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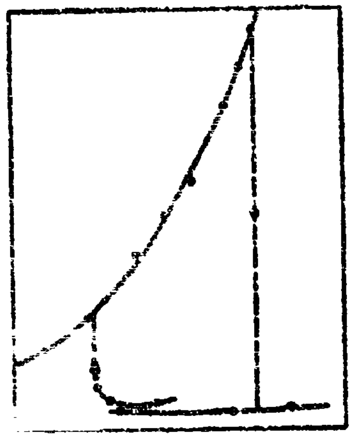

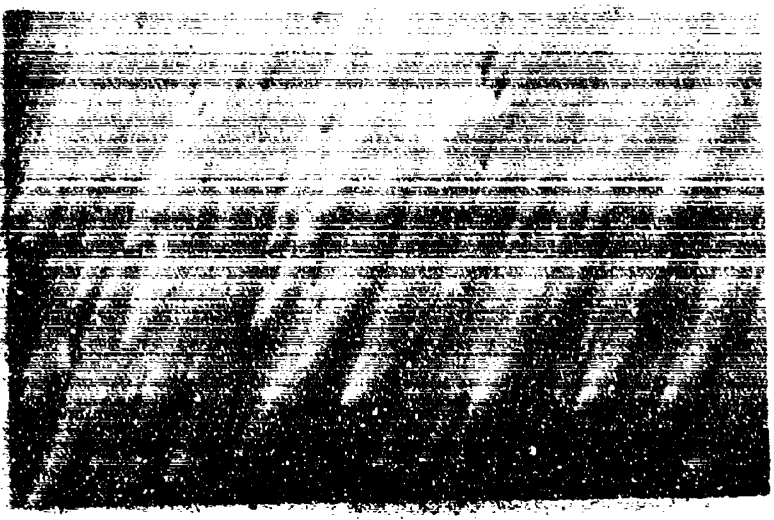

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On Wave Propagation in one Dimensional Rubberlike Materials

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ON WAVE PROPAGATION IN ONE DIMENSIONAL RUBBERLIKE MATERIALS

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1. Introduction

The evolution of discontinuities in solutions of nonlinear hyperbolic equations possessing smooth initial data was first examined in a simple problem by Riemann [1]. His conjecture concerned the conditions for a simple wave to develop a discontinuity. Ludford [2] re-examined this conjecture in the context of the initial value problem for unsteady isentropic flow of a perfect gas. This unfolding process, in the hodograph plane, provides an asymptotic estimate of the time to breakdown of the solution. Zabusky [3] employed the unfolding method to determine an estimate of the time to breakdown for the transverse oscillations of a nonlinear model string. The work of Lax [4] and Jeffrey [5] also employs the Riemann invariants to develop comparison theorems which provide upper and lower bounds for the critical time of singularity occurrence. An alternate method, characterized by its simplicity has been introduced by Ames [6]. Large classes of quasilinear equations can be obtained by differentiation of first order equations. The general solutions of these are also solutions of the corresponding second order equations. These solutions display a critical time of singularity occurrence.

This critical time analysis is applied herein to nonlinear wave equations which result from rubber-like materials characterized by Mooney-Rivlin and Neo-Hookean bodies. The times to discontinuity evolution from smooth initial data are computed and used to ascertain the region of validity of the generalized Lagrange series solution.

2. Fundamental Equations

The present investigation pertains to straight bars or wires with negligible transverse dimensions and possessing a uniform cross section of finite area. In agreement with Nowinski [7] we make the following additional assumptions:

(i) Transverse inertia during the bar motion is neglected.

(ii) In compression and tension zones the bar does not experience material instability.

(iii) The material is perfectly elastic and incompressible.

(iv) The bar is subjected to simple uni-directional strain in the sense that the only identically nonvanishing stress component is the longitudinal normal stress component which is uniformly distributed over the cross section.

(v) The effect of strain-rate on the constitutive equations is neglected and the static stress-strain relations are extended to the dynamic case.

(vi) The bar is infinitely long so that no reflections of waves occur and other possible wave interferences are discarded.

Adopting the Lagrangian formulation let both the material coordinate X and spatial coordinate x be referred to the same fixed Cartesian system, one of whose axes coincides with the axis of the bar. Let ρ_0 and ρ be the mass densities in the stress free configuration (associated with the X coordinate) and deformed configuration (associated with $x = x\ (X,t)$). If t is time, σ_0 the normal stress referred to the undeformed cross section of the rod, and u the particle displacement, then Cauchy's law of motion becomes (James and Guth [8])

$$\frac{\partial \sigma_0}{\partial X} = \rho_0 \frac{\partial^2 u}{\partial t^2} \tag{1}$$

Since $x = X + u$ the stretch (extension ratio)

$\lambda = \frac{\partial x}{\partial X}$ can be written as

$$\lambda = 1 + \frac{\partial u}{\partial X} . \tag{2}$$

Consequently, Eq. (1) becomes

$$\frac{\partial^2 x}{\partial t^2} = C^2 \frac{\partial^2 x}{\partial X^2} \tag{3}$$

or in terms of the stretch,

$$\frac{\partial^2 \lambda}{\partial t^2} = \frac{\partial}{\partial X} \left\{ C^2 \frac{\partial \lambda}{\partial X} \right\} \tag{4}$$

where

$$C^2 = \frac{1}{\rho_0} \frac{d\sigma_0}{d\lambda}, \quad \sigma_0 = \sigma_0(\lambda)$$

$$\text{(we assume } d\sigma_0/d\lambda > 0 \text{)} \tag{5}$$

Equations (3) and (4) are special cases of the general forms treated by Ames [6].

Under the assumption that the strain energy exists the theory of finite elastic deformations (Truesdell [8], Eq. 42.11) furnishes the stress-stretch relation

$$\sigma_0 = 2\left[\frac{\partial W}{\partial I} + \frac{1}{\lambda}\frac{\partial W}{\partial II}\right]\left(\lambda - \frac{1}{\lambda^2}\right) \tag{6}$$

for an incompressible body in simple extension. Here W is the elastic strain energy function and for an incompressible material the strain invariants are

$$I = 2\lambda^{-1} + \lambda^2, \quad II = 2\lambda + \lambda^{-2}, \quad III = 1. \tag{7}$$

From the experimental results of Rivlin and Saunders (see Truesdell [8], p.(214) experimental data is well approximated by

$$W = \alpha(I-3) + f(II-3) \tag{8}$$

where α is a constant and f is an arbitrary function to be obtained. The expanded form

$$W = \alpha(I-3) + \sum_{k=1}^{m} \beta_k (II-3)^k, \tag{9}$$

with β_k constants, has been employed. Retention of only the linear term leads to (10)

$$W = \alpha(I-3) + \beta(II-3), \quad (\alpha, \beta > 0)$$

corresponding to the Mooney-Rivlin material. If $\beta=0$ we obtain Rivlin's Neo-Hookean material.

If W takes the general form, Eq. (8), then Eq. (4) becomes

$$\frac{\partial^2\lambda}{\partial t^2} = \frac{\partial}{\partial X}\left\{\frac{2}{\rho_0}\left[\alpha(1+2\lambda^{-3}) + 3\lambda^{-4}\frac{\partial f}{\partial II} + 2(1-\lambda^{-3})^2\frac{\partial^2 f}{\partial II^2}\right]\frac{\partial\lambda}{\partial X}\right\} \tag{11}$$

Alternately, we may write

$$\frac{\partial^2 x}{\partial t^2} =$$

$$\frac{2}{\rho_o}\left\{\alpha\left[1+2\left(\frac{\partial x}{\partial X}\right)^{-3}\right] + 3\left(\frac{\partial x}{\partial X}\right)^{-4}\frac{\partial f}{\partial II} + 2\left[1-\left(\frac{\partial x}{\partial X}\right)^{-3}\right]^2\frac{\partial^2 f}{\partial II^2}\right\}\frac{\partial^2 x}{\partial X^2} \tag{12}$$

For the Mooney-Rivlin material Eq. (12) becomes

$$\frac{\partial^2 x}{\partial t^2} = \frac{2}{\rho_o}\left\{\alpha\left[1+2\left(\frac{\partial x}{\partial X}\right)^{-3}\right] + 3\beta\left(\frac{\partial x}{\partial X}\right)^{-4}\right\}\frac{\partial^2 x}{\partial X^2} \tag{13}$$

And for the Neo-Hookean material, Eq. (12) becomes

$$\frac{E}{3\rho_o}\left\{1 + 2\left(\frac{\partial x}{\partial X}\right)^{-3}\right\}\frac{\partial^2 x}{\partial X^2} = \frac{\partial^2 x}{\partial t^2} \tag{14}$$

In Eq. (14) 2α has been replaced by $E/3$ a value which is suggested by the desireability of obtaining the familiar infinitesimal strain relation $\sigma_0 = E\epsilon$ from Eq. (6).

From his own theory of finite elasticity Seth [9] obtains the corresponding equation

$$\frac{E}{\rho_o}\left\{1 + \frac{\partial x}{\partial X}\right\}^{-3}\frac{\partial^2 x}{\partial X^2} = \frac{\partial^2 x}{\partial t^2} \tag{15}$$

which differs fundamentally from Eq. (14).

3. Reduction to First Order Equations

Equation (11) and its specializations are of the general form given by Eq. (4) while Eqs. (12-15) are of the general form specified by Eq. (3). As previously observed both are special cases of the general quasilinear equation

$$u_{ss} - (F_p/F_q)^2 u_{rr} + (F_u/F_q^2)(F_q u_s - F_p u_r) + (1/F_q^2)(F_q F_s - F_p F_r) = 0 \tag{16}$$

treated by Ames [6]. In that work it is shown that Eq. (16) results if one calculates r and s derivatives of the general first order equation

$$F(r, s, u, p, q) = 0, \quad p = u_r, \quad q = u_s \tag{17}$$

and eliminates the cross partial derivative term u_{rs}. Equations of the form

$$u_{ss} - \{\phi^2(u)\, u_r\}_r = 0 \tag{18}$$

are obtained if $F = F(u, p, q)$, and $(F_p/F_q)^2 = \phi^2(u)$. In that case the two equations for F become $F_p \pm \phi(u) F_q = 0$. Clearly $F = q \mp \phi(u) p$ are solutions for F. Upon integrating F=0, we obtain the general solutions

$$H[u, r + \phi(u) s] = 0 \tag{19}$$

and

$$G[u, r - \phi(u) s] = 0 \tag{20}$$

respectively, where H and G are arbitrary.

Alternatively, equations of the form

$$u_{ss} - \phi^2(u_r)\, u_{rr} = 0 \tag{21}$$

are obtained if $F = F(p, q)$ and $(F_p/F_q)^2 = \phi^2(p)$ (the assumption $(F_p/F_q)^2 = \phi^2(q)$ generates a similar form with $\phi^2(u_s)$ appearing). Consequently, the equations for F become $F_p \pm \phi(p) F_q = 0$. Solutions for F are $F = q \mp g(p)$, $g' = \phi$ and upon integrating F=0 we obtain

$$H[u_r, r + s\phi(u_r)] = 0 \tag{22}$$

and

$$G[u_r, r - s\phi(u_r)] = 0. \tag{23}$$

4. Calculation of Breakdown "Time"

The finite "time" to the evolution of a discontinuity in u_r or higher order derivatives can be calculated from Eqs. (19), (20), (22) and (23). For example, from the total r derivative of Eq. (20) we obtain

$$u_r = -\frac{G_r + G_\omega}{G_u - s G_\omega \phi'(u)}, \quad \omega = r - \phi(u) s \tag{24}$$

which becomes arbitrarily large when

$$s = G_u / G_\omega \phi'(u). \tag{25}$$

Since the initial "time" of this occurrence is usually of most interest we write

$$s_c = \min \frac{G_u}{G_\omega \phi'(u)} \tag{26}$$

where the minimum is estimated over the appropriate range of the quantities in Eq. (26). (For real problems we are interested in positive values s_c). On occasion the general solutions are employable in simpler forms. For example, in some problems the general solution may be used as

$$u = h[r + \phi(u) s] \tag{27}$$

instead of Eq. (19). In this case the r derivative becomes

$$u_r = \frac{h'(\omega)}{1 - h'(\omega)\phi'(u) s}, \quad \omega = r + \phi(u) s \tag{28}$$

which is unbounded when

$$s = 1/h'(\omega)\phi'(u)$$

and the critical (minimum) time is

$$s_c = \min \frac{1}{h'(\omega)\phi'(u)} \tag{29}$$

Similar calculations for the equation $u_{ss} - \phi^2(u_r)\,u_{rr} = 0$ can be carried out from Eqs. (22) and (23). A discontinuity in the first derivative can be discovered by inquiring when the second derivative becomes unbounded. From Eq. (23) we find

$$u_{rr} = -\frac{G_r + G_\omega}{G_{u_r} - s\,G_\omega\,\phi'(u_r)}, \quad \omega = r - \phi(u)s \tag{30}$$

and, consequently,

$$s_c = \min \frac{G_{u_r}}{G_\omega\,\phi'(u_r)}. \tag{31}$$

5. Lagrange Series Solution

While useful for the determination of breakdown times the implicit nature of the general solutions (Eqs. (19), (20), (22) and (23)) inhibits their use in the determination of the solutions. Alternately, we can obtain a series solution to the equations F=0 which are

$$u_s \pm \phi(u)\,u_r = 0 \tag{32}$$

in the first case discussed and

$$u_s \pm g(u_r) = 0 \tag{33}$$

in the second case. Lagrange expansions are discussed in various contexts in Goursat [10], Bellman [11], Banta [12] (for finite amplitude sound waves) and Ames and Jones [13] (for a Monge-Ampere equation).

A Lagrange series is now constructed for $u_s + \phi(u)\,u_r = 0$ as a typical example of the methodology. Suppose u has a Taylor's series expansion about s=0,

$$u(r,s) = \Gamma_0(r) + \sum_{m=1}^{\infty} \Gamma_m(r)\frac{s^m}{m!} \tag{34}$$

$$\Gamma_m(r) = \left.\frac{\partial^m u(r,s)}{\partial s^m}\right|_{s=0}$$

This form is inconvenient since the derivatives are with respect to s and not r. Replacement of the s derivatives is carried out by using the differential equation and an inductive scheme due to Goursat [10, p.405] (see also Banta [12]). If $u_s + \phi(u)\, u_r = 0$

then

$$\frac{\partial^m u}{\partial s^m} = (-1)^m \frac{\partial^{m-1}}{\partial r^{m-1}} \left[\phi^m \frac{\partial u}{\partial r} \right] . \tag{35}$$

Consequently, from Eqs. (34) and (35), for $m \geq 1$

$$\Gamma_m(r) = (-1)^m \frac{\partial^{m-1}}{\partial r^{m-1}} \left\{ \phi^m \frac{\partial u}{\partial r} \right\} \Bigg|_{s=0} \tag{36}$$

a form which contains only r derivatives of u. If the process of r differentiation and evaluation at s=0 are interchangeable the series takes the form

$$u(r,s) = f(r) + \sum_{m=1}^{\infty} \frac{s^m}{m!} \frac{d^{m-1}}{dr^{m-1}} \left\{ \left[\phi(f(r)) \right]^m \frac{df}{dr} \right\} \tag{37}$$

where $u(r,0) = f(r)$, the "initial" condition. This series is valid out to the first singularity - that is to the smallest breakdown value s_c.

To integrate $u_s + g(u_r) = 0$ from Eq. (33) we note that it becomes

$$u_{sr} + g'(u_r)\, u_{rr} = 0 \tag{38}$$

upon differentiation with respect to r. With $v = u_r$ Eq. (38) becomes

$$v_s + \phi(v)\, v_r = 0 \tag{39}$$

an equation taking the same form previously analyzed. Upon solving for v, u is recovered by integrating with respect to r.

The Lagrange series, Eq. (37), or its integrated form describes the waveform in its transition from the smooth initial form f(r) to the onset of breakdown.

6. Application to Rubberlike Materials

Results of the preceding analyses are applicable to a wide variety of problems. Herein we will investigate Eq. (4) as it applies to Mooney-Rivlin materials. For that application Eq. (4) becomes

$$\frac{\partial^2\lambda}{\partial t^2} = \frac{\partial}{\partial X}\left\{\frac{2}{\rho_0}\left[\alpha\left(1+2\lambda^{-3}\right) + 3\beta\lambda^{-4}\right]\frac{\partial\lambda}{\partial X}\right\} \tag{40}$$

From Section 3 the first order equations generating Eq. (40) are obtained by setting $u = \lambda$, $s = t$, $r = X$ and $\phi = \left\{\frac{2}{\rho_0}\left[\alpha\left(1+2\lambda^{-3}\right) + 3\beta\lambda^{-4}\right]\right\}^{1/2}$, whereupon

$$\lambda_t \pm \phi(\lambda)\lambda_X = 0. \tag{41}$$

If the initial stretch is provided by $\lambda(X,0) = f(X)$ then the general solution of Eqs. (41) become (see Eq. (27))

$$\lambda = f\left[X \mp \phi(\lambda)t\right] \tag{42}$$

If $0 < \lambda_0 < \lambda < \lambda_1$ the breakdown time can be calculated from the positive minimum of

$$t = \pm\,(2\rho_0)^{1/2}\,\frac{\left[\alpha(1+2\lambda^{-3}) + 3\beta\lambda^{-4}\right]^{1/2}}{\left[6\alpha\lambda^{-4} + 12\beta\lambda^{-5}\right]f'(\omega)}\,,\quad \omega = X \mp \phi(\lambda)t \tag{43}$$

If $f'(\omega) \leq A$ and the material is Neo-Hookean $(\beta = 0)$

then (recall $\alpha = E/6$)

$$t_c = \left(\frac{\rho_0}{3E}\right)^{1/2} \frac{\lambda_0^4 (1+2\lambda_0^{-3})^{1/2}}{A}$$

which becomes

$$t_c = \frac{1}{A}\left(\frac{\rho_0}{E}\right)^{1/2}$$

if $\lambda_0 = 1$ -i.e. the material is always in tension.

The Lagrange series becomes, in this case,

$$\lambda(X,t) = f(X) + \sum_{m=1}^{\infty} \frac{t^m}{m!} \frac{d^{m-1}}{dX^{m-1}}\left\{\left[\phi(f(X))\right]^m \frac{df}{dX}\right\}$$

which can be employed to study the onset of the discontinuity. Since $\lambda = \frac{\partial x}{\partial X} = 1 + \frac{\partial u}{\partial X}$ the quantities x and u are recovered by integrations.

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13. ABSTRACT

Two methods for studying the evolution of dicontinuities in solutions of nonlinear hyperbolic equations possessing smooth initial data are extant. A Third method, characterized by its simplicity, results from the observation that large classes of quasilinear equations can be obtained by differentiation of first order nonlinear equations. The first order equations are employed to calculate the time to singularity occurrence. This result is generalized and applied herein to nonlinear wave equations from one-dimensional rubber-like materials. The time to discontinuity evolution from smooth initial data is computed and used to ascertain the region of validity for a generalized Lagrange series solution.

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