AVERAGE COST SEMI-MARKOV DECISION PROCESSES

by

SHELDON M. ROSS

OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING
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Sheldon M. Ross
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

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The Semi-Markov Decision model is considered under the criterion of long-run average cost. A new criterion, which for any policy considers the limit of the expected cost incurred during the first \( n \) transitions divided by the expected length of the first \( n \) transitions, is considered. Conditions guaranteeing that an optimal stationary (non-randomized) policy exist are then presented. It is also shown that the above criterion is equivalent to the usual one under certain conditions.
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1. INTRODUCTION

A process is observed at time $0$ and classified into some state $x \in X$. After classification, an action $a \in A$ must be chosen. Both the state space $X$ and the action space $A$ are assumed to be Borel subsets of complete, separable metric spaces.

If the state is $x$ and action $a$ is chosen, then

(i) the next state of the process is chosen according to a known regular conditional probability measure $P(\cdot | x, a)$ on the Borel sets of $X$, and

(ii) conditional on the event that the next state is $y$, the time until the transition from $x$ to $y$ occurs is a random variable with known distribution $F(\cdot | x, a, y)$. After the transition occurs, an action is again chosen and (i) and (ii) are repeated. This is assumed to go on indefinitely.

We further suppose that a cost structure is imposed on the model in the following manner: If action $a$ is chosen when in state $x$ and the process makes a transition $t$ units later, then the cost incurred by time $s \leq t$ after the action was taken is given by a known real-valued Baire function $C(s | x, a)$.

*If one allows the cost to also depend upon the next state visited, then $C(s | x, a)$ should be interpreted as an expected cost.*
In order to ensure that transitions do not take place too quickly, we shall need to assume the following:

**Condition 1:**

There exists $\delta > 0$, $\epsilon > 0$, such that

$$\int_{y \leq x} F(\delta \mid x, a, y) dP(y \mid x, a) < 1 - \epsilon$$

for all $x, a$.

In other words, Condition 1 asserts that for every state $x$ and action $a$ there is a positive probability of at least $\epsilon$ that the transition time will be greater than $\delta$.

A policy $\pi$ is any measurable rule for choosing actions. The problem is to choose a policy which minimizes the expected average cost per time. When the time between transitions is identically 1, then the process is called a Markov decision process and has been extensively studied (see, for instance, [2], [5] and [6]). When this restriction is lifted, we have a semi-Markov decision process and results have only previously been given for the case where $A$ and $S$ are finite (see [3] and [4]).
2. EQUALITY OF CRITERIA

Let $X_n$ and $a_n$ be respectively the nth state of the process and the nth action chosen, $n = 1, 2, \ldots$. Also, let $\tau_n$ be the time between the $(n-1)$st and the nth transition, $n \geq 1$.

Furthermore, let $Z(t)$ denote the total cost incurred by $t$, and let $Z_n$ be the cost incurred during the nth transition interval; and define for any policy $\pi$

$$\phi_1^\pi(x) = \lim_{t \to \infty} \mathbb{E}_{\pi} \left[ \frac{Z(t)}{t} \mid X_1 = x \right]$$

and

$$\phi_2^\pi(x) = \lim_{n \to \infty} \mathbb{E}_{\pi} \left[ \frac{\sum_{i=1}^{n} Z_i}{\sum_{i=1}^{n} \tau_i} \mid X_1 = x \right].$$

Thus $\phi_1$ and $\phi_2$ both represent, in some sense, the average expected cost. Though $\phi_1$ is clearly more appealing, it will be criterion $\phi_2$ that we shall deal with. Fortunately, it turns out that under certain conditions both criterions are identical.

Definition:

A policy is said to be stationary if the action it chooses only depends on the present state of the system.

The reader should note at this point that if a stationary policy is employed then the process $\{X(t), t \geq 0\}$ is a semi-Markov process, where $X(t)$ represents the state of the process at time $t$.

$^\dagger$Of course, $Z(t)$ and $Z_n$ are determined by $X_i, a_i, \tau_i, i \geq 1$. 
For any initial state $x$, let
\[ T = \inf \{ t > 0 : X(t) = x, X(t^-) \neq x \}, \]
and
\[ N = \min \{ n > 0 : X_{n+1} = x \}. \]

Hence, $T$ is the time of the first return to state $x$ and $N$ is the number of transitions that it takes.

**Lemma 1:**

If Condition 1 holds, and if $E_n[T | X_1 = x] < \infty$, then $E_n[N | X_1 = x] < \infty$ and $T = \sum_{n=1}^{N} \tau_n$.

**Proof:**

By the definition of $T$ and $N$ it follows that $T \geq \sum_{n=1}^{N} \tau_n$, with equality holding if $N < \infty$. Now, if we let
\[
\bar{\tau}_n = \begin{cases} 
0 & \text{if } \tau_n \leq \delta \\
\delta & \text{with probability } \frac{\varepsilon}{\int (1 - F(\delta | x, y, a))dP(y | x, a)} \text{ if } \tau_n > \delta, \\
0 & \text{with probability } 1 - \frac{\varepsilon}{\int (1 - F(\delta | x, y, a))dP(y | x, a)} \text{ if } \tau_n > \delta,
\end{cases}
\]

then it follows from Condition 1 that $\bar{\tau}_n, n = 1, 2, \ldots$ are independent and identically distributed with

[If the set in brackets is empty then take $N$ to be $\infty$, and similarly for $T$.]
\[
P(T_n = \delta) = \epsilon = 1 - P(T_n = 0).
\]

Now, from Wald's equation it follows that if \( EN = \infty \) then \( E \sum_1^N T_n = \infty \), and hence that \( ET \geq E \sum_1^N \tau_n > E \sum_1^N \bar{r}_n = \infty \) (since \( \tau_n \leq \bar{r}_n \)).

Q.E.D.

**Theorem 1:**

Assume Condition 1. If \( \pi \) is a stationary policy, and if \( E_\pi [T \mid X_1 = x] < \infty \), then

\[
\phi_1^\pi(x) = \phi_2^\pi(x) = \frac{E_\pi [Z(T) \mid X_1 = x]}{E_\pi [T \mid X_1 = x]}.
\]

**Proof:**

Suppose throughout the proof that \( X_1 = x \). Now, under a stationary policy \( (X(t), t \geq 0) \) is a regenerative process with regeneration (or cycle) point \( T \).

Hence, by a well known result

\[
\phi_1^\pi(1) = \frac{E_\pi [cost incurred during a cycle]/E_\pi [length of cycle]}{E_\pi [T]/E_\pi [T]}.
\]

Also, it is easy to see that \( (X_n, n = 1, 2, \ldots) \) is a discrete time regenerative process with regeneration time \( N \). Hence, by regarding \( Z_1 + \ldots + Z_N \) as the "cost" incurred during the first cycle of this process, it follows by the same well known result that

\[
E_\pi \sum_{n=1}^m Z_n/m \to E_\pi \sum_{n=1}^N \frac{Z_n}{E_N \pi} \quad \text{as} \quad m \to \infty.
\]
where we have used Lemma 1 to assert that \( E_N < \infty \). However, we may also regard \( \tau_1 + \cdots + \tau_N \) as the "cost" incurred during the first cycle and hence, by the same reasoning,

\[
\begin{align*}
E N & \equiv \sum_{n=1}^{m} \tau_n / m \to E N \sum_{n=1}^{N} \tau_n / E N \quad \text{as } m \to \infty.
\end{align*}
\]

By combining (1) and (2) we obtain

\[
\phi^2_\pi(x) = \frac{E \sum_{n=1}^{N} Z_n}{E \sum_{n=1}^{N} \tau_n}.
\]

However, since \( N < \infty \) (Lemma 1) it is easy to see that \( \sum_{n=1}^{N} Z_n = Z(T) \) and \( \sum_{n=1}^{N} \tau_n = T \), and the result follows.

Q.E.D.

Remarks:

It also follows from the above proof that, with probability 1,

\[
\lim_{t \to \infty} \frac{Z(t)}{T} = \lim_{m \to \infty} \frac{\sum_{n=1}^{m} Z_n}{E[Z(T)]} = \frac{E[Z(T)]}{E[T]}.
\]

Also, suppose that the initial state is \( y, y \neq x \). When is it true that \( \phi^1_\pi(y) = \phi^2_\pi(y) = \phi^1_\pi(x) \)? One answer is that if, with probability 1, the process will eventually enter state \( x \), then \( \{X(t), t \geq 0\} \) is a delayed (or general) regenerative process, and the proof goes through in an identical manner.

Let
\[ \bar{r}(x,a) = \int_{y \in \mathbb{X}} \int_{0}^{\infty} t dF(t \mid x,a,y) dP(y \mid x,a) \]

and

\[ \bar{C}(x,a) = \int_{y \in \mathbb{X}} \int_{0}^{\infty} C(t \mid x,a) dF(t \mid x,a,y) dP(y \mid x,a) . \]

We shall suppose that both \( \bar{C}(x,a) \) and \( \bar{r}(x,a) \) exist and are finite for all \( x, a \).

We also note that the expected cost incurred during a transition interval and the expected length of a transition interval only depend on the parameters of the process through \( \bar{r}(x,a) \), \( \bar{C}(x,a) \) and \( P(\cdot \mid x,a) \); and, as a result, \( \phi^2 \) only depends on the parameters of the process through these three functions. Thus, we may choose the cost and transition time distributions in as convenient a manner as possible; and hence for the remainder of this paper, let us suppose that

\[ F(t \mid x,a,y) = \begin{cases} 1 & t \geq \bar{r}(x,a) \\ 0 & t < \bar{r}(x,a) \end{cases} \]

and

\[ C(t \mid x,a) = \begin{cases} 0 & t < \bar{r}(x,a) \\ \bar{C}(x,a) & t \geq \bar{r}(x,a) \end{cases} . \]

That is, we suppose that the time until transition is (with probability 1) \( \bar{r}(x,a) \) and that a cost of \( \bar{C}(x,a) \) is incurred at the time of transition.
3. AVERAGE COST RESULTS

Theorem 2:

Assuming Condition 1, if there exists a bounded Baire function \( f(x), x \in \mathbb{X} \), and a constant \( g \), such that

\[
(3) \quad f(x) = \min_{a} \left\{ \bar{C}(x,a) + \int_{\mathbb{X}} f(y) dP(y \mid x,a) - gT(x,a) \right\} \quad x \in \mathbb{X},
\]

then, for any policy \( \pi^* \) which, when in state \( x \), selects an action minimizing the right side of (3), we have

\[
\phi^2_{\pi^*}(x) = g = \min_{\pi} \phi^2_{\pi}(x) \quad \text{for all} \quad x \in \mathbb{X}.
\]

Proof:

Let \( S_i = (X_1, a_1, \ldots, X_i, a_i), i = 1, 2, \ldots \). For any policy \( \pi \)

\[
E_{\pi} \left[ \sum_{i=2}^{n} [f(X_i) - E_{\pi}(f(X_i) \mid S_{i-1})] \right] = 0.
\]

But,

\[
E_{\pi}[f(X_1) \mid S_{1-1}] = \int_{\mathbb{X}} f(y) dP(y \mid X_{1-1}, a_{1-1})
\]

\[
\begin{align*}
&= \bar{C}(X_{1-1}, a_{1-1}) + \int_{\mathbb{X}} f(y) dP(y \mid X_{1-1}, a_{1-1}) - gT(X_{1-1}, a_{1-1}) \\
&\quad - \bar{C}(X_{1-1}, a_{1-1}) + gT(X_{1-1}, a_{1-1}) \\
&= \min_{a} \left\{ \bar{C}(X_{1-1}, a) + \int_{\mathbb{X}} f(y) dP(y \mid X_{1-1}, a) - gT(X_{1-1}, a) \right\} \\
&\quad - \bar{C}(X_{1-1}, a_{1-1}) + gT(X_{1-1}, a_{1-1}) \\
&= f(X_{1-1}) - \bar{C}(X_{1-1}, a_{1-1}) + gT(X_{1-1}, a_{1-1}),
\end{align*}
\]
with equality for \( \pi^* \), since \( \pi^* \) takes the minimizing actions. Hence,

\[
0 \leq \mathbb{E}_\pi \left( \sum_{i=2}^{n} [f(X_i) - f(X_{i-1}) + \bar{c}(X_{i-1}, a_{i-1}) - g\bar{r}(X_{i-1}, a_{i-1})] \right)
\]

or

\[
g \leq \frac{\mathbb{E}_\pi \left( \sum_{i=2}^{n} \bar{c}(X_{i-1}, a_{i-1}) \right)}{\mathbb{E}_\pi \left( \sum_{i=2}^{n} \bar{r}(X_{i-1}, a_{i-1}) \right)} + \frac{\mathbb{E}_\pi [f(X_n) - f(X_1)]}{\mathbb{E}_\pi \left( \sum_{i=2}^{n} \bar{r}(X_{i-1}, a_{i-1}) \right)}
\]

with equality for \( \pi^* \). By letting \( n \to \infty \) and using the boundedness of \( f \) and the fact that Condition 1 implies that \( \mathbb{E}_\pi \left( \sum_{i=1}^{n} \bar{r}(X_{i-1}, a_{i-1}) \right) \geq n \in \delta + \infty \), we obtain

\[
g \leq \lim_{n \to \infty} \frac{\mathbb{E}_\pi \left( \sum_{i=2}^{n} \bar{c}(X_{i-1}, a_{i-1}) \right)}{\mathbb{E}_\pi \left( \sum_{i=2}^{n} \bar{r}(X_{i-1}, a_{i-1}) \right)} = \phi^2(\pi, X_{i-1})
\]

with equality for \( \pi^* \) and for all possible values of \( X_1 \).

**Remarks:**

The above proof is an adaptation of one given in [6] for Markov decision processes. We have tacitly assumed that a rule minimizing the right side of (3) may be chosen in a measurable manner. Clearly a sufficient (but by no means necessary) condition is that the action space \( A \) be countable.

In order to determine sufficient conditions for the existence of a bounded function \( f(x) \) and a constant \( g \) satisfying (3), we introduce a discount factor \( \alpha, 0 < \alpha < \infty \), and continuously discount costs. That is, we suppose that
a cost of \( C \) incurred at time \( t \) is equivalent to a cost \( C e^{-\alpha t} \) incurred at time 0.

Let \( V_{\pi, \alpha}(x) \) denote the total expected discounted cost when \( \pi \) is employed, and the initial state is \( x \); and let \( V_{\alpha}(x) = \inf_{\pi} V_{\pi, \alpha}(x) \). Then, it may be shown by standard arguments (see [1]) that

\[
V_{\alpha}(x) = \min_{\alpha} \left\{ e^{-\alpha \tau(x, a)} \left[ \bar{C}(x, a) + \int_0^\infty V_{\alpha}(y) dP(y | x, a) \right] \right\}.
\]

Now, fix some state--call it 0--and define

\[
f_{\alpha}(x) = V_{\alpha}(x) - V_{\alpha}(0).
\]

From (4), we obtain

\[
V_{\alpha}(0) + f_{\alpha}(x) = \min_{\alpha} \left\{ e^{-\alpha \tau(x, a)} \left[ \bar{C}(x, a) + \int_0^\infty f_{\alpha}(y) dP(y | x, a) + V_{\alpha}(0) \right] \right\}.
\]

We shall need the following condition:

**Condition 2:**

There exists an \( M < \infty \), such that

\[
\bar{C}(x, a) \leq M \tau(x, a) \quad \text{for all } x, a.
\]
Theorem 3:

Under Conditions 1 and 2, if the action space $A$ is finite, and if \( \{f_\alpha(x), 0 < \alpha < c\} \) is a uniformly bounded equicontinuous family of functions for some $0 < c < \infty$, then

(i) there exists a bounded continuous function $f(x)$ and a constant $g$ satisfying (3);

(ii) for some sequence $\alpha_n \to 0$, $f(x) = \lim_{n \to \infty} f_\alpha(x)$;

(iii) $\lim_{\alpha \to 0} \alpha V_\alpha(x) = g$ for all $x \in X$.

Proof:

From (5), we obtain that

$$f_\alpha(x) = \min_a \left\{ e^{-\alpha \tau(x,a)} \left[ \delta(x,a) + \int_0^\infty f_\alpha(y) dP(y | x,a) \right] - V_\alpha(0)(\alpha \tau(x,a) + o(\alpha)) \right\}. \quad (6)$$

Now, by the Arzela-Ascoli theorem there exists a sequence $\alpha_n \to 0$ and a continuous function $f$ such that $\lim_{n \to \infty} f_\alpha(x) = f(x)$ for all $x$. Also, it follows from Conditions 1 and 2 that $\alpha V_\alpha(0)$ is bounded, and hence we can require that $\lim_{n \to \infty} \alpha_n V_\alpha(0) = g$ exists. The results (i) and (ii) then follow by letting $\alpha_n \to 0$ in (6) and using Lebesgue's dominated convergence theorem.

The proof of (iii) is identical with the one given in [6].
4. AN EXAMPLE

Suppose that batches of letters arrive at a post office at a Poisson rate $\lambda$. Suppose further that each batch consists of $j$ letters with probability $P_j$, $j \geq 1$, independently of each other. At any time, a truck may be dispatched to deliver the letters. Assume that the cost of dispatching the truck is $K$, and also that the cost rate when there are $j$ letters present is $C_j$, an increasing, positive, bounded sequence, $j \geq 1$. The problem is to choose a policy minimizing the long-run average cost.

The above may be regarded as two action semi-Markov decision process with states $1, 2, 3, \ldots$; where state $i$ means that there are $i$ letters presently in the post office. Action 1 is "dispatch a truck" and action 2 is "don't dispatch a truck." (Note that since a truck would never be dispatched if there were no letters in the post office, we need not have a state 0.)

The parameters of the process are:

- $P(j/i, 1) = P_j$, $P(i + j/i, 2) = P_j$
- $\tau(i, 1) = 1/\lambda$, $\tau(i, 2) = 1/\lambda$
- $\bar{C}(i, 1) = K + \frac{C(0)}{\lambda}$, $\bar{C}(i, 2) = \frac{C(i)}{\lambda}$

Now, if we let

$$e^{\alpha/\lambda} V_\alpha(i, 1) = \min \left\{ K + \frac{C(0)}{\lambda}; \frac{C(i)}{\lambda} \right\},$$

and for $n > 1$

$$e^{\alpha/\lambda} V_\alpha(i, n) = \min \left\{ K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} P_j V_\alpha(j, n - 1); \frac{C(i)}{\lambda} + \sum_{j=1}^{\infty} P_j V_\alpha(i + j, n - 1) \right\},$$

then it follows by induction that $V_\alpha(i, n)$ is increasing in $i$ for each $n$.

Also, since costs are bounded and the discount factor $e^{-\alpha/\lambda} < 1$, it follows that
\( V_a(i) = \lim_{n} V_a(i,n) \), and hence \( V_a(i) \) is increasing. Also, \( V_a(i) \) satisfies

\[
(7) \quad e^{\alpha/\lambda} V_a(i) = \min \left\{ K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} P_j V_a(j) ; \frac{C(i)}{\lambda} + \sum_{j=1}^{\infty} P_j V_a(i+j) \right\}.
\]

We will now show that \( V_a(i) - V_a(1) \) is uniformly bounded and hence Theorem 3 is applicable. To do this, we consider two cases:

Case i:

\[
e^{\alpha/\lambda} V_a(1) = K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} P_j V_a(j).
\]

In this case, we have by (7) that \( V_a(i) \leq V_a(1) \) and hence, by monotonicity,

\[
V_a(i) = V_a(1) \quad \text{for all } i.
\]

Case ii:

\[
e^{\alpha/\lambda} V_a(1) = \frac{C(1)}{\lambda} + \sum_{j=1}^{\infty} P_j V_a(1+j).
\]

In this case, we have by (7) that

\[
e^{\alpha/\lambda} V_a(1) \leq e^{\alpha/\lambda} V_a(i) \leq K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} P_j V_a(j)
\]

\[
\leq K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} P_j V_a(i+1)
\]

\[
= K + \frac{C(0)}{\lambda} - \frac{C(1)}{\lambda} + e^{\alpha/\lambda} V_a(1).
\]

Thus, in either case \( V_a(i) - V_a(1) \) is uniformly bounded and hence by Theorem 3 there exists an increasing function \( f(i) \) and a constant \( g \) such that
and the policy which chooses the minimizing actions is optimal.

Now, if we let

\[ i^* = \min \left\{ i : \frac{C(i)}{\lambda} + \sum_{j=1}^{\infty} P_j h(j + i) > K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} P_j h(j) \right\}, \]

then it follows from the monotonicity of \( C(i) \) and \( h(i) \) that the optimal policy is to dispatch a truck whenever the number of letters in the post office is at least \( i^* \); and hence, the structure of the optimal policy is determined.
REFERENCES


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**AUTHOR**
Sheldon M. Ross

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