AVERAGE COST SEMI-MARKOV DECISION PROCESSES

by

SHELDON M. ROSS

OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA • BERKELEY
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Sheldon M. Ross
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

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The Semi-Markov Decision model is considered under the criterion of long-run average cost. A new criterion, which for any policy considers the limit of the expected cost incurred during the first $n$ transitions divided by the expected length of the first $n$ transitions, is considered. Conditions guaranteeing that an optimal stationary (non-randomized) policy exist are then presented. It is also shown that the above criterion is equivalent to the usual one under certain conditions.
1. INTRODUCTION

A process is observed at time 0 and classified into some state $x \in X$. After classification, an action $a \in A$ must be chosen. Both the state space $X$ and the action space $A$ are assumed to be Borel subsets of complete, separable metric spaces.

If the state is $x$ and action $a$ is chosen, then

(i) the next state of the process is chosen according to a known regular conditional probability measure $P(\cdot \mid x,a)$ on the Borel sets of $X$, and

(ii) conditional on the event that the next state is $y$, the time until the transition from $x$ to $y$ occurs is a random variable with known distribution $F(\cdot \mid x,a,y)$. After the transition occurs, an action is again chosen and (i) and (ii) are repeated. This is assumed to go on indefinitely.

We further suppose that a cost structure is imposed on the model in the following manner: If action $a$ is chosen when in state $x$ and the process makes a transition $t$ units later, then the cost incurred by time $s(s \leq t)$ after the action was taken is given by a known real-valued Baire function $C(s \mid x,a)$.

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*If one allows the cost to also depend upon the next state visited, then $C(s \mid x,a)$ should be interpreted as an expected cost.*
In order to ensure that transitions do not take place too quickly, we shall need to assume the following:

**Condition 1:**

There exists \( \delta > 0 \), \( c > 0 \), such that

\[
\int_{y \in S} F(\delta \mid x, a, y) dP(y \mid x, a) < 1 - c \quad \text{for all } x, a.
\]

In other words, Condition 1 asserts that for every state \( x \) and action \( a \) there is a positive probability of at least \( c \) that the transition time will be greater than \( \delta \).

A policy \( \pi \) is any measurable rule for choosing actions. The problem is to choose a policy which minimizes the expected average cost per time. When the time between transitions is identically 1, then the process is called a Markov decision process and has been extensively studied (see, for instance, [2], [5] and [6]). When this restriction is lifted, we have a semi-Markov decision process and results have only previously been given for the case where \( A \) and \( S \) are finite (see [3] and [4]).
2. EQUALITY OF CRITERIA

Let \( X_n \) and \( a_n \) be respectively the \( n \)th state of the process and the \( n \)th action chosen, \( n = 1, 2, \ldots \). Also, let \( \tau_n \) be the time between the \((n - 1)\)st and the \( n \)th transition, \( n \geq 1 \).

Furthermore, let \( Z(t) \) denote the total cost incurred by \( t \), and let \( Z_n \) be the cost incurred during the \( n \)th transition interval; and define for any policy \( \pi \)

\[
\phi_1^\pi(x) = \lim_{t \to \infty} \mathbb{E}_{\pi} \left[ Z(t) \mid X_1 = x \right]
\]

and

\[
\phi_2^\pi(x) = \lim_{n \to \infty} \mathbb{E}_{\pi} \left[ \sum_{i=1}^{n} \tau_i \mid X_1 = x \right]
\]

Thus \( \phi_1 \) and \( \phi_2 \) both represent, in some sense, the average expected cost. Though \( \phi_1 \) is clearly more appealing, it will be criterion \( \phi_2 \) that we shall deal with. Fortunately, it turns out that under certain conditions both criterions are identical.

**Definition:**

A policy is said to be stationary if the action it chooses only depends on the present state of the system.

The reader should note at this point that if a stationary policy is employed then the process \( \{X(t), t \geq 0\} \) is a semi-Markov process, where \( X(t) \) represents the state of the process at time \( t \).

\( ^{\dagger} \) Of course, \( Z(t) \) and \( Z_n \) are determined by \( X_i, a_i, \tau_i, i \geq 1 \).
For any initial state \( x \), let

\[
T = \inf \{ t > 0 : X(t) = x, X(t^-) \neq x \},
\]

and

\[
N = \min \{ n > 0 : X_{n+1} = x \}. \tag{1}
\]

Hence, \( T \) is the time of the first return to state \( x \) and \( N \) is the number of transitions that it takes.

Lemma 1:

If Condition 1 holds, and if \( E_n[T | X_1 = x] < \infty \), then \( E_n[N | X_1 = x] < \infty \) and \( T = \sum_{n=1}^{N} \tau_n \).

Proof:

By the definition of \( T \) and \( N \) it follows that \( T > \sum_{n=1}^{N} \tau_n \), with equality holding if \( N = \infty \). Now, if we let

\[
\bar{\tau}_n = \begin{cases} 
0 & \text{if } \tau_n \leq \delta \\
\delta & \text{with probability } \frac{\epsilon}{\int (1 - F(\delta | x,y,a))dP(y | x,a)} \text{ if } \tau_n > \delta, \\
0 & \text{with probability } 1 - \frac{\epsilon}{\int (1 - F(\delta | x,y,a))dP(y | x,a)} \text{ if } \tau_n > \delta, \\
\end{cases}
\]

then it follows from Condition 1 that \( \bar{\tau}_n \), \( n = 1,2, \ldots \) are independent and identically distributed with

\[
\bar{\tau}_n = a
\]

†If the set in brackets is empty then take \( N \) to be \( \infty \), and similarly for \( T \).
\[ P(\tau_n = \delta) = \epsilon = 1 - P(\tau_n = 0). \]

Now, from Wald's equation it follows that if \( EN = \infty \) then \( E \sum_{n=1}^{N} \tau_n = \infty \), and hence that \( ET > E \sum_{n=1}^{N} \tau_n > E \sum_{n=1}^{N} \tau_n = \infty \) (since \( \tau_n < \tau_n \)).

Q.E.D.

Theorem 1:

Assume Condition 1. If \( \pi \) is a stationary policy, and if \( E_{\pi}[T \mid X_1 = x] < \infty \), then

\[ \phi_{\pi}^{1}(x) = \phi_{\pi}^{2}(x) = \frac{E_{\pi}[Z \mid X_1 = x]}{E_{\pi}[T \mid X_1 = x]} . \]

Proof:

Suppose throughout the proof that \( X_1 = x \). Now, under a stationary policy \( (X(t), t > 0) \) is a regenerative process with regeneration (or cycle) point \( T \). Hence, by a well known result

\[ \phi_{\pi}^{1}(1) = E_{\pi}[\text{cost incurred during a cycle}] / E_{\pi}[\text{length of cycle}] \]

\[ = E_{\pi}[Z_{\pi}]/E_{\pi}(T) . \]

Also, it is easy to see that \( (X_n, n = 1, 2, \ldots) \) is a discrete time regenerative process with regeneration time \( N \). Hence, by regarding \( Z_1 + \ldots + Z_N \) as the "cost" incurred during the first cycle of this process, it follows by the same well known result that

\[ E_{\pi} \frac{1}{m} \sum_{n=1}^{m} Z_n \rightarrow E_{\pi} \frac{1}{N} \sum_{n=1}^{N} Z_n / E_{\pi} N \quad \text{as} \quad m \to \infty , \]
where we have used Lemma 1 to assert that $E_{\pi} N < \infty$. However, we may also regard $\tau_1 + \ldots + \tau_N$ as the "cost" incurred during the first cycle and hence, by the same reasoning,

$$E_{\pi} \sum_{n=1}^{m} \frac{\tau_n}{m} \rightarrow E_{\pi} \sum_{n=1}^{N} \frac{\tau_n}{E_{\pi} N} \quad \text{as} \quad m \rightarrow \infty. \quad (2)$$

By combining (1) and (2) we obtain

$$\phi_{\pi}^2(x) = \frac{E_{\pi} \sum_{n=1}^{N} Z_n}{E_{\pi} \sum_{n=1}^{N} \tau_n}.$$ 

However, since $N < \infty$ (Lemma 1) it is easy to see that $\sum_{n=1}^{N} Z_n = Z(T)$ and $\sum_{n=1}^{N} \tau_n = T$, and the result follows. Q.E.D.

Remarks:

It also follows from the above proof that, with probability 1,

$$\lim_{t \rightarrow \infty} \frac{Z(t)}{t} = \lim_{n \rightarrow \infty} \frac{\sum_{n=1}^{m} Z_n}{m} = \frac{E_{\pi}[Z(T)]}{E_{\pi} T},$$

Also, suppose that the initial state is $y$, $y \neq x$. When is it true that $\phi_{\pi}^1(y) = \phi_{\pi}^2(y) = \phi_{\pi}^1(x)$? One answer is that if, with probability 1, the process will eventually enter state $x$, then \{X(t), t \geq 0\} is a delayed (or general) regenerative process, and the proof goes through in an identical manner.

Let
\[ \overline{\tau}(x,a) = \int \int_0^\infty t \, dF(t \mid x,a,y) \, dp(y \mid x,a) \]

and

\[ \overline{C}(x,a) = \int \int_0^\infty C(t \mid x,a) \, dF(t \mid x,a,y) \, dp(y \mid x,a) . \]

We shall suppose that both \( \overline{C}(x,a) \) and \( \overline{\tau}(x,a) \) exist and are finite for all \( x, a \).

We also note that the expected cost incurred during a transition interval and the expected length of a transition interval only depend on the parameters of the process through \( \overline{\tau}(x,a) \), \( \overline{C}(x,a) \) and \( P(\cdot \mid x,a) \); and, as a result, \( \overline{\tau} \) only depends on the parameters of the process through these three functions. Thus, we may choose the cost and transition time distributions in as convenient a manner as possible; and hence for the remainder of this paper, let us suppose that

\[ F(t \mid x,a,y) = \begin{cases} 1 & t \geq \overline{\tau}(x,a) \\ 0 & t < \overline{\tau}(x,a) \end{cases} \]

and

\[ C(t \mid x,a) = \begin{cases} 0 & t < \overline{\tau}(x,a) \\ \overline{C}(x,a) & t \geq \overline{\tau}(x,a) \end{cases} \]

That is, we suppose that the time until transition is (with probability 1) \( \overline{\tau}(x,a) \) and that a cost of \( \overline{C}(x,a) \) is incurred at the time of transition.
3. AVERAGE COST RESULTS

**Theorem 2:**

Assuming Condition 1, if there exists a bounded Baire function $f(x), x \in \mathcal{X}$, and a constant $g$, such that

$$f(x) = \min_a \left\{ \bar{c}(x,a) + \int_{\mathcal{X}} f(y) dP(y \mid x,a) - g\bar{T}(x,a) \right\} \quad x \in \mathcal{X}, \tag{3}$$

then, for any policy $\pi^*$ which, when in state $x$, selects an action minimizing the right side of (3), we have

$$\phi^2_{\pi^*}(x) = g = \min_{\pi} \phi^2_{\pi}(x) \quad \text{for all } x \in \mathcal{X}.\]$$

**Proof:**

Let $S_i = (X_i, a_i, \ldots, X_1, a_1), i = 1, 2, \ldots$ For any policy $\pi$

$$E_{\pi} \left[ \sum_{i=2}^{n} \left[ f(X_i) - E_{\pi}(f(X_i) \mid S_{i-1}) \right] \right] = 0.$$

But,

$$E_{\pi}[f(X_i) \mid S_{i-1}] = \int_{\mathcal{X}} f(y) dP(y \mid X_{i-1}, a_{i-1})$$

$$= \bar{c}(X_{i-1}, a_{i-1}) + \int_{\mathcal{X}} f(y) dP(y \mid X_{i-1}, a_{i-1}) - g\bar{T}(X_{i-1}, a_{i-1})$$

$$- \bar{c}(X_{i-1}, a_{i-1}) + g\bar{T}(X_{i-1}, a_{i-1})$$

$$= \min_a \left\{ \bar{c}(X_{i-1}, a) + \int_{\mathcal{X}} f(y) dP(y \mid X_{i-1}, a) - g\bar{T}(X_{i-1}, a) \right\}$$

$$- \bar{c}(X_{i-1}, a_{i-1}) + g\bar{T}(X_{i-1}, a_{i-1})$$

$$= f(X_{i-1}) - \bar{c}(X_{i-1}, a_{i-1}) + g\bar{T}(X_{i-1}, a_{i-1})$$,
with equality for \( \pi^* \), since \( \pi^* \) takes the minimizing actions. Hence,

\[
0 \leq E_\pi \sum_{i=2}^{n} [f(X_i) - f(X_{i-1}) + c(X_{i-1}, a_{i-1}) - g^+(X_{i-1}, a_{i-1})]
\]

or

\[
g \leq \frac{E_\pi \sum_{i=2}^{n} c(X_{i-1}, a_{i-1})}{E_\pi \sum_{i=2}^{n} \tau(X_{i-1}, a_{i-1})} + \frac{E_\pi [f(X_1) - f(X_n)]}{E_\pi \sum_{i=2}^{n} \tau(X_{i-1}, a_{i-1})}
\]

with equality for \( \pi^* \). By letting \( n \to \infty \) and using the boundedness of \( f \) and the fact that Condition 1 implies that \( E_\pi \sum_{i=1}^{\infty} \tau(X_{i-1}, a_{i-1}) = n \in \delta + \infty \), we obtain

\[
g \leq \frac{E_\pi \sum_{i=2}^{n} c(X_{i-1}, a_{i-1})}{\lim_{n \to \infty} E_\pi \sum_{i=2}^{n} \tau(X_{i-1}, a_{i-1})} = \phi^2_\pi(X_1)
\]

with equality for \( \pi^* \) and for all possible values of \( X_1 \).

**Remarks:**

The above proof is an adaptation of one given in [6] for Markov decision processes. We have tacitly assumed that a rule minimizing the right side of (3) may be chosen in a measurable manner. Clearly a sufficient (but by no means necessary) condition is that the action space \( A \) be countable.

In order to determine sufficient conditions for the existence of a bounded function \( f(x) \) and a constant \( g \) satisfying (3), we introduce a discount factor \( \alpha, 0 < \alpha < 1 \), and continuously discount costs. That is, we suppose that
a cost of \( C \) incurred at time \( t \) is equivalent to a cost \( Ce^{-\alpha t} \) incurred at time 0.

Let \( V_{\pi,\alpha}(x) \) denote the total expected discounted cost when \( \pi \) is employed, and the initial state is \( x \); and let \( V_{\alpha}(x) = \inf_{\pi} V_{\pi,\alpha}(x) \). Then, it may be shown by standard arguments (see [1]) that

\[
(4) \quad V_{\alpha}(x) = \min_{\alpha} \left\{ e^{-\alpha T(x,\alpha)} \left[ \bar{C}(x,\alpha) + \int_0^\infty V_{\alpha}(y) dP(y \mid x,\alpha) \right] \right\}.
\]

Now, fix some state—call it 0—and define

\[
f_{\alpha}(x) = V_{\alpha}(x) - V_{\alpha}(0).
\]

From (4), we obtain

\[
(5) \quad V_{\alpha}(0) + f_{\alpha}(x) = \min_{\alpha} \left\{ e^{-\alpha T(x,\alpha)} \left[ \bar{C}(x,\alpha) + \int_0^\infty f_{\alpha}(y) dP(y \mid x,\alpha) + V_{\alpha}(0) \right] \right\}.
\]

We shall need the following condition:

**Condition 2:**

There exists an \( M < \infty \), such that

\[
\bar{C}(x,\alpha) \leq M\bar{r}(x,\alpha) \quad \text{for all } x, \alpha.
\]
Theorem 3:

Under Conditions 1 and 2, if the action space $A$ is finite, and if
$
\{f_{\alpha}(x), 0 < \alpha < c\}
$ is a uniformly bounded equicontinuous family of functions for
some $0 < c < \infty$, then

(i) there exists a bounded continuous function $f(x)$ and a constant $g$
satisfying (3);

(ii) for some sequence $\alpha_n \to 0$, $f(x) = \lim_{\alpha \to 0} f_{\alpha}(x)$;

(iii) $\lim_{\alpha \to 0} \alpha V_{\alpha}(x) = g$ for all $x \in X$.

Proof:

From (5), we obtain that

$$
f_{\alpha}(x) = \min_{\alpha} \left\{ e^{-\alpha(x,a)} \left[ \delta(x,a) + \int_0^\alpha f_{\alpha}(y) dP(y | x,a) \right] - V_{\alpha}(0)(a(x,a) + o(\alpha)) \right\}.
$$

(6)

Now, by the Arzela-Ascoli theorem there exists a sequence $\alpha_n \to 0$ and a
continuous function $f$ such that $\lim_{\alpha \to 0} f_{\alpha}(x) = f(x)$ for all $x$. Also, it
follows from Conditions 1 and 2 that $\alpha V_{\alpha}(0)$ is bounded, and hence we can require
that $\lim_{\alpha \to 0} \alpha V_{\alpha}(0) = g$ exists. The results (i) and (ii) then follow by letting
$\alpha_n \to 0$ in (6) and using Lebesgue's dominated convergence theorem.

The proof of (iii) is identical with the one given in [6].
4. AN EXAMPLE

Suppose that batches of letters arrive at a post office at a Poisson rate $\lambda$. Suppose further that each batch consists of $j$ letters with probability $p_j$, $j \geq 1$, independently of each other. At any time, a truck may be dispatched to deliver the letters. Assume that the cost of dispatching the truck is $K$, and also that the cost rate when there are $j$ letters present is $C_j$, an increasing, positive, bounded sequence, $j \geq 1$. The problem is to choose a policy minimizing the long-run average cost.

The above may be regarded as two action semi-Markov decision process with states $1, 2, 3, \ldots$; where state $i$ means that there are $i$ letters presently in the post office. Action 1 is "dispatch a truck" and action 2 is "don't dispatch a truck." (Note that since a truck would never be dispatched if there were no letters in the post office, we need not have a state 0.)

The parameters of the process are:

\[
P(j/1, 1) = p_j, \quad P(i + j/1, 2) = p_j
\]

\[
T(i, 1) = 1/\lambda, \quad T(i, 2) = 1/\lambda
\]

\[
\overline{C}(i, 1) = K + \frac{C(0)}{\lambda}, \quad \overline{C}(i, 2) = \frac{C(i)}{\lambda}.
\]

Now, if we let

\[
e^{\alpha/\lambda} v_\alpha(i, 1) = \min \left\{ K + \frac{C(0)}{\lambda} ; \frac{C(i)}{\lambda} \right\},
\]

and for $n > 1$

\[
e^{\alpha/\lambda} v_\alpha(i, n) = \min \left\{ K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} p_j v_\alpha(j, n - 1) ; \frac{C(i)}{\lambda} + \sum_{j=1}^{\infty} p_j v_\alpha(i + j, n - 1) \right\},
\]

then it follows by induction that $v_\alpha(i, n)$ is increasing in $i$ for each $n$.

Also, since costs are bounded and the discount factor $e^{-\alpha/\lambda} < 1$, it follows that
\[ V_a(i) = \lim_{n \to \infty} V_a(i,n), \] and hence \( V_a(i) \) is increasing. Also, \( V_a(i) \) satisfies

\[ e^{a/\lambda}V_a(i) = \min \left\{ K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} p_j V_a(j), \frac{C(i)}{\lambda} + \sum_{j=1}^{\infty} p_j V_a(i+j) \right\}. \]

We will now show that \( V_a(i) - V_a(1) \) is uniformly bounded and hence Theorem 3 is applicable. To do this, we consider two cases:

**Case i:**

\[ e^{a/\lambda}V_a(1) = K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} p_j V_a(j). \]

In this case, we have by (7) that \( V_a(i) \leq V_a(1) \) and hence, by monotonicity,

\[ V_a(i) = V_a(1) \quad \text{for all } i. \]

**Case ii:**

\[ e^{a/\lambda}V_a(1) = \frac{C(1)}{\lambda} + \sum_{j=1}^{\infty} p_j V_a(1+j). \]

In this case, we have by (7) that

\[
e^{a/\lambda}V_a(1) \leq e^{a/\lambda}V_a(i) \leq K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} p_j V_a(i) \\
\leq K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} p_j V_a(i+1) \\
= K + \frac{C(0)}{\lambda} - \frac{C(1)}{\lambda} + e^{a/\lambda}V_a(1).
\]

Thus, in either case \( V_a(i) - V_a(1) \) is uniformly bounded and hence by Theorem 3 there exists an increasing function \( f(i) \) and a constant \( g \) such that
\[ f(i) = \min \left \{ K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} P_j h(j) - \frac{K}{\lambda} + \frac{C(i)}{\lambda} + \sum_{j=1}^{\infty} P_j h(j + i) - \frac{K}{\lambda} \right \}, \]

and the policy which chooses the minimizing actions is optimal.

Now, if we let

\[ i^* = \min \left \{ i : \frac{C(i)}{\lambda} + \sum_{j=1}^{\infty} P_j h(j + i) > K + \frac{C(0)}{\lambda} + \sum_{j=1}^{\infty} P_j h(j) \right \}, \]

then it follows from the monotonicity of \( C(i) \) and \( h(i) \) that the optimal policy is to dispatch a truck whenever the number of letters in the post office is at least \( i^* \); and hence, the structure of the optimal policy is determined.
REFERENCES


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