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MEMORANDUM
RM-5905-ARPA
SEPTEMBER 1969

IRRADIANCE STATISTICS, MODULATION TRANSFER FUNCTION, AND PHASE STRUCTURE FUNCTION OF AN OPTICAL WAVE IN A TURBULENT MEDIUM

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PREPARED FOR:
ADVANCED RESEARCH PROJECTS AGENCY

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PREFACE

This Memorandum, prepared for the Advanced Research Projects Agency, is part of a study of those phenomena which affect the performance of optical or infrared reconnaissance and guidance equipment. The objective of these studies is to provide sufficient understanding for the system analyst to compute performance estimates under various operational conditions.

A quantitative understanding of the effect of atmospheric turbulence on a beam of light of finite cross section is required for the prediction of the performance of various devices employing lasers for target acquisition or guidance in tactical missions. Such applications are characterized by near-horizontal propagation paths near the ground of the order of one to tens of kilometers in length. This Memorandum discusses the evidence in support of a log-normal distribution for the fluctuating irradiance of light after propagation through a turbulent atmosphere, calculates the modulation transfer function and phase structure function for the wave, demonstrates reasonable agreement between the computed results and experiments, and points out the significance of these results in determining the signal-to-noise ratio of an optical heterodyne communications system and in seeing through the atmosphere.

These results should be of use to those interested in tactical applications of laser range finders, laser line scanners, and the various guidance systems employing an illuminating beam.

SUMMARY

Tatarski's analysis of the first-order (single scattering) distribution function based on the Rytov approximation is shown to be in error, and the recent analysis of deWolf, which yields a Rayleigh distribution for the fluctuating intensity of light in a turbulent atmosphere in the limit of long optical propagation paths, is disputed. The experimental evidence for a log-normal distribution for the intensity fluctuations is pointed out.

We conclude that none of the theories proposed can account, in a consistent fashion, for the experimentally observed far-field irradiance statistics.

Expressions are derived for the modulation transfer function and the phase structure function. The dependence of these wave properties on the turbulence parameters within the Kolmogorov inertial subrange is found and, in particular, it is demonstrated that certain optical measurements of the phase structure function and the modulation transfer function can be interpreted in terms of an outer scale of turbulence of the order of 10 to 50 cm, values which typically can be expected under inversion conditions. For sufficiently short paths, it is pointed out that the theoretically achievable resolution through the atmosphere and the signal-to-noise ratio of an optical heterodyne detection system can be larger than previous calculations have implied.

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I. INTRODUCTION

The properties of an optical wave which has propagated along a path through the atmosphere are of considerable interest in communications, laser radar systems, and general applications where a knowledge of the degradation of resolution affects the performance of a system.

The probability distribution for the intensity fluctuations at a fixed point in the turbulent medium is discussed in the next section. No new theoretical results are established; however, the analysis of Tatarski is shown to be in error, the results of deWolf, yielding a Rayleigh intensity distribution, are also disputed, and the experimental evidence in support of a log-normal distribution is pointed out.

In the third section we consider wave characteristics related to the correlation of amplitude and phase over distances transverse to the direction of propagation. Expressions are derived for the wave phase structure function, defined as the mean of the squared phase difference as a function of transverse direction, and the modulation transfer function (MTF), which is the autocorrelation function of the complex field in the transverse direction. This latter quantity is important in determining the limiting resolution obtainable in forming an image through a turbulent medium, and enters into calculations of the signal-to-noise ratio of an optical heterodyne system. The dependence of these wave properties on the turbulence parameters within the Kolmogorov inertial subrange is obtained and, in particular, it is demonstrated that certain optical measurements of the phase structure function and the MTF can be interpreted in terms of an outer scale of turbulence of the order of 10 to 50 cm, which can be expected under inversion conditions. This result of the analysis suggests that, for sufficiently short paths, the theoretically achievable resolution and signal-to-noise ratios can be larger than previous calculations implied.

II. IRRADIANCE STATISTICS

The probability distribution for the intensity fluctuations of an optical wave in a turbulent medium has received considerable attention in the recent literature.[1-5] The two theoretical treatments of the problem of determining the distribution function which have received the most attention are those due to Tatarski and deWolf. Tatarski has claimed the distribution function in I for plane waves in the far field of the scattering eddies (where geometric optics is no longer applicable), but still in the single scattering zone, to be log-normal (i.e., $\ell n I$ is normally distributed). deWolf, on the other hand, has claimed the distribution function to be a Rice distribution, in general, which approaches a Rayleigh distribution in the limit of many scatterings. The experimental evidence, under conditions where the wind speed is sufficiently high that the turbulence is fully developed, does seem to bear out the log-normal relationship in both the single and multiple scattering regimes.

In this section we would like to take issue with both of the theoretical treatments mentioned above, notwithstanding the fact that Tatarski's calculations appear to predict the same form of the distribution as indicated by experiments.

Tatarski's solution is based on assuming a solution to the scalar wave equation

$$\left[\nabla^2 + k^2(1 + n_1)^2\right] U = 0 \tag{1}$$

of the form $U = e^{\psi}$, where k is the wave number of the unperturbed wave, n_1 ($|n_1| << 1$) is the fluctuating part of the index of refraction, and U represents a component of the electric or magnetic field. The function ψ is then expanded as $\Sigma_{i=0}^{\infty}\psi_i$, where the i^{th} term ψ_i is proportional to the i^{th} power of n_1. The function ψ_1, the so-called first-order Rytov solution, can be recognized as U_1/U_0, where U_i is the i^{th} Born solution to Eq. (1) obtained by expanding $U = \sum_{i=0}^{\infty} U_i$. For plane waves, $U_0 = e^{ikz}$, and

$$U_1(\underline{r}) = \frac{k^2}{2\pi} \int n_1(\underline{r}') \; e^{ikz'} \frac{e^{ik|\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|} \; d^3r' \qquad (2)$$

Applying the central limit theorem to Eq. (2), Tatarski concluded that U_1, and hence ψ_1, was normal. The intensity

$$I = UU^* = |U_0|^2 \left[1 + 2\mathrm{Re}(U_0 U_1^*)\right] \qquad (3a)$$

$$= |U_0|^2 \left[e^{2\mathrm{Re}\psi_1}\right] \qquad (3b)$$

will be normal if Eq. (3a) is used, and log-normal if Eq. (3b) is considered. Obviously, at least one of these expressions yields an incorrect distribution to order n_1. Yura[(5)] has recently shown that it is necessary to retain the second-order Rytov term in order to compute the standard deviation of the intensity fluctuations correct to first order in n_1. It follows that the irradiance statistics and the distribution function can only be determined to first order in n_1 in the Rytov formalism if the dependence of I on ψ_2 is obtained.

In order to make the argument more general, let $I = f(x)$, where $f(x)$ depends on the form of the substitution used to solve the wave equation (e.g., in the Rytov formalism, $f(x) = e^{\mathrm{Re}\,x}$) and expand $x = \Sigma_{i=0}^{\infty} x_i$, where $x_i \sim n_1^i$. Then, correct to second order in n_1,

$$f(x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0} (x_1 + x_2) + \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} x_1^2 \tag{4a}$$

and

$$f^2(x) = f^2(x_0) + 2f(x_0) \left.\frac{\partial f}{\partial x}\right|_{x_0} (x_1 + x_2)$$

$$+ \left(\left.\frac{\partial f}{\partial x}\right|_{x_0}\right)^2 x_1^2 + f(x_0) \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} x_1^2 \tag{4b}$$

Conservation of energy requires $\overline{f(x)} = I_0 = f(x_0)$ (to all orders in n_1), and utilizing $\overline{x_1} = 0$ yields

$$\left.\frac{\partial f}{\partial x}\right|_{x_0} \overline{x_2} = - \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} \overline{x_1^2} \tag{5}$$

Substituting from Eq. (5) yields for the standard deviation

$$\sigma_I = \left\{ \overline{[f(x) - \overline{f(x)}]^2} \right\}^{\frac{1}{2}} = \left(\left.\frac{\partial f}{\partial x}\right|_{x_0}\right)\left(\overline{x_1^2} \right)^{\frac{1}{2}} \tag{6}$$

One can then inquire if there is any consistent first-order function $f(x_0 + x_1)$, where $\overline{x_1} = 0$, which will satisfy energy conservation and yield the correct standard deviation to order n_1, Eq. (6). Obviously, energy can be conserved only if $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} = 0$.

Substituting in Eqs. (4a) and (4b) then yields Eq. (6) for the dispersion. Hence, the only physically consistent first-order function which yields the correct standard deviation to order n_1 is a linear one, corresponding to the Born approximation. Therefore, Tatarski's derivation of a log-normal distribution for propagation paths dominated by single scattering is invalid. The correct first-order statistics are given only by the first-order Born approximation, which implies a normally distributed intensity distribution. This result should be valid for propagation paths small compared to the single-scattering length, L_c,[5] but, as will be pointed out in further detail, the experimental evidence supports a log-normal distribution.

A calculation supposedly valid for all propagation distances has been made by deWolf.[1] Starting with the high-frequency approximation to the Fourier-transformed wave equation, and assuming the refractive index fluctuations to be a gaussian random process, deWolf purports to have calculated the distribution function by computing the intensity moments of all orders. With the gaussian assumption, the 2m-point correlation breaks down into a sum of products of two-point correlations. J. P. Laussade* claims that deWolf arbitrarily discards all but one of the products in the sum. In Laussade's formulation, the integrals for the higher moments could not be represented in terms of the lower order moments, and no formula corresponding to deWolf's Eq. (22) could be found.

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*Laussade, J.P., "Theoretical Study of Optical Wave Propagation Through a Turbulent Medium and its Applications to Optical Communication," Doctoral Thesis, California Institute of Technology, Pasadena, California, November 1968.

For strong turbulence, or long paths, deWolf's moments tend toward those of a Rayleigh distribution. deWolf has considered the function $F_o^{-1}[F(\ell nI)]$ where F_o is the cumulative distribution for a normalized gaussian random variable and F is the same function for an arbitrary random variable. Plotted versus log I, this function will be a straight line if, and only if, I is log-normally distributed. deWolf then "manufactures" data which are supposedly Rayleigh distributed in I, and claims the above function cannot be distinguished from a straight line over the four-octave range of data of Gracheva.(2) We do not know how he manufactured his data, but he could have plotted

$$\mathrm{cerf}^{-1}\left\{1 - \exp\left[-\frac{e^{2v}}{2\sigma_R^2}\right]\right\} \text{ versus } v\ (= \ell nI)$$

which is the exact expression for $F_o^{-1}[F(v)]$, and observed the range over which the curve appeared to be linear.

Rather than follow this procedure, we plotted the cumulative distributions for I distributed according to a log-normal, Rayleigh, and exponential distribution (the latter corresponding to a Rayleigh amplitude distribution) in Fig. 1. The curves were plotted on the same axis over a two-decade range for which Fried, *et al.*,(4) have taken data. There is only one parameter in the Rayleigh and exponential distributions, and in each case it was chosen such that $\bar{I}$, the mean intensity, corresponded to the mean in Fried's data.

Energy conservation requires $\bar{I}$ to equal the intensity in the absence of turbulence. The difference between these distributions

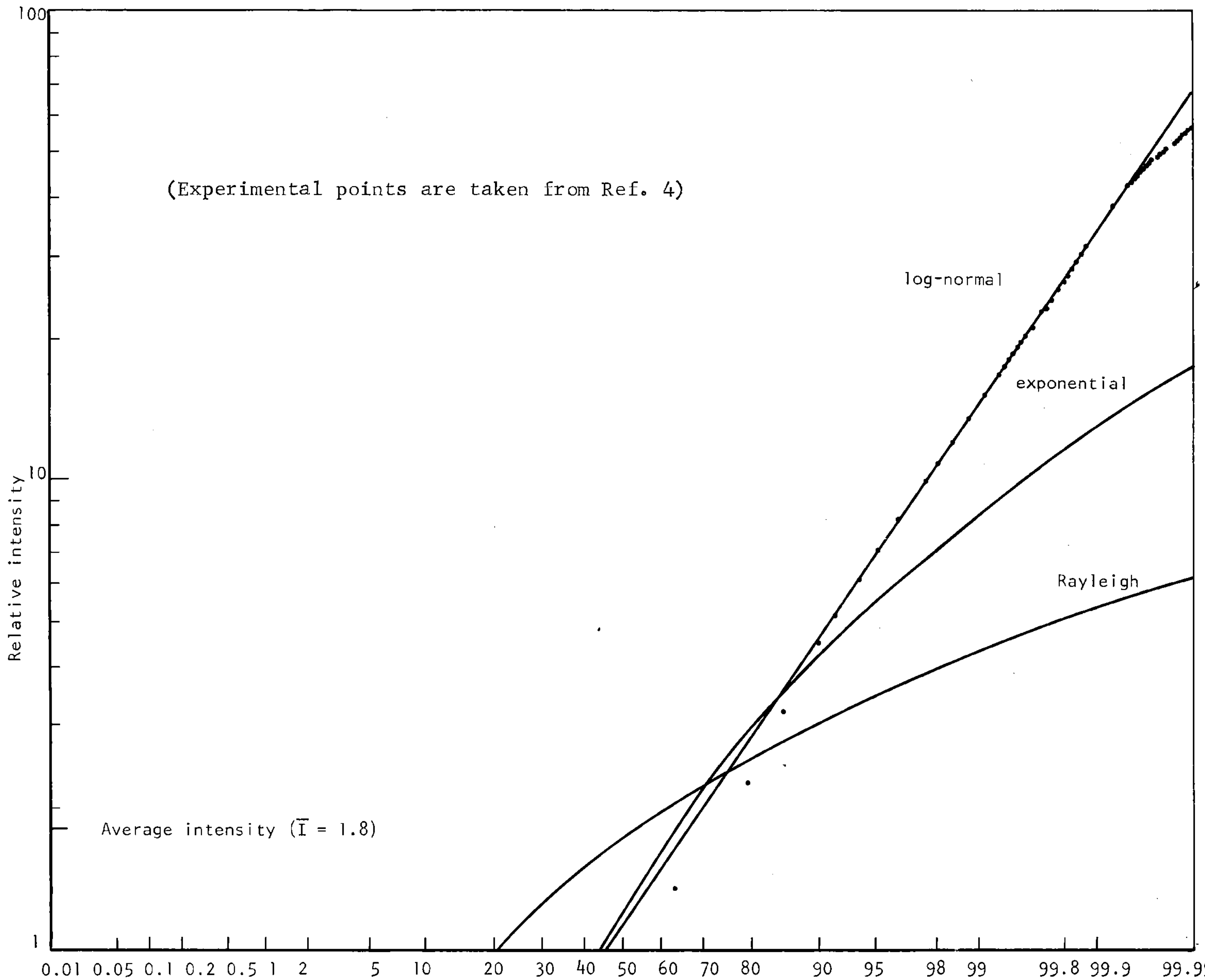

Fig. 1--Cumulative log-normal, exponential, and Rayleigh distributions with equal means

for intensities greater than two standard deviations from the mean is obvious. A quantitative measure of these differences could be obtained from a chi-squared test for each of these distributions, but this hardly seems necessary, as in this region we are essentially comparing

$$e^{-I^2} \text{ with } e^{-(\ell n I)^2}$$

a readily observable difference.

In conclusion, we feel that at present the only consistent theoretical treatment of irradiance statistics in the far field is obtained from the Born approximation, which yields a normal distribution. It should be pointed out that geometric optics, valid in the near field, does yield a log-normal intensity distribution. However, in addition to the experiments reported here, experiments at Boulder* over propagation paths of up to 145 km bear out the log-normal distribution. Hence, a modified geometric optics approach might lead to a better understanding of this phenomenon.

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*Private communication from Dr. R. S. Lawrence, Environmental Science Services Administration, Boulder, Colo.

III. WAVE STRUCTURE AND MODULATION TRANSFER FUNCTIONS

With regard to our interpretation of the measurements of D_φ and the MTF, we note the following formulas: First, the complex field at the point $\underline{r}_i$ can be written, correct through terms of second order, in the fluctuations of refractive index, n_1 [5]

$$U(\underline{r}_i) = U_o \exp\left[\psi_1(\underline{r}_i) + \varphi_2(\underline{r}_i) - \frac{1}{2}\psi_1^2(\underline{r}_i)\right] \tag{7}$$

where U_o is the field in the absence of turbulence, ψ_1 is the (first-order) Rytov approximation ($= \frac{U_1}{U_o}$ where U_1 is the first-order Born approximation), and $\varphi_2 = \frac{U_2}{U_o}$, where U_2 is the second-order Born approximation. Noting that conservation of the average energy through second order in n_1 implies $\overline{\mathrm{Re}\ \varphi_2} + \frac{1}{2}\overline{|\psi_1|^2} = 0$, we obtain for the MTF, to second order in n_1,

$$M = \overline{U(\underline{r}_1)\ U(\underline{r}_2)^*} = \exp\left\{-\frac{1}{2}\overline{|\psi_1(\underline{r}_1) - \psi_1(\underline{r}_2)|^2}\right\} \tag{8}^*$$

where, for convenience, we have taken $|U_o|^2 = 1$, and the asterisk denotes the complex conjugate of the appropriate quantity. Writing $\psi_1(\underline{r}_i) = \ell(\underline{r}_i) + i\varphi(\underline{r}_i)$, where ℓ is the log-amplitude and φ is the phase of the wave, we obtain

$$M = \exp\left\{-\frac{1}{2}(D_\ell + D_\varphi)\right\} \tag{9}$$

where D_ℓ and D_φ are the log-amplitude and phase structure functions,

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*A bar over a quantity indicates the ensemble average of that quantity.

respectively. In the case of isotropic homogeneous turbulence, Eq. (9) can be written as

$$M = \exp\left\{- \left[C(0) - C(\rho)\right]\right\} \tag{10}$$

where $C(\rho)$ is the sum of the log-amplitude and phase covariance functions, and $\rho = |\underline{r}_1 - \underline{r}_2|$.

Tatarski[(3)] calculates the phase and log-amplitude structure functions in a plane perpendicular to the direction of propagation of a plane wave, with wave number k which has propagated a distance L, to be

$$D_{\substack{\varphi \\ \ell}}(\rho) = 4\pi^2 k^2 L \int_0^\infty \left[1 - J_o(K\rho)\right]\left[1 \pm \frac{k}{K^2 L} \sin\left(\frac{K^2 L}{k}\right)\right] \Phi_n(K)\ KdK \tag{11}$$

where J_o is the Bessel function of zero order, and $\Phi_n(K)$ is the three-dimensional spectral density of refractive index. Substituting Eq. (11) into Eq. (9) yields the MTF for plane waves:

$$M(\rho) = \exp\left\{- 4\pi^2 k^2 L \int_0^\infty \left[1 - J_o(K\rho)\right] \Phi_n(K)\ KdK\right\} \tag{12}$$

Equation (12) has been shown to be correct to all orders in n_1 for a gaussian random process,* and has been demonstrated valid through second order in n_1 for arbitrary $\Phi_n(K)$. For spherical waves, $J_o(K\rho)$ in Eq. (12) is replaced by $\int_0^1 J_o(K\rho x)dx$.

The Kolmogorov theory yields for the refractive index structure function:

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*Laussade, *op. cit.*

$$D_n(r) = \begin{cases} C_n^2 \, r^{2/3} & \text{for } \ell_o \ll r \ll L_o \\ C_n^2 \, \ell_o^{2/3} \left(\frac{r}{\ell_o}\right)^2 & \text{for } r \ll \ell_o \end{cases} \tag{13}$$

where ℓ_o and L_o are the inner and outer scales of turbulence, respectively, and the multiplicative coefficients were chosen to make $D_n(r)$ continuous at $r = \ell_o$. Tatarski demonstrates the insensitivity of D_ℓ and D_φ to the behavior of $\Phi_n(K)$ for $K \gtrsim 1/L_o$ when $\sqrt{\lambda L} \ll L_o$. Hence the refractive index spectral density (assuming isotropic turbulence)

$$\Phi_n(K) = \frac{0.033 C_n^2 \, e^{-(K\ell_o/5.92)^2}}{\left(K^2 + Ł_o^{-2}\right)^{11/6}} \tag{14}$$

extrapolated from Tatarski, which implies a flat spectrum for $K < 1/Ł_o$ ($Ł_o = L_o/2\pi$), should give an adequate quantitative dependence of D_φ and M on the turbulence parameters.

In Figs. 2 and 3 we have reproduced some experimental curves[6] of D_φ for propagation paths of 0.5 and 3.5 km, respectively. On the same axis we have plotted D_φ, obtained by numerically integrating Eq. (11) for plane waves (which correspond to the experimental situation in Ref. 6). The results are very insensitive to ℓ_o, and a nominal value of 0.1 cm was assumed. The phase structure function will scale as C_n^2, and $Ł_o$ will be roughly determined by the transverse distance corresponding to the inflection point in D_φ. Comparison shows that reasonable agreement can be obtained with $C_n^2 \sim (5 \times 10^{-16})$ cm$^{-2/3}$ and

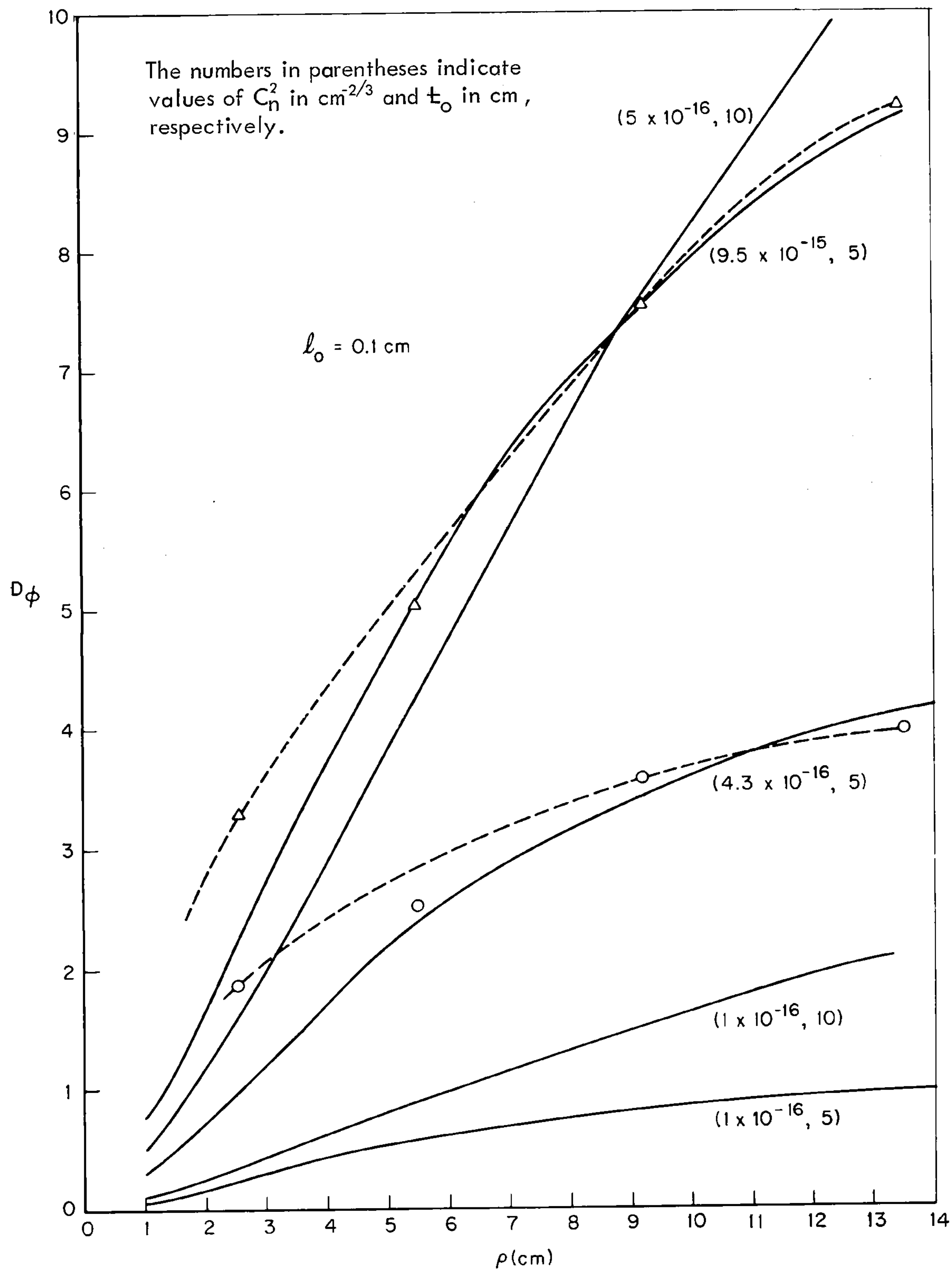

Fig. 2--Comparison of computed phase structure function with data of Bertolotti, et al., at a range of 0.5 km

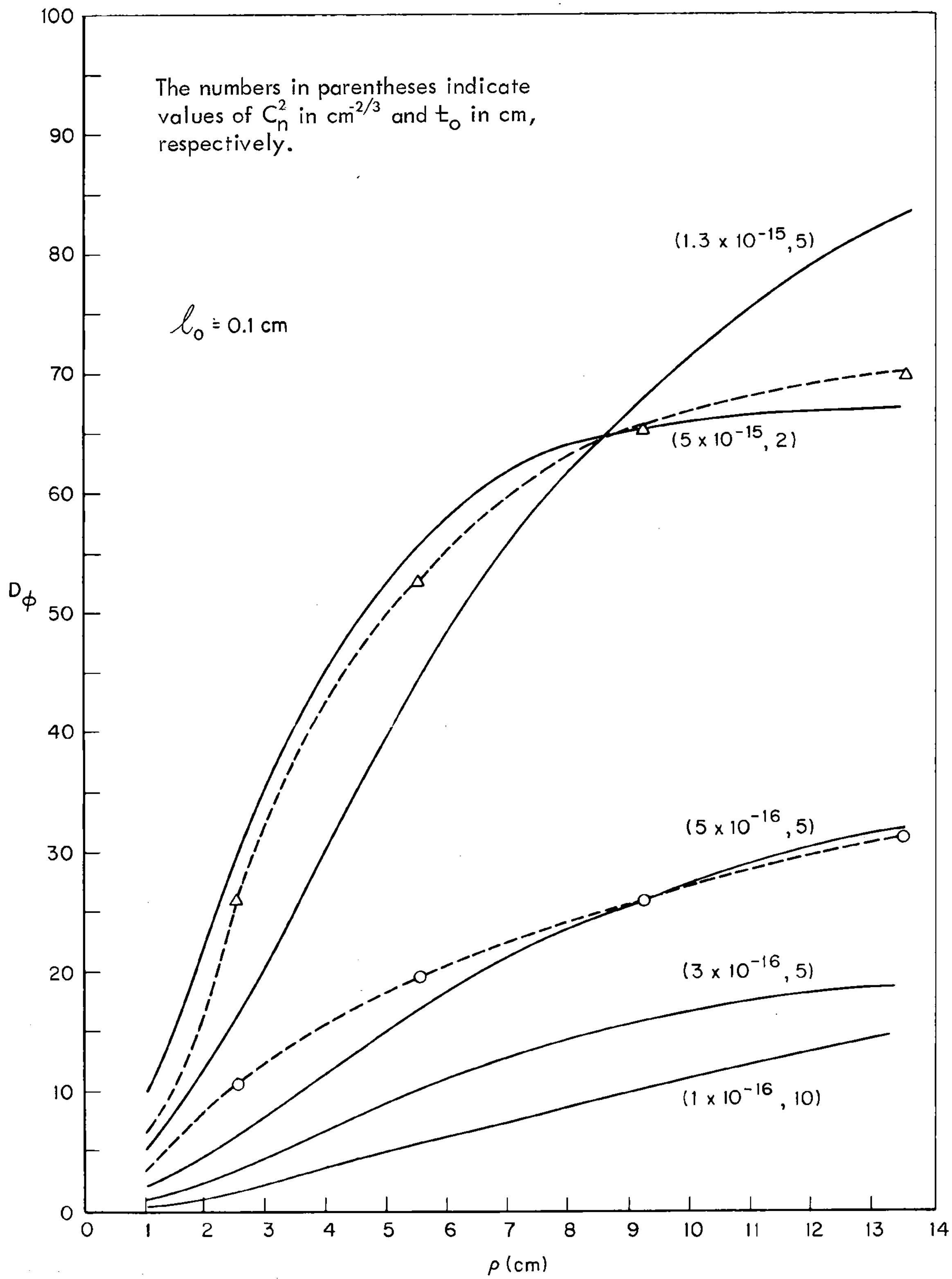

Fig. 3--Comparison of computed phase structure function with data of Bertolotti, et al., at a range of 3.5 km

$L_o \sim 12\text{-}30$ cm. Larger values of L_o would not yield a leveling off of D_φ within ~ 10 cm to the extent obtained in the experiment. If the measurements had been made in the daytime rather than after sunset, larger L_o's and a correspondingly greater range of validity for the "5/3 law" would be expected.

In Fig. 4, Djurle and Bäck's[7] experimental curves of the MTF for a path length of 11 km have been reproduced. With $\ell_o = 0.1$ cm, the curves can be fairly well represented by using spherical waves and choosing $C_n^2 \sim 10^{-16}$ cm$^{-2/3}$ and $L_o \sim 25\text{-}50$ cm. The interpretation of the shape of the MTF's implying outer scales of the magnitude suggested depends on the experimental values being correct to within ~ 10 percent. It is noteworthy that no estimates of the experimental errors are given for either D_φ or the MTF. This seems to be characteristic of most optical experiments, which makes theoretical interpretation difficult.

Measurements made under temperature lapse conditions[8] are usually inferred to yield an outer scale of turbulence of the order of a meter (when the path is ~ 1 meter above the ground). However, most optical experiments, including the ones discussed here, are performed after dark when the probability of temperature inversion is high. During the semi-stable conditions occurring during inversion, a reduction in turbulent energy generation takes place. The usual picture presented is the cascading of energy from larger to smaller turbulent eddies, and with a decrease in energy generation a reduction of the outer scale size would occur. This phenomena is supported by the experiments of Deitz[8] and Tsvang.[9] Nighttime

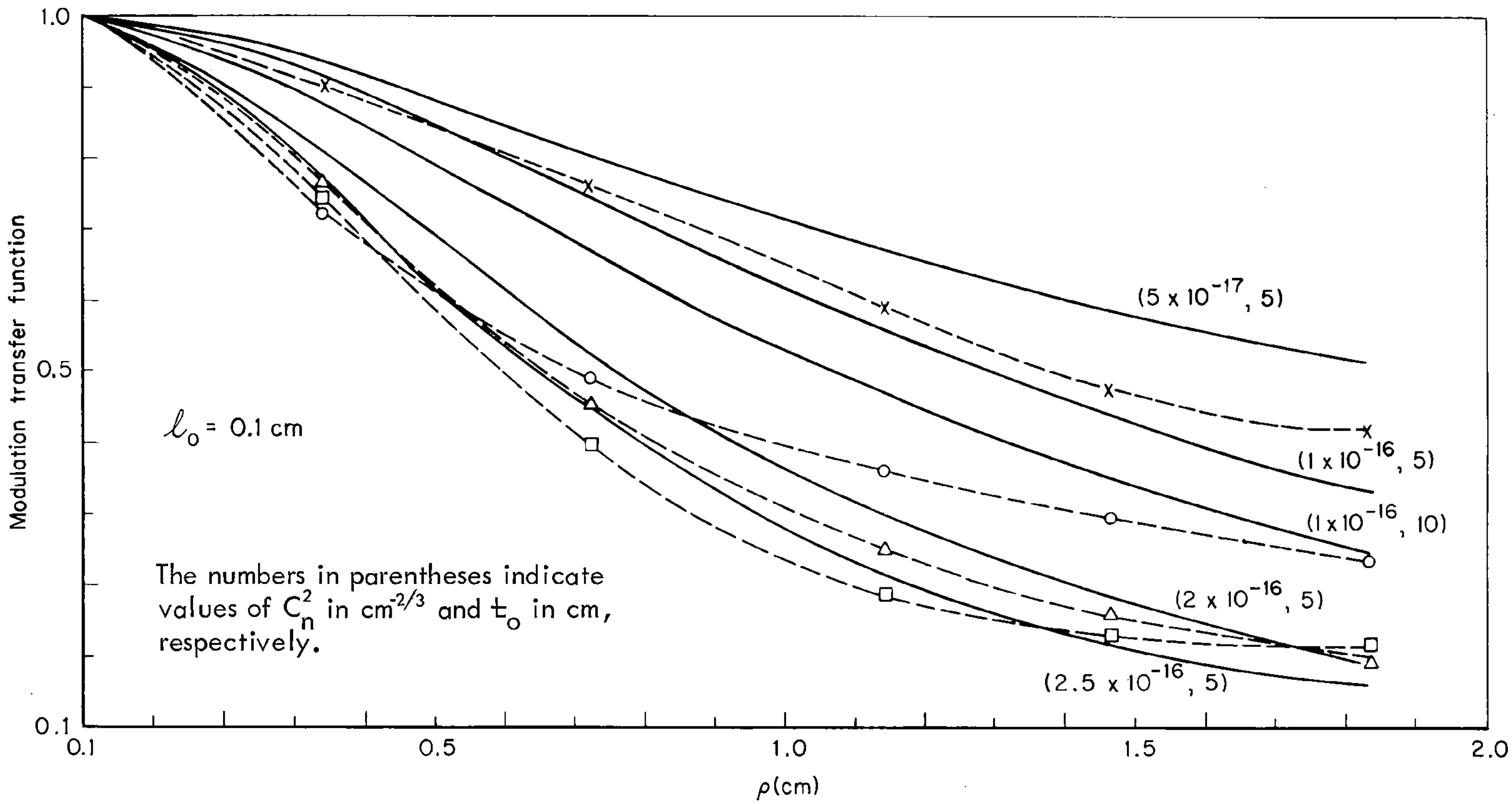

Fig. 4--Comparison of computed modulation transfer function with data of Djurle and Bäck at a range of 11 km

measurements of the temperature structure function between two sensors lead Deitz to infer an outer scale of 30 cm or less. Tsvang's measurements display a shifting of the center of gravity of Φ_n to larger K as the vertical temperature gradient changed from lapse to inversion conditions.

One can argue on physical grounds that a lower limit to the MTF must be obtained when the transverse distance ρ is sufficiently large that the optical wave arriving at the points $\underline{r}_1$ and $\underline{r}_2$ has been scattered through (statistically) independent media. In this case, as $\rho \to \infty$, $M = \overline{U(\underline{r}_1)\ U(\underline{r}_2)^*} \to \overline{U(\underline{r}_1)}\ \overline{U(\underline{r}_2)}^* \simeq e^{-2L/L_c}$.* Hence, if the optical path is sufficiently short, the medium does not limit the resolution which can be obtained.

A decrease in the inertial subrange could thus explain the "saturation" in the phase structure function and the MTF. If the limiting value of the MTF were sufficiently high, optical resolution could be improved by employing larger receiver optics.

A similar consideration is involved in evaluating the performance of an optical heterodyne detector, which involves a double integral of the MTF over the receiving aperture. Based on an MTF which rapidly tends to zero, Fried[10] has predicted a limit to the achievable average signal-to-noise ratio, no matter how large the

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*The quantity L_c is the propagation distance where the average field is down by the factor e^{-1} and is determined from the equation $\overline{\frac{1}{2}\ |\psi_1(L_c)|^2} = 1.$[5]

detection collection aperture. A high saturation value of the MTF suggests that further improvement of the performance with larger apertures might still be a reasonable consideration for sufficiently short paths.

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