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NORMAL INTEGRAL BY THE METHOD OF DAS

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A note on the evaluation of a multivariate normal integral by the method of Das

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1. Introduction

Das (1956) presents a method of evaluating the integral

\[ I = \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} f(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \, \cdots \, dx_n \]

(where \( f(x_1, x_2, \ldots, x_n) \) is the joint multivariate normal density function with zero means and nonsingular variance-covariance matrix \( \Sigma \)) through the combining of \( n + k \) independent normal variables with zero means and unit variances. Later Marsaglia (1963) shows that this is a special case of a convolution formula. The complexity of implementing the solution is highly dependent upon the size of \( k \) and Marsaglia (1963) notes that \( k \) equal to \( n \) minus the multiplicity of the smallest latent root of \( \Sigma \) can always be achieved. This note investigates properties of \( \Sigma \) that will allow smaller values of \( k \).

2. The equivalent expression

As a method for evaluating \( I \), Das (1956) considers two row vectors \( y' = (y_1, y_2, \ldots, y_n) \) and \( z' = (z_1, z_2, \ldots, z_k) \) all of whose elements are normally and independently distributed with zero means and unit variances. The problem is to choose a positive constant \( c \) and an \( n \times k \) real matrix \( B \) such that

\[ \Sigma = c^2 I_n + BB' \]
for then \( x' = (x_1, x_2, \ldots, x_n) \) can be expressed as

\[
x = cy - Bz
\]

I can now be expressed as

\[
I = \text{Pr}(x_1 \geq a_1, x_2 \geq a_2, \ldots, x_n \geq a_n)
\]

\[
= \text{Pr}
\left[
\begin{array}{l}
y_1 \geq \left( a_1 + \sum_{j=1}^{k} b_{1j} z_j \right)/c, \\
y_2 \geq \left( a_2 + \sum_{j=1}^{k} b_{2j} z_j \right)/c, \\
y_n \geq \left( a_n + \sum_{j=1}^{k} b_{nj} z_j \right)/c
\end{array}
\right]
\]

\[
= (2\pi)^{-k/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P\left[\left( a_i + \sum_{j=1}^{k} b_{ij} z_j \right)/c\right] \exp\left(-z'z/2\right) \prod_{j=1}^{k} dz_j
\]

where \( P[a] = \text{Pr}(y_1 \geq a) \). It is evident that a small \( k \) is advantageous.

The slight change in 1)

\[
\xi = C^2 + BB',
\]

where \( C \) is a diagonal matrix with positive diagonal elements \( c_i \), results in

\[
x = Cy - Bz
\]

and 2) of the form

\[
I = (2\pi)^{-k/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P\left[\left( a_i + \sum_{j=1}^{k} b_{ij} z_j \right)/c_i\right] \exp\left(-z'z/2\right) \prod_{j=1}^{k} dz_j
\]

3. Latent vector properties

If \( \xi \) can be expressed in the form 3) and \( \mathbf{w}_l \) is a latent vector
of \( \mathbf{T} \) then

\[
\lambda_j \mathbf{w}_\ell = \mathbf{T} \mathbf{w}_\ell \\
= C^2 \mathbf{w}_\ell + BB' \mathbf{w} \\
= C^2 \mathbf{w}_\ell + \sum_{j=1}^k b_j b_j' \mathbf{w}_\ell
\]

where \( b_j \) is the \( j \)th column of \( B \). From expression 5)

\[
\lambda_j \mathbf{w}_i = C_i^2 \mathbf{w}_i + \sum_{j=1}^k \Theta_j \mathbf{w}_i b_{ij}
\]

where \( \mathbf{w}_i \) is the \( i \)th element of \( \mathbf{w} \) and \( \Theta_j = b_j' \mathbf{w}_i \). If the \( C_i^2 \) are not all equal the \( n \) equations of 6) are parametric equations for a \( k \) dimension variety in \( n \) space of order two, \( V_k \); a brief discussion of this may be found in Kendall (1956), page 5. That is given a vector \( \mathbf{w}_i \), algebraic operations to eliminate \( \lambda_i \) and the \( \Theta_j \) will result in \( n-k \) second degree homogeneous equations in the \( \mathbf{w}_i \). If the \( C_i^2 \) are all equal say to \( c \) then 6) can be written as

\[
\mathbf{w}_i = \sum_{j=1}^k \Theta_j b_{ij}
\]

where \( \Theta_j = b_j' \mathbf{w}_i (\lambda_i - c^2) \), which are the parametric equations of a \( k \) dimension flat, \( S_k \).

Conversely if the latent vectors of \( \mathbf{T} \) all fall in a \( V_k \) of the form of 6) with

\[
\Theta_j \mathbf{w}_i b_{ij} \geq 0 \quad \text{and} \quad \lambda_i \geq C_i^2 > 0
\]

then \( \mathbf{T} \) can be written in the form 3) . If the latent vectors of \( \mathbf{T} \) all
fall in a $S_k$ with

$$\theta_{j\ell} w_j b_j \geq 0$$

9)

then it can be written in the form 1).

Thus the smallest value of $k$ for evaluating I through form 4) is determined by the smaller of the minimum dimension of the $V_k$'s of the form 6) satisfying conditions 8) and the minimum dimension of the $S_k$ satisfying the condition 9) that contain the latent vectors of $\xi$.

Alternative conditions can be given for the special case of $k = 1$.

6) becomes

$$\lambda_\ell w_{1\ell} = c_1^2 w_{1\ell} + \theta_\ell b_i$$

or

$$\frac{1}{w_{1\ell}} = -\frac{1}{\theta_\ell b_i} + \frac{\lambda_\ell}{\theta_\ell b_i}$$

10)

which is the parametric expression of a plane, $S_2$, passing through the origin. Thus if the $s_{1\ell}' = (1/w_{1\ell}, 1/w_{2\ell}, \ldots, 1/w_{nt})$ all fall in a plane with parametric conditions 10) satisfying

$$(\lambda_\ell - c_1^2) \theta_\ell w_{1\ell} b_j > 0$$

only one auxiliary variable is necessary. For the case of $c_1^2 = c^2$ for all $\ell$, $n-1$ of the $\lambda_\ell$ equal $c^2$ and their corresponding $\theta_\ell$ are zero. That is the minimum latent root of $\xi$ has multiplicity $k-1$.

4. Summary

In general the minimum number of auxiliary variables necessary to evaluate a multivariate normal integral through the method of Das (1956)
can be determined by the dimensions of the second order varieties with parametric form \(6\) and the dimensions of the flats that contain the latent vectors of the variance-covariance matrix.

This problem could also be looked at as determining the diagonal matrix \( D \) such that \( D \mathbf{X} D \) has maximum multiplicity of its smallest latent root. Then Marsaglia's (1963) solution suffices. It is interesting to note that the difficulty of the solution, the dimension of the integration in \( I \), is affected by change of scales of the normal variables.

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Marsaglia, G. (1963). Expressing the normal distribution with covariance matrix \( \mathbf{A} + \mathbf{B} \) in terms of one with covariance matrix \( \mathbf{A} \). Biometrika, 50, 535-538.
Das (1956) presents a method of evaluating the integral

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