermal couples

$\tilde{M}$ defined by equation (41)

$N_{x}, N_{\theta}$ stress resultants

$N_{Tx}, N_{T0}$ thermal resultants defined by equation (41)

$p_{1i}$ $\sigma_{Z}(h_{i}/2) \ (i = a,b)$

$p_{2i}$ $\sigma_{Z}(-h_{i}/2) \ (i = a,b)$

$p(x)$ $p_{1a} - p_{2b}$

$P_{j}$ joint normal stress ($=\sigma_{Z}(-\frac{h_{a}}{2}) = \sigma_{Z}(\frac{h_{b}}{2})$)

$Q_{j}$ shear resultant

$Q$ defined by equation (41)

$R$ Radius to shell reference surface

$T$ temperature measured from the stress-free temperature of the material

$u_{x}, u_{z}$ deflections in the "x" and "z" directions respectively

$u_{0j}$ axial deflection of lamina middle surface

$u_{j}$ virtual displacement of shell middle surface

$w_{i}$ radial displacement of lamina middle surface

$\tilde{w}_{i}$ $\int_{0}^{Z} (\alpha_{c}T)_{i} \ dz \ (i = a,b)$

$w_{i}$ virtual radial displacement of lamina middle surface.

$x$ axial coordinate for cylindrical shell

$z$ cylindrical coordinate

$\alpha_{x}, \alpha_{c}$ thermal expansion coefficients in the "x" and "z" directions respectively
\( \beta_i \) \hspace{1cm} \text{rotation of the normal to the undeformed lamina middle surface due to deformation}

\( \Gamma \) \hspace{1cm} \text{defined by equation (C.10)}

\( \lambda_1, \lambda_2, \lambda_3 \) \hspace{1cm} \text{defined by equation (C.9)}

\( \beta_i^* \) \hspace{1cm} \text{virtual rotation defined by equation (21)}

\( \Delta_i \) \hspace{1cm} \text{strain component}

\( \varepsilon_{ij} \) \hspace{1cm} \text{Poisson's ratio in the } x - \theta \text{ plane (} \nu_{x\theta} = \nu_{\theta x} \text{)}

\( \nu_c \) \hspace{1cm} \text{Poisson's ratio (} \nu_{xz} = \nu_{\theta z} \text{)}

\( \nu_{ij} \) \hspace{1cm} \text{Poisson's ratio defined as the negative of the ratio of the strain in the } j \text{-direction to the strain in the } i \text{-direction due to a stress in the } i \text{-direction}

\( \sigma_{ij} \) \hspace{1cm} \text{stress component}

\( \tau_j \) \hspace{1cm} \text{joint shear stress (} = \sigma_{zx} (-h_a/2) = \sigma_{zx} (+h_b/2) \text{)}

\( \theta \) \hspace{1cm} \text{cylindrical coordinate in the circumferential direction}

\( \theta_1, \theta_2 \) \hspace{1cm} \text{defined by equation (C.10)}

\( \frac{d}{dx} ( ) \)
I. INTRODUCTION

The ever-expanding missile and space technology continually demands materials capable of maintaining structural integrity at very high temperatures. Of late, attention has been focused on refractory materials, their anisotropy in physical and mechanical properties making them ideally suited for a wide range of insulation and/or structural applications.

Of the many refractory materials possible, pyrolytic graphite (PG) has probably received the most attention of late although it was known to Edison (1)* in 1883 who described methods for its manufacture, the technique involving formation of carbon deposits onto substrates heated in carbon-containing gases. For structural use, pyrolytic graphite is generally deposited at temperature from 3500°F to 4000°F in a stream of hydrocarbon gas, such as methane, onto a substrate of commercial graphite maintained at temperatures of 1500°F to 5000°F. The rate at which the material is produced depends on a number of factors which include the temperature, the reaction pressure, the hydrocarbon flow rate and

*Numbers in parenthesis indicate corresponding references in the bibliography
the surface to volume ratio of the substrate surface (2), (3), (4), (5). X-ray analysis of the resultant deposit shows a well-crystallized structure having much in common with the single graphite crystal (6). Growth is always normal to the substrate surface and after a thickness of 0.1" - 0.5" is reached, the deposition process is stopped and the deposit allowed to cool for several weeks.

The result of such a formation process is a material highly anisotropic in physical properties. The PG has one plane of isotropy parallel to the mandrel surface (x,θ direction -- see figure II-1 for geometry) and a single preferred direction (z), a state commonly referred to as transverse isotropy. With thermal conductivity between 100 and 1000 times greater in the (x,θ) direction than in the (z) direction, the material acts as an excellent conductor along its surface but also as a good insulator in the thickness (z) direction. The coefficient of thermal expansion in the (z) direction is from 10 to 30 times greater than that in the plane of isotropy (x,θ) so that thermal expansion through-the-thickness must be considered in many analyses of the material's thermal behavior.

Other curious effects due to the anisotropy are manifest in the Poisson's ratio which is negative in the plane of isotropy ($v_{xθ} = v_{θx} = -0.21$) but large and positive in the preferred direction.
Furthermore, the ratio of the elastic modulus in the isotropic plant to the shear modulus in the transverse plane \( E_x/G_{xz} \) may range from 20 to 50, compared to an \( E/G \) ratio of 2.5 for an isotropic material with \( v = 0.25 \). Therefore, in an analysis of a structure composed of such material, transverse shear deformation even for thin cross-sections must be considered.

Thermal and mechanical properties can be found readily \(^{(2)}\), \(^{(5)}\), \(^{(7)}\), \(^{(8)}\), \(^{(9)}\), \(^{(10)}\), \(^{(11)}\), \(^{(12)}\). Typical properties, given in Table I-1, are taken from \(^{(10)}\), which are reasonably close to those given in other references, discrepancies most probably being due to variations in the deposition process.

Among the earliest analyses of structures of pyrolytic materials were those of Garber \(^{(13)}\) and Levy \(^{(14)}\) who treated thermal stresses in cylindrical and spherical shells and also the residual stresses caused by the general anisotropy of pyrolytic graphite, but neglected transverse shear deformation and did not account for the high thermal expansion coefficient in the \((z)\) direction. McDonough \(^{(15)}\) has considered thermal stresses in shells of revolution of pyrolytic graphite type materials subjected to axially symmetric loads, including transverse shear deformation and thermal expansion through the thickness. He was able to show that neglect of transverse shear deformation would lead to an over-estimate of the stiffness coefficient.
TABLE I-1 PHYSICAL PROPERTIES OF PYROLYTIC GRAPHITE

MECHANICAL PROPERTIES

1. Young's Modulus (PSI)

<table>
<thead>
<tr>
<th>TEMP</th>
<th>(x,θ) Direction</th>
<th>(z) Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>70°F</td>
<td>5.4x10^6</td>
<td>1.5 x10^6</td>
</tr>
<tr>
<td>1000°F</td>
<td>4.3x10^6</td>
<td>1.29x10^6</td>
</tr>
<tr>
<td>2000°F</td>
<td>3.5x10^6</td>
<td>1.05x10^6</td>
</tr>
<tr>
<td>3000°F</td>
<td>2.7x10^6</td>
<td>0.81x10^6</td>
</tr>
</tbody>
</table>

2. Poisson's Ratio

<table>
<thead>
<tr>
<th>TEMP</th>
<th>νxθ = 0.21</th>
<th>νxz = 0.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>70°F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

THERMAL PROPERTIES

1. Thermal Expansion in/in - °F

<table>
<thead>
<tr>
<th>TEMP</th>
<th>Thermal Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>70°F</td>
<td>0.0</td>
</tr>
<tr>
<td>1000°F</td>
<td>0.6x10^-6</td>
</tr>
<tr>
<td>2000°F</td>
<td>1.2x10^-6</td>
</tr>
<tr>
<td>3000°F</td>
<td>1.7x10^-6</td>
</tr>
</tbody>
</table>

2. Conductivity, BTU/hr-ft-°F

<table>
<thead>
<tr>
<th>TEMP</th>
<th>Conductivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>70°F</td>
<td>290.0</td>
</tr>
<tr>
<td>1000°F</td>
<td>165.0</td>
</tr>
<tr>
<td>2000°F</td>
<td>100.0</td>
</tr>
<tr>
<td>3000°F</td>
<td>60.0</td>
</tr>
</tbody>
</table>
for properties representative of PG while excluding thermal expansion in the (z) direction leads not only to erroneous stress predictions but that even the sign of the stress (tension or compression) may be wrong. Kliger (16), (17) has extended McDonough's work for the case of conical shells in that he derives equations for non-axially symmetric mechanical and thermal loadings. Raju (18) studied the case of shallow shells of pyrolytic graphite type materials subjected to a variety of axially symmetric and non-axially symmetric loads. Daugherty (19), (20) treated the case of non-circular cylindrical shells of pyrolytic materials.

The preceding deal with single-layer shells. Anisotropic laminated shells of revolution with elastic properties symmetric about the middle surface of the composite shell are extensively treated by Ambartsumian (21), whereas Dong (22), (23), et al (24) treat layered shells wherein the structure is assumed to be composed of an arbitrary numbers of bonded layers each of different constant thickness, different orientation of elastic axes and different anisotropic elastic properties. Since Dong does not assume elastic symmetry about the middle surface, flexural and extensional deformations are coupled and solution techniques for homogeneous shells do not carry over directly for anisotropic elastic shells. Hence, alternate methods of solution are developed.

Radkowski et al (25) considered laminated isotropic shells of revolution with variable thickness using E. Reissner's for-
formulation (26). Radkowski extended this work to include variable laminated orthotropic material properties (27). Both formulations were restricted to axisymmetric loads. In Radkowski’s works and that of Sepetoski (28), the governing equations were cast in finite difference form and solved with the aid of a digital computer. The introduction to Dong’s paper (23) makes interesting reading regarding the hazards of this perfectly valid technique.

Other treatments of laminated cylinders have been by Jones and Whittier (29), Tsai and Azzi (30), Paul (31), Au (32), Keffe and Windholz (33). These, and most other references cited herein are characterized by neglect of transverse shear deformation. A recent work of great theoretical elegance, even though it neglects transverse shear deformation, is that of Zudans (34) which presents a theory for arbitrarily loaded (mechanically & thermally) shells of revolution with internal masses and ring stiffeners, derived under the Kirchoff Hypothesis and consistent with balance of energy as well as linear and angular momentum and invariance under transformation of middle surface coordinate systems and rigid body displacements. The elegance, unfortunately, does not carry over to the computational techniques (35).

Laminated isotropic plates have been considered by Vinson (36) who treated thermal stresses in circular plates, neglecting transverse shear deformation. Summers (37) treats thick and thin isotropic and orthotropic laminated plates including transverse shear deformation.
Mehta (34) considers orthotropic and isotropic laminated as well as single-layered rectangular plates of pyrolytic graphite type materials under static mechanical and thermal loads. Wu (39), (40), and (41) treats the lateral vibrations of both small and large amplitude for rectangular plates of pyrolytic and graphite materials.

Except for those references dealing with pyrolytic graphite type materials, all the works reviewed which analyze multi-layered structures are characterized by their neglect of either transverse shear deformation or thermal expansion through-the-thickness or both.

Prompted by the absence of a definitive treatment of laminated shells applicable to pyrolytic graphite type materials, this thesis was undertaken. It is an extension of McDonough's work in that layered cylindrical shells, including the effects of transverse shear deformation and thermal expansion through-the-thickness, subjected to arbitrary axi-symmetric loading are considered.
II. DERIVATION OF GOVERNING EQUATIONS

The coordinate system used is shown in Figure (II-1). The three-dimensional equations of thermoelasticity (uncoupled) for the case of axial symmetry are given by:

Stress-Strain Relations (transversely isotropic material)

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} \left( \sigma_x - \nu \sigma_\theta - \frac{1}{C} \sigma_z \right) + \alpha T \\
\varepsilon_\theta &= \frac{1}{E} \left( \sigma_\theta - \nu \sigma_x - \frac{1}{C} \sigma_z \right) + \alpha T \\
\varepsilon_z &= \frac{\sigma_z}{E} - \frac{\nu C}{E} \left( \sigma_x + \sigma_\theta \right) + \alpha C T \\
\varepsilon_{xz} &= \frac{\sigma_{xz}}{2G C} 
\end{align*}
\]

(1)

where \( \varepsilon_{x\theta} = \varepsilon_{\theta z} = \sigma_{x\theta} = \sigma_{\theta z} = 0 \) by symmetry.

\( \varepsilon_{ij} \) and \( \sigma_{ij} \) are the physical components of the strain and stress tensors respectively, and for brevity \( \sigma_{ij} = \sigma_i \) when \( i = j \).

\( E, E_C, \nu_z, \nu_C, G_C \) are five independent elastic constants where again for brevity \( \nu = \nu_{x\theta} = \nu_{\theta x} \) and \( \nu_C = \nu_{xz} = \nu_{\theta z} \). \( \alpha, \alpha_C \) are the coefficients of thermal expansion in the \( x \) and \( z \) directions of the materials. \( T \) is the temperature measured from the stress-free temperature of the material in units consistent with the \( \alpha \)'s.
The preferred direction for the material is everywhere coincident with the z coordinate. This restriction is carried throughout this work.

Equilibrium Equations

The equilibrium equations when applied to the present problem become for each lamina:

\[ R(1 + \frac{z}{R}) \frac{\partial \sigma_x}{\partial x} + R(1 + \frac{z}{R}) \frac{\partial \sigma_{xz}}{\partial z} + \sigma_{xz} = 0 \] (2)

\[ R(1 + \frac{z}{R}) \frac{\partial \sigma_z}{\partial z} + R(1 + \frac{z}{R}) \frac{\partial \sigma_{xz}}{\partial x} + \sigma_z - \sigma_\theta = 0 \]

The third equation is identically zero from symmetry considerations.

Strain-Deformation Equations

The strain-deformation relations can be written correspondly as:

\[ \varepsilon_x = \frac{\partial u_x}{\partial x} \]
\[ \varepsilon_\theta = \frac{1}{R(1+\frac{z}{R})} u_z \]
\[ \varepsilon_z = \frac{\partial u_z}{\partial z} \]
\[ \varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \] (3)

\[ u_\theta = 0 \text{ by symmetry} \]
The displacements are positive in the direction of the positive corresponding coordinate (See Figure II-1).

**Assumptions:**

1. The thickness of the shell is small compared with other dimensions, hence Love's First Approximation is applicable:

   \[ \frac{h}{R_{\text{min}}} \ll 1 \quad (4) \]

2. The displacements are small compared to the thickness of the shell and the angles of rotation are small compared to unity.

3. The transverse normal stress is small compared with other normal stress components and is neglected in the stress-strain equations.

4. A linear element normal to the undeformed middle surface undergoes translation and rotation and remains straight, implying deformations of the form

   \[ u_x = u_0(x) + z\beta(x) \quad (5) \]

5. Transverse normal strain due to thermal expansion will be included, that due to mechanical (or isothermal) loads will be neglected.

   This implies a deformation of the form

   \[ u_z(x,z) = w(x) + \tilde{w}(x,z) \quad (6) \]

where

\[ \tilde{w}(x,z) = \int_0^z T(x,z) \, dz \]
6. Material properties are constant for each lamina.

With these assumptions, the equation (1) becomes:

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E_x} (\sigma_x - \nu \sigma_\theta) + \alpha T \\
\varepsilon_\theta &= \frac{1}{E_\theta} (\sigma_\theta - \nu \sigma_x) + \alpha T, \\
\varepsilon_z &= \alpha \varepsilon_c T \\
\varepsilon_{xz} &= \frac{\sigma_{xz}}{2G_c}
\end{align*}
\]  

Making use of equations (5) and (6) and neglecting \(z/R\) in comparison to unity, the equations (3) become:

\[
\begin{align*}
\varepsilon_x &= u_0^{"} + z \beta^{"} \\
\varepsilon_\theta &= \frac{1}{R} (w + \bar{w}) \\
\varepsilon_z &= \frac{\partial w}{\partial z} \\
\varepsilon_{xz} &= 1/2(\beta + w^{"})
\end{align*}
\]  

where \((\ )^{"} = D (\ ) = \frac{d}{dx} (\ )\)
Integrated Equations

The stress resultants and couples are defined as follows:

\[ N_{x_1} = \int_{-h_1}^{h_1} \frac{h_1}{2} \sigma_{x_1} \, dz \]

\[ N_{\theta_1} = \int_{-h_1}^{h_1} \frac{h_1}{2} \sigma_{\theta_1} \, dz \]

\[ Q_1 = \int_{-h_1}^{h_1} \frac{h_1}{2} \sigma_{x_2} \, dz \]

\[ M_{x_1} = \int_{-h_1}^{h_1} \frac{h_1}{2} z \sigma_{x_1} \, dz \]

\[ M_{\theta_1} = \int_{-h_1}^{h_1} \frac{h_1}{2} z \sigma_{\theta_1} \, dz \]

\[ N_{T_{x_1}} = \int_{-h_1}^{h_1} \frac{h_1}{2} E_i a_i T_{dz} \]

\[ N_{T_{\theta_1}} = \int_{-h_1}^{h_1} \frac{h_1}{2} E_i a_i T_{dz} \]

\[ M_{T_{x_1}} = \int_{-h_1}^{h_1} \frac{h_1}{2} z E_i a_i T_{dz} \]

\[ M_{T_{\theta_1}} = \int_{-h_1}^{h_1} \frac{h_1}{2} z E_i a_i T_{dz} \]

\[ i = a, b \]
The equilibrium equations in terms of stress resultants and couples are readily obtained. Integrating (2) directly yields

\[ N_{x_i} + \tau_{11} - \tau_{21} = 0 \]

\[ M_{x_i} - Q_i + \frac{h_i}{2} \tau_{11} + \frac{h_i}{2} \tau_{21} = 0 \]

\[ Q_i - \frac{N_{\theta i}}{R_i} + P_{1i} - P_{2i} = 0 \]

(10)

Solving equations (7) for normal stresses, first integrating them from \(-h_i/2\) to \(+h_i/2\); then multiplying through the equations by \(z\) and integrating once again between the same limits, making use of the definitions of the stress resultants and couples (9) and simplifying, the following stress-strain relations result:
\[ N_{x_1} = \frac{E_i h_1}{(1-v_1^2)} \left( u_{o1} + \frac{v_i w_i}{R_1} \right) - \frac{N_{T x_1}}{1-v_1} + \frac{E_i v_1}{R_1(1-v_1^2)} \int_{-\frac{h_1}{2}}^{\frac{h_1}{2}} w_i dz \]

\[ N_{\theta_1} = \frac{E_i h_1}{(1-v_1^2)} \left( v_i u_{o1} + \frac{w_i}{R_1} \right) - \frac{N_{T \theta_1}}{1-v_1} + \frac{E_i}{R_1(1-v_1^2)} \int_{-\frac{h_1}{2}}^{\frac{h_1}{2}} w_i dz \]

\[ M_{x_1} = \frac{E_i h_1^3}{12(1-v_1^2)} \beta_1 - \frac{M_{T x_1}}{1-v_1} + \frac{E_i v_1}{R_1(1-v_1^2)} \int_{-\frac{h_1}{2}}^{\frac{h_1}{2}} z w_i dz \]

\[ M_{\theta_1} = \frac{E_i h_1^3}{12(1-v_1^2)} \beta_1 - \frac{M_{T \theta_1}}{1-v_1} + \frac{E_i}{R_1(1-v_1^2)} \int_{-\frac{h_1}{2}}^{\frac{h_1}{2}} z w_i dz \]

\[ i = a, b \]
An integrated shear stress strain equation is required in
addition to (11). The necessary expression is obtained using weighted
integration, the procedure being analogous to that in reference 15,
and for convenience given in Appendix D.

\[ Q_i = \frac{m_i}{6} + \frac{5}{6} h_i G^i_c (\beta_i + \omega_i) + A_i \]

where \( m_i = \frac{h_i}{2} (\tau_{11i} + \tau_{22i}) \) . \hspace{1cm} (11A)

Solving for \( Q_a \) and \( Q_b \) from the second of the integrated
equilibrium equations (10) and substituting into the third, making
use of \( M_{a_i} \) and \( N_{ao} \) from (11) and the definitions

\[ C_i = \frac{E_i h_i}{(1-v_i^2)} \; ; \; D_i = \frac{E_i h_i^3}{12(1-v_i^2)} \; (i = a, b) \]

\[ \tau_j - \sigma_{zx}(-\frac{h_a}{2}) = \sigma_{zx} (+\frac{h_b}{2}) \]

\[ P_j = \sigma_z (-\frac{h_a}{2}) = \sigma_z (+\frac{h_b}{2}) \]

Two useful relations are obtained:

\[ D_a D^3 B_a - \frac{C_a \nu_a h_a}{2R_a} D_B_a - \frac{C_a \nu_a h_b}{2R_a} D_B_b + \frac{h_a}{2} D_{ij} - P_j - \frac{C_a \nu_a a_{ij}}{R_a} U_{o_b} - \frac{C_a}{2R_a} \omega_b \]

\[ = a_{17} + \nu_{1} \] \hspace{1cm} (12)
$$D_b D^3 \beta_b + \frac{h_b}{2} D \tau_j + p_j - \frac{C_b v_b}{R_b} D \omega_o - \frac{C_b}{R_b^2} w_b = a_{27} + \pi_2$$ (13)

Note that in the above, use has been made of the conditions that the laminae are bonded together and that no slippage occurs in the joints between laminae. The former is given by

$$u_z(-\frac{h_a}{2}) = u_z(\frac{h_b}{2})$$ which implies that $$\bar{w}_a(x, -\frac{h_a}{2}) + w_a = \bar{w}_b(x, \frac{h_b}{2}) + w_b$$

or $$w_a = w_b + \bar{w}_b(x, \frac{h_b}{2}) - \bar{w}_a(x, -\frac{h_a}{2})$$

= $$w_b + \hat{E}$$ (14)

where $$\hat{E} = \bar{w}_b(x, \frac{h_b}{2}) - \bar{w}_a(x, -\frac{h_a}{2})$$.

The latter condition is expressed by

$$u_x(-\frac{h_a}{2}) = u_x(\frac{h_b}{2})$$

$$u_{o_a} - \frac{h_a}{2} \beta_a = u_{o_b} + \frac{h_b}{2} \beta_b$$

$$u_{o_a} = u_{o_b} + \frac{h_a}{2} \beta_a + \frac{h_b}{2} \beta_b$$ (15)

Making use of (11), (14) and (15) in the first integrated equilibrium equation, two further relations are obtained:
\[
\frac{C_a h_a}{2} D_a^2 b_a + \frac{C_b h_b}{2} D_b^2 b_b - \tau_j + C_a D^2 u_{o_b} + \frac{C_a v_a}{R_a} D w_b = \alpha_{37} + \pi_3 \tag{16}
\]

\[
\tau_j + C_b D^2 u_{o_b} + \frac{C_b v_b}{R_b} D w_b = \alpha_{47} + \pi_4 \tag{17}
\]

Using (11A) and the second of the integrated equilibrium equations together with (14) and (15), two final relations are obtained:

\[
D_a D^2 b_a - \frac{5}{6} G^a h_a b_a + \frac{5}{12} h_a \tau_j - \frac{5}{6} G^a h_a D w_b = \alpha_{57} + \pi_5 \tag{18}
\]

\[
D_b D^2 b_b - \frac{5}{6} G^b h_b b_b + \frac{5}{12} h_b \tau_j - \frac{5}{6} G^b h_b D w_b = \alpha_{67} + \pi_6 \tag{19}
\]

where in the above

\[
\alpha_{17} = \frac{N_T e_a}{R_a (1 - v_a)} - \frac{C_a E_a}{R^2 a} - \frac{E_a}{R^2 a} \left( \frac{h_a}{2} \int w_a dz \right)
\]

\[
+ D^2 \left( \frac{1}{1 - v_a} - \frac{E_a v_a}{R_a (1 - v_a^2)} \right) \left( \frac{h_a}{2} \int z w_a dz \right)
\]
\[ a_{27} = R_b (1-v_b) \frac{N_{T \theta_b}}{R^2 b (1-v_b)^2} \int h_b \frac{\bar{w}_b}{2} dz \]

\[ + D^2 (1-v_b) \frac{E_b v_b}{R_b (1-v_b)^2} \int h_b \frac{\bar{w}_b}{2} dz \]

\[ a_{37} = D (1-v_a) \frac{C_{a v_a}}{R_a} \hat{E} - \frac{E_a v_a}{R_a (1-v_a^2)} \int h_a \frac{\bar{w}_a}{2} dz \]

\[ a_{47} = D (1-v_b) \frac{E_b v_b}{R_b (1-v_b)^2} \int h_b \frac{\bar{w}_b}{2} dz \]

\[ a_{57} = \Delta_a + D \left( \hat{E} + \frac{M_{T \alpha_a}}{1-v_a} \right) \frac{h_a}{2} \frac{z w_a}{dz} \]

\[ a_{67} = \Delta_b + D \left( \frac{E_b v_b}{R_b (1-v_b)^2} \right) \frac{h_b}{2} \frac{z w_b}{dz} \]

(20)
The governing equations may be transformed to a more useful form for solution. Since they are cumbersome for manipulation in their present form, a matrix notation was used to define the coefficients of the unknowns. These are tabulated in Appendix C. After some manipulation of the governing equations, the following are obtained:
(g_1 D^7 + g_2 D^5 + g_3 D^3 + g_4 D) w_b = L_I(x) \quad (22)

(b_{11} D^4 + b_{12} D^2 + b_{13}) \beta_b = L_{II}(x) - (b_{14} D^3 + b_{15} D) w_b \quad (23)

a_{23} D b^2_a + L_{III}(x) - (a_{21} D^2 c + a_{22} c \epsilon) \beta_b - a_{24} D w_b \quad (24)

D^2 u_{ob} = L_{IV}(x) - k_1 D^2 \beta_a - k_2 D \beta_b - k_3 D w_b \quad (25)

\tau_j = L_V(x) - k_4 D^2 u_{ob} - k_5 D w_b \quad (26)

p_j = L_{VI}(x) - k_6 D^3 \beta_b - k_7 D \tau_j + k_8 D u_{ob} + k_9 w_a \quad (27)
III. SOLUTION OF GOVERNING EQUATIONS

Homogeneous Solution:

Consider first $W_b$. Assuming a solution of the form $W_b = e^{sX}$ and letting $y = s^2$, (22) then takes the form

$$y^3 + \frac{q_2}{9_1} y^2 + \frac{q_3}{9_1} y + \frac{q_4}{9_1} = 0$$

(28)

for the homogeneous solution.

The solution of equations in the form (28) is developed in reference (42) and leads to three possibilities for the roots:

Case 1: there are two conjugate imaginary roots and one real root

Case 2: there are three real and unequal roots

Case 3: there are three real roots of which at least two are equal.

Case 1 leads to a solution of (22) which is of the form

$$W_{bH} = V_1 e^{c_1 X} + V_2 e^{-c_1 X} + e^{c_2 X} (V_3 \cos c_3 x + V_4 \sin c_3 x)$$

$$+ e^{-c_2 X} (V_5 \cos c_3 x + V_6 \sin c_3 x)$$

(29)

where $V_1 - V_6$ are constants to be evaluated through boundary conditions. The constants $c_1 - c_3$ are defined in Appendix C.

The Case 2 solution can take on several forms depending on whether the roots of (28) are positive or negative. The final forms
for the case of one, two and three positive real roots respectively, are:

\[ W_{bh} = V_1 e^{c_4 x} + V_2 e^{-c_4 x} + V_3 \cos c_5 x + V_4 \sin c_5 x + V_5 \cos c_6 x + V_6 \sin c_6 x \]  
(30)

\[ W_{bh} = V_1 e^{c_4 x} + V_2 e^{-c_4 x} + V_3 e^{c_5 x} + V_4 e^{-c_5 x} + V_5 \cos c_6 x + V_6 \sin c_6 x \]  
(31)

\[ W_{bh} = V_1 e^{c_4 x} + V_2 e^{-c_4 x} + V_3 e^{c_5 x} + V_4 e^{-c_5 x} + V_5 e^{c_6 x} + V_6 e^{-c_6 x} \]  
(32)

where, again, the \( V_1 - V_6 \) are boundary value constants and \( c_4 - c_6 \) are defined in Appendix C.

Case 3 represents the degenerate forms (30) - (32) where two of the roots are equal. The equations then take the form:

\[ W_{bh} = V_1 e^{c_4 x} + V_2 e^{-c_4 x} + (V_3 + V_5 x) \cos c_5 x + (V_4 + V_6 x) \sin c_5 x \]  
(33)

\[ W_{bh} = (V_1 + V_3 x) e^{c_4 x} + (V_2 + V_4 x) e^{-c_4 x} + V_5 \cos c_6 x + V_6 \sin c_6 x \]  
(34)

\[ W_{bh} = (V_1 + V_3 x) e^{c_4 x} + (V_2 + V_4 x) e^{-c_4 x} + V_5 e^{c_6 x} + V_6 e^{-c_6 x} \]  
(35)

In treating single-layered cylinders, it is possible to write the \( c_1 - c_6 \) explicitly in terms of the physical quantities.
involved and thus have a feel for the physical behavior of the shell. In the present derivation, however, these expressions are so lengthy and involved that, unless one has a specific example in mind, their explicit form for the general case is of dubious utility. A point is reached where one must decide whether to obscure the physical situation with mathematics or obscure the mathematics with the physical quantities involved. The former course was chosen to show where the mathematical formulation is analogous to that for the single-layered cylinder (Case 1) and where it diverges (Cases 2 and 3). To import greater physical significance to the equations it becomes necessary to either consider a specific case and determine the form and constants listed in Appendix C will take or resort to numerical evaluation of the constants.

Consider next equation (25). This may be integrated directly to give

\[ U_{ob} = \int_{IV} L(x) \, dx - k_1 \beta_a - k_2 \beta_b - k_3 \int w_b \, dx + V_7 x + V_8 \]  

(36)

where \( V_7 \) and \( V_8 \) are also boundary value constants. Since there are also eight boundary conditions, four at each edge, it follows that the homogeneous solutions for the remaining unknowns \( \beta_a \) and \( \beta_b \) are not required. Their particular solutions will suffice to satisfy the governing equations. This is equivalent to setting the boundary value constants of the \( \beta_a \) and \( \beta_b \) homogeneous solutions to zero.
**Total Solution:**

The total solution will consist of the homogeneous solutions given in the previous section together with whatever particular solutions are called for due to the mechanical and thermal loading for any given problem. Total solutions for displacements will take the form:

\[ W_b(x) = W_{bH} + W_{b\text{ part}} \]

\[ \beta_b(x) = \beta_{b\text{ part}} \]

\[ \beta_a(x) = \beta_{a\text{ part}} \]

where \( W_{b\text{ part}} \) is the particular solution of (22) due to \( L_1(x) \), and \( \beta_{b\text{ part}} \) is the particular solution of (23) due to \( L_{II}(x) - (b_{14}D^3 + b_{15}D)W_b(x) \), and \( \beta_{a\text{ part}} \) is the particular solution of (24) due to \( L_{III}(x) - (a_{21}D^2 + a_{22})\beta_b(x) - a_{24}DW_b(x) \).

Once (37) - (39) are known, \( u_{ob} \) may be found from (36) and the joint shear and normal stresses from (26) and (27) respectively.
IV. BOUNDARY CONDITIONS

Boundary conditions for plates and shells are listed in many sources (21), (34), (37), (36), (37). For the axi-symmetric case, they are usually stated as:

At the edges $x = 0$ and $x = L$

either $w$ or $N$ prescribed

either $w$ or $Q$ prescribed

either $\beta$ or $M$ prescribed

(40)

For multilayered problems, the same boundary conditions usually apply if $N$, $M$ and $Q$ are interpreted as resultants $\bar{N}$, $\bar{M}$, $\bar{Q}$. Considering a two-layered cylinder for example, lamina $a$ and $b$, the resultants would be:

\[
\bar{N} = N_a + N_b \\
\bar{Q} = Q_a + Q_b \\
\bar{M} = M_a + M_b + \left(\frac{h_a}{2} + \frac{h_b}{2}\right)N_a
\]  

(41)

However, (40), (41) do not make use of the no slip - no delamination conditions. These provide two constraints not only on displacements but on boundary conditions as well.

Appropriate boundary conditions can be derived using the principle of virtual displacements of the layer middle surfaces.
\[ U_i^* \]
\[ B_i^* \quad (i = a, b) \]  
\[ W_i^* \]

where the asterisk denotes a virtual quantity. Multiplying equations (10) by the virtual displacements (42) in the order listed, adding the products, integrating over shell length and summing over the layers, we get:

\[
\sum_{i=a}^{b} \int_{C} \left( [N_{x_1} u_i^* + (\tau_{11}^i - \tau_{21}^i)] u_i^* + \right. \\
[Q_i - (1/R)N_{\theta_1} + (P_{11} - P_{21})] w_i^* \right) \, dc = 0
\]

since the virtual work done by a shell in equilibrium through a virtual displacement is equal to zero.

After integration by parts, the virtual work principle for the multilayered cylinder takes the form:

\[
\sum_{i=a}^{b} \left( [N_{x_1} u_i^* + M_{x_1} \theta_i^* + Q_i w_i^*] \right)_{0}^{L} \\
+ \int_{0}^{L} \left( (\tau_{11} - \tau_{21}) u_i^* + 1/2 h_i \left( \tau_{11} + \tau_{21} \right) \theta_i^* \right) \, dc = 0
\]
\[ + (p_{1i} - p_{2i}) w_i^* \] dx \] = 0

\[ \sum_{i=a}^{b} \int_{0}^{L} \left[ N_{x_i} u_i^* + M_{x_i} \beta_i^* + Q_i (\beta_i^* + \dot{w}_i^*) \right] dx \] (44)

\[ + \frac{N_{b_i}}{R} w_i \] dx

The first quantity in brackets on the left hand side of (44) involves terms which are specified at the boundaries, namely:

\[ \sum_{i=a}^{b} \left[ N_{x_i} u_i^* + M_{x_i} \beta_i^* + Q_i \dot{w}_i^* \right]_0^L \]

Summing over layers gives

\[ \left[ N_{x_a} u_a^* + M_{x_a} \beta_a^* + Q_a \dot{w}_a^* + N_{x_b} u_b^* + M_{x_b} \beta_b^* + Q_b \dot{w}_b^* \right]_0^L \]

Since this derivation does not allow for slip or delamination, the virtual displacements must be constrained to:

\[ u_a^* = u_b^* + (1/2)h_a \beta_a^* + (1/2)h_b \beta_b^* \] (45)

\[ w_a^* = w_b^* + (\overline{w}_b - \overline{w}_a)^* \]
Substituting into (49) and collecting terms, the quantities to be specified at \(x = 0\) and \(x = L\) are found to be:

- either \(N_a x_a' + N_b x_b'\) or \(u_b\) specified
- either \(1/2 h_a N_a x_a' + M_a x_a\) or \(\beta_a\) specified
- either \(1/2 h_b N_a x_a' + M_b x_b\) or \(\beta_b\) specified
- either \(Q_a + Q_b\) or \(w_b\) specified

\[ Q_a (\bar{w}_b - \bar{w}_a) \text{ specified} \]

Note that the first four conditions are not a unique set. Other possibilities are \((u_a, u_b, \beta_a, \beta_b, w_b)\) or \((u_a, u_b, \beta_b, w_b)\) specified, but the set (46) seems to be a good choice.

The last condition of (46) is the result of retaining thermal expansion through-the-thickness while dropping terms of order \(h/R\) throughout the remainder of the derivation. For cases where \(\bar{w}\) must be retained, the following procedure can be employed. For simplicity, free-free boundary conditions are considered, though the treatment is analogous for any set of conditions specified.

Since the last of (46) is a temperature dependent, \(\bar{w}_b - \bar{w}_a\)
is a priori specified. Hence, for free boundaries, either \( Q_a = 0 \) or \( (\overline{w}_b - \overline{w}_a) = 0 \). \( Q_a = 0 \) is acceptable for special cases but is not generally true. \( (\overline{w}_b - \overline{w}_a) \) can be rigorously satisfied by redefining the reference surface location in layer \( a \), i.e.,

\[
\overline{w}_a (x, Z_0) = \int_0^{Z_0} \alpha_c T_a (x, z) \, dz
\]

\[
\overline{w}_b (x, h_b) = \int_{h_b/2}^{h_b/2} \alpha_c T_b (x, z) \, dz
\]

Under these conditions, the no-slip requirement becomes

\[
u_o = u_{o_b} + (-Z_0 \beta_a + 1/2 h_b \beta_b)
\]

and all other results remain the same if \( 1/2 h_a \) is replaced by \( -Z_0 \).

Note that when this approach is used, the second strain definition of equation (8) must be used in the form:

\[
\epsilon_i = \frac{w_i + \overline{w}_i}{R_i + 2} \quad (i = a, b)
\]

in order to have a zero stress state for the case of an isotropic material with the same \( \alpha_c \) in both layers and \( T = \text{constant} \).
V. SAMPLE PROBLEM

The problem selected was the slow cooling of pyrolytic graphite. Because of the difference in thermal expansion coefficients in both "a" and "c" directions of the pyrolytic graphite and mandrel material and also because of curvature effects, normal and shear stresses at the deposit-mandrel interface will be formed during the cooling process. These may be of sufficient magnitude to cause flaking or delamination. An investigation of their behavior with changing material and geometric properties is therefore of interest.

The test case considered a laminated cylinder with free-free edges and a constant temperature $T_0 = -1000^\circ F$. The material properties used were averaged in the range $3000^\circ F - 2000^\circ F$. Layer a, the top-most layer, was taken to be pyrolytic graphite, layer b commercial (ATJ) graphite. The properties used are given in Table (B.1). The calculations were performed with the aid of a CDC 6600 computer. The program tabulation is given in Appendix A. Results showing the behavior of $\sigma_j$ and $p_j$ due to variation in lamina thickness and/or $E/G_c$ ratio are in Appendix B. From these, the following conclusions may be drawn:

1. Figures (B.1) and (B.2) indicate a decrease in normal and shear stresses at the mandrel-PG interface as the $E/G_c$ ratio is increases, implying that a material weak in shear is desirable to minimize stresses and therefore the possibility of debonding.
2. Figures (B.3) and (B.4) show results when \( h = 0.50 \), \( E/C = 20 \) and \( 5 > h_a/h_b > 1 \). Figures (B.5) and (B.6) show the case where \( h_b = 0.25 \), \( E/G_c = 20 \), \( 4 > h_a/h_b > 1 \). All the curves indicate that a high \( h_a/h_b \) ratio is the desirable, for given material properties, for minimization of normal and shear stresses at the joint. This implies that a thin graphite mandrel is preferable to a thick mandrel. Note that when \( h_a/h_b = 1 \), the ultimate stress (\( \sigma_{ult} = 18,000 \) psi) in tension for pyrolytic graphite is exceeded.

3. To determine the behavior of the roots of equation (22) and establish a range wherein the various forms of solution (29)-(35) will occur, computations were also made for \( 1000 > h_a/h_b > 0.001 \) with \( E/G_c \) at 50, 20 and 2.6. Results show that the Case 1 solution in the form (29) occurs whenever \( h_a/h_b > 1 \) independent of the \( E/G_c \) ratio. The constants \( c_1 - c_3 \) are affected by \( E/G_c \), the constant \( c_1 \) being much more sensitive to a change in \( E/G_c \) than \( c_2 \) or \( c_3 \).

4. The first two terms of (29) have their principal effects at the edge only. It was also observed that for \( h_a/h_b >> 1 \), the value \( c_1 \) becomes so very large that the boundary value constants \( V_1 \) and \( V_2 \) tend to become very small, and as a first approximation the terms containing them can be dropped from the solution functions. In this case, (29) takes on a form similar to that for the solution for radial displacement of a single-layered cylinder, with suitably defined constants.

5. It should be kept in mind that all the above remarks are based on the numerical results and are valid in the range.

\[ 1000 > h_a/h_b > 0.001; \ 400 > L/n > 26; \ 0.05 > h/R > 0.0033 \]
BIBLIOGRAPHY


APPENDIX A

Computer Program
FORTRAN IV PROGRAM PCE(INPUT, OUTPUT, TAPE5=INPUT, TAPE9=OUTPUT)  PCE  1
DIMENSION ICO(6), IMM(6), IRE(6)  PCE  2
DIMENSION ORG(10), MPH(10), HQ(10), COEFS(5), G3AC0(15), ROOTS(2,6)  PCE  3
COMMON /SHA/ QA, OR, FO  PCE  4
COMMON /KLP/ ICI, IC2, ICA, IC5, IC6, IM1, IM2, IM3, IM4, IM5, IM6, IM7  PCE  5
IM2, IM3, IM4, IM5, IM6  PCE  6
COMMON DLTEM  PCE  7
COMMON TEMP, EA, EAC, NUA, NUAC, EB, ERC, NUB, NUBC, MA, MB, G3A, G3B, CA, CR  PCE  8
COMMON ORG, HQ, MPH, CAPC, NUM, GAM, GAMR, K1(40), K5(1), G7(17), RT(35), PMTOA, FMTOA  PCE  9
210R, BDE(501), LOAD1, AM(51), LOAD2, ENU(171), FXA, FNK, FNXA, FNKB, FXE, FNL  PCE  10
SIX, YCRA, YCRO, AC(3), ALIP, ELF, DELPHI(201), V(161), AL(181), MM, DM, DMM, W, W0, DWP  PCE  11
CALL DDB, DDBM, MA, DM, DMM, DMP, DDP, DUM, TAU, TDA, WP, WM, WMA, DMA, NUA, NUAC, UWP  PCE  12
SUA  PCE  13
EQUIVALENCE (ICI, ICO(1)), (IM1, IMM(1)), (IM2, IRE(1))  PCE  14
REAL NUA, NUAC, MA, MB, NUB, NUBC, NUM, K, LOADI, LOAD2  PCE  15
READ (5, 12) (KORD(I), I=1,4)  PCE  16
READ (5, 12) (MP(I), I=1,6)  PCE  17
READ (5, 12) (HQ(I), I=1,4)  PCE  18
READ (5, 11) TEMP, DLTMP  PCE  19
READ (5, 11) EA, EAC, NUA, NUAC  PCE  20
READ (5, 11) EB, ERC, NUB, NUBC  PCE  21
READ (5, 9) BT(1,1), BT(1,2), BT(1,3), BT(1,4)  PCE  22
READ (5, 10) (DELPHI(J), J=1,11)  PCE  23
WRITE (9, 13) (HQ(I), I=1,4)  PCE  24
WRITE (9, 14) TEMP  PCE  25
WRITE (9, 15) EA, EAC, NUA, NUAC  PCE  26
WRITE (9, 16) EB, ERC, NUB, NUBC  PCE  27
COEFS(2)=50.  PCE  28
COEFS(2)=26.  PCE  29
DO B (I=4), I=6  PCE  30
DO B (I=4), I=6  PCE  31
DO B (I=4), I=6  PCE  32
HM=HP(I)  PCE  33
MB=HQ(I)  PCE  34
R=30.  PCE  35
G3AC0(1)=EA/COEFS(1)  PCE  36
G3A=G3AC0(1)  PCE  37
EB=EB/2.+(I=1,NUBC1)  PCE  38
CALL PELIM(E, N, K)  PCE  39
CALL POLY(6, G, ROOTS(0))  PCE  40
CALL RICF(HOUT5, AC, KLP)  PCE  41
CALL THERM  PCE  42
CALL MISC  PCE  43
DO 7 I=1,4  PCE  44
ELL=CRK(IUP)  PCE  45
IF (ELL.EQ.0.0) GO TO 7  PCE  46
AM(1)=AC(2)*2-AC(3)  PCE  47
AM(2)=AC(2)*2-AC(3)  PCE  48
AM(3)=AC(2)*2-AC(3)  PCE  49
AM(4)=AC(2)*2-AC(3)  PCE  50
AM(5)=AC(2)*2-AC(3)  PCE  51
AM(6)=AC(2)*2-AC(3)  PCE  52
AM(7)=AC(2)*2-AC(3)  PCE  53
AM(8)=AC(2)*2-AC(3)  PCE  54
AM(9)=AC(2)*2-AC(3)  PCE  55
AM(10)=AC(2)*2-AC(3)  PCE  56
AM(11)=AC(2)*2-AC(3)  PCE  57
IF (KLP=5) 1,2,3  PCE  58
1 CONTINUE  PCE  59
IF (KLP=2) 4,5,6  PCE  60
SUBROUTINE UCOFF (FNTMW, CONST, X)
COMMON OLTMP
COMMON OLTEMP

FORMATT(4(E15.7))
FORMAT (4(E12.5))
FORMAT (4(E15.0))
FORMAT (1(EFA.0))
FORMAT (AN MB+12F9.3)
FORMAT (1X,6TEMP=15.5)
FORMAT (1X,6HAPPOE=15.7)
FORMAT (1X,6HPROP**=15.7)

END

SUBROUTINE PRELIM (FNTMW)
COMMON OLTMP

REAL NUA,NUAC,NUC,NUMU(1),LOAD1,LOAD2
PPS=-(ELL/2.)*AC(1)
PPS=-(ELL/2.)*AC(2)
PPB=-(ELL/2.)*AC(3)
CONST=BUE(19)*MA*HDE(20)*WB*BDE(21)*FNTMW-BT(20)*X/CAPCB
RETURN
END

$UQ
REAL NUA,NUAC,NUC,NUMU(1),LOAD1,LOAD2
CA=EA*MA/(1.-NUA**2)
CB=EB/(1.-NUB**2)
DA=EA*(H**2)/(12.+(1.-NUA**2))
DB=EB*(H**2)/(12.+(1.-NUB**2))
H=MA*MB
CAPC=CA+CB
CAPCB=CA-CB
NUMU=NUA-NUB
GAMA=5.0/(G3A*MA)
GAMB=5.0/(G3B*MB)
K(15)=LC+CB/CA
K(16)=LC+CA/CB
K(17)=MA/NB
PREL1 8
PREL1 9
PREL1 10
PREL1 11
PREL1 12
PREL1 13
PREL1 14
PREL1 15
PREL1 16
PREL1 17
PREL1 18
PREL1 19
PREL1 20
PREL1 21
PREL1 22
DISC+(MP**2)/4.*FRIMP**3)/27.

WRITE (9,2) HA,HR,RATIO,DISC

DISC**M**N*71.LT.0 IMPLIES KLIP = 2,3,4, OR 5 DEPENDING ON RPREL185
DISCRIMINANT,EQ.0 IMPLIES DEGENERATE CASE
DISCRIMINANT,GT.0 IMPLIES MODIFIED CLASSICAL SOLUTION PREL187
RETURN
PREL188
PREL169
TIM

TIB TKFM22

EXPANSION IN THICKNESS DIRECTION - B

MODIFIED AXIAL THERMAL RESULTANT A

AXIAL THERMAL RESULTANT (MODIFIED) - B

TEMP. EA, EAC, UA, NWAC, EB, EBC, NUB, NWB, HA, HR, G3A, G3B, CA, CB, EKLIP
WRITE (9,3) (BDE(I),I=1,100)  
WRITE (9,5)  
WRITE (9,6) (AM(I),I=1,SOI)  
WRITE (9,10)  
WRITE (9,6) (EN(I,J),I=1,7,J=1,6)  
CALL FIGLE (EN,Y)  
WRITE (9,7) (Y(I),I=1,16)  

AM(12)=ALTER/CHAP-BDE(12)*BDE(49)  
PPE=AC(J)=CELL(2)  
PPEP=AC(J)=CELL(/2.  
CALL CHECKS (V:AC, PPE, PP2, PP3)  
WRITE (9,7) (Y(I),I=1,16)  

AL(1)=AM(13)+Y(31)  
AL(2)=AM(13)+Y(12)  
AL(3)=ROE(11)*V(31)-BDE(11)*V(41)  
AL(4)=BDE(11)*V(31)-BDE(10)*V(41)  
AL(5)=ROE(11)*V(41)-ROE(11)*V(6)  
AL(6)=ROE(11)*V(65)-BDE(10)*V(6)  
AL(7)=BDE(12)*V(11)  
AL(8)=BDE(12)*V(12)  
AL(9)=ROE(11)*V(13)-BDE(14)*V(4)  
AL(10)=ROE(11)*V(13)-BDE(14)*V(6)  
AL(11)=ROE(17)*V(15)+ROE(18)*V(6)  
AL(12)=BDE(18)*V(15)+BDE(18)*V(6)  

AL(13)=AM(13)+Y(13)  
AL(14)=AM(13)+Y(13)  
AL(15)=AM(13)+Y(15)  
AL(16)=AM(13)+Y(16)  
AL(17)=AM(13)+Y(17)  
AL(18)=AM(13)+Y(18)  
WRITE (9,4)  
WRITE (9,4) (AL(I),I=1,18)  

DO 2 K=1,11  
2=DELPM(K)  
IF (KLE,CELL) GO TO 2  
P1=AC(1)*X  
P2=AC(21)*X  
P3=AC(3)*X  

CALL DIFF (P1,P2,P3)  

CALL IN1 (EXP(P1),V11,EXP(-P1),EXP(P2),V12,EXP(-P2),EXP(P3),V13,EXP(-P3))  

55
56
WRITE (9,3) (BDE(I),I=1,100) EKLP
WRITE (9,5) EKLP
WRITE (9,4) (AM(J),J=1,50) EKLP
WRITE (9,12) EKLP
WRITE (9,6) (EN(I),I=1,7),J=1,6) EKLP
CALL FINGLE (EN,V) EKLP
WRITE (9,7) (V(I),I=1,6) EKLP

A(I2) = 8T(2) / CAPCB - BDE(21) * BDE(49) EKLP
PP1 = AC(1) * ELL/2. EKLP
PP2 = AC(2) * ELL/2. EKLP
PP3 = AC(3) * ELL/2. EKLP
CALL CHECK2 (V,AC,PPI,PP2,PPI) EKLP
WRITE (9,7) (V(I),I=1,6) EKLP

AL(I1) = AM(30) * V(1) EKLP
AL(I2) = AM(31) * V(2) EKLP
AL(I3) = BDE(10) * V(3) - BDE(11) * V(4) EKLP
AL(I4) = BUE(11) * V(5) - UEF(10) * V(6) EKLP
AL(I5) = BUE(11) * V(1) - UEF(10) * V(2) EKLP
AL(I6) = BDE(12) * V(1) EKLP
AL(I7) = BDE(12) * V(2) EKLP
AL(I8) = BUE(12) * V(1) EKLP
AL(I9) = BUE(12) * V(2) EKLP
AL(I10) = BDE(15) * V(3) - BDE(17) * V(4) EKLP
AL(I11) = BUE(15) * V(3) - UEF(10) * V(4) EKLP
AL(I12) = BDE(18) * V(5) - BDE(17) * V(6) EKLP
WRITE (9,8) EKLP
WRITE (9,9) (AL(I),I=1,12) EKLP
DO 2 KF=1,11 EKLP
X = DELPH(KF) EKLP
IF (X.GT.ELL) GO TO 2 EKLP
P1 = AC(1) * X EKLP
P2 = AC(2) * X EKLP
P3 = AC(3) * X EKLP

CALL DIFF (P1,P2,P3) EKLP
WMV = V(1) * COS(P1) + V(2) * SIN(P1) + EXP(P2) * VS1(3) * COS(P3) + V(4) * SIN(P3) + EKLP
EXP = P2 * V(1) * COS(P1) + V(2) * SIN(P1) + UEF(49) EKLP
GMMV = V(1) * BDE(13) * V(2) * BDE(13) * V(3) * ODC(12) * V(4) * BDE(13) * V(5) * EKLP

59
WRITE (9,2) (BDE(I),I=1,120)
WRITE (9,3) (BDE(I),I=1,120)

CALL BINGO (E1,I)

WRITE (9,5) (I,(V(I),I=1,6)

PP1=AC(1)*ELL/2.
PP2=AC(2)*ELL/2.
PP3=AC(3)*ELL/2.

CALL CHECK (V,AC,PP1,PP2,PP3)

WRITE (9,6) (I,(V(I),I=1,12)

DO 1 KF=1,11

X=DELPH(KF)

P1*AC(1)*X
P2*AC(2)*X
P3*AC(3)*X

VW=Y1*EXP(-P1)+V2*EXP(-P1)+V31*Cos(P3)+V41*Sin(P1)+V51*Cos(P3)+
   V61*Sin(P3)+V71*Cos(P3)+V81*Sin(P3)+V91*Cos(P3)+V101*Sin(P3)+V111*

1 WRITE (9,7) (AL(I),I=1,12)

EN(J)=RDE(85)*AC(J)-RDE(3)*BDE(84)*AC(J)+BDE(12)*AC(J)+BDE(12)*AC(J)
CALL RSLT (K)
CALL RSLT (K)

2 FORMAT (1X,11H8AOI ARE ) / (10E12.5))
3 FORMAT (1X,2OH EN(I,J) ARE )
4 FORMAT (1X,TELL,4)
5 FORMAT (4(1X,ZHV(ll,2H)=E12.5I/2(1X,2HV(ll,1,2H)=E12.5I)
6 FORMAT (1X,2OH AL(I) ARE )
7 FORMAT (6E12.5)
8 FORMAT (11,1H3G8=El2.5,2X,3HGB=E12.5,2X,5HGBAR=E12.5)
END

SUBROUTINE ERLIF

DIMENSION JIF(K)

COMMON 01

COMMON TEMP.f

A.EAC,A,NUA.0AC,EB,HA,HS,3A,3B,CB,EB,HA,HC

REAL NUA,NUA,EB,HA,HS,3A,3B,CB,EB,HA,HC

CASE OF THREE REAL KNOTS,UNEQUAL,NEGATIVE

BDE(1)=A(4)+AC(1)+A(4)+AC(1)
BDE(2)=A(4)+AC(1)+A(4)+AC(1)+A(4)+AC(1)
BDE(3)=BDE(1)/BDE(2)

BDE(4)=A(4)+AC(2)+A(4)+AC(2)
BDE(5)=A(4)+AC(2)+A(4)+AC(2)+A(4)
BDE(6)=BDE(1)/BDE(5)

BDE(7)=A(4)+AC(3)+A(4)+AC(3)
BDE(8)=A(4)+AC(3)+A(4)+AC(3)+A(4)
BDE(9)=BDE(1)/BDE(8)

BDE(10)+BDE(3)+AC(1)+A(2)+A(2)+AC(1)+A(2)+A(2)+AC(1)
BDE(11)+A(2)+A(2)+AC(1)+A(2)+A(2)+AC(1)
BDE(12)=BDE(10)/BDE(11)

BDE(13)=BDE(6)+A(2)+A(2)+AC(2)+A(2)+A(2)+AC(2)
BDE(14)=A(2)+A(2)+AC(2)+A(2)+A(2)+AC(2)
BDE(15)=BDE(13)/BDE(14)

BDE(16)=BDE(9)+A(2)+A(2)+AC(3)+A(2)+A(2)+AC(3)
BDE(17)=A(2)+A(2)+AC(3)+A(2)+A(2)+AC(3)
BDE(18)=BDE(16)/BDE(17)

2 FORMAT (1X,11H8AOI ARE ) / (10E12.5))
3 FORMAT (1X,2OH EN(I,J) ARE )
4 FORMAT (1X,TELL,4)
5 FORMAT (4(1X,ZHV(ll,2H)=E12.5I/2(1X,2HV(ll,1,2H)=E12.5I)
6 FORMAT (1X,2OH AL(I) ARE )
7 FORMAT (6E12.5)
8 FORMAT (11,1H3G8=El2.5,2X,3HGB=E12.5,2X,5HGBAR=E12.5)
END

SUBROUTINE ERLIF

DIMENSION JIF(K)

COMMON 01

COMMON TEMP.f

A.EAC,A,NUA.0AC,EB,HA,HS,3A,3B,CB,EB,HA,HC

REAL NUA,NUA,EB,HA,HS,3A,3B,CB,EB,HA,HC

CASE OF THREE REAL KNOTS,UNEQUAL,NEGATIVE

BDE(1)=A(4)+AC(1)+A(4)+AC(1)
BDE(2)=A(4)+AC(1)+A(4)+AC(1)+A(4)+AC(1)
BDE(3)=BDE(1)/BDE(2)

BDE(4)=A(4)+AC(2)+A(4)+AC(2)
BDE(5)=A(4)+AC(2)+A(4)+AC(2)+A(4)
BDE(6)=BDE(1)/BDE(5)

BDE(7)=A(4)+AC(3)+A(4)+AC(3)
BDE(8)=A(4)+AC(3)+A(4)+AC(3)+A(4)
BDE(9)=BDE(1)/BDE(8)

BDE(10)+BDE(3)+AC(1)+A(2)+A(2)+AC(1)+A(2)+A(2)+AC(1)
BDE(11)+A(2)+A(2)+AC(1)+A(2)+A(2)+AC(1)
BDE(12)=BDE(10)/BDE(11)

BDE(13)=BDE(6)+A(2)+A(2)+AC(2)+A(2)+A(2)+AC(2)
BDE(14)=A(2)+A(2)+AC(2)+A(2)+A(2)+AC(2)
BDE(15)=BDE(13)/BDE(14)

BDE(16)=BDE(9)+A(2)+A(2)+AC(3)+A(2)+A(2)+AC(3)
BDE(17)=A(2)+A(2)+AC(3)+A(2)+A(2)+AC(3)
BDE(18)=BDE(16)/BDE(17)
BDE(104) * BDE(105) * AC(1)

BDE(111) = -AC(1) * EXP(-P1)
BDE(112) = -BDE(111) * AC(1)
BDE(113) = -BDE(112) * AC(1)
BDE(114) = -BDE(113) * AC(1)

BDE(121) = EXP(P2) * (AC(2) * COS(P3) - AC(3) * SIN(P3))
BDE(122) = EXP(P2) * (AM(1) * COS(P3) - AM(2) * SIN(P3))
BDE(123) = EXP(P2) * (AM(3) * COS(P3) - AM(4) * SIN(P3))
BDE(124) = EXP(P2) * (AM(5) * COS(P3) - AM(6) * SIN(P3))

BDE(131) = EXP(P2) * (AC(3) * COS(P3) + AC(2) * SIN(P3))
BDE(132) = EXP(P2) * (AM(2) * COS(P3) + AM(3) * SIN(P3))
BDE(133) = EXP(P2) * (AM(4) * COS(P3) + AM(5) * SIN(P3))
BDE(134) = EXP(P2) * (AM(6) * COS(P3) + AM(7) * SIN(P3))

BDE(141) = EXP(-P2) * (AC(2) * COS(P3) - AC(3) * SIN(P3))
BDE(142) = EXP(-P2) * (AM(1) * COS(P3) + AM(2) * SIN(P3))
BDE(143) = EXP(-P2) * (AM(3) * COS(P3) + AM(4) * SIN(P3))
BDE(144) = EXP(-P2) * (AM(5) * COS(P3) + AM(6) * SIN(P3))

BDE(151) = EXP(-P2) * (AC(3) * COS(P3) - AC(2) * SIN(P3))
BDE(152) = EXP(-P2) * (AM(2) * COS(P3) - AM(3) * SIN(P3))
BDE(153) = EXP(-P2) * (AM(4) * COS(P3) - AM(5) * SIN(P3))
BDE(154) = EXP(-P2) * (AM(6) * COS(P3) - AM(7) * SIN(P3))

BDE(161) = EXP(P2) * AC(2)
BDE(162) = EXP(P2) * BDE(161)
BDE(163) = EXP(P2) * BDE(162)
BDE(164) = EXP(P2) * BDE(163)

BDE(171) = EXP(-P2) * AC(2)
BDE(172) = EXP(-P2) * BDE(171)
BDE(173) = EXP(-P2) * BDE(172)
BDE(174) = EXP(-P2) * BDE(173)

BDE(181) = AC(1) * SIN(P1)
BDE(182) = AC(1) * COS(P1)
BDE(183) = AC(1) * SIN(P1)
BDE(184) = AC(1) * COS(P1)

BDE(191) = AC(1) * COS(P1)
BDE(192) = AC(1) * SIN(P1)
BDE(193) = AC(1) * SIN(P1)
BDE(194) = AC(1) * COS(P1)

BDE(201) = AC(2) * SIN(P2)
BDE(202) = AC(2) * COS(P2)
BDE(203) = AC(2) * SIN(P2)
BDE(204) = AC(2) * COS(P2)
BDE(211) = AC(2) * COS(P2)
BDE(212) = -(AC(2) * P2) * SIN(P2)
BDE(213) = -(AC(2) * P2) * COS(P2)
BDE(214) = (AC(2) * P2) * SIN(P2)

BDE(221) = AC(3) * COS(P3)
BDE(222) = -(AC(3) * P2) * COS(P3)
BDE(223) = -(AC(3) * P2) * COS(P3)
BDE(224) = (AC(3) * P2) * SIN(P3)

BDE(231) = AC(3) * COS(P3)
BDE(232) = -(AC(3) * P2) * SIN(P3)
BDE(233) = -(AC(3) * P2) * COS(P3)
BDE(234) = (AC(3) * P2) * SIN(P3)

BDE(241) = AC(3) * EXP(P3)
BDE(242) = AC(3) * BDE(241)
BDE(243) = AC(3) * BDE(242)
BDE(244) = AC(3) * BDE(243)

BDE(251) = AC(3) * EXP(-P3)
BDE(252) = AC(3) * BDE(251)
BDE(253) = AC(3) * BDE(252)
BDE(254) = AC(3) * BDE(253)

RETURN
END

SUBROUTINE POLYR(N, COEFF, ROOTS, D)
DIMENSION A(51, 3), IA(51), ROOTS(2, 4), D(1), COEFF(11)
INTEGER DEGREE
Ni=DEGREE+1
M=30
MAX=15
DELTAD0.0001
EPSILON=0.00001
DO I=1,N1
10 A(I,1)=COEFF(I)
11 A(I,1)=0
CALL SCALE (A(I,1),A(1,1))
2 CONTINUE
2 RETURN
3 PRINT 4
RETURN
4 FORMAT (21H0SOME ROOTS NOT FOUND)
END

SUBROUTINE RSSR(A, IA, ROOTS, DEGREE, M, MAX, DELTA, EPSILON, D)
DIMENSION A(51, 3), IA(51), ROCTS(2, 50), D(51), ROMGS(50), HROMGS(50), NONRT(50)
INTEGER DEGREE
M=DEGREE

1 DEGREE = NCUR
2 RETURN
3 N1 = N + 1
4 DO 5 I = 1, N
5 K = N2 - I
6 IF (IA(K, 11)) # 4, 6
7 J = N1 - I
8 ROOTS(1, J) = 0.0
9 ROOTS(2, J) = 0.0
10 CONTINUE
11 DEGREE = 0
12 GO TO 2
13 M = K
14 N = K - 1
15 NCUR = N
16 N0 = NCUR
17 DO 10 N0 = NCUR, NL, -1
18 M = NCUR + 1
19 CALL ROOTS (A, IA, NCUR, M)
20 CALL REALRT (A, IA, NCUR, DELTA, EPSILON, ROMOD, MRQMOD, NONRT, MNONRT, N)
21 IF (NCUR = 12) EXIT
22 CALL COMPRT (A, IA, ROMOD, ROOTS, M, MNONRT, NONRT, MRQMOD, NCO, DELTA, EPSILON, NCUR)
23 IF (NCUR = 12) EXIT
24 CALL COMPRT (A, IA, ROMOD, ROOTS, M, MNONRT, NONRT, MRQMOD, NCO, DELTA, EPSILON, NCUR)
25 IF (NCUR = 12) EXIT
26 CALL COMPRT (A, IA, ROMOD, ROOTS, M, MNONRT, NONRT, MRQMOD, NCO, DELTA, EPSILON, NCUR)
27 CALL COMPRT (A, IA, ROMOD, ROOTS, M, MNONRT, NONRT, MRQMOD, NCO, DELTA, EPSILON, NCUR)
28 CALL COMPRT (A, IA, ROMOD, ROOTS, M, MNONRT, NONRT, MRQMOD, NCO, DELTA, EPSILON, NCUR)
29 CALL COMPRT (A, IA, ROMOD, ROOTS, M, MNONRT, NONRT, MRQMOD, NCO, DELTA, EPSILON, NCUR)
30 CALL COMPRT (A, IA, ROMOD, ROOTS, M, MNONRT, NONRT, MRQMOD, NCO, DELTA, EPSILON, NCUR)
31 CALL COMPRT (A, IA, ROMOD, ROOTS, M, MNONRT, NONRT, MRQMOD, NCO, DELTA, EPSILON, NCUR)
32 CALL COMPRT (A, IA, ROMOD, ROOTS, M, MNONRT, NONRT, MRQMOD, NCO, DELTA, EPSILON, NCUR)
33 CONTINUE
34 IF (NCUR NE 1) GOTO 12
35 CALL RECON (ROOTS, A(1, 1), IA(1, 1), D, DEGREE)
36 GO TO 1
37 CALL KROOTS (A, IA, NCUR, MM)
38 END
IX=IA(JL,2)+IA(JLP,2)
CALL SCALE (IX,IX)
CALL SBTRT (AI(J,3),IA(J,3),X,IX,ADJ,3,IA(J,3))
5 CONTINUE
   CALL SCALE (AI(J,3),IA(J,3))
   X=AI(J,2)**2
   IX=IA(J,2)+IA(J,2)
   CALL SCALE (IX,IX)
   CALL ADD (AI(J,3),IA(J,3),X,IX,ADJ,3)
   JR=XMDF(J,2)
   IF (JR) 6,6,7
5 CONTINUE
6 A(J,3)=A(J,3)
7 CONTINUE
IF (MK-M) 10,10,8
8 DO 9 J=1,ML
   A(J,2)=A(J,3)
   AI(J,2)=IA(J,3)
   AI(J,3)=0.0
   IAJ=3=0
9 CONTINUE
10 CONTINUE
RETURN
END

SUBROUTINE REALT (A,A,M,NCUR,DELTA,EPSILON,ROMOD,ROMOD,MONRT,MONRT

DIMENSION A(SL,1),A(SL,3),ROOTS(2,50),ROMOD(50),ROMOD(50),NRLRT 3
MONRT(50),MONRT(50),RATIO(S1),PIVIS(S1),ARED(S1),ARED(S1)
RATIO(1)=1.0
DO 6 1=2,NCUR
   II=XMDF(F,2)
IF (II) 6,6,7
1 RATIO(1)=RATIO(1)
6 GO TO 7
7 T=AI(1,2)*AI(1,2)
   IT=AI(1,1)+AI(1,2)
   CALL SCALE (IT,IT)
   T=IT/AI(1,3)
   IT=IT-AI(1,3)
IF (IT) 7,7,3
   IF (IT) 7,7,3
   3 IF (IT)*2 1.0,4,4
   CALL UNSCALE (T,IT)
   RATIO(1)=T
   IF (IT) 6,6,6
   6 RATIO(1)=RATIO(1)
5 CONTINUE
   RATIO(NCUR+1)=1.0
   PIVIS(1)=1
   PIVIS(NCUR+1)=1
   DO 9 1=2,NCUR
        II=ABS(RATI(1)-1.0)
        IF (II) 7,7,7
   7 PIVIS(1)=1
   GO TO 9
8 PIVIS(1)=0
9 CONTINUE
   NCUR=NCUR+1
11=0
   MULT=0
11=1
14=1
10 11=1+1
12=1+1
MULT=MULT+1
IF (IPIV(12)) 10,10,11
11 ROMOD(14)=A(12,3)/A(1,3)
IROMOD=A(12,3)-A(1,3)
CALL SCALE (ROMOD(14),IROMOD)
IF (IROMOD(14)) 17,13,13
12 ROMOD(14)=ROMOD(14)
13 CALL DOUPLOG (ROMOD(14),IROMOD,KN,IXN)
T=2*#*#1
KN=IXN
CALL SCALE (KN,IXN)
CALL OLEXP (KN,IXN,ROMOD(14),IROMOD)
IF (IROMOD-74) 14,14,15
14 IF (IROMOD=74) 15,15,16
15 ROMOD(14)=0.0
IROMOD=0
GO TO 17
16 CALL UNSCALE (ROMOD(14),IROMOD)
17 IRROMD(14)=MULT
IF (NCUR=1-12) 19,19,18
18 I=12
14=14+1
MULT=0
I=0
GO TO 10
19 Q=0.0
NCUR=J
DO 20 1=1,14
KL=14+1-I
W=ROMOD(KL)
IS=ROMOD(KL)
D0 26 J=1,15
J=J
GO TO 20
20 CALL TEST (A,A,=0,NCUR,ROMOD(KL),EPSILON,K)
IF (J) 23,23,21
21 ROOTS(1,NCUR)=W
ROOTS(2,NCUR)=Q
ARED(I)=A(1,1)
IARED(I)=A(1,1)
D0 22 L=2,NCUR
Y=ARED(L-1)*W
JY=IARED(L-1)
CALL SCALE (Y,Y)
CALL SCALE (AI,L2),TAIL,1,Y,Y,IARED(L),IARED(L)
AIL,1)=IARED(L)
TAIL,1)=IARED(L)
CONTINUE
GO TO 25
23 IF (M) 24,27,27
24 M=W
GO TO 20
25 NCUR=NCUR-1
26 CONTINUE
GO TO 28
27 NCUR=KCO+1
NOWART(1,1)=KL
NOWRT(1,1)=15+1-J
CONTINUE
RETURN
END

SUBROUTINE CCMAPT (A, IA, ROMOD, ROOTS, M, NONAT, NONRT, KROMOD, KNO, DELT**** 1
A• EPSILON, NONRT
*** 2
DIMENSION A(N1,3), IA(N1,3), ROMOD(90), \( SRT(2,50), \) SR(151,3), ISR, SROOTS(2,50), MNONRT(50), NONRT(50), KROMOD(5)
201; NSONRT(4), MSORT(44), MROMOD(50), U(2), R(2), B(44)

DO 28 I=1,NCO
J=NONAT(I)
I1=NONAT(I)
N(I)=I/2
IF (I1) 1,1,2
1 I1=1
2 IF (ROMOD(JA) 3,20,3
3 Q=ROMOD(JA)
DO 27 J=1,11
CALL SUBRES (A, IA, NCUR, SR, ISR, XR)
15 IF (NCUR-4) 5,4,6
4 NSCUR=2
GO TO 7
5 NSCUR=1
J2=1
GO TO 8
6 NSCUR=NSCUR-3
7 J2=NSCUR
8 LL=NSCUR+1
9 IF (NSCUR-1) 9,9,11
IF (NRUR1) 10,12,10
10 X=SR1(1,1)
IX=ISR1(1,1)
Y=ISR2(1,1)
YV=ISR2(1,1)
CALL UNSCALE (X, XR)
31 CALL UNSCALE (Y, YR)
32 ROOTS(1,1+1,1,1) = Y/X
33 NSCUR=0
GO TO 13
11 CALL ROOTS (SR, ISR, NSCUR, M)
36 CALL REALAT (SR, ISR, NSCUR, DELTA, EPSILON, KROMOD, KROMOD, NSNONRT, MSORT, NSCUR, SROOTS)
37 IF (J2-NSCUR) 12,12,13
12 SROOTS(1, J2) = 0.0
40 SROOTS(1, J2) = SROOTS(1, J2) * ROMOD(JA)
13 T=ROMOD(JA) * KROMOD(JA)
41 IF (SROOTS(1, J2)-T) 14,21,21
14 W=SROOTS(1, J2)
44 WE=ROMOD(JA) * KROMOD(JA)
45 CALL TEST (A, IA, M, WE, NCUR, KROMOD(JA), EPSILON, XR)
46 IF (K) 20,29,15
47 IF (K) 20,29,15
15 ROOTS(1, NCUR) = W/2.0
T=4.0*WE
U=WE
T=T-U
51 IF (T) 16,16,17
16 T=T
U=SORT(T)
ROOTS(1, NCUR) = ROOTS(1, NCUR) - W/2.0
ROOTS(1, NCUR) = (W-U)/2.0
ROOTS(2, NCUR) = 0.0
74
ROOTS(2, NCUR-1)=0.0
GO TO 18
17 U=SORT(I)
ROOTS(2, NCUR)=U/2.0
ROOTS(1, NCUR-1)=ROOTS(1, NCUR)
ROOTS(2, NCUR-1)=ROOTS(2, NCUR)
10 D(1)=W
D(2)=WE
CALL QADIV (NCUR, A, IAR, D, B)
JX=NCUR-1
DO 19 JY=1, JX
A(JY, 1)+=B(JY)
I[A(JY, 1)]=0
CALL SCALE (A(JY, 1), I[A(JY, 1)])
19 CONTINUE
NCUR=NCUR-2
GO TO 27
20 W=W
CALL TEST (A, IAR, WE, NCUR, ROMOD(IJA), EPSILON, K)
IF (K) 21, 21, 15
21 IF (J2-(NSCUR+1)) 28, 22, 24
22 IF (J2-(J2-1)) 26, 26, 23
23 J2=J2-1
SR0OTS(1, J2)=0.0
GO TO 14
24 IF (SR00TS(1, J2)-SR0OTS(1, J2-1)) 25, 26, 25
25 J2=J2-1
GO TO 21
26 J2=J2-1
GO TO 21
27 CONTINUE
28 CONTINUE
RETURN
END
SUBROUTINE TEST (A, IAR, WE, NCUR, ROMOD(IJA), EPSILON, K)
DIMENSION A(51, 3), IA(51, 3), B(1), B(2), T(2), E(2), C(51)
B(1)=0.0
IX=0
IW=0
IB(1)=0
B(2)+A(1, 1)
IB(2)+I[A(1, 1)]
DO 2 I=1, 4
X=W+B(2)
2 IX=IB(2)
CALL SCALE (X, IX)
Y=0\#A(1)
IV=I(1)
CALL SCALE (Y, IV)
CALL ADD (X, IX, Y, IV, Z, IZ)
CALL SBTRT (A(I[A(1,1)], IA[I[A(1,1)], Z, IZ, B(1)], IB(1), IB(2))
IF (IV-1) 2, 2, 1
1 B(1)=E(2)
IB(1)=IB(2)
B(2)+A(3)
IB(2)+IB(3)
2 CONTINUE
K0UT1=1
CEPSIL=EPSILON
Y(1)=0.0
ST(2)=0.0
**** 1
TEST 2
TEST 3
TEST 4
TEST 5
TEST 6
TEST 7
TEST 8
TEST 9
TEST 10
TEST 11
TEST 12
TEST 13
TEST 14
TEST 15
TEST 16
TEST 17
TEST 1d
TEST 19
TEST 20
TEST 21
TEST 22
TEST 23
TEST 24
TEST 25
TEST 26
75
**TEST 27**

**EI0N1**

**EI2=CEPSILON**

**EI=0.0**

**EI0**

**EI2=CEPSILON**

**EI0**

**DO I=1,11**

**IF** (A(I),11) 3,4,6

**3**

**EI0**

**GO TO 5**

**4**

**CALL UNSCALE (EI1,IC)**

**T1=T1+1**

**T2=T2+1**

**CALL (EI1)**

**6 CONTINUE**

**DIFF=Y1-Y2**

**IF** (0) 16,7,10

**7**

**IF** (R31) 8,9,9

**8**

**IF** (183) 10,10,12

**9**

**IF** (183) 12,11,11

**10**

**CALL UNSCALE (EI1,IR31)**

**IF** (DIFF-B13) 12,12,17

**11**

**K=0**

**IF** (KOUNT=2) 13,14,16

**12**

**IF** (1) 15,14,15

**13**

**IF** (R41) 16,16,16

**14**

**SENS1**

**15**

**KOUNT=KOUNT+1**

**16 RETURN**

**17**

**K=0**

**GO TO 16**

**18**

**IF** (182) 19,19,12

**19**

**IF** (182) 12,20,20

**20**

**CALL UNSCALE (EI2,IR12)**

**IF** (183) 21,21,12

**21**

**CALL UNSCALE (EI1,IR11)**

**X=0B12*B12**

**Y=0B3*B3**

**Z=0B4*B4**

**24**

**DIFF=DIFF-DEF**

**IF** (DIFF-V) 12,17,17

**END**

**SUBROUTINE SUBRES (A,I,N,M,SR,ISR,RMOD)**

**DIMENSION AISJ,ATJ,SR(IS1J),ISR(IS1J),SR1J1,J1J1,A150,11J2**

**M=1**

**N=1**

**X=0.0**

**DO L=1,14**

**J=M+1**

**CALL RMOD**

**IC*=-141J**

**IC*=-141J**

**CALL UNSCALE (C1J1,IC)**

**1 CONTINUE**

**C1J1=A1J1**

**IC*=-141J1**

**CALL UNSCALE (C1J1,IC)**

**76**
IF (4-2) 17,17,2
2 N2=N-2
DO 3 I=1,N2
B(I,1)=.0
B(I,2)=.0
3 CONTINUE
I=2
B(I,2)=C(I)
4 B(I,3)=C(I)-B(I,1)
DO 5 J=2,N2
B(J,3)=B(J-1,2)-B(J,1)
5 CONTINUE
IF (4-(3+11)) 8,6,6
6 I=1
DO 7 J=1,N2
B(J,1)=B(J,2)
B(J,2)=B(J,3)
7 CONTINUE
GO TO 4
8 IF (4-4) 19,9,13
9 IF (4-(2*1)) 12,10,10
10 I=I+1
DO 11 J=1,2
B(J,1)=B(J,2)
B(J,2)=B(J,3)
11 CONTINUE
GO TO 4
12 B(3,3)=M(2,2)
SR(3,1)=C(5)*B(1,3)
ISR(3,1)=0
SR(2,1)=B(2,3)
ISR(2,1)=0
SR(1,1)=D(3,3)
ISR(1,1)=0
CALL SCALE (SR(1,1),ISR(1,1))
CALL SCALE (SR(2,1),ISR(2,1))
CALL SCALE (SR(3,1),ISR(3,1))
GO TO 16
13 SR(N2,1)=C(4)-B(1,3)
ISR(N2,1)=0
SR(N2-1,1)=C(N1)-B(2,3)
ISR(N2-1,1)=0
CALL SCALE (SR(N2,1),ISR(N2,1))
CALL SCALE (SR(N2-1,1),ISR(N2-1,1))
IF (42-2) 16,16,14
14 DO 15 J=3,N2
K=N2+1-J
SR(K,1)=B(J,3)
ISR(K,1)=0
CALL SCALE (SR(K,1),ISR(K,1))
15 CONTINUE
16 RETURN
17 SR(1,1)=C(1)
ISR(1,1)=0
SR(2,1)=C(2)
ISR(2,1)=0
18 CALL SCALE (SR(1,1),ISR(1,1))
CALL SCALE (SR(2,1),ISR(2,1))
GO TO 16
19 SR(1,1)=C(4)
SUBROUTINE RECON (ROOTS, A, N)
DIMENSION ROOTS(2,50), D(51)
X=0
I=1
CALL UNSCALE (X, I)
DO 1 I=2, N
D(I)=0.0
1 CONTINUE
D(IN+1)=0.0
I=1
NL=N-1
2 IF (ROOTS(I,1)) 3, 7, 3
3 T=ROOTS(I,1)*ROOTS(I,1)
U=ROOTS(I,2)*ROOTS(I,2)
T=T+U
U=2.0*ROOTS(I,1)
DO 5 J=1, NL
IF (I+J-1 < N, 5, 4)
4 D(I+J)=D(I+J)+T*D(J)
5 CONTINUE
D(IN)+D(IN)+D(IN+1)
D(IN+1)=T*D(IN)
I=I+2
6 IF (I+2 < N, 10, 2.2)
7 DO 9 J=1, N
IF (J+I-1 < N, 9, 8)
8 D(I+J)=D(I+J)+D(I+J)*ROOTS(I,1)
9 CONTINUE
D(IN+1)=D(IN+1)*ROOTS(I,1)
I=I+1
10 CONTINUE
GO TO 6
10 NS=NS+1
DO 11 I=1, NS
D(I)=D(I)*2
11 CONTINUE
RETURN
END
SUBROUTINE QUADIV (N, A, R, D, B)
DIMENSION A(N, 3), R(N), D(N), B(N)
R(I)=A(I,1)
B(I)=A(I,1)
CALL UNSCALE (B(1), B)
IF (N-2 < 4, 4, 1)
1 AA=A(I,2)
IA=IA-1
CALL UNSCALE (AA, IA)
B(I+2)=AA+R(I+1)*D(I)
IF (I-3 < 4, 4, 2)
2 NT=NT-1
DO 3 I=3, N
3 AA=R(I-2)*D(I)
AA=AA+1
END
SUBROUTINE OLEXP (X,IX,IX,1Z,12)
  IZ=0
  IF ( IX ) 1,6,6
1  IZ=1
  IF =4P1
  GO TO 4
  K=XPDDF(IX,IZ)
  IF (K) 2,3,3
2  IX=IZ-1
  Z=64.0*Z
  Z=SORF(Z)
  CALL SCALE (Z,IZ)
  CONTINUE
3  IX=IZ/2
4  CONTINUE
9  RETURN
6  I=6P1
  GO TO 7
7  Z=Z2
  Z=IZ+12
  CALL SCALE (Z,IZ)
  CONTINUE
GO TO 5
8  CALL SCALE (Z,IZ)
  GO TO 5
END
SUBROUTINE ADD (X,IX,IX,1Y,IX,1Z,12)
  IF (IX) 2,3,3
1  Z=Y

SUBROUTINE DOUBLOG (X,IX,IX,1Y,1Y)
  T=64.0
  IF (IX) 1,2,3
1  PRINT 5
  Y=0.0
  IF =6
    7  GO TO 4
3  TO+X
  Y=ALOG1(X)+TOMLOG1(T)
4  IF =0
  CALL SCALE (Y,1Y)
  11  RETURN
4  FORMAT (+640THE LOG OF A NON-POSITIVE NUMBER IS REQUESTED)
END
SUBROUTINE OLEXP (X,IX,IX,1Z,12)
  IZ=0
  IF ( IX ) 1,6,6
1  IZ=1
  IF =4P1
  GO TO 4
  K=XPDDF(IX,IZ)
  IF (K) 2,3,3
2  IX=IZ-1
  Z=64.0*Z
  Z=SORF(Z)
  CALL SCALE (Z,IZ)
  CONTINUE
3  IX=IZ/2
4  CONTINUE
9  RETURN
6  I=6P1
  GO TO 7
7  Z=Z2
  Z=IZ+12
  CALL SCALE (Z,IZ)
  CONTINUE
GO TO 5
8  CALL SCALE (Z,IZ)
  GO TO 5
END
SUBROUTINE ADD (X,IX,IX,1Y,IX,1Z,12)
  IF (IX) 2,3,3
1  Z=Y

79
IZ=1
2 RETURN
3 IF (Y) 5,4,5
4 I=K
IZ=IX
GO TO 2
5 IDIFF=IX-Y
IF (IDIFF) 6,7,7
6 IA=1Y
A=Y
B=X
IDIFF=IDIFF
GO TO 6
7 IA=IX
A=X
B=Y
8 IF (16-IDIFF) 9,9,10
9 J=A
IZ=1A
GO TO 2
10 IF (IDIFF) 11,13,11
11 DO 12 I=1,11,1
12 CONTINUE
13 CONTINUE
Z=A+B
IZ=1A
CALL SCALE (Z,IZ)
GO TO 2
END
SUBROUTINE SBYTF (X,IX,Y,IY,IZ,IZ)
W=Y
CALL ADD (X,IX,W,1Y,2,IZ)
RETURN
END
SUBROUTINE SCALE (X,IX)
REC6=1.0/66.3
IF (IX) 1,11,2
1 Y=-X
GO TO 3
2 Y=X
3 IF (64,0-Y) 4,5,5
4 Y=Y/64.0
IX=IX+1
GO TO 3
5 IF (IX-REC64) 6,7,7
6 Y=Y*64.0
IX=IX-1
GO TO 9
7 IF (IX) 8,9,9
8 X=--Y
GO TO 10
9 X=Y
10 RETURN
11 IX=0
GO TO 10
END
SUBROUTINE UVSCLAE (X,IX)
IF (1X=K4) 1,2,2
1 X=0.0
IX=0
GO TO 6
2 IF (IX=94) 4,4,3
3 X=1,00E+133
IX=0
PRINT 7
GO TO 6
4 IF (IX) 5,6,5
5 X=X+64,30×IX
IX=0
6 RETURN

7 FORMAT (25H0EXP., OVERFLOW IN UNSCALE)
END
SUBROUTINE CHECK1 (V,AC,PP1,PP2,PP3)
DIMENSION V(I6), AC(1)
V11=V(11)EXP(PP1)
V12=V(12)EXP(-PP1)
AV3=V(3)EXP(PP2)×V(11)×COS(PP3)×V(4)×SIN(PP3)
AV4=V(4)EXP(PP3)×V(4)×COS(PP3)×V(3)×SIN(PP3)
AV5=V(5)×COS(PP3)×V(6)×SIN(PP3)
AV6=V(6)×COS(PP3)×V(5)×SIN(PP3)
V(3)=AV3
V(4)=AV4
V(5)=AV5
V(6)=AV6
RETURN
END
SUBROUTINE CHECK2 (V,AC,PP1,PP2,PP3)
DIMENSION V(I6), AC(1)
AV1=V(1)×COS(PP1)×V(2)×SIN(PP1)
AV2=V(1)×SIN(PP1)×V(2)×COS(PP1)
V(1)=AV1
V(2)=AV2
AV3=V(3)EXP(PP2)×V(1)×COS(PP3)×V(4)×SIN(PP3)
AV4=V(4)EXP(PP3)×V(4)×COS(PP3)×V(3)×SIN(PP3)
AV5=V(5)×COS(PP3)×V(6)×SIN(PP3)
AV6=V(6)×COS(PP3)×V(5)×SIN(PP3)
V(3)=AV3
V(4)=AV4
V(5)=AV5
V(6)=AV6
RETURN
END
SUBROUTINE CHECK4 (V,AC,PP1,PP2,PP3)
DIMENSION V(I6), AC(1)
V11=V(11)EXP(PP1)
V12=V(12)EXP(-PP1)
AV3=V(3)EXP(PP2)×V(11)×COS(PP2)×V(4)×SIN(PP2)
AV4=V(4)EXP(PP3)×V(4)×COS(PP3)×V(3)×SIN(PP3)
AV5=V(5)×COS(PP3)×V(6)×SIN(PP3)
AV6=V(6)×COS(PP3)×V(5)×SIN(PP3)
V(3)=AV3
V(4)=AV4
V(5)=AV5
V(6)=AV6
RETURN
END
SUBROUTINE CHECK5 (V,AC,PP1,PP2,PP3)
END
AC(2) = ABS(Root(2, IM1))
IF (AC(2) .NE. ABS(ROOT(2, IM2))) GO TO 8
AC(3) = ABS(ROOT(2, IM3))
RETURN
8 AC(3) = ABS(ROOT(2, IM2))
RETURN

9 KLIP = 3
AC(1) = ABS(ROOT(2, IM1))
IF (ABS(ROOT(2, IM1)) .NE. ABS(ROOT(2, IM2))) GO TO 10
AC(2) = ABS(ROOT(2, IM3))
IF (ABS(ROOT(2, IM3)) .NE. AC(1)) AND (ABS(ROOT(2, IM4)) .NE. AC(2)) GO TO 10
ELSE
GO TO 14
10 AC(2) = ABS(ROOT(2, IM2))
IF (ABS(ROOT(2, IM3)) .EQ. AC(1)) GO TO 11
IF (ABS(ROOT(2, IM4)) .EQ. AC(2)) GO TO 13
AC(3) = ABS(ROOT(2, IM6))
RETURN
11 IF (ABS(ROOT(2, IM4)) .NE. AC(2)) GO TO 14
12 AC(3) = ABS(ROOT(2, IM4))
RETURN
13 IF (ABS(ROOT(2, IM6)) .NE. AC(1)) GO TO 15
AC(3) = ABS(ROOT(2, IM5))
RETURN
15 AC(3) = ABS(ROOT(2, IM4))
RETURN

16 KLIP = 4
AC(1) = ABS(ROOT(2, IM1))
AC(2) = ABS(ROOT(1, IC1))
AC(3) = ABS(ROOT(1, IC1))
RETURN

17 KLIP = 5
AC(1) = ABS(ROOT(1, IM1))
AC(3) = ABS(ROOT(1, IM2))
IF (ABS(ROOT(1, IM2)) .EQ. AC(1)) GO TO 18
AC(2) = ABS(ROOT(1, IM2))
RETURN
18 AC(2) = ABS(ROOT(1, IM3))
19 KLIP = 1776
RETURN

20 FORMAT (10x, I0H, ROOTS )
21 FORMAT (1x, E12.5, 12x, E12.5)
22 FORMAT (10x, 3H(R-14, 3x, 3H11-14, 3x, 3HIC=14/1)
END

SUBROUTINE REDUCTION (EN, V)

DIMENSION FEN(3, 3), VPI(3)
DIMENSION D(3, 3), EN(6, 7), VI(6), DUDU(6), C(3, 1)
DO 1 J = 1, 3
DO 1 J = 1, 3
DEK(1, J) = 0.6
1 CONTINUE
DC 2 I = 1, 3

83
DEN(1,1)=EN(1,3,1)
DEN(1,2)=EN(1,3,3)
DEN(1,3)=EN(1,3,4)
C(1,1)=EN(1,3,7)

2 CONTINUE
WRITE (9,9) (DEN(I,J),J=1,3),I=1,3
CALL LEQ (DEN,C,3,3,3,DEF)
V(1)=C(1,1)
V(3)=C(2,1)
V(4)=C(3,1)
WRITE (9,10) V(1),V(3),V(4)
DO 3 I=1,3
DO 4 J=1,3
FEN(I,J)=0.0
3 CONTINUE
WRITE 14,10 V(1),V(3),V(4)
DO 4 I=1,3
FEN(I,1)=EN(I,2)
FEN(I,2)=EN(I,5)
FEN(I,3)=EN(I,6)
4 CONTINUE
CALL LEQ (FEN,C,3,1,3,DEF)
V(2)=C(1,1)
V(5)=C(2,1)
V(6)=C(3,1)
WRITE (9,11) V(1),V(5),V(6)
DO 5 I=1,6
DOD(I)=EN(I,1)*V(1)+EN(I,2)*V(2)+EN(I,3)*V(3)+EN(I,4)*V(4)+EN(I,5)
5 CONTINUE
WRITE 14,7 (UUUUIII,J=1,6)
RETURN

6 FORMAT (1X,25H FOR THE FEN MATRIX /10X,THC(1,1)=E12.5,THC(1,2)=E12.5,THC(1,3)=E12.5,THC(2,1)=E12.5,THC(2,2)=E12.5,THC(2,3)=E12.5)
7 FORMAT (1X,35H SOLUTION CHECK BY SUBSTITUTION IS /16E12.5)
8 FORMAT (1X,25H FEN(I,J) BY ROWS /S/E15.7)
9 FORMAT (1X,25H FEN(I,J) BY COLUMNS /S/E15.7)
10 FORMAT (1X,5HV(1)=E12.5,2X,5HV(2)=E12.5,2X,5HV(3)=E12.5)
11 FORMAT (1X,5HV(1)=E12.5,2X,5HV(2)=E12.5,2X,5HV(3)=E12.5)
END

SUBROUTINE LEQ (A,B,NEOS,NSOLNS,IA,IB,DEF)
LINEAR EQUATIONS SOLUTIONS FORTRAN II VERSION
SOLVE A SYSTEM OF LINEAR EQUATIONS OF THE FORM AX=B BY A MODIFIED
GAUSS ELIMINATION SCHEME
NEOS = NUMBER OF EQUATIONS AND unknowns
NSOLNS = NUMBER OF VECTOR solutions desired
IA = NUMBER OF ROWS OF A AS DEFINED BY DIMENSION STATEMENT ENTRY
IB = NUMBER OF ROWS OF B AS DEFINED BY DIMENSION STATEMENT ENTRY
ADEF = DETERMINANT OF A, AFTER EXIT FROM LEQ
DIMENSION A(IA,IA), B(1B,IB)
NSIZ=NEQ
NBSIZ=NSOLNS
NORMALIZE EACH ROW BY ITS LARGEST ELEMENT. FORM PARTIAL DETERM
DEG=1.0
DO 6 1=1,NSIZ
BIG=A(I,1)
1 IF (NSIZ-1) LT I LT NSIZ
2 BIG=A(I,J)
3 CONTINUE
BG=1.0/BIG
DO 4 J=1,NSIZ
4 A(I,J)=A(I,J)/BIG
DO 5 J=1,NSIZ
5 B(I,J)=B(I,J)/BIG
6 CONTINUE
START SYSTEM REDUCTION
NUMSYS=NSIZ-1
DO 10 I=1,NUMSYS
SCAN FIRST COLUMN OF CURRENT SYSTEM FOR LARGEST ELEMENT
CALL THE ROW CONTAINING THIS ELEMENT, ROW NBGRW
MN=I+1
BIG=A(I,1)
NBGRW=I
DO 8 J=MV,NSIZ
8 IF (ABS(BIG)-ABS(A(J,I))) GT B
BIG=A(J,I)
NBGRW=J
9 DO 10 J=1,NSIZ
10 DET=DET*BIG
DO 11 J=1,NSIZ
11 A(I,J)=TEMP
DET=DET
SWAP A-MATRIX ROWS
DO 11 J=1,NSIZ
11 A(I,J)=B(J,I)
12 DO 13 I=1,NSIZ
13 ARG=A(I,I)
14 B(I,1)=PMULT*A(I,1)
15 CONTINUE
16 CONTINUE
DO BACK SUBSTITUTION
WITH A-MATRIX COLUMN = NCOLB
17 DO 22 NCOLB+1,NSIZ
18 DO FOR ROW = NROW
19 DO 22 NROW+1,NSIZ
NROW=NSIZ+1
20
TEMP=0.0
NUMBER OF PREVIOUSLY COMPUTED UKNOWNS = NXS
NXS=NS12-NROW
ARE WE DOING THE BOTTOM ROW
IF (NXS) 18,20,18
NO
18 DO 19 K=1,NXS
19 KK=NXS+K-1,NKS
19 WRITE(KK,101)
20 B=K(NROW,NCOLB)-TEMP/A(NROW,NROW)
HAVE WE FINISHED ALL ROWS FOR B-MATRIX COLUMN = NCOLB
21 CONTINUE
22 YES
23 WRITE(14,N(1,1))
24 WRITE(14,N(1,1))
25 WRITE(14,N(1,1))
26 WRITE(14,N(1,1))
27 RETURN
10 FORMAT (5X,2F8.2)

20 FORMAT (15X,2F8.2)
30 FORMAT (15X,2F8.2)
40 FORMAT (15X,2F8.2)
50 FORMAT (15X,2F8.2)
60 FORMAT (15X,2F8.2)
70 FORMAT (15X,2F8.2)
80 FORMAT (15X,2F8.2)
90 FORMAT (15X,2F8.2)
100 FORMAT (15X,2F8.2)
110 FORMAT (15X,2F8.2)
120 FORMAT (15X,2F8.2)
130 FORMAT (15X,2F8.2)
140 FORMAT (15X,2F8.2)
150 FORMAT (15X,2F8.2)
160 FORMAT (15X,2F8.2)
170 FORMAT (15X,2F8.2)
180 FORMAT (15X,2F8.2)
190 FORMAT (15X,2F8.2)
200 FORMAT (15X,2F8.2)
210 FORMAT (15X,2F8.2)
SUBROUTINE KSLT (x)

COMMON DTEMP
COMMON TEMP, EA, NU, NUA, EB, NUB, NUMC, HA, HR, CA, CB, RSLT

10 COMMON CH, CAPC, NUMU, GAMMA, GAMB, K(40), A(55), G(71), K(370), F(70), F(RSLT)

20 COMMON REAL NA, NUB, CA, CB, CA.CB.RSLT

30 COMMON /SHEAR/ QA, QH, QF

REAL NU, NUA, NUB, NUMC, NUMU, R, LOAD1, LOAD2

WNA = WW + RT(1)

DWW + DMW

WUA = WUA + (HA/2, 1) * WA + (HR/2, 1)

DWW + DMU

FNA = CA + (CA * NUA/R) * WNA - AT(5) / (1 - NUA) + RT(21)

FNS = CA + (CA * NUB/R) * WNB - AT(6) / (1 - NUB) + RT(22)

FNA = FNA + FNXA

FNA = FNA - AT(12) / (1 - NUA) + RT(1)

FMR = DMW - RT(13) / (1 - NUB) + RT(19)

FMR = FMR + FNXA

QA + QA - QA + QA

END

SUBROUTINE PINCO (FN, VI)

DIMENSION EN(6, 7), REN(6, 6), C(6, 1), VI(6)

DO 1 I = 1, 6

1 J = I - 1, 6

RE = C(I, J) * VI(I)

C(I, J) = VI(J)

1 CONTINUE

CALL LEQ (REV, C, 6, 1, 6, 6, DET)

DO 2 I = 1, 6

2 VI(I) = C(I, 1)

2 CONTINUE

DO 3 I = 1, 6

3 DUMMY(I) = C(I, 1) + C(I, 2) + C(I, 3) + C(I, 4) + C(I, 5) + C(I, 6) - EN(I, 1)

3 CONTINUE

WRITE (9, 4) (DUMMY(I), I = 1, 6)

RETURN

4 FORMAT 45H SOLUTION CHECK BY SUBSTITUTION 45H

9 FORMAT (14, HM(AR(4)), H12, 3, 12, HM(AR(4)) = 12, 5, 12, HM(AR(4)) = 12, 5, 12)

11, 36HM(AR(4)) = 12, 5, 12, HM(AR(4)) = 12, 5, 12, HM(AR(4)) = 12, 5, 12)

END

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TABLE B.1
AVERAGE MATERIAL PROPERTIES FOR PG AND ATJ GRAPHITE

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<th>PG</th>
<th>ATJ</th>
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<td>$E$</td>
<td>$3.1 \times 10^6 \text{ psi}$</td>
<td>$2.26 \times 10^6 \text{ psi}$</td>
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<tr>
<td>$\nu$</td>
<td>-0.21</td>
<td>+0.30</td>
</tr>
<tr>
<td>$\nu_c$</td>
<td>+0.90</td>
<td>+0.25</td>
</tr>
<tr>
<td>$R$</td>
<td>varies with case</td>
<td>30 in.</td>
</tr>
<tr>
<td>$L$</td>
<td>40 in.</td>
<td>40 in.</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$1.43 \times 10^{-6} \text{ in} / \text{in} - \text{0}^\circ \text{F}$</td>
<td>$4.25 \times 10^{-6} \text{ in} / \text{in} - \text{0}^\circ \text{F}$</td>
</tr>
<tr>
<td>$\alpha_c$</td>
<td>$13.1 \times 10^{-6} \text{ in} / \text{in} - \text{0}^\circ \text{F}$</td>
<td>$4.25 \times 10^{-6} \text{ in} / \text{in} - \text{0}^\circ \text{F}$</td>
</tr>
</tbody>
</table>
Figure B1: Effect of E/G_c on Joint Shear Stress.
Figure 8.2 Effect of $E/G_c$ on Joint Normal Stress

$T_0 = 0.5, N_R = 0.25$
$T_0 = 0.0000 + F$

15$\sigma - 0.01 = f_d$
Figure B.3 Behavior of Joint Shear Stress With Varying Mandrel Thickness
Figure B.5: Variation of Joint Shear Stresses With Increasing PG Deposition Thickness
Figure B.6: Variation of Joint Normal Stresses with Increasing PG Deposition Thickness.

- $h_0 = 0.25$
- $T_0 = -1000\,^\circ F$
- $\frac{E}{G_c} = 20$

Graph shows the variation of normal stresses $N_1$ with $x$ for different values of $h_0$. The stresses are plotted in ksi (thousands of pounds per square inch) against the coordinate $x$ in inches.
APPENDIX C

DEFINITION OF TERMS
\[ Z_1 = (C_a + C_b) \left( \left( \alpha_{a17} + \alpha_{a27} + \pi_1 + \pi_2 \right) - \frac{\hbar}{2} D \left( \alpha_{a47} + \pi_4 \right) \right) \]

\[ + \frac{\hbar}{2} C_b D^2 + \left( \frac{C_a v_a + C_b v_b}{R_a R_b} \right) \left( a_{a37} + a_{a47} + \pi_3 + \pi_4 \right) \]

\[ Z_2 = (C_a + C_b) \left( \alpha_{a67} + \pi_6 - \frac{5}{12} h_b \left( \alpha_{a47} + \pi_4 \right) \right) \]

\[ + \frac{5}{12} h_b C_b \left( a_{a37} + a_{a47} + \pi_3 + \pi_4 \right) \]

\[ Z_3 = (C_a + C_b) \left( \alpha_{a57} + \pi_5 - \frac{5}{12} h_a \left( \alpha_{a47} + \pi_4 \right) \right) \]

\[ + \frac{5}{12} h_a C_b \left( a_{a37} + a_{a47} + \pi_3 + \pi_4 \right) \]

\[ Z_4 = a_{31} Z_1 - \left( a_{11} D^2 + a_{15} \right) Z_2 \]

\[ Z_5 = a_{23} D^2 Z_3 - \left( a_{31} D^2 + a_{32} \right) Z_2 \]

\[ (C.1) \]

\[ a_{11} = D_a \left( C_a + C_b \right) + \frac{C_a C_b h_a h}{4} \]

\[ a_{12} = -\frac{C_a C_b h_a}{2} \left( \frac{v_a}{R_a} - \frac{v_b}{R_b} \right) \]

\[ a_{13} = D_b \left( C_a + C_b \right) + \frac{C_a C_b h_b h}{4} \]
\[ a_{14} = \frac{h_b}{h_a} a_{12} \]

\[ a_{15} = \frac{C_a C_b h}{2} \left( \frac{v_a}{R_a} - \frac{v_b}{R_b} \right) \]

\[ a_{16} = \left( \frac{C_a v_a + C_b v_b}{R_a} \right)^2 - \left( \frac{C_a + C_b}{R_a} \right)^2 \]

\[ a_{21} = D_b (C_a + C_b) + \frac{5}{24} C_a C_b h \]

\[ a_{22} = - (C_a + C_b) \frac{5}{6} G^b h_b \]

\[ a_{23} = \frac{5}{24} C_a C_b h_a h_b \]

\[ a_{24} = \frac{5}{12} C C h \left( \frac{v_a}{R_a} - \frac{v_b}{R_b} \right) + (C_a + C_b) \frac{5}{6} G^b h_b \]

\[ a_{31} = D_a (C_a + C_b) + \frac{5}{24} C_a C_b h_a^2 \]

\[ a_{32} = - (C_a + C_b) \frac{5}{6} G^a h_a \]

\[ a_{33} = a_{23} \]

\[ a_{34} = \frac{5}{12} C_a C_b h_a \left( \frac{v_a}{R_a} - \frac{v_b}{R_b} \right) - \frac{5}{6} G^a h_a (C_a + C_b) \]
\[ b_{11} = a_{13} a_{23} - a_{11} a_{21} \]

\[ b_{12} = a_{14} a_{23} - a_{11} a_{22} - a_{21} a_{12} \]

\[ b_{13} = - a_{12} a_{22} \]

\[ b_{14} = a_{15} a_{23} - a_{11} a_{24} \]

\[ b_{15} = a_{16} a_{23} - a_{12} a_{24} \]

\[ b_{21} = a_{23} a_{23} - a_{31} a_{21} \]

\[ b_{22} = - \left( a_{31} a_{22} + a_{32} a_{21} \right) \]

\[ b_{23} = - a_{32} a_{22} \]

\[ b_{24} = a_{23} a_{34} - a_{31} a_{24} \]

\[ b_{25} = - a_{32} a_{24} \]

\[(c.3)\]
\( g_1 = b_2 b_1 - b_{14} b_{24} \)

\( g_2 = b_2 b_1 + b_{15} b_{14} - b_{21} b_{12} - b_{24} b_{11} - b_{25} b_{10} \)

\( g_3 = b_2 b_1 + b_{15} b_{14} - b_{21} b_{12} - b_{24} b_{11} - b_{25} b_{10} \)

\( g_4 = b_2 b_1 - b_{16} b_{13} \)

\[ L_I(x) = (b_{21} D^4 + b_{22} D^2 + b_{23}^4) Z - (b_{21} D^4 + b_{22} D^2 + b_{23}^4) Z \]

\( L_{II}(x) = Z_4 \)

\( L_{III}(x) = Z_2 \)

\( L_{IV}(x) = 1/(C_a + C_b) (\alpha_{37} + \alpha_{47} + \pi_3 + \pi_4) \)

\( L_{V}(x) = \alpha_{47} + \pi_4 \)

\( L_{VI}(x) = \alpha_{27} + \pi_2 \)

\( k_1 = C_a h_a / (C_a + C_b) \)
\[ k_2 = \frac{C_a h_b}{2}(C_a + C_b) \]

\[ k_3 = \frac{1}{R} \left( C_a \nu a + C_b \nu b \right)/(C_a + C_b) \]

\[ k_4 = C_b \]

\[ k_5 = k_8 = \frac{C_b \nu b}{R_b} \]

\[ k_6 = C_b h_b^2/12 \]

\[ k_7 = h_b/2 \]

\[ k_9 = \frac{C_b}{R_b}^2 \]  \hspace{1cm} (C.6)

\[ m = \frac{1}{3} \left\{ 3 \left( \frac{g_2}{g_1} \right)^3 - \left( \frac{g_2}{g_1} \right) \right\} \]

\[ n = \frac{1}{27} \left\{ 2 \left( \frac{g_2}{g_1} \right)^3 - 9 \left( \frac{g_2}{g_1} \right)^2 \left( \frac{g_3}{g_1} \right) + 27 \left( \frac{g_4}{g_1} \right) \right\} \]  \hspace{1cm} (C.7)

\[ A = \left\{ -\frac{n}{2} + \frac{n^2}{4} + m^2/27 \right\}^{1/3} \]

\[ B = \left\{ -\frac{n}{2} - \frac{n^2}{4} + m^2/27 \right\}^{1/3} \]  \hspace{1cm} (C.8)

\[ \lambda_1 = A + B - 1/3 \left( g_2/g_1 \right) \]

\[ \lambda_2 = +1/2 \left( A + B \right) + 1/3 \left( g_2/g_1 \right) \]  \hspace{1cm} (C.9)
\[ \lambda_3 = \frac{3}{2} (A - B) \]

\[ r = (\lambda_2)^2 + (\lambda_3)^2 \]

\[ \tan \theta_1 = -\frac{\lambda_3}{\lambda_2} \]

\[ \tan \theta_2 = \frac{\lambda_3}{\lambda_2} \]  

\[ c_1 = (\lambda_1)^{1/2} \]  

\[ c_2 = r^{1/4} \cos (\theta_1/2) \]  

\[ c_3 = r^{1/4} \sin (\theta_1/2) \]  

\[ q = \left| \frac{n^2}{4} + \frac{m^3}{27} \right|^{1/2} \]  

\[ c_4 = (-n)^{1/3} \]  

\[ c_5 = -1/2 \left\{ c_4 + 3 (2q)^{1/3} \right\} \]  

\[ c_6 = -1/2 \left\{ c_4 - 3 (2q)^{1/3} \right\} \]
APPENDIX D

Derivation of Integrated Shear Stress Resultant
The required shear stress-strain relation can be developed by weighted integration in order to obtain the factor $5/6$ that is generally accepted for isotropic plates and shells. The method follows that in reference (15). For convenience, the procedure for cylindrical shells is reproduced here.

Expressions for normal stress distributions with $z$ can be obtained by replacing the strains in the stress-strain relations (1) by the approximate forms (8) and neglecting $(z/R)$ in comparison to unity in the same terms. It is to be understood that the following hold for each lamina.

\[
\sigma_x = \frac{E}{1-\nu^2} \left( u_0 + \frac{\nu w}{R} + z \beta \right) - \frac{E a T}{\nu} + \frac{\nu E w}{R(1-\nu^2)} \tag{D.1}
\]

\[
\sigma_0 = \frac{E}{1-\nu^2} \left( \nu u_0 + \frac{w}{R} + \nu z \beta \right) - \frac{E a T}{1-\nu} + \frac{\nu E w}{R(1-\nu^2)} \tag{D.2}
\]

Expressions for the stress distributions in terms of stress resultants and couples can be obtained by using equations (11) in (D.1) and (D.2).

\[
\sigma_x = \frac{N_x}{h} + \frac{N_{Tx}}{(1-\nu)h} - \frac{\nu E}{Rh(1-\nu^2)} \int_{-h/2}^{h/2} \overline{w dz} + \frac{12}{h^3} z \left[ \frac{M_x}{1-\nu} + \frac{M_{Tx}}{R(1-\nu^2)} \right] \int_{-h/2}^{h/2} \overline{w dz}
\]

\[
- \frac{\nu E w}{R(1-\nu^2)} \tag{D.3}
\]
The shear stress is related to the normal stress by the equilibrium equations (2). If the first of these is integrated with respect to \( z \), the following is obtained.

\[
\sigma_{xz} - \tau_{2i} = \frac{1}{R} \int_{-h/2}^{h/2} \frac{3}{2} \left( R_{ij} \right) dz
\]  

(D.5)

Using (D.3) in (D.5)

\[
\sigma_{xz} - \tau_{2i} = -\frac{1}{R} \int_{-h/2}^{h/2} \left\{ \frac{3}{2} \left( R_{ij} \right) + \frac{12}{h^2} z R_{ij} \right\} dz + \frac{1}{R} \int_{-h/2}^{h/2} \omega dz
\]  

(D.6)

where

\[
\omega = \frac{R}{1-\nu} \int_{-h/2}^{h/2} \left\{ \frac{N_{Tx}}{h} + \frac{12z}{h^3} M_{Tx} - E\alpha T - \frac{\nu E}{(1+\nu)} \left[ \frac{h}{2} \int_{-h/2}^{h/2} \omega dz + \frac{12z}{h^3} \int_{-h/2}^{h/2} \omega dz \right] \right\}
\]

Refering to the integrated equilibrium equations (10) and replacing the normal stress resultants and couples in (D.5) by their
equivalents in terms of the shear stress resultant.

\[ \sigma_{xz} - \tau_{2i} = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ -\frac{12z}{h^2} Q + (1 + \frac{6z}{h}) \tau_{1i} - (1 - \frac{6z}{h}) \tau_{2i} \right] \, dz + \frac{1}{R} \int_{-\frac{h}{2}}^{\frac{h}{2}} \omega \, dz \]  \hspace{1cm} (D.8)

Performing the indicated integration

\[ \sigma_{xz} - \tau_{2i} = \frac{3}{2} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] \frac{Q}{h} - \frac{1}{4} \left[ 1 - \frac{4z}{h} - 3\left( \frac{2z}{h} \right)^2 \right] - \frac{1}{4} \left[ 1 + \frac{4z}{h} - 3\left( \frac{2z}{h} \right)^2 \right] \] 

\[ -3\left( \frac{2z}{h} \right)^2 + \frac{1}{R} \int_{-\frac{h}{2}}^{\frac{h}{2}} \omega \, dz \]  \hspace{1cm} (D.9)

From (D.6) and equations (9), it can readily be shown that

\[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \omega \, dz = 0 \]  \hspace{1cm} (D.10)

\[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \omega \, dz = 0 \]

The shear stress distribution (D.9) satisfies shear stress boundary conditions and the definition of the shear stress resultant.

To prove the latter it is necessary to make use of (D.10) and
\[ \int_{-h/2}^{h/2} \int_{-h/2}^{z} f(y) \, dy \, dz = \int_{-h/2}^{h/2} (h/2 - z) f(z) \, dz \]  

(D.11)

McDonough (15) points out that the effect of the last term in (D.9) is to modify the classical quadratic shear stress distribution but not the magnitude of the shear stress resultant. The modification is due to the nonlinearity of the normal stress distribution, primarily its distribution with \( x \).

Proceeding with the weighted integration as in (15), the shear stress-strain relation of the set (1) is multiplied by the weighting function \[ 1 - \left( \frac{2z}{h} \right)^2 \] and then integrated through the thickness of the shell to yield:

\[ \int_{-h/2}^{h/2} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] \sigma_{xz} \, dz = 2G_c \int_{-h/2}^{h/2} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] G_{xz} \, dz \]  

(D.17)

The integral on the left hand side is evaluated using (D.9) with the aid of (D.10) and (D.11). The integral on the right hand side is evaluated by using the shear strain given in equation (8). After integration, rearrangement and simplification, equation (114) results:
\[ Q = \frac{m}{6} + \frac{5}{6} G_c h \left( \beta_i + w_i' \right) + \frac{5}{4} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ 1 - (\frac{2z}{h})^2 \right] G_c \tilde{v}_d z + \frac{5}{4h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{2z}{h} \right)^2 \mu dydz \] (D.13)
Analysis of Transversely Isotropic Laminated Cylinders

Jonas A. Zukas

November 1969

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Report No. 1457

This document has been approved for public release and sale; its distribution is unlimited.

A theory for the analysis of stresses in laminated circular cylindrical shells subjected to arbitrary axisymmetric mechanical and thermal loadings has been developed. This theory, specifically for use with pyrolytic graphite type materials, differs from the classical thin shell theory in that it includes the effects of transverse shear deformation and transverse isotropy, as well as thermal expansion through the shell thickness.

Solutions in several forms are developed for the governing equations. The form taken by the solution function is governed by geometric considerations. A range in which the various solution forms occur was determined numerically.

As a sample problem, the slow cooling of pyrolytic graphite deposited onto a commercial graphite mandrel was considered. Investigation of normal and shear stress behavior at the pyrolytic graphite - mandrel interface showed that these stresses decrease in magnitude with increasing E/G ratio and increasing deposit to mandrel thickness (h/d) ratio. This implies that a thin mandrel and a material weak in shear are desirable to minimize the possibilities of flaking and delamination of the pyrolytic graphite.
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