

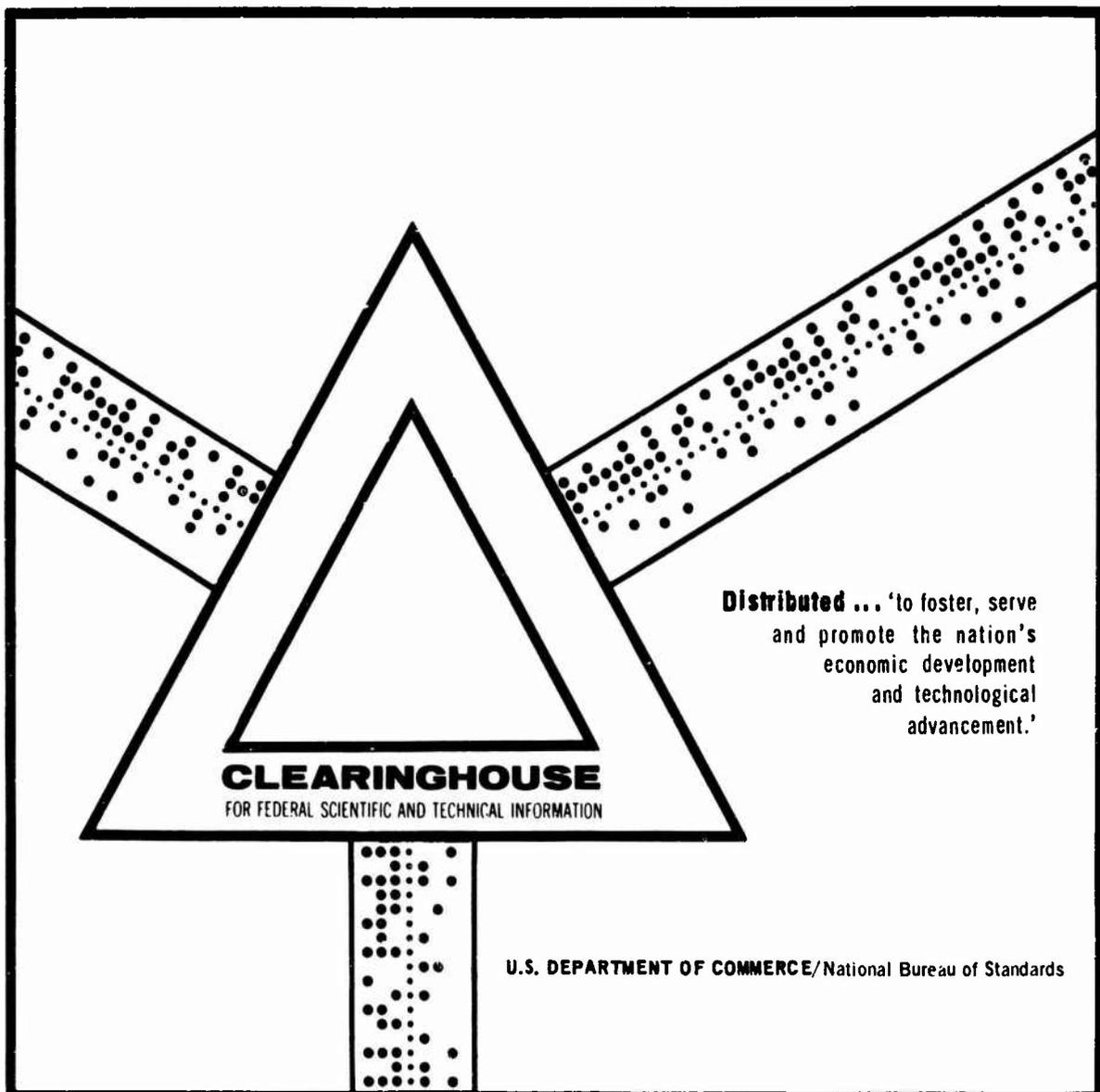
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AN APPROXIMATE METHOD FOR DETERMINING STRESSES
IN AN ELASTIC ANISOTROPIC PLATE NEAR AN OPENING
WHICH IS ALMOST CIRCULAR

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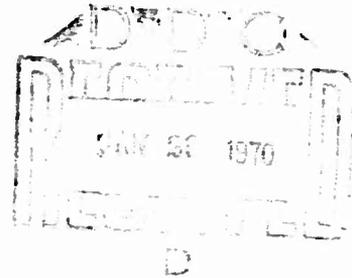
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by

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EDITED TRANSLATION

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AN APPROXIMATE METHOD FOR DETERMINING STRESSES IN AN ELASTIC
ANISOTROPIC PLATE NEAR AN OPENING WHICH IS ALMOST CIRCULAR

S. G. Lekhnitskiy
(Saratov)

The two-dimensional problem of the theory of elasticity for an anisotropic plate with an opening has only been solved for the case of an elliptical or circular opening. All other shapes of openings, including many of practical importance, as yet have not been sufficiently investigated. In this paper an approximate method is suggested for solving the two-dimensional problem for an infinite, anisotropic plate with an almost circular opening. The method is based on the introduction of a small parameter (characterizing the deviation of the opening from circular), the highest powers of which (beginning, for example, with the third or fourth) are discarded during the investigation. The problem is reduced to the well-known one concerning the equilibrium of an anisotropic plate with a circular opening. Chief attention is given to an opening having four axes of symmetry (with the proper parameter selection it can differ only slightly from a square with rounded corners). Approximate solutions for a plate with such an opening are derived for both the general case of loading and for two particular cases when the plate is orthotropic and is deformed by: 1) tensile stresses and 2) bending moments in the middle plane.

1. General equations for the two-dimensional problem of the theory of elasticity of an anisotropic body. In this and the following paragraphs we shall use the common designations for component stresses, projections of displacement, elastic constants, and quantities related to them [1, 2].

We shall recall the basic equations of a two-dimensional problem for an anisotropic body. Let an elastic homogeneous anisotropic body be found in a generalized two-dimensional stressed state or in a state of two-dimensional deformation relative to the plane xy . It is assumed that three-dimensional forces are absent, deformations are small, and the material follows a generalized Hooke law, with which at each point there is a plane of elastic symmetry parallel to the xy plane. Then, as is known, the component stresses which are parallel to this plane are expressed through stress function $F(x, y)$:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 F}{\partial x \partial y} \quad (1.1)$$

The general expression for the stress function has the form

$$F = 2 \operatorname{Re} [F_1(z'_1) + F_2(z'_2)] \quad (1.2)$$

where Re is the designation for the real part of the complex expression; F_1 and F_2 are arbitrary analytic functions of the complex variables; $z'_1 = x + \mu_1 y$ and $z'_2 = x + \mu_2 y$, while μ_1 and μ_2 are complex parameters, i.e., roots of equation

$$\beta_{11} \mu^4 - 2\beta_{16} \mu^3 + (2\beta_{12} + \beta_{66}) \mu^2 - 2\beta_{26} \mu + \beta_{22} = 0. \quad (1.3)$$

In the last equation $\beta_{ij} = a_{ij}$ for a generalized two-dimensional stressed state or $\beta_{ij} = a_{ij} - a_{i3}a_{j3}/a_{33}$ for a two-dimensional deformation ($i, j = 1, 2, 6$; a_{ij} are the elastic constants of the equations expressing the generalized Hooke law¹).

Let us introduce the new parameters:

$$\lambda_1 = \frac{1 + i\mu_1}{1 - i\mu_1}, \quad \lambda_2 = \frac{1 + i\mu_2}{1 - i\mu_2} \quad (i = \sqrt{-1}) \quad (1.4)$$

Here λ_1 and λ_2 are real or complex numbers, in absolute value less than unity (or, at least, equal to 1). Instead of the complex variables z'_1 and z'_2 and functions F_1 and F_2 , it is convenient to introduce

¹See [1], pages 16, 27, 31, 34 (in our book variables z'_1 and z'_2 are designated by z_1 and z_2 , respectively).

$$z_1 = z + \lambda_1 \bar{z}, \quad z_2 = z + \lambda_2 \bar{z}, \quad z = x + iy, \quad \bar{z} = x - iy \quad (1.5)$$

$$\theta_1(z_1) = F_1(z_1'), \quad \theta_2(z_2) = F_2(z_2'). \quad (1.6)$$

Then instead of (1.2) we shall have

$$F = 2\text{Re}[\theta_1(z_1) + \theta_2(z_2)]. \quad (1.7)$$

Designating

$$\Phi_i(z_i) = (1 + \lambda_i) \frac{d\theta_i}{dz_i}, \quad \Phi_i'(z_i) = \frac{d\Phi_i}{dz_i} \quad (i = 1, 2) \quad (1.8)$$

we shall derive the following general expressions:

$$\frac{\partial F}{\partial x} = 2\text{Re}[\Phi_1(z_1) + \Phi_2(z_2)], \quad \frac{\partial F}{\partial y} = 2\text{Re}[\mu_1 \Phi_1(z_1) + \mu_2 \Phi_2(z_2)] \quad (1.9)$$

$$\begin{aligned} \sigma_x &= 2\text{Re}[\mu_1^2(1 + \lambda_1)\Phi_1'(z_1) + \mu_2^2(1 + \lambda_2)\Phi_2'(z_2)] \\ \sigma_y &= 2\text{Re}[(1 + \lambda_1)\Phi_1'(z_1) + (1 + \lambda_2)\Phi_2'(z_2)] \\ \tau_{xy} &= -2\text{Re}[\mu_1(1 + \lambda_1)\Phi_1'(z_1) + \mu_2(1 + \lambda_2)\Phi_2'(z_2)] \end{aligned} \quad (1.10)$$

Knowing how the component stresses are expressed, it is easy to find from the equations of the generalized Hooke law (by integration) the general expressions for the components of the displacement. They have the form:

$$\begin{aligned} u &= 2\text{Re}[p_1 \Phi_1(z_1) + p_2 \Phi_2(z_2)] - \omega y + u_0 \\ v &= 2\text{Re}[q_1 \Phi_1(z_1) + q_2 \Phi_2(z_2)] + \omega x + v_0 \end{aligned} \quad (1.11)$$

Here

$$p_i = \beta_{11} \mu_i^2 + \beta_{12} - \beta_{16} \mu_i, \quad q_i = \beta_{12} \mu_i + \frac{\beta_{22}}{\mu_i} - \beta_{26} \quad (i = 1, 2) \quad (1.12)$$

the integration constants which characterize "rigid displacements" in the plane parallel to the xy plane are designated by ω , u_0 , and v_0 .

In studying the stressed state of a plate with an opening, of greatest interest is the stress σ_0 near the opening on the small areas normal to its contour. It is determined by formula

$$\sigma_{\theta} = \sigma_x \cos^2(n, y) + \sigma_y \cos^2(n, x) - 2\tau_{xy} \cos(n, x) \cos(n, y) \quad (1.13)$$

Expressing the cosines of angles, formed by the normal n to the contour of the opening with the coordinate axes, through derivatives of the coordinates of contour points x and y along its arc s and using formula (1.10), we derive

$$\sigma_{\theta} = 2 \operatorname{Re} \left\{ (1 + \lambda_1) \left(\frac{dy}{ds} - \mu_1 \frac{dx}{ds} \right)^2 \Phi_1'(z_1) + (1 + \lambda_2) \left(\frac{dy}{ds} - \mu_2 \frac{dx}{ds} \right)^2 \Phi_2'(z_2) \right\} \quad (1.14)$$

Let the components of internal forces X_n and Y_n be given on the contour of the region occupied by the body (first basic problem). We shall take counterclockwise stress as positive. Then boundary conditions will have the form:

$$\frac{\partial F}{\partial x} = \int_0^s Y_n ds + c_1, \quad \frac{\partial F}{\partial y} = - \int_0^s X_n ds + c_2 \quad (1.15)$$

Arc of the contour s is calculated from a certain point on the contour, which is taken as the initial point; c_1 and c_2 are integration constants which can be assumed to be arbitrary in the case of a simply connected region. Considering (1.9), we shall write the boundary conditions as:

$$\begin{aligned} 2 \operatorname{Re} [\Phi_1(z_1) + \Phi_2(z_2)] &= \int_0^s Y_n ds + c_1 \\ 2 \operatorname{Re} [\mu_1 \Phi_1(z_1) + \mu_2 \Phi_2(z_2)] &= - \int_0^s X_n ds + c_2 \end{aligned} \quad (1.16)$$

The two-dimensional problem is reduced to a determination of two functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ in the region of body S , which satisfy (at prescribed internal forces) conditions (1.16), do not have singularities within region S , and give single-valued displacements and stresses. In other words, these functions must be determined in regions S_1 and S_2 obtained from S by affine transformation.¹ Naturally, a solution found for a plate can be transferred to the case of two-dimensional deformation.

¹See [1], pages 35-38.

2. An approximate method of solving a two-dimensional problem for an infinite plane with a notch. Let the region of the body be an infinite plane with a notch in the form of a figure which closely resembles a circle with radius a . We shall give the equation for the contour of the notch the following form:

$$\begin{aligned} x &= a \left[\cos \vartheta + \varepsilon \sum_{n=1}^N (\alpha_n \cos n\vartheta + \beta_n \sin n\vartheta) \right] \\ y &= a \left[\sin \vartheta + \varepsilon \sum_{n=1}^N (-\alpha_n \sin n\vartheta + \beta_n \cos n\vartheta) \right] \end{aligned} \quad (2.1)$$

where ε is the small parameter and α_n and β_n are constants; during one complete passage along the contour ϑ changes from 0 to 2π . When $\varepsilon = 0$ we obtain the equation of a circle. Finding a precise solution for an anisotropic plate with an opening bounded by a contour in the form of (2.1) involves considerable difficulties and has not, as yet, been accomplished. However, making use of the fact that ε is small, it is comparatively easy to find an approximate solution, considering this quantity as the small parameter. The "small parameter method" has been used by many authors to construct approximate solutions for various problems of the theory of elasticity (for example, G. Yu. Dzhanelidze, N. V. Zvolinskiy, A. I. Lur'ye, D. Yu. Panov, and P. M. Riz).¹

The function which maps conformally the exterior of a unit circle found in the plane of complex variable $\zeta = \rho e^{i\theta}$ on an infinite region with notch (2.1) has the form:

$$z = \omega(\zeta) = a [\zeta + \varepsilon \varphi(\zeta)] \quad (2.2)$$

Here

$$\varphi(\zeta) = \sum_{n=1}^N (\alpha_n + i\beta_n) \zeta^{-n} \quad (2.3)$$

¹A brief survey of works of this nature performed before 1948 and other related literature can be found in reference [5] (pages 182-190).

Branch points are determined from equation

$$1 + \varepsilon \varphi'(\zeta) = 0; \quad (2.4)$$

they must all be within the unit circle or inside the notch on plane z ; otherwise the mapping will not be one-to-one. This imposes certain conditions (bounds) on the quantities of coefficients α_n , β_n , and parameter ε ; we consider them fulfilled.

Passing to plane ζ , we replace z and \bar{z} in the arguments of functions ϕ_1 and ϕ_2 by ω and $\bar{\omega}$. It is easy to see by simple checking that function

$$\Phi_1(z_1) = \Phi_1[\omega(\zeta) + \lambda_1 \bar{\omega}(\bar{\zeta})] \quad (2.5)$$

satisfies equation

$$\lambda_1 \bar{\omega}'(\bar{\zeta}) \frac{\partial \Phi_1}{\partial \bar{\zeta}} - \omega'(\zeta) \frac{\partial \Phi_1}{\partial \zeta} = 0 \quad (2.6)$$

(in which ζ and $\bar{\zeta}$ are considered independent variables), or, in greater detail, equation

$$\lambda_1 [1 + \varepsilon \bar{\varphi}'(\bar{\zeta})] \frac{\partial \Phi_1}{\partial \bar{\zeta}} - [1 + \varepsilon \varphi'(\zeta)] \frac{\partial \Phi_1}{\partial \zeta} = 0 \quad (2.7)$$

If we take ρ and $\sigma = e^{\theta i}$ for the independent variables, equation (2.6)-(2.7) assumes the form

$$[\omega'(\zeta) - \frac{\lambda_1}{\sigma^2} \bar{\omega}'(\bar{\zeta})] \frac{\partial \Phi_1}{\partial \rho} - [\varepsilon \omega'(\zeta) + \frac{\lambda_1}{\sigma} \bar{\omega}'(\bar{\zeta})] \frac{1}{\rho} \frac{\partial \Phi_1}{\partial \sigma} = 0 \quad (2.8)$$

We shall seek an expression for ϕ_1 in the form of a power series of ε :

$$\Phi_1 = \Phi_{10} + \varepsilon \Phi_{11} + \varepsilon^2 \Phi_{12} + \dots \quad (2.9)$$

where ϕ_{1k} does not depend on ε . Substituting (2.9) into equation (2.7) and equating to zero terms which do not depend on ε and coefficients at various powers of ε , we derive an infinite system of recurrent equations:

$$\begin{aligned} \lambda_1 \frac{\partial \Phi_{10}}{\partial \zeta} - \frac{\partial \Phi_{10}}{\partial \bar{\zeta}} &= 0 \\ \lambda_1 \frac{\partial \Phi_{1k}}{\partial \zeta} - \frac{\partial \Phi_{1k}}{\partial \bar{\zeta}} + \lambda_1 \bar{\varphi}(\bar{\zeta}) \frac{\partial \Phi_{1, k-1}}{\partial \zeta} - \varphi(\zeta) \frac{\partial \Phi_{1, k-1}}{\partial \bar{\zeta}} &= 0 \quad (k=1,2,3,\dots) \end{aligned} \quad (2.10)$$

Integrating successively these equations in partial derivatives, beginning with the first, we obtain:

$$\begin{aligned} \Phi_{10} &= f_{10}(\zeta + \lambda_1 \bar{\zeta}) \\ \Phi_{11} &= f_{11}(\zeta + \lambda_1 \bar{\zeta}) + [\varphi(\zeta) + \lambda_1 \bar{\varphi}(\bar{\zeta})] f_{10}'(\zeta + \lambda_1 \bar{\zeta}) \\ \Phi_{12} &= f_{12}(\zeta + \lambda_1 \bar{\zeta}) + [\varphi(\zeta) + \lambda_1 \bar{\varphi}(\bar{\zeta})] f_{11}'(\zeta + \lambda_1 \bar{\zeta}) + \\ &\quad + \frac{1}{2!} [\varphi(\zeta) + \lambda_1 \bar{\varphi}(\bar{\zeta})]^2 f_{10}''(\zeta + \lambda_1 \bar{\zeta}) \end{aligned} \quad (2.11)$$

etc., where the quantities f_{1k} are arbitrary analytic functions of the argument $\zeta + \lambda_1 \bar{\zeta}$, and the derivatives of these functions throughout the argument are designated by primes. Consequently:

$$\begin{aligned} \Phi_1 &= f_{10} + \varepsilon [f_{11} + (\varphi + \lambda_1 \bar{\varphi}) f_{10}'] + \varepsilon^2 [f_{12} + (\varphi + \lambda_1 \bar{\varphi}) f_{11}' + \\ &\quad + \frac{1}{2!} (\varphi + \lambda_1 \bar{\varphi})^2 f_{10}''] + \dots + \varepsilon^k [f_{1k} + (\varphi + \lambda_1 \bar{\varphi}) f_{1, k-1}' + \\ &\quad + \frac{1}{2!} (\varphi + \lambda_1 \bar{\varphi})^2 f_{1, k-2}'' + \dots + \frac{1}{k!} (\varphi + \lambda_1 \bar{\varphi})^k f_{10}^{(k)}] + \dots \end{aligned} \quad (2.12)$$

Here, for the sake of brevity, we omit argument $\zeta + \lambda_1 \bar{\zeta}$ of functions $f_{1i}^{(n)}$ and arguments ζ and $\bar{\zeta}$ of functions φ and $\bar{\varphi}$.

A fully analogous expression is also obtained for Φ_2 , replacing the first subscripts 1 in (2.12) with 2 and λ_1 with λ_2 . We shall note that expression (2.12) is an expansion in power series of a function depending on parameter ε :

$$\Phi_1(z_1) = f_1\left(\frac{z_1}{a}; \varepsilon\right) = f_1\left\{\zeta + \lambda_1 \bar{\zeta} + \varepsilon [\varphi(\zeta) + \lambda_1 \bar{\varphi}(\bar{\zeta})]; \varepsilon\right\} \quad (2.13)$$

When argument ζ passes along the contour of the unit circle, argument $\zeta_1 = \zeta + \lambda_1 \bar{\zeta}$ passes along the contour of a "unit ellipse" obtained from the circle by affine transformation, while argument $\zeta_2 = \zeta + \lambda_2 \bar{\zeta}$ passes along the contour of another "unit ellipse" which corresponds to parameter λ_2 . Consequently, the regions of variation in functions $f_{1k}(\zeta_1)$ and $f_{2k}(\zeta_2)$ are infinite planes with

notches in the form of "unit ellipses" and the problem is reduced to determining these functions based on the prescribed boundary conditions on the contours of the notches, in other words, to a well-known problem.

Let us examine an infinite anisotropic plate with an opening whose contour is determined by equation (2.1). We shall assume that internal forces X_n and Y_n are distributed along the edge of the opening, their principal vector being zero.¹ Obviously X_n and Y_n will be periodic functions of θ . Assuming that they can be expanded in a Fourier series, we, under the limitation taken, derive the integrals which go into the boundary conditions (1.16) in the form of Fourier series. Let us assume that the forces also depend on parameter ϵ and can be expanded into an ϵ power series. Integrating them along an arc of the notch's contour, we derive:

$$\begin{aligned} \int_0^s Y_n ds + c_1 &= \sum_{k=0}^{\infty} \epsilon^k \left[\bar{\alpha}_{k0} + \sum_{m=1}^{\infty} (\alpha_{km} \sigma^m + \bar{\alpha}_{km} \sigma^{-m}) \right] \\ - \int_0^s X_n ds + c_2 &= \sum_{k=0}^{\infty} \epsilon^k \left[\bar{\beta}_{k0} + \sum_{m=1}^{\infty} (\beta_{km} \sigma^m + \bar{\beta}_{km} \sigma^{-m}) \right] \end{aligned} \quad (2.14)$$

Here α_{km} and β_{km} are known coefficients; $\bar{\alpha}_{km}$ and $\bar{\beta}_{km}$ are conjugate quantities; $\bar{\alpha}_{k0}$ and $\bar{\beta}_{k0}$ are constants which can be considered arbitrary.

In this case, functions ϕ_1 and ϕ_2 will be holomorphic and unique² in their regions S_1 and S_2 . Substituting their boundary values into (1.16) and equating coefficients at identical powers of ϵ in the left and right sides, we derive an infinite series of pairs of conditions, each of which corresponds to a certain power ϵ . We shall write them in abbreviated form, omitting the boundary values of the arguments

$$\zeta = \sigma, \quad \bar{\zeta} = \frac{1}{\sigma}, \quad \zeta_1 = \sigma + \frac{\lambda_1}{\sigma}, \quad \zeta_2 = \sigma + \frac{\lambda_2}{\sigma}; \quad (2.15)$$

¹This limitation is not essential, but we assume it in order not to write out elementary, but very cumbersome expressions.

²See [1], page 37.

we have

$$\begin{aligned}
 f_{10} + f_{20} + \bar{f}_{10} + \bar{f}_{20} &= \bar{a}_{00} + \sum_{m=1}^{\infty} (\bar{a}_{0m} \sigma^m + \bar{a}_{0m} \sigma^{-m}) \\
 \mu_1 f_{10} + \mu_2 f_{20} + \bar{\mu}_1 \bar{f}_{10} + \bar{\mu}_2 \bar{f}_{20} &= \bar{\beta}_{00} + \sum_{m=1}^{\infty} (\bar{\beta}_{0m} \sigma^m + \bar{\beta}_{0m} \sigma^{-m}) \\
 f_{1k} + f_{2k} + (\varphi + \lambda_1 \bar{\varphi}) f'_{1,k-1} + (\varphi + \lambda_2 \bar{\varphi}) f'_{2,k-1} + \dots + \frac{1}{k!} (\varphi + \lambda_1 \bar{\varphi})^k f_{10}^{(k)} + \\
 &+ \frac{1}{k!} (\varphi + \lambda_2 \bar{\varphi})^k f_{20}^{(k)} + \text{conj. quantities} = \bar{a}_{20} + \sum_{m=1}^{\infty} (\bar{a}_{km} \sigma^m + \bar{a}_{km} \sigma^{-m}) \\
 \mu_1 f_{1k} + \mu_2 f_{2k} + \bar{\mu}_1 (\varphi + \lambda_1 \bar{\varphi}) f'_{1,k-1} + \bar{\mu}_2 (\varphi + \lambda_2 \bar{\varphi}) f'_{2,k-1} + \dots \\
 &\dots + \frac{\mu_1}{k!} (\varphi + \lambda_1 \bar{\varphi})^k f_{10}^{(k)} + \frac{\mu_2}{k!} (\varphi + \lambda_2 \bar{\varphi})^k f_{20}^{(k)} + \text{conj. quantities,} \\
 &= \bar{\beta}_{k0} + \sum_{m=1}^{\infty} (\bar{\beta}_{km} \sigma^m + \bar{\beta}_{km} \sigma^{-m}) \quad (k = 1, 2, 3, \dots)
 \end{aligned}$$

This problem was reduced to the problem concerning the elastic equilibrium of an anisotropic plate with a circular opening. Its solution is known. Functions $f_{1k}(\zeta_1)$ and $f_{2k}(\zeta_2)$, which correspond to the case when forces are applied to the edge of the opening and their principal vector is equal to zero, have the form:¹

$$\begin{aligned}
 f_{1k} &= A_{k0} + \sum_{m=1}^{\infty} A_{km} \left(\frac{2}{\zeta_1 + \sqrt{\zeta_1^2 - 4\lambda_1}} \right)^m \\
 f_{2k} &= B_{k0} + \sum_{m=1}^{\infty} B_{km} \left(\frac{2}{\zeta_2 + \sqrt{\zeta_2^2 - 4\lambda_2}} \right)^m
 \end{aligned} \tag{2.16}$$

On the contour of the opening

$$f_{1k} = A_{k0} + \sum_{m=1}^{\infty} A_{km} \sigma^{-m}, \quad f_{2k} = B_{k0} + \sum_{m=1}^{\infty} B_{km} \sigma^{-m}. \tag{2.17}$$

Substituting values (2.17) into the first pair of boundary conditions ($k = 0$), we find coefficients A_{0m} and B_{0m} easily (A_{00} , B_{00} remain arbitrary and do not influence the distribution of stresses).

¹See our work [1], page 90 (the designations which we use here differ somewhat from the designations in the referenced work). If the principal vector of forces X_n and Y_n does not equal zero, expressions (2.14) will contain components in the form of $A \ln \sigma$; according to this, logarithmic terms with undetermined coefficients must be added to functions (2.16).

Derivatives from the found functions f_{10} and f_{20} multiplied, respectively, by $\phi + \lambda_1 \bar{\phi}$ and $\phi + \lambda_2 \bar{\phi}$ enter into the second pair of boundary conditions ($k = 1$). These derivatives will contain both negative and positive powers of σ . Substituting boundary conditions f_{11} and f_{21} into conditions corresponding to $k = 1$, we find A_{1m} and B_{1m} by a comparison of coefficients in the left and right sides; they will be expressed through $\bar{\alpha}_{1m}$, $\bar{\beta}_{1m}$ and through A_{0m} , B_{0m} found earlier. Functions f_{10} , f_{20} , f_{11} , and f_{21} determine the solution to the problem in the first approximation. Desiring to obtain a second approximation, we keep two powers of ϵ in the expressions for ϕ_1 and ϕ_2 . Functions f_{12} and f_{22} are determined from the third pair of boundary conditions corresponding to $k = 2$; their coefficients are expressed through A_{0m} , B_{0m} , A_{1m} , and B_{1m} found earlier and through $\bar{\alpha}_{2m}$ and $\bar{\beta}_{2m}$.

Proceeding this way, we can construct (at least formally) any approximation. Let us mention here that we are not studying the question of the convergence of a process of successive approximations, but are limiting ourselves to particular cases by the second and third approximations. It also remains unclear what the highest value of ϵ is which can still be considered small in each particular case of opening. A comparison of the numerical results found in the second and third approximations furnishes a basis for concluding that these approximations are sufficiently accurate for practical use even when the parameter ϵ is not very small in comparison with unity.

In a perfectly analogous manner we can construct approximate solutions for an infinite plate with an opening closely resembling elliptical, the equation of whose contour has the form:

$$\begin{aligned} x &= a \left[\cos \vartheta + \epsilon \sum_{n=1}^N (\alpha_n \cos n\vartheta + \beta_n \sin n\vartheta) \right] \\ y &= a \left[c \sin \vartheta + \epsilon \sum_{n=1}^N (-\alpha_n \sin n\vartheta + \beta_n \cos n\vartheta) \right] \end{aligned} \quad (2.18)$$

Here a and b are semi-axes of an ellipse, $c = b/a$. Function ϕ_1 , represented in the form of series (2.9), will have the form (2.12) only when the following expression is the argument of function f_{1k} :

$$\zeta_1 = \frac{1+c}{2}\zeta + \frac{1-c}{2}\frac{1}{\zeta} + \lambda_1 \left(\frac{1+c}{2}\bar{\zeta} + \frac{1-c}{2}\frac{1}{\bar{\zeta}} \right) \quad (2.19)$$

The problem is reduced to a determination of the pressure distribution in an anisotropic plate with an elliptical opening (with semi-axes 1 and c), the solution of which is known.

3. Coefficients of boundary values for functions ϕ_1 and ϕ_2 for a plate weakened by an opening with four axes of symmetry. Let us examine an infinite anisotropic plate with an opening whose contour is determined by equation

$$\begin{aligned} x &= a(\cos \vartheta + \varepsilon \cos 3\vartheta) \\ y &= a(\sin \vartheta - \varepsilon \sin 3\vartheta) \end{aligned} \quad (3.1)$$

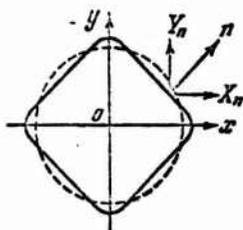


Fig. 1.

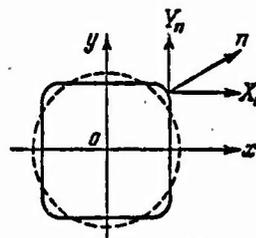


Fig. 2.

Here ε is the small parameter (in any case $|\varepsilon| < \frac{1}{3}$).

The opening is a figure with four axes of symmetry; when ε is positive it is located, with respect to the coordinate system, as in Fig. 1, and when negative as in Fig. 2. When $|\varepsilon| = \frac{1}{9}$ or $|\varepsilon| = \frac{1}{6}$ this figure will have the shape of a square with rounded corners and slightly curved sides.¹ In this case

¹Solutions to certain problems concerning the elastic equilibrium of an isotropic plate with an opening bounded by a contour such as (3.1) can be found in the work of M. I. Nayman [3] and in the book of G. N. Savin [4].

$$\varphi(\zeta) = \frac{1}{\zeta^2}, \quad \omega(\zeta) = a\left(\zeta + \frac{c}{\zeta}\right) \quad (2.2)$$

Owing to the simplicity of function ϕ , it is easy to set up here the structure of boundary conditions of products

$$\frac{1}{n!} (\varphi + \lambda_1 \bar{\varphi})^n f_{1k}^{(n)} \quad (3.3)$$

in the left sides of conditions (2.15) for any n and k and determine all coefficients at various powers of σ depending on coefficients A_{km} of the same function f_{1k} [see (2.16) and (2.17)]. Formulas for coefficients of expression (3.3) are necessary when we determine the first, second, and third approximations (and also higher approximations which we are not considering in this work).

The boundary value of the derivative of the n -th order of function f_{1k} is derived on the basis of the boundary value of $n - 1$ - its first derivative according to formula

$$f_{1k}^{(n)} = \frac{df_{1k}^{(n-1)}}{d\sigma} : \frac{d}{d\sigma} \left(\sigma + \frac{\lambda_1}{\sigma} \right) = \frac{df_{1k}^{(n-1)}}{d\sigma} \frac{1}{1 - \lambda_1/\sigma^2} \quad (3.4)$$

or

$$f_{1k}^{(n)} = \frac{df_{1k}^{(n-1)}}{d\sigma} \left(1 + \frac{\lambda_1}{\sigma^2} + \frac{\lambda_1^2}{\sigma^4} + \dots \right) \quad (3.5)$$

Assuming $n = 1$ and using the first expression (2.17), we find f_{1k}^1 , and then based on the first derivative we find the second; based on the second, the third; etc. Multiplying the derivatives by the corresponding powers $\varphi + \lambda_1 \bar{\varphi}$, we arrive at the conclusion that the product (3.3) has the following structure

$$\frac{1}{n!} (\varphi + \lambda_1 \bar{\varphi})^n f_{1k}^{(n)} = \frac{1}{n!} \left(\frac{1}{\sigma^2} + \lambda_1 \sigma^2 \right)^n f_{1k}^{(n)} = \sum_{m=0}^{2n-1} A_{k,-m}^n \sigma^m + \sum_{m=1}^{\infty} A_{km}^n \sigma^{-m} \quad (3.6)$$

Formulas for coefficients A_{km}^n and $A_{k,-m}^n$ will be more complex the larger m is and the superscript n which indicates the order of the derivative. If we make the stipulation, then they can all be written in the form of one

$$A_{km}^n = \sum_{i=1} A_{k, m+2n+2-2i} g_{mi}^n(\lambda_1) \quad (3.7)$$

Here under the summation sign A indicates coefficients of function f_{lk} , while g_{mi}^n is an integral polynomial to the power $i + n - 1$ relative to λ_1 , having the form¹

$$\begin{aligned} g_{mi}^n(\lambda_1) = & \frac{(-1)^n}{(n-1)!n!} (m+2n+2-2i)\lambda_1^{i-3n-1} \times \\ & \times \{(m+2n+2-i)(m+2n+3-i)\dots(m+3n-i)i(i+1)\dots \\ & \dots(i+n-2)\lambda_1^{4n} + n(m+2n-1-i)(m+2n-i)\dots \\ & \dots(m+3n-3-i)(i-3)(i-2)\dots(i+n-5)\lambda_1^{4n-4} + \\ & + \binom{n}{2}(m+2n-4-i)(m+2n-3-i)\dots(m+3n-6-i)(i-6)(i-5)\dots \\ & \dots(i+n-8)\lambda_1^{4n-8} + \dots + \binom{n}{i}(m-n+8-i)(m-n+9-i)\dots \\ & \dots(m+6-i)(i-3n+6)(i-3n+7)\dots(i-2n+4)\lambda_1^8 + \\ & + n(m-n+5-i)(m-n+6-i)\dots(m+3-i)(i-3n+3) \times \\ & \times (i-3n+4)\dots(i-2n+1)\lambda_1^4 + (m-n+2-i)(m-n+3-i) \dots \\ & \dots(m-i)(i-3n)(i-3n+1)\dots(i-2n-2) \end{aligned} \quad (3.8)$$

In order to determine A_{km}^n with these formulas, at the given values for n , k , and m , after substituting these values into (3.7) and (3.8) we must discard all A with the second subscript negative and all terms with negative powers of λ_1 . The formulas are also valid for determining coefficients at positive powers of σ (m must be replaced in them by the quantity $-m$).

Specifically, for coefficients which correspond to derivatives of the first, second, and third orders ($n = 1, 2, 3$) we derive from (3.7)-(3.8):

$$A_{km}^1 = - \sum_{i=1} A_{k, m+4-2i} (m+4-2i)\lambda_1^{i-4}(\lambda_1^4 + 1) \quad (3.9)$$

¹ $\binom{n}{2}$ and others are binomial coefficients.

$$A_{km}^3 = \frac{1}{2} \sum_{i=1} A_{k, m+6-2i} (m+6-2i) \lambda_1^{i-7} \times \\ \times [(m+6-i) i \lambda_1^6 + 2(m+3-i)(i-3) \lambda_1^4 + (m-i)(i-6)] \quad (3.10)$$

$$A_{km}^3 = -\frac{1}{12} \sum_{i=1} A_{k, m+8-2i} (m+8-2i) \lambda_1^{i-10} \times \\ \times [(m+8-i)(m+9-i) i (i+1) \lambda_1^{12} + 3(m+5-i)(m+6-i)(i-3)(i-2) \lambda_1^8 + \\ + 3(m+2-i)(m+3-i)(i-6)(i-5) \lambda_1^4 + (m-1-i)(m-i)(i-9)(i-8)] \quad (3.11)$$

Coefficients B_{km}^n are found by the same formulas (3.7)-(3.11) in which we must substitute B instead of A and λ_2 instead of λ_1 .

Formulas similar to these are also easy to derive for an opening of another shape in which the contour is given by equation

$$x = a(\cos \vartheta + \varepsilon \cos 2\vartheta), \quad y = a(\sin \vartheta - \varepsilon \sin 2\vartheta) \quad (3.12)$$

With the proper selection of ε the opening will differ only slightly from an equilateral triangle with rounded corners, but we shall not consider this case here.¹

4. Approximate solution for an anisotropic plate weakened by an opening with four axes of symmetry, with an arbitrary distribution of forces along the contour. Let us find an approximate solution to the problem concerning the elastic equilibrium of an anisotropic plate with the opening whose contour is given by equation (3.1).

Let internal forces X_n and Y_n be distributed arbitrarily along the edge of the opening, but their principal vector be equal to zero.

From the beginning let us limit ourselves to the second approximation, discarding the highest powers of ε in expression (2.9) beginning with the third. Then

¹Solutions to problems concerning stress distribution in an isotropic plate with such an opening can be found in references [3, 4].

$$\Phi_1 = \Phi_{10} + \epsilon \Phi_{11} + \epsilon^2 \Phi_{12}, \quad \Phi_2 = \Phi_{20} + \epsilon \Phi_{21} + \epsilon^2 \Phi_{22} \quad (4.1)$$

Boundary conditions for functions independent of ϵ are written in the following manner [see (1.16) and (2.14)]:

$$\begin{aligned} \Phi_{1k} + \Phi_{2k} + \bar{\Phi}_{1k} + \bar{\Phi}_{2k} &= \bar{a}_{k0} + \sum_{m=1}^{\infty} (\bar{a}_{km} \sigma^m + \bar{a}_{km} \sigma^{-m}) \\ \mu_1 \Phi_{1k} + \mu_2 \Phi_{2k} + \mu_1 \bar{\Phi}_{1k} + \mu_2 \bar{\Phi}_{2k} &= \bar{\beta}_{k0} + \sum_{m=1}^{\infty} (\bar{\beta}_{km} \sigma^m + \bar{\beta}_{km} \sigma^{-m}) \end{aligned} \quad (4.2)$$

($k = 0, 1, 2$)

Using the formulas from the preceding paragraph, we conclude that functions Φ_{1k} and Φ_{2k} will have the following form on the contour of the opening:

$$\Phi_{10} = A_{00} + \sum_{m=1}^{\infty} A_{0m} \sigma^{-m}, \quad \Phi_{20} = B_{00} + \sum_{m=1}^{\infty} B_{0m} \sigma^{-m} \quad (4.3)$$

$$\begin{aligned} \Phi_{11} &= -A_{01} \lambda_1 \sigma - 2A_{02} \lambda_1 + A_{10} + \sum_{m=1}^{\infty} (A_{1m} + A_{0m}^1) \sigma^{-m} \\ \Phi_{21} &= -B_{01} \lambda_2 \sigma - 2B_{02} \lambda_2 + B_{10} + \sum_{m=1}^{\infty} (B_{1m} + B_{0m}^1) \sigma^{-m} \end{aligned} \quad (4.4)$$

$$\begin{aligned} \Phi_{12} &= A_{01} \lambda_1^2 \sigma^3 + 3A_{02} \lambda_1^2 \sigma^2 + (3A_{01} \lambda_1^3 + 6A_{03} \lambda_1^2 - A_{11} \lambda_1) \sigma + \\ &\quad + 8A_{02} \lambda_1^3 + 10A_{04} \lambda_1^2 - 2A_{12} \lambda_1 + A_{20} + \sum_{m=1}^{\infty} (A_{2m} + A_{1m}^1 + A_{0m}^2) \sigma^{-m} \\ \Phi_{22} &= B_{01} \lambda_2^2 \sigma^3 + 3B_{02} \lambda_2^2 \sigma^2 + (3B_{01} \lambda_2^3 + 6B_{03} \lambda_2^2 - B_{11} \lambda_2) \sigma + \\ &\quad + 8B_{02} \lambda_2^3 + 10B_{04} \lambda_2^2 - 2B_{12} \lambda_2 + B_{20} + \sum_{m=1}^{\infty} (B_{2m} + B_{1m}^1 + B_{0m}^2) \sigma^{-m} \end{aligned} \quad (4.5)$$

Satisfying conditions (4.2) corresponding to $k = 0$, we derive equations:

$$A_{0m} + B_{0m} = \bar{a}_{0m}, \quad \mu_1 A_{0m} + \mu_2 B_{0m} = \bar{\beta}_{0m} \quad (4.6)$$

(and an analogous system for conjugates \bar{A}_{0m} and \bar{B}_{0m}). Hence

$$A_{0m} = \frac{\bar{\beta}_{0m} - \mu_2 \bar{a}_{0m}}{\mu_1 - \mu_2}, \quad B_{0m} = -\frac{\bar{\beta}_{0m} - \mu_1 \bar{a}_{0m}}{\mu_1 - \mu_2} \quad (4.7)$$

Substituting boundary values ϕ_{11} and ϕ_{12} and conjugate functions into conditions (4.2) for $k = 1$ and comparing the coefficients at identical powers of σ in the left and right sides, we derive the following system of equations:

$$\begin{aligned} (A_{11} + A_{01}^1) + (B_{11} + B_{01}^1) &= \bar{\alpha}_{11} + \bar{A}_{01}\bar{\lambda}_1 + \bar{B}_{01}\bar{\lambda}_2 \\ \mu_1(A_{11} + A_{01}^1) + \mu_2(B_{11} + B_{01}^1) &= \bar{\beta}_{11} + \bar{A}_{01}\bar{\mu}_1\bar{\lambda}_1 + \bar{B}_{01}\bar{\mu}_2\bar{\lambda}_2 \end{aligned} \quad (4.8)$$

$$\begin{aligned} (A_{1m} + A_{0m}^1) + (B_{1m} + B_{0m}^1) &= \bar{\alpha}_{1m} \\ \mu_1(A_{1m} + A_{0m}^1) + \mu_2(B_{1m} + B_{0m}^1) &= \bar{\beta}_{1m} \end{aligned} \quad (m = 2, 3, 4, \dots) \quad (4.9)$$

In solving them we find:

$$\begin{aligned} A_{11} + A_{01}^1 &= \frac{\bar{\beta}_{11} - \mu_2\bar{\alpha}_{11}}{\mu_1 - \mu_2} + \frac{\bar{A}_{01}\bar{\lambda}_1(\bar{\mu}_1 - \mu_2) + \bar{B}_{01}\bar{\lambda}_2(\bar{\mu}_2 - \mu_2)}{\mu_1 - \mu_2} \\ B_{11} + B_{01}^1 &= -\frac{\bar{\beta}_{11} - \mu_1\bar{\alpha}_{11}}{\mu_1 - \mu_2} + \frac{\bar{A}_{01}\bar{\lambda}_1(\mu_1 - \bar{\mu}_1) + \bar{B}_{01}\bar{\lambda}_2(\mu_2 - \bar{\mu}_1)}{\mu_1 - \mu_2} \end{aligned} \quad (4.10)$$

$$\begin{aligned} A_{1m} + A_{0m}^1 &= \frac{\bar{\beta}_{1m} - \mu_2\bar{\alpha}_{1m}}{\mu_1 - \mu_2} \\ B_{1m} + B_{0m}^1 &= -\frac{\bar{\beta}_{1m} - \mu_1\bar{\alpha}_{1m}}{\mu_1 - \mu_2} \end{aligned} \quad (m = 2, 3, 4, \dots) \quad (4.11)$$

Coefficients A_{1m} and B_{1m} together with A_{0m} and B_{0m} determine the solution to the problem in the first approximation.

Substituting the boundary values of ϕ_{12} and ϕ_{22} into conditions (4.2), corresponding to $k = 2$, and comparing the coefficients at identical powers of σ , we derive systems of equations somewhat more complex than the preceding ones. Without writing them out, we shall introduce the final results:

$$\begin{aligned} A_{21} + A_{11}^1 + A_{01}^2 &= \frac{\bar{\beta}_{21} - \mu_2\bar{\alpha}_{21}}{\mu_1 - \mu_2} + \frac{\bar{A}_{11}\bar{\lambda}_1(\bar{\mu}_1 - \mu_2) + \bar{B}_{11}\bar{\lambda}_2(\bar{\mu}_2 - \mu_2)}{\mu_1 - \mu_2} \\ &- 3 \frac{\bar{A}_{01}\bar{\lambda}_1^2(\bar{\mu}_1 - \mu_2) + \bar{B}_{01}\bar{\lambda}_2^2(\bar{\mu}_2 - \mu_2)}{\mu_1 - \mu_2} - 6 \frac{\bar{A}_{01}\bar{\lambda}_1^2(\bar{\mu}_1 - \mu_2) + \bar{B}_{01}\bar{\lambda}_2^2(\bar{\mu}_2 - \mu_2)}{\mu_1 - \mu_2} \end{aligned} \quad (4.12)$$

$$A_{22} + A_{12}^1 + A_{02}^2 = \frac{\bar{\beta}_{22} - \mu_2\bar{\alpha}_{22}}{\mu_1 - \mu_2} - 3 \frac{\bar{A}_{02}\bar{\lambda}_1^2(\bar{\mu}_1 - \mu_2) + \bar{B}_{02}\bar{\lambda}_2^2(\bar{\mu}_2 - \mu_2)}{\mu_1 - \mu_2} \quad (4.13)$$

$$A_{23} + A_{13}^1 + A_{03}^2 = \frac{\bar{\beta}_{23} - \mu_2 \bar{\alpha}_{23}}{\mu_1 - \mu_2} - \frac{\bar{A}_{01} \bar{\lambda}_1^2 (\bar{\mu}_1 - \mu_2) + \bar{B}_{01} \bar{\lambda}_2^2 (\bar{\mu}_2 - \mu_1)}{\mu_1 - \mu_2} \quad (4.14)$$

$$A_{2m} + A_{1m}^1 + A_{0m}^2 = \frac{\bar{\beta}_{2m} - \mu_2 \bar{\alpha}_{2m}}{\mu_1 - \mu_2} \quad (m = 4, 5, 6, \dots) \quad (4.15)$$

In accordance with the derived formulas, the boundary values of functions ϕ_{10} , ϕ_{11} , and ϕ_{12} are expressed through σ in the following manner:

$$\Phi_{10} = \sum_{m=1}^{\infty} \frac{\bar{\beta}_{0m} - \mu_2 \bar{\alpha}_{0m}}{\mu_1 - \mu_2} \sigma^{-m} + C_{10} \quad (4.16)$$

$$\Phi_{11} = \sum_{m=1}^{\infty} \frac{\bar{\beta}_{1m} - \mu_2 \bar{\alpha}_{1m}}{\mu_1 - \mu_2} \sigma^{-m} + \frac{\bar{A}_{01} \bar{\lambda}_1 (\bar{\mu}_1 - \mu_1) + \bar{B}_{01} \bar{\lambda}_2 (\bar{\mu}_2 - \mu_2)}{\mu_1 - \mu_2} \sigma^{-1} - A_{01} \lambda_1 \sigma + C_{11} \quad (4.17)$$

$$\begin{aligned} \Phi_{12} = & \sum_{m=1}^{\infty} \frac{\bar{\beta}_{2m} - \mu_2 \bar{\alpha}_{2m}}{\mu_1 - \mu_2} \sigma^{-m} + \\ & + \frac{(\bar{A}_{11} \bar{\lambda}_1 - 3\bar{A}_{01} \bar{\lambda}_1^2 - 6\bar{A}_{03} \bar{\lambda}_1^3) (\bar{\mu}_1 - \mu_2) + (\bar{B}_{11} \bar{\lambda}_2 - 3\bar{B}_{01} \bar{\lambda}_2^2 - 6\bar{B}_{03} \bar{\lambda}_2^3) (\bar{\mu}_2 - \mu_1)}{\mu_1 - \mu_2} \sigma^{-1} - \\ & - 3 \frac{\bar{A}_{02} \bar{\lambda}_1^2 (\bar{\mu}_1 - \mu_2) + \bar{B}_{02} \bar{\lambda}_2^2 (\bar{\mu}_2 - \mu_1)}{\mu_1 - \mu_2} \sigma^{-2} - \frac{\bar{A}_{01} \bar{\lambda}_1^3 (\bar{\mu}_1 - \mu_2) + \bar{B}_{01} \bar{\lambda}_2^3 (\bar{\mu}_2 - \mu_1)}{\mu_1 - \mu_2} \sigma^{-3} + \\ & + (3A_{01} \lambda_1^3 + 6A_{03} \lambda_1^2 - A_{11} \lambda_1) \sigma + 3A_{02} \lambda_1^2 \sigma^2 + A_{01} \lambda_1^2 \sigma^3 + C_{12} \end{aligned} \quad (4.18)$$

We take the abbreviated designations C_{10} , C_{11} , and C_{12} for constant components which do not affect stress distribution.

The boundary values of functions ϕ_{20} , ϕ_{21} , and ϕ_{22} are obtained from (4.16)-(4.18) by simple transposition: it is necessary to insert B , A , μ_2 , μ_1 , λ_2 , λ_1 in the written formulas instead of A , B , μ_1 , μ_2 , λ_1 , λ_2 . To calculate stresses in the second approximation clear expressions for A_{11} and B_{11} are required; let us introduce the first of them:

$$A_{11} = \frac{\bar{\beta}_{11} - \mu_2 \bar{\alpha}_{11}}{\mu_1 - \mu_2} + \frac{\bar{A}_{01} \bar{\lambda}_1 (\bar{\mu}_1 - \mu_2) + \bar{B}_{01} \bar{\lambda}_2 (\bar{\mu}_2 - \mu_1)}{\mu_1 - \mu_2} + A_{01} \lambda_1^2 + 3A_{03} \lambda_1 \quad (4.19)$$

If it is necessary to determine stresses not only near the edge of the opening but also at other points on the plate, then we must find coefficients A_{km} and B_{km} of functions f_{1k} and f_{2k} . Knowing expressions (4.7), (4.10)-(4.15), we can easily do this by using (3.9)-(3.11) [or, in general, (3.7)].

For the third approximation we must keep the third power of ϵ in (2.9) and find Φ_{13} and Φ_{23} . While omitting the operations, let us introduce the boundary value of Φ_{13} (the value of Φ_{23} is obtained by transposition):

$$\begin{aligned}
 \Phi_{13} = & \sum_{m=1}^{\infty} \frac{\bar{\mu}_1^{2m} - \bar{\mu}_2^{2m}}{\bar{\mu}_1 - \bar{\mu}_2} \sigma^{-m} + \\
 & + [(\bar{A}_{21}\bar{\lambda}_1 - 3\bar{A}_{11}\bar{\lambda}_1^3 - 6\bar{A}_{13}\bar{\lambda}_1^2 + 20\bar{A}_{01}\bar{\lambda}_1^5 + 45\bar{A}_{03}\bar{\lambda}_1^4 + 35\bar{A}_{05}\bar{\lambda}_1^3)(\bar{\mu}_1 - \bar{\mu}_2) + \\
 & + (\bar{B}_{21}\bar{\lambda}_2 - 3\bar{B}_{11}\bar{\lambda}_2^3 - 6\bar{B}_{13}\bar{\lambda}_2^2 + 20\bar{B}_{01}\bar{\lambda}_2^5 + 45\bar{B}_{03}\bar{\lambda}_2^4 + \\
 & + 35\bar{B}_{05}\bar{\lambda}_2^3)(\bar{\mu}_2 - \bar{\mu}_2)] \frac{\sigma^{-1}}{\bar{\mu}_1 - \bar{\mu}_2} + [(-3\bar{A}_{13}\bar{\lambda}_1^2 + 20\bar{A}_{02}\bar{\lambda}_1^4 + 20\bar{A}_{04}\bar{\lambda}_1^3)(\bar{\mu}_1 - \bar{\mu}_2) + \\
 & + (-3\bar{B}_{13}\bar{\lambda}_2^2 + 20\bar{B}_{02}\bar{\lambda}_2^4 + 20\bar{B}_{04}\bar{\lambda}_2^3)(\bar{\mu}_2 - \bar{\mu}_2)] \frac{\sigma^{-2}}{\bar{\mu}_1 - \bar{\mu}_2} + \\
 & + [(-\bar{A}_{11}\bar{\lambda}_1^2 + 6\bar{A}_{01}\bar{\lambda}_1^4 + 10\bar{A}_{03}\bar{\lambda}_1^3)(\bar{\mu}_1 - \bar{\mu}_2) + \\
 & + (-\bar{B}_{11}\bar{\lambda}_2^2 + 6\bar{B}_{01}\bar{\lambda}_2^4 + 10\bar{B}_{03}\bar{\lambda}_2^3)(\bar{\mu}_2 - \bar{\mu}_2)] \frac{\sigma^{-3}}{\bar{\mu}_1 - \bar{\mu}_2} + \\
 & + 4 \frac{\bar{A}_{02}\bar{\lambda}_1^2(\bar{\mu}_1 - \bar{\mu}_2) + \bar{B}_{02}\bar{\lambda}_2^2(\bar{\mu}_2 - \bar{\mu}_2)}{\bar{\mu}_1 - \bar{\mu}_2} \sigma^{-4} + \frac{\bar{A}_{01}\bar{\lambda}_1^2(\bar{\mu}_1 - \bar{\mu}_2) + \bar{B}_{01}\bar{\lambda}_2^2(\bar{\mu}_2 - \bar{\mu}_2)}{\bar{\mu}_1 - \bar{\mu}_2} \sigma^{-5} + \\
 & + (-A_{21}\lambda_1 + 3A_{11}\lambda_1^3 + 6A_{13}\lambda_1^2 - 20A_{01}\lambda_1^5 - 45A_{03}\lambda_1^4 - 35A_{05}\lambda_1^3)\sigma + \\
 & + (3A_{12}\lambda_1^2 - 20A_{02}\lambda_1^4 - 20A_{04}\lambda_1^3)\sigma^2 + (A_{11}\lambda_1^2 - 6A_{01}\lambda_1^4 - 10A_{03}\lambda_1^3)\sigma^3 - \\
 & - 4A_{02}\lambda_1^2\sigma^4 - A_{01}\lambda_1^2\sigma^5 + C_{13}
 \end{aligned} \tag{4.20}$$

Here we add expressions for certain coefficients which are necessary for calculating stresses near the edge of the opening:

$$\begin{aligned}
 A_{12} = & \frac{\bar{\mu}_1 - \bar{\mu}_2 \bar{\alpha}_{12}}{\bar{\mu}_1 - \bar{\mu}_2} + 2A_{02}\lambda_1^2 + 4A_{04}\lambda_1 \\
 A_{13} = & \frac{\bar{\mu}_1 - \bar{\mu}_2 \bar{\alpha}_{12}}{\bar{\mu}_1 - \bar{\mu}_2} + A_{01}\lambda_1^3 + 3A_{03}\lambda_1^2 + 5A_{05}\lambda_1
 \end{aligned} \tag{4.21}$$

$$\begin{aligned}
 A_{21} = & \frac{\bar{\mu}_1 - \bar{\mu}_2 \bar{\alpha}_{21}}{\bar{\mu}_1 - \bar{\mu}_2} + \frac{\bar{A}_{11}\bar{\lambda}_1(\bar{\mu}_1 - \bar{\mu}_2) + \bar{B}_{11}\bar{\lambda}_2(\bar{\mu}_2 - \bar{\mu}_2)}{\bar{\mu}_1 - \bar{\mu}_2} - \\
 & - 3 \frac{\bar{A}_{01}\bar{\lambda}_1^2(\bar{\mu}_1 - \bar{\mu}_2) + \bar{B}_{01}\bar{\lambda}_2^2(\bar{\mu}_2 - \bar{\mu}_2)}{\bar{\mu}_1 - \bar{\mu}_2} - 6 \frac{\bar{A}_{02}\bar{\lambda}_1^2(\bar{\mu}_1 - \bar{\mu}_2) + \bar{B}_{02}\bar{\lambda}_2^2(\bar{\mu}_2 - \bar{\mu}_2)}{\bar{\mu}_1 - \bar{\mu}_2} + \\
 & + A_{11}\lambda_1^2 + 3A_{12}\lambda_1 - 6A_{01}\lambda_1^4 - 15A_{02}\lambda_1^2 - 15A_{05}\lambda_1^2
 \end{aligned} \tag{4.22}$$

Formulas for coefficients B_{12} , B_{13} , and B_{21} are analogous; we derive them from (4.21)-(4.22) by transposing letters and subscripts.

Stress components are expressed through derivatives of functions Φ_1 and Φ_2 with respect to their arguments. Assuming independent variables ρ and $\sigma = e^{\theta_1}$, we obtain

$$\Phi_i'(z_i) = \frac{d\Phi_i}{dz_i} = \frac{(\partial\Phi_i/\partial\rho)d\rho + (\partial\Phi_i/\partial\sigma)d\sigma}{[\sigma\omega'(\zeta) + \sigma^{-1}\lambda_i\bar{\omega}'(\bar{\zeta})]d\rho + [\omega'(\zeta) - \sigma^{-1}\lambda_i\bar{\omega}'(\bar{\zeta})]\rho d\sigma} \quad (4.23)$$

(i = 1, 2)

Using equations (2.8), $\delta\Phi_1/\delta\rho$ can be expressed by $\delta\Phi_1/\delta\sigma$ and vice versa. After this substitution, instead of (4.23) we obtain two equivalent formulas with which we can find the derivative of $\Phi_1'(z_1)$ at point (ρ, σ) :

$$\Phi_i'(z_i) = \frac{\partial\Phi_i/\partial\rho}{\sigma\omega'(\zeta) + \sigma^{-1}\lambda_i\bar{\omega}'(\bar{\zeta})} \quad (i = 1, 2) \quad (4.24)$$

$$\Phi_i'(z_i) = \frac{\rho^{-1}\partial\Phi_i/\partial\sigma}{\omega'(\zeta) - \sigma^{-1}\lambda_i\bar{\omega}'(\bar{\zeta})} \quad (4.25)$$

When the "small parameter method" is used, we derive approximate expressions for functions Φ_1 and Φ_2 while their precise values remain unknown. For determining the approximate values of the derivatives Φ_1' and Φ_2' we use formulas (4.24) and (4.25) where, instead of Φ_1 and Φ_2 , we substitute their approximate values.

The formula which determines approximate values for derivatives on the contour is derived in this case from (4.25) where it is necessary to assume $\rho = 1$ and, instead of Φ_1 , to substitute the boundary values indicated in this paragraph for the functions (approximate). For points on the contour we have

$$\Phi_i'(z_i) = \frac{d\Phi_i}{d\sigma} = a \left[1 - \frac{3\epsilon}{\sigma^4} + \lambda_i \left(-\frac{1}{\sigma^2} + 3\epsilon\sigma^2 \right) \right] \quad (4.26)$$

Other quantities which enter the formula for stresses σ_θ on the contour [see (1.14)] have the form:

$$\begin{aligned} dx &= -a(\sin\vartheta + 3\epsilon\sin 3\vartheta)d\vartheta \\ dy &= a(\cos\vartheta - 3\epsilon\cos 3\vartheta)d\vartheta \\ ds^2 &= a^2(1 + 9\epsilon^2 - 6\epsilon\cos 4\vartheta)d\vartheta^2 \end{aligned} \quad (4.27)$$

For an orthotropic plate all formulas are somewhat simplified. If the opening is cut so that its axes of symmetry, taken as x and

y , are normal to the planes of elastic symmetry, then, depending upon the elastic constants, three cases of complex parameters μ_1 and μ_2 are possible:

- I. $\mu_1 = \beta i, \mu_2 = \delta i, \bar{\mu}_1 = -\beta i, \bar{\mu}_2 = -\delta i$ ($\beta > 0, \delta > 0$)
- II. $\mu_1 = \mu_2 = \beta i, \bar{\mu}_1 = \bar{\mu}_2 = -\beta i$ ($\beta > 0$)
- III. $\mu_1 = \alpha + \beta i, \mu_2 = -\alpha + \beta i, \bar{\mu}_1 = \alpha - \beta i, \bar{\mu}_2 = -\alpha - \beta i$ ($\beta > 0$)

Next we examine in more detail two cases of elastic equilibrium in an orthotropic plate with an opening - expansion and bending by moments acting in the middle plane.

5. Expansion of an orthotropic plate with an opening. Let us examine an infinite orthotropic plate with an opening whose contour is given by equation (3.1) (the opening is cut so that the x and y axes are normal to the planes of elastic symmetry). Let the edge of the opening be free of internal forces, and at a great distance from it (in theory - at infinity) there are tensions distributed uniformly with intensity p parallel to the axis of symmetry which has been taken as the x axis.

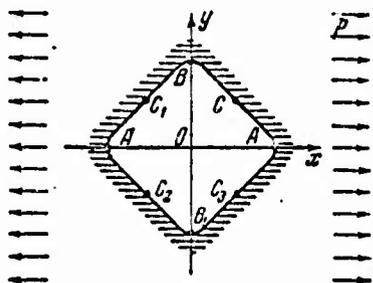


Fig. 3.

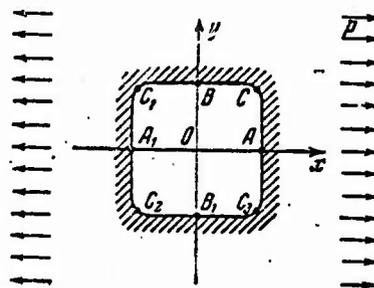


Fig. 4.

The location of the opening with respect to the axes and the forces is shown in Fig. 3 for $\epsilon > 0$ and in Fig. 4 for $\epsilon < 0$.

It is sufficient to examine only case I where the complex parameters are purely imaginary; then

$$\lambda_1 = \frac{1-\beta}{1+\beta}, \quad \lambda_2 = \frac{1-\delta}{1+\delta} \quad (5.1)$$

are real numbers, not exceeding unity in absolute magnitude. The formulas for the other cases II and III we obtain from the formulas derived for case I, assuming that $\delta = \beta$ everywhere in them or (in case III) replacing β and δ by the quantities $\beta + \alpha i$ and $\beta - \alpha i$.

Stress distribution in a solid plate being stretched by forces p will be:

$$\sigma_x^\circ = p, \quad \sigma_y^\circ = \tau_{xy}^\circ = 0 \quad (5.2)$$

The following forces act on the lines corresponding to the contour of the opening:

$$X_n^\circ = p \cos(n, x) = -p \frac{dy}{ds}, \quad Y_n^\circ = 0 \quad (5.3)$$

Stress distribution in a plate with an opening is found by applying the solution to (5.2) and the solution for a case of internal forces acting on the edge of the opening and equal to

$$X_n = -X_n^\circ, \quad Y_n = 0 \quad (5.4)$$

Functions ϕ_1 and ϕ_2 which correspond to load (5.4) satisfy conditions (1.16) in which

$$\int_0^s Y_n ds = 0$$

$$-\int_0^s X_n ds = -pa(\sin \vartheta - \varepsilon \sin 3\vartheta) = \frac{pai}{2} \left[\sigma - \frac{1}{\sigma} + \varepsilon \left(-\sigma^3 + \frac{1}{\sigma^3} \right) \right] \quad (5.5)$$

Consequently, in the formulas of the preceding paragraph which were found for the arbitrary distribution of forces along the edge of the opening, we should assume

$$\bar{\beta}_{01} = -\frac{1}{2} pai, \quad \bar{\beta}_{13} = \frac{1}{2} pai \quad (5.6)$$

and the remaining $\bar{\beta}_{km}$ and all $\bar{\alpha}_{km}$ are equal to zero.

The boundary values for the functions which make up the second approximation, on the basis of (4.16)-(4.18) and (4.7), (4.19), will have the form:

$$\begin{aligned}\Phi_1 &= \frac{\rho a}{2(\beta-\delta)} \left[-\frac{1}{\sigma} + \varepsilon \left(\lambda_1 \sigma + \frac{m_1}{\sigma} + \frac{1}{\sigma^2} \right) + \varepsilon^2 \left(-\lambda_1^2 \sigma^2 + n_1 \sigma + \frac{h_1}{\sigma} + \frac{k_1}{\sigma^2} \right) \right] + C_1 \\ \Phi_2 &= \frac{\rho a}{2(\beta-\delta)} \left[\frac{1}{\sigma} - \varepsilon \left(\lambda_2 \sigma + \frac{m_2}{\sigma} + \frac{1}{\sigma^2} \right) + \varepsilon^2 \left(\lambda_2^2 \sigma^2 - n_2 \sigma - \frac{h_2}{\sigma} - \frac{k_2}{\sigma^2} \right) \right] + C_2\end{aligned}\quad (5.7)$$

Here C_1 and C_2 are constant components which do not affect stress distribution, while $h_1, k_1, m_1,$ and n_1 are coefficients expressed by β and δ in the following manner:

$$\begin{aligned}h_1 &= \frac{1}{(1+\beta)^2(1+\delta)^2} [-3 + 7\beta + 19\delta - 5\beta^2 + 13\beta\delta - 9\delta^2 + \beta^3 - 19\beta^2\delta - \\ &\quad - 59\beta\delta^2 + \delta^3 + 3\delta(3\beta^2 - 31\beta\delta - \delta^2) + 3\beta^2\delta^2(\beta + 5\delta) + \beta^2\delta^3] \\ k_1 &= \frac{1}{(1+\beta)^2(1+\delta)^2} [-1 + 2(1-\beta)(\beta+3\delta) - \beta^2 + 4\beta\delta - \delta^2 - \beta^2\delta^2] \\ m_1 &= \frac{1}{(1+\beta)(1+\delta)} (1 - \beta - 3\delta - \beta\delta) \\ n_1 &= \frac{\beta-1}{(1+\beta)^2(1+\delta)} (3 - 4\beta - \delta + \beta^2 - 8\beta\delta + \beta^2\delta)\end{aligned}\quad (5.8)$$

The coefficients $h_2, k_2, m_2,$ and n_2 are obtained from (5.8) by transposition of the quantities β and δ . On the contour of the opening we find the following (approximate) expression for the derivative

$$\Phi_1' = \frac{\rho}{2(\beta-\delta)} \frac{\frac{1}{\sigma^2} + \varepsilon \left(\lambda_1 - \frac{m_1}{\sigma^2} - \frac{3}{\sigma^4} \right) + \varepsilon^2 \left(-3\lambda_1 \sigma^2 + n_1 - \frac{h_1}{\sigma^2} - \frac{3k_1}{\sigma^4} \right)}{1 - \frac{3\varepsilon}{\sigma^4} + \lambda_1 \left(-\frac{1}{\sigma^2} + 3\varepsilon\sigma^2 \right)}\quad (5.9)$$

and an analogous expression for Φ_2' .

We show only the formula for stresses σ_θ on the edge of the opening [see (1.14)] and at specific points on the contour - at "corners" and on the middles of the "sides" (A, A₁, B, B₁, C, C₁, C₂, C₃ on Figs. 3 and 4).

Let us introduce the further designations:

$$\begin{aligned}
 A &= \cos \vartheta - 3\varepsilon \cos 3\vartheta, & B &= \sin \vartheta + 3\varepsilon \sin 3\vartheta \\
 C^2 &= A^2 + B^2 = 1 + 9\varepsilon^2 - 6\varepsilon \cos 4\vartheta \\
 D^2 &= -A^2\beta\delta + A^2B^2(1 - 2\beta\delta - \beta^2\delta^2) + B^4(2 - \beta\delta - \beta^2 - \delta^2) \\
 L &= \beta_{11}(B^2 + \beta^2A^2)(B^2 + \delta^2A^2) = \beta_{22}A^4 + (2\beta_{12} + \beta_{66})A^2B^2 + \beta_{11}B^4
 \end{aligned} \tag{5.10}$$

$$\begin{aligned}
 \xi &= \frac{8(1-\beta\delta)}{(1+\beta)^2(1+\delta)^2} \\
 k &= \frac{2}{(1+\beta)^2(1+\delta)^2} [1 + 2(\beta+\delta)(\beta\delta-1) + \beta^2 - 4\beta\delta + \delta^2 + \beta^2\delta^2] \\
 k &= \frac{4}{(1+\beta)(1+\delta)}, & l &= \frac{2}{(1+\beta)(1+\delta)} (1 - \beta - \delta - \beta\delta) \\
 m &= \frac{8}{(1+\beta)^2(1+\delta)^2} [(\beta+\delta)(1+\beta\delta) - 3(1+\beta^2\delta^2) + 10\beta\delta] \\
 n &= \frac{2}{(1+\beta)^2(1+\delta)^2} [(\beta+\delta)(7 - 10\beta\delta + 3\beta^2\delta^2) - 3 + 17\beta\delta - 27\beta^2\delta^2 + \beta^2\delta^2 + \\
 &\quad + (\beta^2 + \delta^2)(3\beta\delta - 5) + \beta^2 + \delta^2]
 \end{aligned} \tag{5.11}$$

Stress σ_θ on the edge of the opening is determined in the second approximation by the formula

$$\begin{aligned}
 \sigma_\theta &= p \frac{B^2}{C^2} + \frac{p\beta_{11}}{LC^2} [AD^2 \cos \vartheta + BC^2(\beta+\delta) \sin \vartheta] - \\
 &- \varepsilon \frac{p\beta_{11}}{LC^2} [AC^2 k \beta \delta (\beta+\delta) \cos \vartheta + 3AD^2 \cos 3\vartheta + BC^2(\beta+\delta)(l \sin \vartheta + 3 \sin 3\vartheta)] + \\
 &+ \varepsilon^2 \frac{p\beta_{11}}{L} (\beta+\delta) C^2 [(-m \cos \vartheta + 3g \cos 3\vartheta)\beta\delta A + (-n \sin \vartheta + 3h \sin 3\vartheta)B].
 \end{aligned} \tag{5.12}$$

At points A and A_1 at the ends of the opening's diameter, parallel to the forces (Figs. 3 and 4), which correspond to $\theta = 0$ and $\theta = \pi$, we obtain

$$(\sigma_\theta)_A = \frac{p}{1-3\varepsilon} \frac{1}{\beta\delta} \{-1 + \varepsilon[\beta - (\beta+\delta)k] + \varepsilon^2(\beta+\delta)(3g-m)\} \tag{5.13}$$

At points B and B_1 at the ends of the diameter perpendicular to the forces ($\theta = (1/2)\pi$, $\theta = (3/2)\pi$), we shall have the following expression

$$(\sigma_\theta)_B = \frac{p}{1-3\varepsilon} \{1 + \beta + \delta + \varepsilon[(\beta+\delta)(5-k) - 3] - \varepsilon^2(\beta+\delta)(n+3h)\} \tag{5.14}$$

At points C, C_1 , C_2 , and C_3 at the ends of the diameters, which are directed at an angle of 45° to the forces, we shall have

$$(\sigma_\theta)_C = \frac{P}{1+3\epsilon} \frac{2}{(1+\beta^2)(1+\delta^2)} \{1 + \beta + \delta - \beta\delta + \epsilon [3(1-\beta\delta) - (\beta + \delta)(3 + k\beta\delta + l)] + \epsilon^2 (\beta + \delta) [3(h - g\beta\delta) - m\beta\delta - n]\} \quad (5.15)$$

If we desire to calculate the stresses in the third approximation, i.e., to preserve the third power of ϵ , then to stress σ_θ , determined according to formula (5.12), we must add the quantity $\Delta_3\sigma_\theta$ - the correction for the third approximation. Without deriving the formula for this correction, let us state

$$\Delta_3\sigma_\theta = -\epsilon^3 \frac{P\beta_{11}}{L} (\beta + \delta) C^2 (AN_1\beta\delta + BM_1) \quad (5.16)$$

Here the designations are:

$$\begin{aligned} M_1 &= a_1 \sin \vartheta + 3a_3 \sin 3\vartheta + 5a_5 \sin 5\vartheta \\ N_1 &= a_1' \cos \vartheta + 3a_3' \cos 3\vartheta + 5a_5' \cos 5\vartheta \end{aligned} \quad (5.17)$$

$$\begin{aligned} a_1 &= \frac{1}{4} \{12h - 2m\beta\delta(g+k) + 2n(l-l) + 3h(l^2 + k^2\beta\delta) + \\ &+ 3gk^2\beta\delta(\beta + \delta) + 11l(h^2 - g^2\beta\delta) - 11gk\beta\delta[2h + g(\beta + \delta)]\} \\ a_3 &= \frac{1}{2} \{hl + gk\beta\delta + 5(h^2 - g^2\beta\delta)\}, \quad a_5 = \frac{1}{2} (hl - gk\beta\delta) \\ a_1' &= \frac{1}{4} \{12g + 2m[h - l + k(1 - \beta\delta) + g(\beta + \delta)] + 2n(g - k) + 3g(l^2 + k^2\beta\delta) - \\ &- 3k^2(\beta + \delta)[h + g(\beta + \delta)] + 11k(h^2 - g^2\beta\delta) + 11g[l + k(\beta + \delta)][2h + g(\beta + \delta)]\} \\ a_3' &= \frac{1}{2} \{gl + hk + (5g - k)[2h + g(\beta + \delta)]\} \\ a_5' &= \frac{1}{2} \{hk + g[l + k(\beta + \delta)]\} \end{aligned} \quad (5.18)$$

Values of $\Delta_3\sigma_\theta$ at points A, A₁, B, B₁, C, C₁, C₂, and C₃ (Figs. 3 and 4) are equal to:

$$(\Delta_3\sigma_\theta)_A = -P \frac{\epsilon^3}{1-3\epsilon} \frac{\beta + \delta}{\beta\delta} (a_1' + 3a_3' + 5a_5') \quad (5.19)$$

$$(\Delta_3\sigma_\theta)_B = -P \frac{\epsilon^3}{1-3\epsilon} (\beta + \delta) (a_1 - 3a_3 + 5a_5) \quad (5.20)$$

$$(\Delta_3\sigma_\theta)_C = -P \frac{\epsilon^3}{1+3\epsilon} \frac{2(\beta + \delta)}{(1+\beta^2)(1+\delta^2)} [a_1 + 3a_3 - 5a_5 + \beta\delta (a_1' - 3a_3' - 5a_5')] \quad (5.21)$$

The reduced formulas containing letter designations do not allow us to judge at what points on the contour pressure is highest. However, when $\epsilon > 0$, it is quite probable that σ_θ achieves its highest values at points B and B₁ [although not excluding the

possibility that at certain values for the elastic constants the stress at points A and A₁, found by formulas (5.13) and (5.19), will appear to be greater in absolute magnitude than the stress on the ends of the diameter at points B and B₁].

For an isotropic plate $\beta = \delta = 1$

$$\lambda_1 = \lambda_2 = 0, \quad g = h = 0, \quad k = m = 1, \quad l = n = -1$$

The stress at points A and A₁ of an isotropic plate is determined in the third approximation according to formula (which is derived from (5.13) and (5.19))

$$(\sigma_\theta)_A = \frac{p}{1-3\epsilon} (-1 + \epsilon - 2\epsilon^2 - 2\epsilon^3) \quad (5.22)$$

The exact value of stress at these points is equal to

$$(\sigma_\theta)_A = \frac{p}{1-3\epsilon} \frac{-1 + 2\epsilon - 3\epsilon^2}{1-\epsilon} \quad (5.23)$$

For the stress at points B and B₁ we obtain from (5.14) and (5.19) the approximate formula

$$(\sigma_\theta)_B = \frac{p}{1-3\epsilon} (3 + 5\epsilon + 2\epsilon^2 + 2\epsilon^3) \quad (5.24)$$

The precise formula has the form

$$(\sigma_\theta)_B = \frac{p}{1-3\epsilon} \frac{3 + 2\epsilon - 3\epsilon^2}{1-\epsilon} \quad (5.25)$$

Expressions (5.22) and (5.24) are obtained from (5.23) and (5.25), respectively, if in them quantity $1:(1-\epsilon)$ is expanded in a power series and the highest powers of ϵ , beginning with the fourth, are eliminated after multiplication.

At points C , C_1 , C_2 , and C_3 of an isotropic plate the approximate formula for stresses coincides with the precise formula¹

$$(\varepsilon_\theta)_C = p \frac{1-3\varepsilon}{1+3\varepsilon} \quad (5.26)$$

We note that for an isotropic plate the errors in approximate formulas (5.22) and (5.24) are very small even when ε is not very small as compared with unity.

If we keep only the first and second powers of ε in the brackets, i.e., take the second approximation, then error in stresses at points A and B when $|\varepsilon| \leq \frac{1}{9}$ does not exceed 0.5% [as compared with quantities found using the precise formulas (5.23) and (5.25)].

Even for $|\varepsilon| = \frac{1}{6}$ the errors in formulas (5.22) and (5.24), in which ε^3 is discarded, are less than 12.5%; naturally, they will be even smaller if we examine the third approximation.

Let us introduce the results of calculations for an orthotropic plate whose principal elastic constants for directions parallel to the middle plane have the following values: Young's modulus - $1.2 \times 10^5 \text{ kg cm}^{-2}$ and $0.6 \times 10^5 \text{ kg cm}^{-2}$, Poisson brackets - 0.071 and 0.036, and shear modulus - $0.07 \times 10^5 \text{ kg/cm}^2$. Such elastic constants (average in thickness) are obtained for one type of plywood.² If the directions of the coordinate axes coincide with the principal directions of elasticity then the complex parameters are purely imaginary:

$$\mu_1 = \beta i, \quad \mu_2 = \delta i.$$

We should distinguish the two basic cases:

¹See reference [3], page 54. All three formulas - (5.23), (5.25), and (5.26) - are derived from formula (80) in this work, found by the method of N. I. Muskhelishvili, at particular values of θ if we assume there that $R = 0$, $h = -p/A$, $\alpha = 0$, $m = \varepsilon$.

²See [2], page 133.

1) a plate is stretched, as shown in Figs. 3 and 4, in direction x for which the Young's modulus is greatest (1.2×10^5), $\beta = 4.11$, $\delta = 0.343$;

2) a plate is stretched in direction x for which the Young's modulus is least (0.6×10^5), $\beta = 0.243$, $\delta = 2.91$.

The formula for stress σ_θ at a given fixed point on the contour is written in the form

$$\sigma_\theta = pK \quad (5.27)$$

where K is a dimensionless coefficient depending on β , δ , and ϵ .

Table 1. Values of coefficients $K \times 10^2$ for certain ϵ . Case 1.

Points	Ap- prox- ima- tion	$\epsilon=0$	$\frac{1}{100}$	$\frac{5}{100}$	$\frac{1}{10}$	$\frac{1}{9}$	$\frac{1}{6}$	$-\frac{1}{100}$	$-\frac{5}{100}$	$-\frac{1}{10}$	$-\frac{1}{9}$	$-\frac{1}{6}$
A	(1)	-71	-73	-81	-97	-101	-132	-69	-63	-57	-55	-50
	(2)	-71	-73	-82	-100	-105	-142	-69	-63	-58	-57	-54
	(3)	-71	-73	-82	-100	-105	-143	-69	-63	-58	-57	-54
B	(1)	545	579	740	1017	1096	1646	513	402	291	270	178
	(2)	545	579	737	1004	1079	1595	513	403	284	261	161
	(3)	545	579	737	1007	1082	1612	513	399	283	260	156
C	(1)	40	38	30	22	20	14	43	55	75	81	121
	(2)	40	38	30	23	22	16	43	55	76	82	126
	(3)	40	38	30	22	21	15	43	55	77	83	129

Table 2. Values of coefficients $K \times 10^2$ for certain ϵ . Case 2.

Points	Ap- prox- ima- tion	$\epsilon=0$	$\frac{1}{100}$	$\frac{5}{100}$	$\frac{1}{10}$	$\frac{1}{9}$	$\frac{1}{6}$	$-\frac{1}{100}$	$-\frac{5}{100}$	$-\frac{1}{10}$	$-\frac{1}{9}$	$-\frac{1}{6}$
A	(1)	-141	-145	-163	-194	-203	-264	-138	-125	-113	-111	-101
	(2)	-141	-145	-163	-195	-205	-270	-138	-126	-114	-112	-103
	(3)	-141	-145	-163	-196	-206	-278	-138	-126	-113	-111	-100
B	(1)	415	439	548	739	792	1170	393	317	241	227	164
	(2)	415	439	547	733	785	1146	393	316	238	223	156
	(3)	415	439	547	733	786	1151	393	316	238	222	155
C	(1)	69	65	54	42	40	31	72	89	118	126	183
	(2)	69	65	54	43	41	32	72	89	119	127	187
	(3)	69	65	54	43	41	31	72	89	119	128	189

Tables 1 and 2 give the numerical values for coefficient K found for values of $|\epsilon| = 0, 0.01, 0.05, 0.1, 1/9, 1/6$ in the first (1), second (2), and third (3) approximations for points A, B, and C.

Eleven values of this parameter are taken in all - zero for a circular opening, five positive, and five negative.

Two decimal places are preserved throughout the tables in the final numbers. The calculation of coefficients K with greater accuracy is hardly sensible since the numerical values of β and δ , indicated for cases 1 and 2, are approximate - they are given with three significant digits [let us remember that β and δ are determined based on the prescribed elastic constants from equation (1.3) where $\beta_{16} = \beta_{26} = 0$].

As seen from the tables, for $|\epsilon| \leq 1/9$ the third approximations, within the accuracy adopted, differ little or not at all from the second approximations. While calculating stresses for such ϵ according to formulas (5.13)-(5.15), we shall venture to obtain an error for coefficient K (absolute) which does not exceed 0.03, and for smaller ϵ in a number of cases only the first approximation is necessary. For $\epsilon = \pm 1/6$ the difference between the third and second approximation is more noticeable; however, in these cases, it is comparatively small.

From these same tables it is apparent that at positive ϵ (Fig. 3) the stress at point B increases with an increase in ϵ (this is understandable since the curvature of the contour at point B increases). Simultaneously, the stress at point A increases in absolute magnitude, while the stress at point C decreases. At negative ϵ (Fig. 4) the opposite pattern is observed: with an increase in $|\epsilon|$ the stresses at points B and A drop in absolute magnitude, while at point C they rise. When $\epsilon \geq 0$ the greatest stress for the entire plate is found at point B (and point B_1). For openings corresponding to negative ϵ the position of the point (in the first quadrant), where the stress σ_θ reaches its highest value, is determined by the quantity ϵ ; with an increase in $|\epsilon|$ this point moves from B in a direction toward C.

We shall write out for a comparison the quantities of stress at points A, B, C on the examined anisotropic and isotropic plates

with an opening which is characterized by parameter $\epsilon = \pm 1/9$
 (coefficients K for an anisotropic plate are taken from Table 1 and
 Table 2 in the third approximation).

a) $\epsilon = 1/9$

Anisotropic plate, case 1:

$$(\sigma_{\theta})_A = -1.05 p, \quad (\sigma_{\theta})_B = 10.82 p, \quad (\sigma_{\theta})_C = 0.21 p \quad (5.28)$$

Anisotropic plate, case 2:

$$(\sigma_{\theta})_A = -2.06 p, \quad (\sigma_{\theta})_B = 7.86 p, \quad (\sigma_{\theta})_C = 0.41 p \quad (5.29)$$

Isotropic plate:

$$(\sigma_{\theta})_A = -1.38 p, \quad (\sigma_{\theta})_B = 5.38 p, \quad (\sigma_{\theta})_C = 0.5 p \quad (5.30)$$

b) $\epsilon = -1/9$

Anisotropic plate, case 1:

$$(\sigma_{\theta})_A = -0.57 p, \quad (\sigma_{\theta})_B = 2.60 p, \quad (\sigma_{\theta})_C = 0.83 p \quad (5.31)$$

Anisotropic plate, case 2:

$$(\sigma_{\theta})_A = -1.11 p, \quad (\sigma_{\theta})_B = 2.22 p, \quad (\sigma_{\theta})_C = 1.28 p \quad (5.32)$$

Isotropic plate:

$$(\sigma_{\theta})_A = -0.85 p, \quad (\sigma_{\theta})_B = 1.85 p, \quad (\sigma_{\theta})_C = 2 p \quad (5.33)$$

When comparing these data, we note that at point B of the
 examined anisotropic plate the stress is greater than at the
 corresponding point on the isotropic plate; on the other hand, the
 presence of anisotropy reduces the stress at point C. Generally
 the stress σ_{θ} in the anisotropic plate changes along the contour

of the opening more sharply than the stress in the isotropic plate, forming "peaks" at certain points (especially noticeable at point B for case 1 when $\epsilon = 1/9$).

6. The bending of an orthotropic plate with an opening by moments which act in its plane. Let a rectangular orthotropic plate be weakened in the center by an opening with a contour of type (3.1); the dimensions of the opening are small in comparison with the dimensions of the plate. It is assumed that the planes of elastic symmetry are parallel to the surfaces of the plate and the opening is cut so that its two x and y axes of symmetry are normal to the planes of elastic symmetry. Moments M acting in the middle plane are applied to the two opposite sides and the edge of the opening is not loaded.

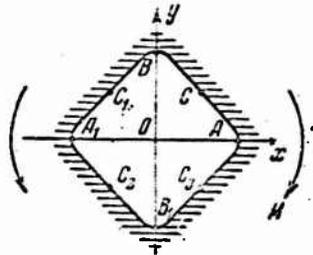


Fig. 5.

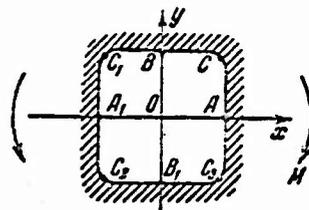


Fig. 6.

When $\epsilon > 0$ we have the pattern shown in Fig. 5; the case of $\epsilon < 0$ corresponds to Fig. 6.

In a plate without an opening, which can be bent by the moments (pure bend), stresses are determined according to the law:

$$\sigma_x^0 = \frac{M}{J} y, \quad \sigma_y^0 = \tau_{xy}^0 = 0 \quad (6.1)$$

Here J is the cross-sectional moment of inertia.

The following forces act in a line whose equation is (3.1):

$$X_n^0 = \frac{M}{J} y \cos(n, x) = -\frac{M}{J} y \frac{dy}{ds}, \quad Y_n^0 = 0 \quad (6.2)$$

We derive an approximate solution for a plate with an opening (which does not satisfy the strict conditions on its sides) by adding to stresses (6.1) the stresses in an infinite plate with an opening along whose edge are distributed forces

$$X_n = -X_n^0, \quad Y_n = 0 \quad (6.3)$$

The integrals in the right side of conditions (1.16), in this case, will be

$$\begin{aligned} \int_0^a Y_n ds &= 0 \\ - \int_0^a X_n ds &= -\frac{M}{2J} y^2 = \\ &= \frac{Ma^2}{8J} \left[-2 + \sigma^2 + \frac{1}{\sigma^2} + 2\varepsilon \left(\sigma^2 - \sigma^4 + \frac{1}{\sigma^2} - \frac{1}{\sigma^4} \right) + \varepsilon^2 \left(-2 + \sigma^6 + \frac{1}{\sigma^6} \right) \right] \end{aligned} \quad (6.4)$$

Consequently,

$$\begin{aligned} \bar{\beta}_{00} &= -\frac{Ma^2}{4J}, & \bar{\beta}_{12} &= \frac{Ma^2}{4J}, & \bar{\beta}_{20} &= -\frac{Ma^2}{4J} \\ \bar{\beta}_{02} &= \frac{Ma^2}{8J}, & \bar{\beta}_{14} &= -\frac{Ma^2}{4J}, & \bar{\beta}_{26} &= \frac{Ma^2}{8J} \end{aligned} \quad (6.5)$$

The remaining $\bar{\beta}_{km}$ and all $\bar{\alpha}_{km}$ are equal to zero.

When we substitute (6.5) into formulas (4.16)-(4.18), we obtain the boundary values for functions of complex variables which determine the second approximation

$$\begin{aligned} \Phi_1 &= \frac{Ma^2 i}{8J(\beta - \delta)} \left[-\frac{1}{\sigma^2} + 2\varepsilon \left(-\frac{1}{\sigma^2} + \frac{1}{\sigma^4} \right) - \varepsilon^2 \left(3\lambda_1^2 \sigma^2 + \frac{3k_1}{\sigma^2} + \frac{1}{\sigma^4} \right) \right] + C_1 \\ \Phi_2 &= \frac{Ma^2 i}{8J(\beta - \delta)} \left[\frac{1}{\sigma^2} + 2\varepsilon \left(\frac{1}{\sigma^2} - \frac{1}{\sigma^4} \right) + \varepsilon^2 \left(3\lambda_2^2 \sigma^2 + \frac{3k_2}{\sigma^2} + \frac{1}{\sigma^4} \right) \right] + C_2 \end{aligned} \quad (6.6)$$

Here C_1 and C_2 are constant components which do not affect stress distribution; k_1 is determined by the second formula (5.8) and k_2 is obtained from k_1 by the transposition of β and δ .

The approximate expression of derivative ϕ_1' on the contour of the opening has the following form [see (4.26)]:

$$\phi_1' = \frac{Ma i}{4J(\beta - \delta)} \frac{\sigma^{-3} + 2\epsilon(\sigma^{-3} - 2\sigma^{-5}) + 3\epsilon^2(-\lambda_1^2\sigma + k_1\sigma^{-3} + \sigma^{-7})}{1 - 3\epsilon\sigma^{-4} + \lambda_1(-\sigma^{-2} + 2\epsilon\sigma^2)}. \quad (6.7)$$

The boundary value of ϕ_2 is found according to an analogous formula.

Using the shortened designations (5.10) and (5.11), we shall write a formula for stress σ_θ near the edge of the opening in the following manner:

$$\begin{aligned} \sigma_\theta = & \frac{Ma}{J} \frac{B^2}{C^2} (\sin \theta - \epsilon \sin 3\theta) + \frac{Ma}{2J} \frac{\beta_{11}}{LC^2} [-BC^4(\beta + \delta) \cos 2\theta + AD^4 \sin 2\theta] + \\ & + \epsilon \frac{Ma}{J} \frac{\beta_{11}}{LC^2} [-BC^4(\beta + \delta) (\cos 2\theta - 2 \cos 4\theta) + AD^4 (\sin 2\theta - 2 \sin 4\theta)] + \\ & + \epsilon^2 \frac{3Ma}{2J} \frac{\beta_{11}}{LC^2} [BC^4(\beta + \delta) (h \cos 2\theta - \cos 6\theta) - \\ & - AC^4 g(\beta + \delta) \beta \delta \sin 2\theta + AD^4 \sin 6\theta]. \end{aligned} \quad (6.8)$$

At points A and A_1 (Figs. 5 and 6) stress is zero. At point B

$$(\sigma_\theta)_B = \frac{Ma}{J(1-3\epsilon)} \left\{ 1 + \frac{\beta + \delta}{2} + \epsilon [3(\beta + \delta) - 2] + 3\epsilon^2 \left[\frac{\beta + \delta}{2} (1 - h) - 1 \right] \right\}. \quad (6.9)$$

Stress at the opposite point B_1 is found to be the same in absolute value but has the opposite sign.

At points C and C_1

$$\begin{aligned} (\sigma_\theta)_C = & \frac{Ma}{J(1+3\epsilon)} \frac{\sqrt{2}}{(1+\beta^2)(1+\delta^2)} \{ 1 - \beta\delta + 2\epsilon [1 - \beta\delta - 2(\beta + \delta)] + \\ & + 3\epsilon^2 [\beta\delta - 1 - g\beta\delta(\beta + \delta)] \}. \end{aligned} \quad (6.10)$$

At points C_2 and C_3 we obtain the same value but the opposite sign.

Let us point out, without derivation, the error which must be added to the stress σ_θ [formula (6.8)] in order to obtain the third approximation:

$$\Delta_3 \sigma_\theta = -\varepsilon^3 \frac{2Ma}{J} \frac{\beta(\beta + \delta)}{L} C^2 (AM_2 \beta \delta + BN_2). \quad (6.11)$$

Here the designations are:

$$\begin{aligned} M_2 &= a_2 \sin 2\theta + 2a_3 \sin 4\theta \\ N_2 &= a_2' \cos 2\theta + 2a_4' \cos 4\theta \end{aligned} \quad (6.12)$$

$$\begin{aligned} a_2 &= \frac{\varepsilon}{4} (6 - 7[2h + g(\beta + \delta)]), & a_3 &= -a_3' \\ a_2' &= \frac{1}{4} [-6h + 7(h^2 - g^2 \beta \delta)], & a_4' &= a_4' \end{aligned} \quad (6.13)$$

At points B and C we obtain

$$(\Delta_3 \sigma_\theta)_B = \varepsilon^3 \frac{2Ma}{J(1-3\varepsilon)} (\beta + \delta) (a_2' - 2a_4') \quad (6.14)$$

$$(\Delta_3 \sigma_\theta)_C = -\varepsilon^3 \frac{\sqrt{2}Ma}{J(1+3\varepsilon)} \frac{\beta + \delta}{(1+\beta^2)(1+\delta^2)} (a_2 \beta \delta - 2a_4') \quad (6.15)$$

(at points A and A_1 $\Delta_3 \sigma_\theta = 0$).

With positive ε it is natural to expect that the stress which is highest in absolute magnitude is found at points B and B_1 .

For an isotropic plate $\Delta_3 \sigma_\theta = 0$ at all points of the contour we have

$$(\sigma_\theta)_B = \frac{2Ma}{J} \frac{1+2\varepsilon}{1-3\varepsilon}, \quad (6.16)$$

$$(\sigma_\theta)_C = -\frac{2\sqrt{2}Ma}{J} \frac{\varepsilon}{1+3\varepsilon} \quad (6.17)$$

These formulas for isotropic material, found by the "small parameter method," are identical with the precise ones.¹

¹See [3], page 54, formula (80), where we must assume the following:

$$h=0, \quad R=a, \quad A=-\frac{M}{J}, \quad \alpha=0, \quad m=\varepsilon, \quad \vartheta=\frac{1}{2}\pi, \quad \varphi=\frac{1}{4}\pi.$$

Tables 3 and 4 give the results of calculating first (1), second (2), and third (3) approximations for the anisotropic (plywood) plate examined in Section 5.

The formula for stress at a given point on the contour can be written as

$$\sigma_{\theta} = \frac{Ma}{J} K_1 \quad (6.18)$$

where K_1 is a dimensionless coefficient depending upon β , δ , and ϵ . The tables are compiled for the same values of ϵ as were examined in Section 5 and their accuracy is the same. Table 3 presents the numerical values of coefficient K_1 for case 1 when the direction of the x-axis (Figs. 5 and 6) coincides with the direction for which the Young's modulus is greatest; Table 4 presents the same for case 2 when the direction of the x-axis coincides with the direction for which Young's modulus is the least.

If we want to calculate coefficient K_1 with two decimal places, then for $|\epsilon| \leq 1/9$ it is sufficient to take only the first approximation; the error which we obtain with this does not exceed 0.02 in the worst case. Even for $\epsilon = 1/6$ the third approximation differs very little from the first and only for $\epsilon = -1/6$ in case 1 does the difference become somewhat larger.

Table 3. Values of coefficients $K_1 \times 10^2$ for certain ϵ . Case 1.

Points	Ap- prox- ima- tion	$\epsilon = 0$	$\frac{1}{103}$	$\frac{5}{100}$	$\frac{1}{10}$	$\frac{1}{9}$	$\frac{1}{6}$	$-\frac{1}{100}$	$-\frac{5}{100}$	$-\frac{1}{10}$	$-\frac{1}{9}$	$-\frac{1}{6}$
B	(1)	323	344	446	623	673	1024	302	231	161	147	89
	(2)	323	344	446	621	671	1016	302	231	160	146	86
	(3)	323	344	446	622	672	1023	302	231	159	145	84
C	(1)	-3	-4	-8	-12	-13	-17	-2	+4	15	18	38
	(2)	-3	-4	-8	-12	-13	-16	-2	+4	15	18	39
	(3)	-3	4	-8	-12	-13	-17	-2	+4	15	18	40

Table 4. Values of coefficients $K_1 \times 10^2$ for certain ϵ . Case 2.

Points	Ap- prox- ima- tion	$\epsilon = 0$	$\frac{1}{100}$	$\frac{5}{100}$	$\frac{1}{10}$	$\frac{1}{9}$	$\frac{1}{\epsilon}$	$-\frac{1}{100}$	$-\frac{5}{100}$	$-\frac{1}{10}$	$-\frac{1}{9}$	$-\frac{1}{\epsilon}$
B	(1)	258	273	347	475	511	764	243	192	141	131	89
	(2)	258	273	347	474	510	762	243	192	140	130	88
	(3)	258	273	347	474	510	763	243	192	140	130	88
C	(1)	4	2	-4	-10	-11	-16	6	15	30	34	65
	(2)	4	2	-4	-10	-11	-16	6	15	30	34	64
	(3)	4	2	-4	-10	-11	-17	6	15	30	34	64

With positive ϵ the stress at point B grows, while with negative ϵ it drops with an increase in $|\epsilon|$. When $\epsilon \geq 0$ the stress at points B and B_1 is the greatest in absolute magnitude for the entire plate. In the case of negative ϵ the point in the first quadrant where stress σ_θ reaches maximum shifts in direction from B toward C with a growth in $|\epsilon|$.

We shall write the numerical results for an anisotropic plate with an opening which corresponds to the values $\epsilon = \pm 1/9$, and for the same type of isotropic plate.

a) $\epsilon = 1/9$

Anisotropic plate, case 1:

$$(\sigma_\theta)_B = 6.72 \frac{Ma}{J}, \quad (\sigma_\theta)_C = -0.13 \frac{Ma}{J} \quad (6.19)$$

Anisotropic plate, case 2:

$$(\sigma_\theta)_B = 5.10 \frac{Ma}{J}, \quad (\sigma_\theta)_C = -0.10 \frac{Ma}{J} \quad (6.20)$$

Isotropic plate:

$$(\sigma_\theta)_B = 3.67 \frac{Ma}{J}, \quad (\sigma_\theta)_C = -0.24 \frac{Ma}{J} \quad (6.21)$$

b) $\epsilon = -1/9$

Anisotropic plate, case 1:

$$(\sigma_\theta)_B = 1.45 \frac{Ma}{J}, \quad (\sigma_\theta)_C = 0.18 \frac{Ma}{J} \quad (6.22)$$

Anisotropic plate, case 2:

$$(\sigma_x)_B = 1.30 \frac{Ma}{J}, \quad (\sigma_x)_C = 0.34 \frac{Ma}{J} \quad (6.23)$$

Isotropic plate:

$$(\sigma_x)_B = 1.17 \frac{Ma}{J}, \quad (\sigma_x)_C = 0.47 \frac{Ma}{J} \quad (6.24)$$

Just as in the case of expansion, the stress at point B of a given anisotropic plate is greater, and at point C is less in absolute magnitude than the stresses at corresponding points on an isotropic plate. This is explained by the sharper increase in stress along the contour of the opening as compared with the isotropic plate.

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ABSTRACT

(U) The plane problem of the theory of elasticity for the anisotropic plate with an aperture is solved only in the case when the aperture has the shape of an ellipse or circle. All the other cases of openings of another shape, including those of great interest for practice, up to now have not been investigated. In this work the author offers an approximate method for the solution of the plane problem for the infinite anisotropic plate with an aperture resembling the circular. The method based on the introduction of a small parameter (characterizing the deviation of the aperture from the circle), whose high degrees (beginning, for instance, with the third or fourth) are rejected in the investigation process. The problem is reduced to the well-known problem of the equilibrium of the anisotropic plate with a circular aperture. The basic attention is given to the aperture, having four axes of symmetry (during the needed selection of the parameter it will be little different from the square with rounded corners). For a plate with such an aperture are deducted approximate solutions for the general case of load, as well as for two cases. when the plate is orthotropic and is deformed. Orig. art. has: 6 figures, 4 tables, and 5 Slavic references.