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SIMPLIFIED FORMS OF
PRELIMINARY TRAJECTORY CALCULATION
FOR GUN-LAUNCHED VEHICLES

by

G.V. PARKINSON

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SPACE RESEARCH INSTITUTE
OF McGINU UNIVERSITY
892 Sherbrooke St. W.
Montreal 2, Quebec
Canada
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## NOMENCLATURE

- **m**: vehicle mass, slugs
- **g**: acceleration of gravity, ft/sec²
- **h**: vehicle vertical altitude, ft
- **X**: vehicle horizontal range, ft
- **θ**: vehicle trajectory elevation angle from horizontal
- **t**: time, secs
- **V**: vehicle velocity, relative to earth, fps
- **V_r**: rocket motor exhaust velocity, fps
- **J**: rocket motor burning rate, slugs/sec
- **d** = \( \frac{C_D A}{\frac{dm}{dt}} \): aerodynamic drag on vehicle, lbs
- **F** = \(-V_j \frac{dm}{dt}\): rocket motor thrust on vehicle, lbs
- **A**: vehicle cross-section reference area, sq ft
- **C_D**: drag coefficient
- **q** = \( \frac{\gamma P}{2} \): dynamic pressure, psf
- **\gamma**: ratio of atmospheric specific heats at constant pressure and constant volume
- **M** = \( \frac{V}{a} \): vehicle Mach number
- **p** = \( \frac{\varrho RT}{a} \): atmospheric pressure, psf
- **a** = \( \sqrt{\gamma RT} \): atmospheric sound speed, fps
- **T**: atmospheric temperature, °Rankine
- **R**: atmospheric gas constant, ft²/sec² °Rankine
- **\varrho**: atmospheric density, slugs/ft³
- **N_R** = \( \frac{Vd}{\gamma} \): vehicle Reynolds number
- **d**: vehicle base diameter, ft
- **\gamma**: atmospheric kinematic viscosity, ft²/sec
NOMENCLATURE (Cont'd)

\( K \)  
constant in analytic approximation to drag coefficient

\( \beta \)  
atmospheric lapse rate, \(^{\circ}\)Rankine/ft

\( \Delta(\quad) \)  
increment in any quantity corresponding to increment in altitude

\( (\quad)_{n-1} \)  
quantity evaluated at beginning of increment in altitude

\( (\quad)_{n} \)  
quantity evaluated at end of increment in altitude

\[ \Delta I_n = \int_{h_{n-1}}^{h_n} \frac{p}{\rho} \, dh \]  
aerodynamic drag integral, lb-sec/ft\(^2\)

\[ \overline{\quad}_{n} \]  
arithmetic mean of quantities \( (\quad)_{n-1} \) and \( (\quad)_{n} \)

\[ \Delta \kappa_n = \frac{2g \Delta h_n}{v_{n-1}^2} \]  
non-dimensional altitude increment

\[ \Delta \chi_n = \frac{2g \Delta x_n}{v_{n-1}^2} \]  
non-dimensional range increment

\[ \Delta \tau_{n} = \frac{g \Delta t_n}{v_{n-1}^2} \]  
non-dimensional time increment

\[ \Delta V_n = \frac{\Delta V_n}{v_{n-1}} \]  
non-dimensional vehicle velocity increment

\[ \Delta V_{g_n} = \frac{\Delta V_{g_n}}{v_{n-1}} \]  
non-dimensional vehicle gravity velocity decrement

\[ \Delta V_{D_n} = \frac{\Delta V_{D_n}}{v_{n-1}} \]  
non-dimensional vehicle aerodynamic drag velocity decrement

\[ \Delta V_{F_n} = \frac{\Delta V_{F_n}}{v_{n-1}} \]  
non-dimensional vehicle thrust velocity increment
NOMENCLATURE (Cont'd)

\( \tilde{\theta}_n \) elevation angle of chord line between points (n-1) and (n) on trajectory

\[ \eta_n = \frac{\Delta h_n}{\sin^2 \theta_{n-1}} \] non-dimensional altitude increment for universal zero-g trajectory

\[ \xi_n = \frac{\Delta \chi_n}{\sin 2 \theta_{n-1}} \] non-dimensional range increment for universal zero-g trajectory

or

\[ \xi_n = \frac{g \triangle t_n}{V_{n-1} \sin \theta_{n-1}} \] non-dimensional time increment for universal zero-g trajectory

\( B = \frac{m \bar{g}}{C_D A} \) ballistic coefficient lbs/ft²

\( \chi = \frac{F}{m \bar{g}} \) non-dimensional thrust parameter

\[ \Delta H = \frac{b_n}{m_{n-1} V_{j_n} \sin \theta_n} \Delta h \] non-dimensional altitude increment

\[ \gamma_{n-1} = \frac{m_{n-1}}{m_n} \] non-dimensional mass

\[ \nu_j = \frac{V_j}{V_{n-1}} \] non-dimensional rocket exhaust velocity

\[ \Delta J_n = \int_{h_{n-1}}^{h_n} \left( \frac{P}{a} \right) dh \] aerodynamic drag integral, lb-sec/ft²

\( c \) coefficient in exponential atmosphere model, /kft

\[ f_n = c - \frac{1.10 b_n}{0.40 \nu_j \frac{m_{n-1} V_{n-1} \sin \theta_n}{A}} \] parameter in evaluation of \( \Delta J_n \)
NOMENCLATURE (Cont'd)

\[ \xi_{n-1} = \frac{v_{n-1} b_n}{g m_{n-1}} \]  
parameter in evaluation of \( \Delta V_n \)

\[ \delta_n = \frac{\sin \theta_n}{\xi_{n-1}} \]  
parameter in evaluation of \( \Delta V_n \)

- \( U \) absolute velocity of vehicle, fps
- \( U_E \) earth surface velocity at gun muzzle, fps
- \( \chi \) angle between vectors \( \vec{V} \) and \( \vec{u}_E \), radians
- \( \psi_v \) azimuth angle of gun, radians
- \( \sigma \) path angle of trajectory (absolute) to local horizontal, radians
- \( \psi_u \) absolute azimuth angle, radians
- \( r \) radius from earth centre, ft
- \( \phi \) angle of radius vector in trajectory plane, measured from perigee, radians
- \( C = r_{n-1} u_{n-1} \cos \sigma_{n-1} \) constant in tangential equation of motion, \( ft^2/sec \)
- \( \Delta h = \frac{Ah}{r} \) non-dimensional altitude increase
- \( S, Q \) parameters introduced in solutions of integrals
- \( \Delta \phi' \) range increment angle, radians
1.0 INTRODUCTION

The precise calculation of the motion of a gun-launched vehicle through and out of the atmosphere depends on a large number of parameters. These include the environmental parameters of atmospheric properties and gravity, which vary with altitude, and effects of the earth’s rotation, which depend also on the launch direction. Then there are the vehicle parameters of mass, external size and shape, motion of control surfaces, specific impulse, mass ratio, and ignition and burning time of any rocket motors, and auxiliary or directional thrust systems. Finally there are the gun launch parameters of initial velocity and direction.

Inclusion of the accurate variation of these parameters along the trajectory leads to equations of motion which cannot be solved exactly. Consequently, various numerical computer programs have been devised, and these satisfactorily determine trajectories for given values of the parameters. In general they also can rapidly display the results of arbitrary variation of the parameters, so that optimum trajectories among those calculated can be selected.

This approach to trajectory calculation, while probably essential for accurate final calculations for a particular vehicle mission, suffers from two serious defects when applied to preliminary trajectory calculations, in which possible missions for existing or proposed vehicles are being considered. First, a computer may not be as readily available as the requirements for a set of rapid preliminary calculations would indicate. Second,
and more serious, as in all numerical programs the output of numbers has to be interpreted and analysed for trends in the light of the parameter input. There are no analytical forms linking input and output, from which trends could be discerned and predictions made, and as a result it is difficult to reach clearcut decisions about new vehicle missions.

It would therefore be useful if simplified forms of the governing equations could be developed which would retain the essential features of the exact equations and produce vehicle trajectories and other performance characteristics accurate to within 5 or even 10 percent, while providing straightforward analytical or graphical links between the various parameters and the resulting performance. That is the main purpose of the present report. An additional purpose is to provide, for reference, derivations of some of the important equations governing the motion of gun-launched vehicles.
2.0 FUNDAMENTAL ASSUMPTIONS

It will be assumed that the vehicles considered are aerodynamically symmetric with respect to their trajectory at all times, so that they experience no lift, merely drag, in their motion through the atmosphere. Any rocket thrust will be assumed to be constant in magnitude, and directed back along the trajectory, so that the vehicles are turned only by gravity. During motion of the vehicle either through the atmosphere or under the action of rocket thrust, or both, it will be assumed that the gravitational force is constant in magnitude and direction. The above assumptions will be accurate to within 3 percent for almost all cases.

2.1 BASIC EQUATION OF MOTION ALONG TRAJECTORY

In the light of the above assumptions, Figure 1 serves to define the equation of vehicle motion along its trajectory. Newton's Second Law of Motion requires 'Resultant External Force along Trajectory = Time Rate of Change of System Momentum along Trajectory'. The external forces are the tangential component of gravitational force mg sin $\theta$ and the aerodynamic drag $D$. The rocket motor thrust $F$ is an internal force accounted for by the system momentum change. Consider the vehicle at time $t$ when its mass is $m$ and its velocity $V$. After an infinitesimal time increment $\Delta t$ the rocket exhaust of constant velocity $V_j$ has expelled mass $-\Delta m$ (the minus sign preserves the algebraic sign convention) and the velocity of the vehicle has increased to $V + \Delta V$. Thus Newton's Second Law becomes:
Figure 1  Vehicle Trajectory Parameters
\[-D - mg \sin \theta = \lim_{\Delta t \to 0} \frac{(m + \Delta m)(V + \Delta V) - \Delta m(V - V_j)}{\Delta t} - \Delta V\]

\[= m \frac{dv}{dt} + V_j \frac{dm}{dt}\]

or

\[F - D - mg \sin \theta = m \frac{dv}{dt}\] \hspace{1cm} (2.1)

where \( F = -V_j \frac{dm}{dt} = V_j b \), and \( b \) is the constant motor burning rate.

If Eq. (2.1) is multiplied through by \( \frac{dt}{m} \) and then integrated from an initial state denoted \( h_0 \) on the trajectory to a final state denoted \( h_n \) the result, using the fact that the rate of increase of altitude,

\[\frac{dh}{dt} = V \sin \theta\] \hspace{1cm} (2.2)

is

\[V_n = V_{n-1} + V_j \ln \frac{m_{n-1}}{m_n} - \int_{h_{n-1}}^{h_n} \frac{D}{mV \sin \theta} \, dh - g \int_{h_{n-1}}^{h_n} \frac{dh}{V}\]

or

\[\Delta V_n = \Delta V_D_n - \Delta V_n - \Delta V_g_n\] \hspace{1cm} (2.3)

In Eq. (2.3), the integrals for the velocity decrements from aerodynamic drag and gravity cannot be evaluated by quadratures without the introduction of further assumptions. The integral for \( \Delta V_D_n \) is considered first.

2.2 AERODYNAMIC DRAG OF GUN-LAUNCHED SYSTEMS

Aerodynamic drag is conventionally expressed in terms of a drag coefficient \( C_D \) through the defining equation

\[D = C_D q A\] \hspace{1cm} (2.4)

where \( A \) = reference area = projected frontal area of body for a rocket or
gun-launched vehicle,

\[ q = \text{dynamic pressure}. \]

In high speed aerodynamics the drag is caused by both the compressibility and friction of the air, and therefore the drag coefficient depends upon both Mach number \( M \) and Reynolds number \( N_R \),

\[ C_D = C_D(M, N_R) \] (2.5)

where

- \( M = \frac{V}{a} \) = Vehicle Mach number
- \( N_R = \frac{Vd}{\nu} \) = Vehicle Reynolds number
- \( a \) = local atmospheric speed of sound
- \( d \) = vehicle base diameter
- \( \nu \) = local atmospheric kinematic viscosity.

It is convenient to express the dynamic pressure in terms of Mach number,

\[ q = \frac{\gamma p}{2} M^2 \] (2.6)

where

- \( \gamma \) = ratio of atmospheric specific heats at constant pressure and constant volume
  - \( = 1.40 \), assumed constant
- \( p \) = local atmospheric pressure.

The dependence of \( C_D \) on \( M \) and \( N_R \) is complex, even for simple vehicle shapes, and it has been common to make approximations in performance analysis. In the present analysis, the nature of gun-launching permits a very simple and useful approximation to be made. All gun-launched vehicles have initial
Mach numbers greater than 3, and even without rocket boost, vehicle Mach numbers will rarely fall much below 3 during motion through atmosphere dense enough to produce significant drag. Therefore it is not necessary to consider the drag variation of the vehicles in subsonic and transonic flight (as is required for conventional rocket-launched systems). Only the drag variation in supersonic and hypersonic motion need be considered and this is much simpler.

Over the range of velocities through the atmosphere experienced by a vehicle on a typical gun-launched mission, the variation of \( C_D \) with \( N \) will be quite small, and it is assumed henceforth that

\[
C_D = C_D (N) \tag{2.7}
\]

only for a particular vehicle. This variation can be approximated quite accurately in the supersonic and hypersonic range by the relation

\[
C_D = \frac{K}{M} \tag{2.8}
\]

where \( K \) is a constant chosen to provide the best fit of Eq. (2.8) to the available data for the vehicle over the Mach number range of interest. In Figure 2 such a fit is shown for the Martlet 2A glide vehicle, the data being obtained from Reference 1. In the figure the maximum deviation of the approximate from the actual curve is 7.7 percent, and since it is clear that positive and negative deviations will tend to cancel in the drag integral of Eq. (2.3), and that in any case the actual values of \( C_D \) may not be accurately known at the time of the preliminary calculations for which this analysis is intended, the approximation of \( C_D \) by Eq. (2.8) is seen to be satisfactory.
Therefore, Eq. (2.4) becomes

\[ D = \frac{KYA}{2} \rho M \]  

and the drag integral can be written

\[ \Delta V_{D_n} = \frac{KYA}{2} \int_{h_{n-1}}^{h_n} \left( \frac{\rho}{a} \right) \frac{dh}{m \sin \theta} \]  

In the integrand in Eq. (2.10), \((\rho/a)\) is a property of the local atmosphere, and is a function of \(h\) only for a given model of the atmosphere. As the vehicle climbs through the atmosphere after gun launch, \(\sin \theta\) is a slowly decreasing function of \(h\) which of course is unknown until the trajectory is determined. However, the initial value is known, and because of the slow decrease with increasing \(h\), a sufficiently accurate average value \(\sin \theta_n\) can be determined from the equivalent zero-drag trajectory between the same altitude limits. The drag integral can then be written

\[ \Delta V_{D_n} = \frac{KYA}{2 \sin \theta_n} \int_{h_{n-1}}^{h_n} \frac{dh}{m} \]  

For vehicles without rocket motors, or for glide sections of any gun-launched vehicle trajectory, \(m\) is constant and Eq. (2.11) can be evaluated directly. During motion through the atmosphere under rocket thrust, \(m\) decreases at a constant time rate, and again the equivalent zero-drag trajectory can be used to approximate \(m\) as an integrable function of \(h\) between the same altitude limits.

The significant observation to be made from Eq. (2.11) is that,
within the accuracy of Eq. (2.8), the decrement in velocity experienced by a supersonic or hypersonic gun-launched vehicle due to atmospheric drag is not increased by increasing the launch velocity of the vehicle, as one might expect intuitively.

On the contrary, for a vehicle without rocket motor, or during a glide section of any vehicle trajectory, the drag velocity decrement is nearly independent of the vehicle velocity, which enters only through its rather small effect on the value of \( \sin \theta_n \) in Eq. (2.11), and here the effect of increased launch velocity is to increase \( \sin \theta_n \) and thus reduce the drag decrement.

For a rocket-powered section of vehicle trajectory with a motor of given burning rate, the effect of increased launch velocity is to reduce the decrease of vehicle mass \( m \) in passing through a given altitude increment, and thus again reduce the drag velocity decrement.

In general, then, Eq. (2.11) shows that for any gun-launched vehicle, the higher the launch velocity, the lower will be the resulting decrement in velocity due to aerodynamic drag.

2.3 THE MODEL ATMOSPHERE

Before Eq. (2.11) can be integrated, the dependence of \( (p/a) \) on \( h \) for the atmosphere under consideration must be put in suitable analytic form. Here the model used assumes a linear variation of temperature \( T \) with altitude,
\[ T = T_{n-1} - \beta (h - h_{n-1}) \]  

(2.12)

where

\[ \beta = \text{lapse rate} = \text{constant}. \]

The air is assumed to obey the equation of state of a perfect gas

\[ p = \rho RT \]  

(2.13)

where \( \rho = \text{air density} \)

\[ R = \text{specific gas constant} \]

\[ = 1716 \text{ ft}^2/(\text{sec}^2 \cdot \circ\text{Rankine})^{-1}. \]

The additional relation needed is the vertical equilibrium equation for the static atmosphere

\[ \frac{dp}{dh} = -g \]  

(2.14)

If \( \rho \) and \( T \) are eliminated from Eqs. (2.12), (2.13), and (2.14) the result is

\[ \frac{dp}{dh} = - \frac{p g}{R} \left[ T_{n-1} - \beta (h - h_{n-1}) \right] \]

or

\[ \int_{p_{n-1}}^{p} \frac{dp}{p} = - \frac{g}{R} \int_{h_{n-1}}^{h} \frac{dh}{T_{n-1} - \beta (h - h_{n-1})} \]

This equation is integrated directly to produce the pressure-altitude relation

\[ \frac{p_n}{p_{n-1}} = \left\{ \frac{T_{n-1} - \beta (h - h_{n-1})}{T_{n-1}} \right\}^{\frac{g}{R}} \]  

(2.15)
The speed of sound in a perfect gas is given by

$$a = \sqrt{\gamma RT}$$

(2.16)

and so, using Eq. (2.12),

$$a = \sqrt{\gamma R T_n^{-1} - \beta(h - h_{n-1})}$$

(2.17)

Therefore, the required function for \((p/a)\) is, with a little rearrangement

$$\left(\frac{p}{a}\right) = \left(\frac{p_{n-1}}{a_{n-1}}\right) \left[\frac{T_{n-1} - \beta(h - h_{n-1})}{T_{n-1}}\right] \frac{8}{\beta R} = \frac{1}{a}$$

(2.18)

For a particular atmosphere, \(\beta\) is chosen to fit the actual temperature variation within given altitude limits. In this report, the reference atmosphere is the Cape Kennedy Standard Atmosphere from Reference 2, and a very good fit is obtained using 3 values of \(\beta\), as follows:

- \(0 < h < 50.8\) kft \(\beta_1 = 3.54^{\circ}R/kft\)
- \(50.8 < h < 162.4\) kft \(\beta_2 = -1.235^{\circ}R/kft\)
- \(162.4 < h < 270.6\) kft \(\beta_3 = 1.574^{\circ}R/kft\)

Figures 3 and 4 show the close agreement between the pressure and sound speed from Reference 2 and the values given by Eqs. (2.15) and (2.17), using the above values of \(\beta\). For this atmosphere, at sea level \((h = 0)\):

\[\begin{align*}
 p_o &= 2125\text{ psf}, a_o = 1138\text{ fps}, g = 32.15\text{ ft/sec}^2
\end{align*}\]
Figure 4: Atmospheric Sound Speed vs Altitude

Figure 5: Atmospheric Pressure vs. Altitude
3.0 GLIDE TRAJECtORIES

When a gun-launched vehicle is not moving under rocket motor thrust, mass \( m \) is constant, and Eq. (2.11) becomes

\[
\Delta V_n = \frac{K \tilde{\gamma} A}{2m \sin \theta_n} \int_{h_{n-1}}^{h_n} \left( \frac{P}{a} \right) dh = \frac{K \tilde{\gamma} A}{2m \sin \theta_n} \Delta I_n
\]

(3.1)

The integral \( \Delta I_n \) is evaluated using Eq. (2.18) and the solution of Eq. (2.3) can then be considered.

3.1 EVALUATION OF DRAG INTEGRAL

\[
\Delta I_n = \left( \frac{a_{n-1}}{a_n} \right) \frac{1}{T_{n-1}^2} \frac{1}{\beta T_{n-1}^2} \left( \frac{g}{R} + \frac{1}{2} \right) \left[ T_{n-1}^2 - \beta(h_n - h_{n-1}) \right] \frac{g}{R} + \frac{1}{2} dh
\]

\[
= \left( \frac{a_{n-1}}{a_n} \right) \frac{1}{\beta T_{n-1}^2} \frac{1}{\left( \frac{g}{R} + \frac{1}{2} \right)} \left[ T_{n-1}^2 - \beta(h_n - h_{n-1}) \right] \frac{g}{R} + \frac{1}{2}
\]

\[
= \left[ \frac{p_{n-1} a_{n-1} - p_n a_n}{g + \frac{\beta R \psi}{2}} \right] \text{lb-sec/ft}^2 \quad \text{using eqs. (2.15) and (2.17).} 
\]

(3.2)

Using the parameters for the model of the Cape Kennedy Standard Atmosphere, this function has been calculated for increments of altitude from sea level until incremental values of the function become negligible. The results are given in Table 1, and the total function \( I_n \) from that Table is plotted in Figure 5. It can be seen that the drag decrement in vehicle velocity becomes negligible above 160 kft.
### TABLE 1

Drag Decrement Integrals $\Delta I_n$ and $I_n$

<table>
<thead>
<tr>
<th>No.</th>
<th>h kft</th>
<th>$\Delta I_n$ lb-sec/ft$^2$</th>
<th>$I_n$ lb-sec/ft$^2$</th>
<th>No.</th>
<th>h kft</th>
<th>$\Delta I_n$ lb-sec/ft$^2$</th>
<th>$I_n$ lb-sec/ft$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>14</td>
<td>65.6</td>
<td>2740</td>
<td>47100</td>
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<tr>
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<td>1370</td>
<td>48500</td>
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<td>2.00</td>
<td>1950</td>
<td>4030</td>
<td>16</td>
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<td>630</td>
<td>49100</td>
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<td>3</td>
<td>3.28</td>
<td>1810</td>
<td>5840</td>
<td>17</td>
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<tr>
<td>4</td>
<td>6.56</td>
<td>5420</td>
<td>11260</td>
<td>18</td>
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<td>141</td>
<td>49500</td>
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<td>4660</td>
<td>15920</td>
<td>19</td>
<td>147.6</td>
<td>67</td>
<td>49600</td>
</tr>
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<td>4240</td>
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<td>49700</td>
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<td>196.8</td>
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<td>213.2</td>
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<td></td>
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<td>229.6</td>
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<td></td>
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<td>44400</td>
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</table>

**TOTAL** 49,700 lb-sec/ft$^2$
Figure 5 Total Drag Decrement Integral

\[ I_n = \int_0^{h_n} \left( \frac{1}{\rho_n} \right) dh \]

CAPE KENNEDY STANDARD ATMOSPHERE

\[ \rho_0 = 14.76 \text{ psi}, \quad a_0 = 1187 \text{ fps} \]
3.2 ZERO-DRAG GLIDE TRAJECTORIES

The equations of motion for a vehicle moving without rocket thrust or atmospheric drag under constant g are very simple. They are useful in the present analysis both in providing a basis for approximating certain functions in Eq. (2.3), and in themselves as giving sufficiently accurate results for the upper parts of vehicle trajectories high enough to neglect aerodynamic drag but not so high as to require the inclusion of g variations.

The first relation needed is obtained directly from the conservation of mechanical potential and kinetic energy:

\[ h_n - h_{n-1} = \frac{V_{n-1}^2 - V_n^2}{2g} \]  \hspace{1cm} (3.3)

The other relations needed are obtained by manipulating the equations of vehicle motion in the X- and h- directions (see Figure 1), and are derived in Appendix A:

\[ \Delta h_n = \tan \frac{\theta_n}{\theta_{n-1}} \Delta X_n - \frac{1}{4} \text{sec}^2 \frac{\theta_n}{\theta_{n-1}} \Delta X_n^2 \]  \hspace{1cm} (3.4)

\[ \sin \theta_n = \sqrt{\frac{\sin^2 \theta_{n-1} - \Delta h_n}{1 - \Delta h_n}} \]  \hspace{1cm} (3.5)

or

\[ \cos \theta_n = \frac{\cos \theta_{n-1} \sqrt{1 - \Delta h_n}}{\sqrt{1 - \Delta h_n}} \]  \hspace{1cm} (3.6)

where

\[ \Delta h_n = \frac{2g}{V_{n-1}^2} (h_n - h_{n-1}), \ \Delta X_n = \frac{2g}{V_{n-1}^2} (X_n - X_{n-1}) \]  \hspace{1cm} (3.7)
3.3 APPROXIMATIONS FOR GLIDE TRAJECTORIES WITH DRAG

For a vehicle moving without rocket thrust, Eq. (2.3) can be written, using Eq. (3.1):

\[ \Delta v_n = - \Delta v_{D_n} - \Delta v_{g_n} \]

\[ = \frac{K \cdot A}{2m \sin \frac{v_n}{n}} \Delta I_n - g \int_{h_n}^{h_{n-1}} \frac{dh}{v} \]  

(3.8)

If there were no aerodynamic drag, Eq. (3.3) could be used, and written in the form showing the velocity decrement caused by gravity:

\[ \Delta v_n = \frac{2g (h_n - h_{n-1})}{v_n + v_{n-1}} \]

or

\[ \Delta v_n = \frac{g \Delta h_n}{\bar{v}_n} = \Delta v_{g_n} \]  

(3.9)

where

\[ \bar{v}_n = \frac{1}{2} (v_{n-1} + v_n) = v_{n-1} + \frac{1}{2} \Delta v_n \]  

(3.10)

It is now assumed that \( \Delta v_{g_n} \) in Eq. (3.8) can be approximated by an expression of the form of Eq. (3.9), with \( \bar{v}_n \) the arithmetic mean of the actual \( v_{n-1} \) and \( v_n \):

\[ \Delta v_n = - \Delta v_{D_n} - \frac{g \Delta h_n}{\bar{v}_n} \]

or, multiplying through by \( \bar{v}_n \) and using Eq. (3.10):

\[ \frac{1}{2} \Delta v_n^2 + (v_{n-1} + \frac{1}{2} \Delta v_{D_n}) \Delta v_n + (v_{n-1} \Delta v_{D_n} + g \Delta h_n) = 0. \]
Solving for \( \Delta V_n \), and dividing through by \( V_{n-1} \):

\[
\frac{\Delta V_n}{V_{n-1}} = - (1 + \frac{1}{2} \frac{\Delta V_D}{V_{n-1}}) + \sqrt{(1 + \frac{1}{2} \frac{\Delta V_D}{V_{n-1}})^2 - 2 \frac{\Delta V_D}{V_{n-1}} - \Delta \theta_n}
\]

or

\[
\Delta v_n = - (1 + \frac{1}{2} \Delta v_D) + \sqrt{(1 - \frac{1}{2} \Delta v_D)^2 - \Delta \theta_n}
\]  \(3.11\)

where

\[
\Delta v_n = \frac{\Delta V_n}{V_{n-1}}, \quad \Delta v_D = \frac{\Delta V_D}{V_{n-1}}
\]  \(3.12\)

Before Eq. (3.11) can be used to calculate velocity decrements along a glide trajectory, a method of determining \( \sin \theta \) in the expression for \( \Delta V_D \) must be given. It can be shown that

\[
\frac{1}{\sin \theta_n} \int_{h_{n-1}}^{h_n} dh
\]

is a good approximation to

\[
\int_{h_{n-1}}^{h_n} \frac{dh}{\sin \theta}
\]

where

\[
\sin \theta_n = \frac{1}{2} (\sin \theta_{n-1} + \sin \theta_n)
\]  \(3.13\)

and \( \sin \theta \) is given by Eq. (3.5).

For example, with \( \sin \theta_{n-1} = 0.50 \) and \( \sin \theta_n = 0.25 \), the approximate formula is only 1% higher than the exact integral.
Therefore \( \sin \theta_n \) as defined by Eq. (3.13) is used in Eq. (3.1).

In the calculation procedure next to be described, the fundamental independent variable is the altitude increment \( \Delta h_n \). Calculation of the vehicle trajectory requires corresponding values of the horizontal range increment \( \Delta X_n \), shown in Figure 6. Since

\[
\Delta X_n = \cot \tilde{\theta}_n \Delta h_n
\]

and \( \tilde{\theta}_n \) will not be appreciably different from \( \theta_n \) as given by Eq. (3.13), it is assumed that a satisfactory approximation is

\[
\Delta X_n = \cot \tilde{\theta}_n \Delta h_n \quad (3.14)
\]

In the same way it is assumed that the time increment \( \Delta t_n \) is given by

\[
\Delta t_n = \frac{1}{V_n \sin \theta_n \Delta h_n} \quad (3.15)
\]

Figure 6

_Basis of Horizontal Range Increment Approximation_
3.4 CALCULATION PROCEDURE FOR CONSTANT-g GLIDE TRAJECTORIES

For a given vehicle launched under given conditions in a given model atmosphere, the following parameters are known:

\[ g, m, A, K, \beta, V_0, \theta_0 \]

The factor \( \frac{K A}{2m} \) in \( \Delta V_n \) can therefore be calculated. The drag integral \( \Delta I_n \) can be determined for each altitude increment \( \Delta h_n \) from Figure 5 or Table 1. The altitude to which the aerodynamic drag is significant is broken down into a few increments, the number increasing with the accuracy desired. Thus four increments should be sufficient for most preliminary calculations, while one may be enough for a rough calculation.

The procedure is then to use the equations of 3.2 and 3.3 to determine the velocity decrement, horizontal range increment, time increment, and final elevation angle for each altitude increment. Above the final atmospheric altitude increment, the equations of 3.2 and Appendix A are used to calculate the remainder of the trajectory to its apogee. Alternatively, Eq. (3.4) for the parabolic zero-drag trajectory can be put in the universal form:

\[ \eta = 2 \xi - \xi^2 \]  \hspace{1cm} (3.16)

where \( \xi \) and \( \eta \) are defined in Appendix A. Eq. (3.16) is plotted in Figure 7 and all zero-drag, constant-g glide trajectories can be determined from it.
Figure 7 Universal Zero-Drag Constant-g Trajectory
In greater detail, the procedure for the atmospheric part of the trajectory, given $h_{n-1}$, $\Delta h_n$, $V_{n-1}$, $\theta_{n-1}$, is to calculate $\Delta \phi_n$ from Eq. (3.7) and use this to calculate $\sin \theta_n$ from Eq. (3.5) or $\cos \theta_n$ from Eq. (3.6). (The latter is more convenient for values of $\theta_{n-1}$ near $90^\circ$.) This determines the new elevation angle $\theta_n$. Next $\sin \theta_n$ is found from Eq. (3.13), giving $\overline{\theta}_n$. With $\Delta I_n$ determined for the given $\Delta h_n$ from Figure 5 or Table 1, $\Delta V_D_n$ is calculated from Eq. (3.8), and $\Delta U_D_n$ from Eq. (3.12). Now $\Delta V_n$ can be calculated from Eq. (3.11), and this gives the velocity change $\Delta V_n$ from Eq. (3.12). The range increment $\Delta X_n$ is calculated from Eq. (3.14) and this gives the new $X_n$ corresponding to $h_n$. The time increment $\Delta t_n$ is calculated from Eq. (3.15), giving the new $t_n$. The new velocity $V_n$ is given by adding $\Delta V_n$ to $V_{n-1}$. The procedure is then repeated for the next altitude increment.

Figure 8 gives the application of the procedure to the trajectory of the Martlet 2A, IOWA shot from the Barbados 16" gun on March 22, 1965. Results of calculations with both four and one atmospheric altitude increments are compared with radar-determined trajectories of the actual shot from Reference 1, and with the standard HARP trajectory from Reference 3. Curves are given for vehicle weights of both 170 lbs and 180 lbs, since the actual vehicle weighed 170 lbs, while the HARP trajectory was calculated for 180 lbs. Figures 9 and 10 show the variation of vehicle velocity and elapsed time with altitude, up to the apogee, for the 180 lb vehicle as calculated by the present method with both four and one atmospheric steps, and by the standard HARP computer program. Calculations for Figures 8, 9 and 10 are tabulated in Appendix B. In Figure 11, the 180 lb trajectories are compared again for greater clarity.
Figure 10: Altitude vs Elapsed Time for Martlet 2A Shot

\[ W = 180 \text{ lbs}, \quad V_0 = 5400 \text{ fps}, \quad \theta_0 = 65.00^\circ \]

1. HARP Computer Program (Ref. 3)
2. Present Theory (4 Atmos. Steps)
3. \( \text{1 Atmos. Step} \)
Figure 11 Martlet 2A Trajectories

W = 180 lbs., \( V_0 = 5.400 \text{ kfps} \), \( \theta_0 = 85.00^\circ \)

1. HARP COMPUTER PROGRAM (REF. 3)
2. PRESENT THEORY (4 ATMOS. STEPS)
3. " " (1 ATMOS. STEP)
A comparison that makes greater demands on the method is shown in Figure 12. Here a projectile is analyzed from launch to impact, with the entire trajectory low enough that measurable aerodynamic drag is experienced throughout. The present method with four atmospheric altitude increments for both climb to and descent from 80,000 ft and altitudes above 80,000 ft assumed non-atmospheric, is compared with the standard HARP computer program, and elapsed time, velocity, and altitude are plotted against range. Calculations are given in Appendix C.

3.5 DISCUSSION

The calculation procedure is very simple and, even for a choice of several altitude increments in the atmosphere, it can be carried out quickly using just a slide rule or a slide rule and trigonometric tables. Despite this simplicity, the curves of Figures 8, 9, 10, 11 and 12 show that it produces accurate results, agreeing closely both with actual measurements and with more exact numerical procedures.

Figure 8 shows that a four-step atmospheric calculation predicts the observed trajectory of Martlet 2A IOWA very closely, with the apogee only 1.8% higher than that observed.

Figures 9, 10 and 11 show that the four-step atmospheric calculation for a 180 lb Martlet 2A gives results nearly identical with those of the HARP computer program. Moreover, even the one-step atmospheric calculation gives good accuracy in this example, the velocity values and elapsed times lying quite close to the HARP curves, and the trajectory having an apogee only 3.0% high.
Figure 12 Trajectory Comparisons for Projectile
Figure 12 shows that even a low-altitude projectile trajectory is calculated from launch to impact with good accuracy by the method. Errors in calculated quantities at points along the trajectory are generally less than 5%, and in some of the more important quantities the errors are:

- range (+1.9%),
- apogee (+4.2%),
- range at apogee (+1.1%),
- velocity at apogee (-3.5%),
- time to impact (+4.1%),
- time to apogee (+3.3%),
- velocity at impact (-9.0%).

The larger error in impact velocity is caused by matching the formula for drag coefficient, \( C_D = \frac{K}{M} \), to the curve used in the computer program at \( M = 4.4 \). This gives excellent agreement at the higher Mach numbers, but produces higher drag at the lower Mach numbers preceding impact.
4.0 ROCKET POWERED TRAJECTORIES

The altitude change during the burning of a rocket motor stage of a vehicle is in general small enough to permit neglect of the change in g during motor burning. For altitudes below about 160 kft, however, effects of aerodynamic drag cannot be neglected, and the vehicle velocity increment \( \Delta V_n \) resulting from rocket thrust is then given by Eq. (2.3), with the drag velocity decrement \( \Delta V_D^n \) given by Eq. (2.11) and the gravity velocity decrement \( \Delta V_g^n \) given by Eqs. (3.9) and (3.10). In this section the evaluation of \( \Delta V_n \) from Eq. (2.3) and the determination of other rocket-powered trajectory parameters are described.

4.1 EVALUATION OF ELEVATION ANGLE

Before \( \Delta V_D^n \) can be evaluated from Eq. (2.11), a suitable determination of the mean value \( \overline{\sin \theta}^n \) must be made for a rocket-powered trajectory, and the dependence of mass \( m \) on altitude \( h \) during thrusting must be expressed so that the integral can be evaluated.

As in the glide trajectories of the previous section, the determination of \( \overline{\sin \theta}^n \) is approached by neglecting the direct effect of drag on elevation angle. It is shown in Appendix D that in zero-drag, constant-g conditions, the differential equation governing the variation of elevation angle \( \theta \) with time \( t \) under the action of thrust \( F \) is

\[
\frac{d^2 \theta}{dt^2} + (2 \tan \theta - \alpha \sec \theta) \left( \frac{d\theta}{dt} \right)^2 = 0
\]  

(4.1)

where

\[ \alpha = \frac{F}{mg} \]
For the constant thrust rocket motors under consideration here, $\alpha$ varies with time and Eq. (4.1) is a fairly complicated nonlinear differential equation. Two special cases of the equation can, however, be treated easily. If $\alpha = 0$, of course, Eq. (4.1) governs zero-drag glide trajectories and the solution of Eqs. (3.5) and (3.6) is obtained. If $\alpha = \text{constant}$, it is shown in Appendix D that Eq. (4.1) is easily reduced to the integral form:

$$\int_{\theta_{n-1}}^{\theta_n} \cos^{n-2} \theta \, d\theta = \frac{\cos^{n-1} \theta_n \Delta \gamma_n}{(1 + \sin \theta)^n}$$  \hspace{1cm} (4.2)

where

$$\Delta \gamma_n = \frac{g \Delta \gamma}{v_{n-1}}$$

This is integrable by quadratures for $\alpha = 0, 1, 2, 3$, and it is readily evaluated graphically or numerically for other values of $\alpha$. Eq. (4.2) shows that $\theta_n = \theta_n (\theta_{n-1}, \alpha, \Delta \gamma_n)$. Now, $\Delta \mathcal{H}_n$, defined by Eq. (3.7) is also given by the functional form $\Delta \mathcal{H}_n = \Delta \mathcal{H}_n (\theta_{n-1}, \alpha, \Delta \gamma_n)$, so that $\theta_n$ can be expressed

$$\theta_n = \theta_n (\theta_{n-1}, \alpha, \Delta \mathcal{H}_n)$$  \hspace{1cm} (4.3)

Eq. (4.3) must reduce to the value given by Eqs. (3.5) or (3.6) for $\alpha = 0$, and must reduce to $\theta_n = \theta_{n-1}$ for $\alpha \to \infty$. It is therefore plausible to try as an approximation suitable for easy calculation the analytic forms

$$\sin \theta_n = \sqrt{\frac{\sin^2 \theta_{n-1} - f(\alpha) \Delta \mathcal{H}_n}{1 - f(\alpha) \Delta \mathcal{H}_n}}$$  \hspace{1cm} (4.4)
or \[ \cos \theta_n = \frac{\cos \theta_{n-1}}{\sqrt{1 - f(\infty)\Delta \kappa_n}} \] (4.5)

where \( f(0) = 1, f(\infty) = 0. \)

An exponential form for \( f(\infty) \) suggests itself, and it was determined empirically as follows. It is shown in Appendix D that, for constant \( \infty \),

\[ \Delta \kappa_n = 2 (\infty + \sin \theta_{n-1}) \int_0^{\Delta \kappa_n} \frac{\sin \theta \, d\theta}{\infty + \sin \theta} + 2 (\infty^2 - 1) \int_0^{\Delta \kappa_n} \frac{\Delta \kappa \sin \theta \, d\theta}{\infty + \sin \theta} \] (4.6)

Eqs. (4.2) and (4.6) were integrated numerically to obtain exact solutions for constant \( \infty \) with which values obtained from Eq. (4.4) with different choices of \( f(\infty) \) could be compared. The final choice of \( f(\infty) \) was

\[ f(\infty) = e^{-0.250 \infty^{0.7}} \] (4.7)

In Fig. 13 this function is compared with the exact solutions at \( \Delta \kappa_n = 0.25 \) for \( \theta_{n-1} = 30^0 \) and 60\(^0\) over the relevant range of \( \infty \). \( f(\infty) \) was chosen to lie somewhat below the exact curves for two reasons. The first is to take some account of the effect of drag, otherwise neglected in this analysis. The second is to relate to the actual constant-thrust rocket motors under consideration, for which \( \infty \) increases steadily from ignition to burn-out. In applying the analysis to such motors, a mean value of \( \infty \),

\[ \overline{\infty}_n = \frac{1}{2} (\infty_{n-1} + \infty_n) \] (4.8)

is used for simplicity, but this tends to make \( f(\overline{\infty}_n) \) too small, and therefore \( \sin \theta_n \) too large. With \( \sin \theta_n \) given by Eq. (4.4) and (4.7), \( \sin \theta_n \) is given by Eq. (3.13).
Figure 13 $\sin \theta_n$ vs $\omega$ for $\Delta \omega_n = 0.25$
4.2 EVALUATION OF DRAG INTEGRAL

Since the drag velocity decrement is a small fraction of the velocity achieved under rocket thrust, the mass-altitude relation needed to determine it can be based with sufficient accuracy on the equivalent zero-drag trajectory, with \( \sin \theta = \frac{\sin \theta}{n} \). Eq. (2.1) can be written

\[
m \frac{dV}{dt} = - \frac{V_i}{j_n} \frac{dm}{dt} - mg \frac{\sin \theta}{n}
\] (4.9)

On multiplication by \( \frac{dt}{m} \) and integration this gives

\[
V = \frac{1}{\sin \theta} \frac{dh}{dt} = V_{n-1} - \frac{V_j}{j_n} \ln \frac{m}{m_{n-1}} - g \frac{\sin \theta}{n} (t - t_{n-1})
\] (4.10)

A second integration, using the motor burning relation

\[
m = m_{n-1} - b_n (t - t_{n-1})
\] (4.11)

gives

\[
h = h_{n-1} + \frac{(V_{n-1} + V_j) \sin \theta}{b_n} + \frac{V_j \sin \theta}{b_n} \ln \frac{m}{m_{n-1}} - \frac{g \sin \theta}{2b_n} (m_{n-1} - m)^2
\] (4.12)

In Eq. (4.12) it should be noted that the last term is very small in comparison with the second term on the right side. Thus, their ratio is

\[
\frac{g \sin \theta}{2b_n (V_{n-1} + V_j) j_n} = \frac{g \sin \theta}{2(V_{n-1} + V_j) j_n} = \text{Order (10)}^{-2}
\]

using typical values for the parameters. Accordingly, the last term can be neglected in the expression for \( m \) for use in Eq. (2.11). The remaining terms
are made dimensionless by multiplication by
\[ b_n \frac{m_{n-1} V_i n \sin \theta_n}{V_{j n}} \]
giving:
\[ \Delta H = b_n \frac{m_{n-1} V_i n \sin \theta_n}{V_{j n}} \left( h - h_{n-1} \right) = \left( \frac{V_{j n}}{V_{j n}} + 1 \right) (1 - \eta) + \eta \ln \eta \]
\[ (4.13) \]
where
\[ \eta = \frac{V_{j n}}{V_{n-1}}, \quad \eta = \frac{m_{n-1}}{m_{n-1}} \]

Eq. (4.13) is plotted for relevant values of \( \eta \) in Fig. 14. (The subscript \( n \) of course refers to the evaluation of the function for an altitude increment \( \Delta h_n = h_n - h_{n-1} \).)

Although a simple expression, Eq. (4.13) is not directly useful in evaluating Eq. (2.11), since it is transcendental and \( \eta \) cannot be given explicitly as a function of \( \Delta H \). However, \( \eta \) is unity for \( \Delta H = 0 \) and \( \eta \) approaches zero as \( \Delta H \) becomes large, so approximation by a function of the form
\[ \eta = e^{-f(\psi_j n)} \Delta H \]
\[ (4.14) \]
is plausible. It was found empirically by comparison with the curves of Fig. 14 that the function
\[ f(\psi_j n) = 1.10 \psi_j n^{0.60} \]
\[ (4.15) \]
gives satisfactory agreement with Eq. (4.13). Thus, Eq. (4.14), using
Figure 14 Altitude-Mass Relation for Zero-Drag, Constant-Thrust Motor
Eq. (4.15), is plotted on Fig. 14 for $\nu_j^{\text{in}} = 1.0$ and 3.0, and the agreement with Eq. (4.13) is seen to be good for $\nu_j$ greater than 0.3, the relevant range. Accordingly, Eq. (2.11) is put in the form

$$\Delta V_n = \frac{k x A}{2 n-1 \sin \theta_n} \Delta J_n$$

(4.16)

where

$$\Delta J_n = \int_{h_{n-1}}^{h_n} \left( \frac{p}{a} \right) \frac{dh}{\nu_j}$$

(4.17)

and $\nu_j$ is given by Eq. (4.14) and (4.15).

Unfortunately, however, the function for $(p/a)$ given by Eq. (2.18) is not integrable by quadratures in combination with Eq. (4.14). It is therefore replaced for this integration by the function

$$\left( \frac{p}{a} \right) = \left( \frac{p}{a} \right)_{n-1} e^{-c(h - h_{n-1})}$$

(4.18)

where $c$ is determined empirically to give good agreement with the true variation of $(p/a)$ over a selected altitude range. Eq. (4.18) would be exact for an isothermal atmosphere, and is a good approximation for an actual atmosphere over limited ranges of altitude. Thus, in Fig. 15, Eq. (4.18) is seen to give excellent agreement with the Cape Kennedy Standard Atmosphere for altitudes between 40 kft and 120 kft, with $c = 0.04808/\text{kft}$.

With the combination of Eq. (4.14) and (4.18), $\Delta J_n$ can be evaluated:
Figure 15: $(\frac{P}{a})$ vs h for Cape Kennedy Standard Atmosphere
\[ \Delta J_n = \left( \frac{p}{a} \right)_{n-1} \int_{h_{n-1}}^{h_n} \left[ -c + \frac{1.10}{r_{j_n} V_{n-1} \sin \theta_n} \right] (h - h_{n-1}) \, dh \]

\[ = \left( \frac{p}{a} \right)_{n-1} \frac{1}{r_n} \left[ 1 - e^{-\int \Delta h_n} \right] \quad (4.19) \]

where

\[ \int_n = c - \frac{1.10 h_n}{r_{j_n} V_{n-1} \sin \theta_n} \quad (4.20) \]

4.3 DETERMINATION OF VELOCITY AND TRAJECTORY PARAMETERS

It is now possible to consider the evaluation of the net velocity increment \( \Delta V_n \) arising from the effects of rocket motor thrust, aero-
dynamic drag, and gravity:

\[ \Delta V_n = \Delta V_F_n - \Delta V_D_n - \Delta V_{g_n} \quad (2.3) \]

or

\[ \Delta V_n = \Delta V_F_n - \Delta V_D_n - \frac{g \Delta h_n}{V_{n-1} + \frac{1}{2} \Delta V_n} \]

using Eq. (3.9) and (3.10). This is made non-dimensional by division by

\[ V_{n-1} : \]

\[ \Delta \bar{V}_n = \Delta \bar{V}_F_n - \frac{\Delta \bar{h}_n}{2 + \Delta \bar{V}_n} - \Delta \bar{V}_{D_n} \quad (4.21) \]

where

\[ \Delta \bar{V}_n = -\bar{V}_j \ln \eta_n, \Delta \bar{V}_F_n = \frac{K \varphi A}{2m_n V_{n-1} \sin \theta_n} \Delta J_n \quad (4.22) \]
Solving Eq. (4.21) for $\Delta V_n$:
\[
\Delta V_n = -1 + \frac{1}{2} \left( \Delta V_{F_n} - \Delta V_{D_n} \right) + \sqrt{1 + \frac{1}{4} \left( \Delta V_{F_n} - \Delta V_{D_n} \right)}^2 - \Delta h_n
\] (4.23)

Eq. (4.23) reduces to Eq. (3.11) if $\Delta V_{F_n} = 0$ (no rocket thrust). The procedure for evaluating Eq. (4.23) is, however, different from the previous one. In evaluating Eq. (3.11), $\Delta h_n$ was selected and therefore $\Delta h_n$ was given, and the time increment $\Delta t_n$ was part of the solution. In the present case, in which the burning of one rocket motor stage constitutes a trajectory increment, the time increment $\Delta t_n$ is known, as is the mass decrement $\Delta m_n$, so $\Delta h_n$ must be found from the mass-altitude relation.

In Eq. (4.12) there is no need to neglect the small final term at this stage of the calculations, since only simple algebra is involved. Therefore, on multiplication by $\frac{2g}{v^2}$, Eq. (4.12) becomes:
\[
\Delta h_n = 2V_{j_n} \frac{\Delta h_n}{\Delta t_n} - (1 - n) \frac{\Delta h_n}{\Delta n}^2 (4.24)
\]

where
\[
\frac{\Delta h_n}{\Delta n} = \frac{g m_{n-1} \sin \gamma_n}{v_{n-1} b_n}
\]

and $\Delta h_n$ is given by Eq. (4.13) or Fig. 14. The calculation procedure can now be described. It is convenient to introduce a parameter $\xi_{n-1}$ defined by:
\[
\xi_{n-1} = \frac{V_{n-1} b_n}{g m_{n-1}}
\] (4.25)
In terms of $E_{n-1}$,
$$\omega_n = \frac{1}{2} (\omega_{n-1} + \omega_n) = \frac{\nu_{j_n} b_n}{2g} \left( \frac{1}{m_{n-1}} + \frac{1}{m_n} \right) = \frac{\nu_{j_n} E_{n-1}}{2} (1 + \frac{1}{m_{n-1}}) \quad (4.26)$$
and
$$\sigma_n = \frac{\sin \theta_n}{E_{n-1}}$$

A complete rocket-powered stage is calculated as one trajectory increment. The known initial parameters are $V_{n-1}$, $\tau_{n-1}$, $h_{n-1}$, $\theta_{n-1}$, $X_{n-1}$, $S$, $K$, $\gamma$, $A$, $m_{n-1}$, $b_n$, $c$, $V_{j_n}$, $(p/a)_{n-1}$. $\Delta t_n$ is known, and this determines $t_n$ and $\Delta m_n$. $\nu_{j_n}$ and $\gamma_{n}$ are calculated and $\Delta H$ is found from Fig. 14. $E_{n-1}$ and $\omega_n$ are calculated, and a trial value of $\sin \theta_n$ is estimated, based on the knowledge of $\sin \theta_{n-1}$ and $\omega_n$. Using this, $\sigma_n$ is calculated from Eq. (4.27) and $\Delta h_n$ from Eq. (4.24). $f(\omega_n)$ is then found from Eq. (4.7) and $\cos \theta_n$ is calculated from Eq. (4.5). This determines $\sin \theta_n$ and leads to a new value of $\sin \theta_n$. If this differs from the estimated value, the process is repeated from that point, using the new value of $\sin \theta_n$. Only the one iteration should ever be required.

Next $\Delta h_n$ is calculated from $\Delta h_n$, giving $h_n$. $\nu_{j_n}$ is calculated from Eq. (4.20) and used to calculate $\Delta J_n$ from Eq. (4.19). This gives $\Delta V_D_n$ from Eq. (4.22), and $\Delta V_D_n$ is then calculated, and $\Delta V_n$ is determined from Eq. (4.23). This gives $\Delta V_n$ and $V_n$. Range increment $\Delta X_n$ can now be calculated from Eq. (3.14) and this gives range $X_n$, completing the stage calculation.

As an example, the method was applied to a Martlet 4, a g-flown three-stage rocket, from launch to burn out of the second stage (the remainder of the trajectory is high enough to require consideration of
the g-variation). It was launched at 6,000 kfps at an elevation angle of 33.0°. The first stage motor was ignited at 40.0 kft, and the second stage motor immediately after burn-out of the first stage. Three altitude increments were used in calculating the glide trajectory before ignition. In Table 2 the results of the calculations are compared with the output of the standard HARP Computer Program for the same example obtained from Ref. 4. Vehicle parameters and trajectory calculations, and the output of the Computer Program are given in Appendix E.

**TABLE 2**
Martlet 4 Trajectory Calculations for HARP Case 1046

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<th>$X_n$</th>
<th>$h_n$</th>
<th>$t_n$</th>
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4.4 DISCUSSION

Table 2 and Appendix E show that rocket-powered trajectories through the atmosphere can be accurately and easily calculated by slide rule by the present method. In the example the velocity and range at burn-out of Stage 2 were given very accurately, the altitude and elevation angle less accurately, both being too high. The time was of course given accurately, since it was known except for the initial glide trajectory.

The high values of $h_n$ and $\theta_n$, although acceptable, lead to over-estimates of apogee, or of orbital height after Stage 3 firing, and this suggests further work in improving the method of determining $\sin \theta_n$, so as to decrease its value towards the correct one while retaining a simple dependence on the relevant parameters.
5.0 **HIGH ALTITUDE AND ORBITAL TRAJECTORIES**

Rocket-powered vehicles will in general reach very high altitudes or go into earth orbits. The upper parts of their trajectories must then be calculated by recognizing that gravitational force is a central force varying inversely as the square of the central distance. No aerodynamic drag need be considered unless such problems as the gradual decay of orbits are being studied, and any high altitude use of rocket thrust can be treated as a constant-g problem using an appropriate value of \( g \), because of the short duration of the firing.

The problem then becomes that of determining zero-drag glide trajectories under the action of an inverse-square central force. The basic equations governing this motion are simple and lead to explicit solutions for trajectory parameters, so there is no need to seek even simpler approximate solutions. In this section the usual solutions are merely derived to complete the set of trajectory equations. No examples are worked, since the solutions are not new.

5.1 **EFFECT OF EARTH ROTATION**

In previous sections of the report, trajectory analysis was given in terms of vehicle velocity \( V \) relative to earth. This was suitable over short distances for which gravity could be considered a constant vector. When the true central force nature of gravity is considered, the trajectory analysis must be based on the vehicle absolute velocity of magnitude \( U \), or in vector form:

\[
\vec{U} = \vec{V} + \vec{U}_E \tag{5.1}
\]
where \( \vec{U}_E \) is the vector velocity of the earth's surface at the gun muzzle. Since \( \vec{U}_E \) and \( \vec{V} \) are in different directions, it is convenient to express relations between components. These relations, in the context of this report, would be applied at the point on the vehicle trajectory where the constant-\( g \) analysis of previous sections is replaced by the analysis of this section. The relations can be obtained from Fig. 16. First, using the cosine law:

\[
U^2 = V^2 + U_E^2 + 2VU_E \cos \psi \tag{5.2}
\]

But, from the right triangles of the diagram it can be seen that

\[
V \cos \theta \cos \psi_v = V \cos \chi \tag{5.3}
\]

so that, cancelling \( V \) and substituting for \( \cos \chi \) in Eq. (5.2):

\[
U = V + U_E + 2VU_E \cos \theta \cos \psi_v \tag{5.4}
\]

where \( \psi_v \) is the azimuth angle of the gun. Eq. (5.4) gives \( U \). The absolute path angle \( \phi \) to the local horizontal is given by

\[
U \sin \phi = V \sin \theta \tag{5.5}
\]

The absolute azimuth angle \( \psi_u \) is given by

\[
U \cos \phi \cos \psi_u = V \cos \theta \cos \psi_v + U_E \tag{5.6}
\]
5.2 FUNDAMENTAL EQUATIONS AND SOLUTIONS

Because only the central force of gravity acts on the vehicle its trajectory remains in a plane passing through the earth's centre, and Fig. 17 defines the variables of the trajectory. The equations of motion of the vehicle for the r- and \( \phi \)-directions are written, for unit mass:

\[
\begin{align*}
\text{r-direction} & \quad \frac{d^2r}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 = - \frac{g_E}{r^2} \\
\text{\( \phi \)-direction} & \quad \frac{d^2\phi}{dt^2} + 2 \frac{dr}{dt} \frac{d\phi}{dt} = 0
\end{align*}
\]

The second terms on the left side of the two equations are the centripetal and Coriolis accelerations, respectively, and \( g_E \) is the value of \( g \) at the earth's surface, used in the previous sections of the report.

If Eq. (5.8) is multiplied by \( r \), it can be written:

\[ \frac{d}{dt} \left( r^2 \frac{d\phi}{dt} \right) = 0 \]

or

\[ r^2 \frac{d\phi}{dt} = C, \text{ constant} = r U \cos \theta \]

where

\[ C = r_{n-1} U_{n-1} \cos \theta_{n-1} \]

\( \frac{d\phi}{dt} \) can be eliminated from Eq. (5.7) using Eq. (5.9):

\[ \frac{d^2r}{dt^2} - \frac{C^2}{r^3} = - \frac{g_E}{r^2} \left( \frac{r_E}{r} \right)^2 \]

Now, \( \frac{dr}{dt} = U \sin \theta = u_r \). Therefore

\[ \frac{d^2r}{dt^2} = \frac{dU_r}{dt} = \frac{U_r}{r} \frac{dU_r}{dt} = \frac{C^3}{r^3} - \frac{g_E r_E^2}{r^2} \]
Figure 16 Vehicle Velocity Diagram
Figure 17 Trajectory Variables
Eq. (5.12) can be integrated directly to give

\[ \frac{u_r^2 - u_{r_{n-1}}^2}{r_{n-1}} = \left( \frac{1}{r_{n-1}} - \frac{1}{r} \right) \left\{ C^2 \left( \frac{1}{r_{n-1}} + \frac{1}{r} \right) - 2g_E \frac{r^2}{r_{n-1}} \right\} \]  

\[ (5.13) \]

Since the altitude change of the vehicle is of primary interest, it is convenient to introduce

\[ \Delta h = \Delta h = 1 - \frac{r_{n-1}}{r} \]  

\[ (5.14) \]

In terms of \( \Delta h \), Eq. (5.13) appears in the form

\[ \frac{u_r^2 - u_{r_{n-1}}^2}{r_{n-1}} = -\Delta h \left\{ \frac{2g_E r_{n-1}^2}{r_{n-1}} - \frac{u_{n-1}^2}{r_{n-1}} (2 - \Delta h) \right\} \]  

\[ (5.15) \]

where

\[ U_\theta = U \cos \theta \]

At apogee, \( U_r = 0 \), so that Eq. (5.15) can be solved for \( \Delta h_{ap} \):

\[ \Delta h_{ap} = \left\{ 1 - \left( \frac{g_E r_{n-1}^2}{r_{n-1} U_\theta^2} \right) \right\} + \sqrt{\left\{ \left( \frac{g_E r_{n-1}^2}{r_{n-1} U_\theta^2} \right) \right\}^2 + \tan^2 \theta_{n-1}} \]  

\[ (5.16) \]

Eqs. (5.15) and (5.9) determine vehicle velocity as a function of altitude.

It is necessary to relate altitude to elapsed time. This can be done using

\[ \frac{dt}{dr} = \frac{U_r}{r_{n-1}} = \frac{r_{n-1} U_r}{r_{n-1} U_r} = \frac{r_{n-1}^d \Delta h_r}{1 - \Delta h_r^2 U_r} \]  

\[ (5.17) \]

Eq. (5.17) can be integrated directly, using Eq. (5.15), to give
\[
\Delta t = \frac{r_{n-1}}{U_0 n_{n-1}} \left[ \tan \phi_{n-1} - \frac{1}{(1 - \Delta h)} \sqrt{\tan^2 \phi_{n-1} - 2 \left( \frac{g_E r_E^2}{r_{n-1} U_0^2} - 1 \right) \Delta h - \Delta h^2} \right]
+ \frac{g_E r_E^2}{U_0^3 s^{3/2}} \sin^{-1} \left( \frac{-2S + \frac{2g_E r_E}{r_{n-1} U_0} (1 - \Delta h)}{2 (1 - \Delta h) Q} \right) - \sin^{-1} \left( \frac{-2S + \frac{2g_E r_E}{r_{n-1} U_0^2}}{Q} \right)
\]

where

\[
S = \frac{2g_E r_E^2}{r_{n-1} U_0^2} - \sec^2 \phi_{n-1}
\]

\[
Q = \sqrt{\tan^2 \phi_{n-1} + \left( \frac{2g_E r_E^2}{r_{n-1} U_0^2} - 2 \right)^2}
\]

In order to find the range of the vehicle, it is necessary to know the change in \( \phi \). This can be found using Eq. (5.9):

\[
\frac{d\phi}{dt} = \frac{d\phi}{dr} \cdot \frac{U_r}{r^2} = \frac{C}{r^2}
\]

Therefore

\[
\frac{d\phi}{dt} = \frac{Cd r}{r^2 U_r} = \frac{C d\Delta h}{r_{n-1} U_r}
\]

Eq. (5.20) can be integrated directly using Eq. (5.15), to give

\[
\Delta \phi = \sin^{-1} \left( \frac{2 \Delta h + 2 \left( \frac{g_E r_E^2}{r_{n-1} U_0^2} - 1 \right)}{Q} \right) - \sin^{-1} \left( \frac{2 \left( \frac{g_E r_E^2}{r_{n-1} U_0^2} - 1 \right)}{Q} \right)
\]

Referring to Fig. 8 the vehicle range increment \( \Delta X \) is the great circle distance between the point 1 intercepted at time \( t \) at the earth's surface by radius \( r \) and the position 2 at time \( t \) of the point 3 on the earth's surface intercepted at time \( t_{n-1} \) by radius \( r_{n-1} \).
\[ \Delta X = r_e \Delta \vartheta' \]  

(5.22)

\( \Delta \vartheta' \) can be obtained by solving the spherical triangle 123. In this, 
\( \Delta \vartheta \) is known from Eq. (5.21), \( \psi_u \) is given by Eq. (5.6) and great circle 
arc 23 and angle 231 can be found, since the latitude of 3 and the small 
circle (eastward) distance 32 travelled by point 3 in the time \( \Delta t \),
given by Eq. (5.18), are known.
Figure 18 Spherical Geometry for Range
REFERENCES


APPENDIX A

Trajectory Equations for Zero Drag, Zero Thrust, Constant g.

Newton's 2nd Law for X-Direction (see Figure 1)

\[ \frac{d}{dt} (m \frac{dx}{dt}) = m \frac{d^2x}{dt^2} = 0 \]

Therefore

\[ \frac{dx}{dt} = \text{constant} = V_{n-1} \cos \theta_{n-1} = V \cos \theta \]

(A-1)

and

\[ \Delta x_n = x_n - x_{n-1} = V_{n-1} \cos \theta_{n-1} (t_n - t_{n-1}) \]

(A-2)

Newton's 2nd Law for h-Direction

\[-mg = \frac{d}{dt} (m \frac{dh}{dt}) = m \frac{d^2h}{dt^2} = \text{constant} \]

Therefore

\[ \frac{dh}{dt} = V_{n-1} \sin \theta_{n-1} - g (t - t_{n-1}) = V \sin \theta \]

(A-3)

and

\[ \Delta h_n = h_n - h_{n-1} = V_{n-1} \sin \theta_{n-1} (t_n - t_{n-1}) - \frac{g}{2} (t_n - t_{n-1})^2 \]

(A-4)

Introduce

\[ \Delta x_n = \frac{2g \Delta x_n}{V_{n-1}} \]

\[ \Delta h_n = \frac{2g \Delta h_n}{V_{n-1}} \]

(A-5)

and substitute for \((t_n - t_{n-1})\) from Eq. (A-2) in Eq. (A-4):

\[ \Delta h_n = \tan \theta_{n-1} \Delta x_n - \frac{1}{2} \sec^2 \theta_{n-1} \Delta x_n^2 \]

(A-6)

\(\sin \) dividing Eq. (A-3) by Eq. (A-1) an expression for \(\tan \theta\) is obtained:

\[ \tan \theta = \tan \theta_{n-1} - \frac{g(t - t_{n-1})}{V_{n-1} \cos \theta_{n-1}} \]

(A-6)
If $\theta$ is set equal to $\theta_n$ and $(t_n - t_{n-1})$ is eliminated using Eq. (A-2), the result can be written:

$$\tan \theta_n = \tan \theta_{n-1} - \frac{1}{2} \sec^2 \theta_{n-1} \Delta \chi_n$$

or

$$\Delta \chi_n = \frac{2(\tan \theta_{n-1} - \tan \theta_n)}{\sec^2 \theta_{n-1}} \quad (A-7)$$

Again, if Eq. (3.4) is solved for $\Delta \chi_n$, the result is:

$$\Delta \chi_n = \frac{2(\tan \theta_{n-1} - \sqrt{\tan^2 \theta_{n-1} - \sec^2 \theta_{n-1} \Delta \chi_n})}{\sec^2 \theta_{n-1}} \quad (A-8)$$

On comparing Eqs. (A-7) and (A-8) it is seen that

$$\tan \theta_n = \sqrt{\tan^2 \theta_{n-1} - \sec^2 \theta_{n-1} \Delta \chi_n} \quad (A-9)$$

With a little rearrangement, using trigonometric identities, this can be put in the form:

$$\sin \theta_n = \sqrt{\frac{\sin^2 \theta_{n-1} - \Delta \chi_n}{1 - \Delta \chi_n}} = \sqrt{1 - \frac{\cos^2 \theta_{n-1}}{1 - \Delta \chi_n}} \quad (3.5)$$

or

$$\cos \theta_n = \frac{\cos \theta_{n-1}}{\sqrt{1 - \Delta \chi_n}} \quad (3.6)$$

If Eq. (A-4) is multiplied through by $\frac{2g}{v_{n-1}^2 \sin^2 \theta_{n-1}}$, the result is

$$\left(\frac{2g \Delta \chi_n}{v_{n-1}^2 \sin^2 \theta_{n-1}}\right)^2 = \left(\frac{\Delta t_n}{v_{n-1} \sin \theta_{n-1}}\right)^2 - \left(\frac{\Delta t_n}{v_{n-1} \sin \theta_{n-1}}\right)^2 \quad (A-10)$$
or, if Eq. (A-2) is used to substitute for $\Delta t_n$:

$$\left(\frac{2g \Delta h_n}{v_{n-1}^2 \sin^2 \theta_{n-1}}\right)^2 \left(\frac{g \Delta X_n}{v_{n-1}^2 \sin \theta_{n-1} \cos \theta_{n-1}}\right)^2$$

(A-11)

Both Eq. (A-10) and (A-11) have the form

$$\dot{\phi} = 2 \ \phi' - \phi''^2$$

(3.16)

giving a universal zero-drag, constant-\(g\) trajectory equation.
Given: \( g = 32.15 \text{ ft/sec}^2 \), \( V = 1.400 \), \( A = \frac{T}{2} \), \( \frac{(5.00)^2}{144} = 0.1364 \text{ sq ft}, \) \( K = 0.920 \)

\[ V_o = 5.400 \text{ kfps}, \theta_o = 85.00^\circ \]

Case 1: \( W = 170.0 \text{ lbs}, \frac{m}{32.15} = 5.29 \text{ slugs}, \)

\[ \frac{K \times \Delta A}{2m} = \frac{0.920(1.400)(0.1364)}{2(5.29)} = 16.62 (10)^{-3} \text{ ft}^3/\text{lb-sec}^2 \]

Choose: \( h_1 = 10 \text{ kft}, h_2 = 30 \text{ kft}, h_3 = 60 \text{ kft}, h_4 = 180 \text{ kft} \)

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For \( h > 180 \text{ kft}, \) Figure 7 is used. Here \( \Delta X_h = \frac{(V_n \sin \theta_n)^2}{2s} \Delta X_h = \frac{(3.107 \times 9920)^2}{2 \times 3.0643} \eta = 147.8 \eta \)

\[ \eta = \frac{37.92 \eta}{0.03215} \]

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</table>

\( \rightarrow \) Apogee
Given: \( g = 32.15 \text{ ft/sec}^2 \), \( \delta = 1.400 \), \( A = 0.1364 \text{ eq. ft.} \), \( K = 0.920 \)
\[ V_o = 5.400 \text{ kfps}, \theta_o = 85.00^\circ \]

Case 2: \( W = 160.0 \text{ lbs, } m = \frac{180.0}{32.15} = 5.60 \text{ slugs, } \frac{KX_A}{2m} = \frac{0.920 (4.600)(0.1364)}{2(5.60)} = 15.70 (10)^{-3} \text{ ft}^3/\text{lb-sec}^2 \)

Choose: \( h_1 = 10 \text{ kft}, h_2 = 30 \text{ kft}, h_3 = 60 \text{ kft}, h_4 = 180 \text{ kft} \)

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<th>4</th>
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For \( h = 180 \text{ kft} \), Figure 7 is used. Here \( \Delta h_5 = \frac{(V_4 \sin \theta_4)^2}{2g} = \frac{(3.168 (.9922))^2}{.0643} = 153.9 \text{ kft} \)
\[ \Delta t_5 = \frac{V_4 \sin \theta_4}{g} = \frac{3.168 (.9922)}{.03215} = 97.9 \text{ sec} \]
\[ \Delta X_5 = \frac{V_4^2 \sin \theta_4 \cos \theta_4}{g} = \frac{(3.168)^2 (.9922) (.1250)}{.03215} = 38.8 \text{ kft} \]

For velocity, Eq.(3.3) is used. Here \( V_5 = \sqrt{\frac{V_4^2 - 2g \Delta h_5}{3.168^2 - 0.0643 \Delta h_5}} \)
Case 3: Data as for Case 1: Choose \( h_1 = 100 \) kft.

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For \( h > 100 \) kft, Figure 7 is used for Trajectory and Eq.(3.3) for Velocity:

Case 3: \( \Delta b_2 = \frac{(\nu_1 \sin \theta_1)^2}{2g} \eta = \frac{(3.888)^2}{.9951} \eta = 233.4 \eta, \Delta x_2 = \frac{\nu_1 \sin \theta_1 \cos \theta_1 \eta \nu_1}{g} = \frac{(3.888)^2}{.9951} \eta = 46.2 \eta \)

Case 4: \( \Delta b_2 = \sqrt{(\nu_1^2 - 2gh_2)^2 - (3.937)^2 \eta_2} \), \( \Delta x_2 = \frac{\nu_1 \sin \theta_1 \eta \nu_1}{g} = \frac{(3.937)^2}{.9951} \eta = 47.4 \eta \)

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--- Apogee
**Projectile Trajectory - Present Method**

| Column | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Fig. 3 | Eq. 3.8 | Eq. 3.12 | Eq. 3.11 | Eq. 3.12 | Eq. 3.14 | Col.15 | Eq. 3.15 | Col.17 |
|--------|---|---|---|---|---|---|---|---|---|-------|--------|----------|----------|----------|--------|----------|
| Source | Given | Given | Col.14 | Eq. 3.7 | Slide Rule | Slide Rule | Slide Rule | Slide Rule | Fig. 3 | Eq. 3.8 | Eq. 3.12 | Eq. 3.11 | Eq. 3.12 | Col.15 | Eq. 3.15 | Col.17 |
| \( n \) | \( h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) | \( \Delta h_n \) |
| - | kft | kft | kft | kft | kft | kft | kft | kft | kft | kft | kft | kft | kft | kft | kft | kft |
| 0 | 0 | 4.700 | 48.0 | 724 | 660 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 10 | 4.085 | 973 | 735 | 679 | 7395 | 47.6 | 16.2 | 0.542 | 0.0576 | 0.1308 | 0.515 | 9.1 | 9.1 | 3.08 |
| 2 | 20 | 3.625 | 936 | 662 | 692 | 7285 | 66.7 | 11.1 | 0.377 | 0.0646 | 0.1129 | 0.561 | 9.4 | 10.5 | 3.57 |
| 3 | 30 | 2.954 | 979 | 63.3 | 686 | 704 | 44.7 | 13.5 | 0.175 | 0.0635 | 0.1049 | 0.670 | 20.2 | 30.7 | 6.66 |
| 4 | 40 | 2.128 | 955 | 29.7 | 496 | 391 | 36.5 | 7.6 | 0.313 | 0.0538 | 0.2793 | 0.826 | 34.5 | 93.2 | 26.65 |
| 5 | 50 | 2.128 | 0 | 29.7 | 496 | 391 | 36.5 | 7.6 | 0.313 | 0.0538 | 0.2793 | 0.826 | 34.5 | 93.2 | 26.65 |
| 6 | 60 | 2.128 | 0 | 0 | 496 | 391 | 36.5 | 7.6 | 0.313 | 0.0538 | 0.2793 | 0.826 | 34.5 | 93.2 | 26.65 |
| 7 | 70 | 2.128 | 0 | 0 | 0 | 496 | 391 | 36.5 | 7.6 | 0.313 | 0.0538 | 0.2793 | 0.826 | 34.5 | 93.2 | 26.65 |
| 8 | 80 | 2.128 | 0 | 0 | 0 | 0 | 496 | 391 | 36.5 | 7.6 | 0.313 | 0.0538 | 0.2793 | 0.826 | 34.5 | 93.2 | 26.65 |
| 9 | 90 | 2.128 | 0 | 0 | 0 | 0 | 0 | 496 | 391 | 36.5 | 7.6 | 0.313 | 0.0538 | 0.2793 | 0.826 | 34.5 | 93.2 | 26.65 |

- \( \Delta h @ \text{Apogee} = \frac{(V_{\Delta} \sin \theta_{\Delta})^2}{2g} = \frac{(2.128 \times 0.496)^2}{2 \times 32.15} = 1.056^2 = 17.3 \text{ kft} \), \( \Delta h_{\text{ap}} = 97.3 \text{ kft} \) (+4.25%)
- \( \Delta X @ \text{Apogee} = \frac{V_{\Delta} \sin \theta_{\Delta} \cos \theta_{\Delta}}{g} = \frac{(2.128 \times 0.496 \times 0.866)}{32.15} = 60.6 \text{ kft} \), \( \Delta X_{\text{ap}} = 153.8 \text{ kft} \) (+1.1%)
- \( \Delta @ \text{Apogee} = \frac{V_{\Delta} \sin \theta_{\Delta}}{g} = \frac{1.056}{32.15} = 32.85 \text{ secs.} \)
- \( t_{\text{ap}} = 48.4 \text{ sec} \) (+3.3%)
- \( V @ \text{Apogee} = V_{\Delta} \cos \theta_{\Delta} = 2.128 \times 0.866 = 1.843 \text{ kfps} \) (-3.3%)

- \( X_{\text{impact}} = 298.9 \text{ kft} \) (+1.9%)
- \( t_{\text{impact}} = 160.8 \text{ sec} \) (+4.1%)
- \( V_{\text{impact}} = 1.700 \text{ kfps} \) (-9.0%)
APPENDIX C

Projectile Trajectory - WARP Computer Program

VEHICLE DESCRIPTION

<table>
<thead>
<tr>
<th>INITIAL ALTITUDE (FEET)</th>
<th>381.00 LBS</th>
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<tbody>
<tr>
<td>GUN ELEVATION (DEGREES)</td>
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<tr>
<td>MUZZLE VELOCITY (FT/SEC)</td>
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DRAG COEFFICIENTS

| Subsonic | 0.3600 | 0.1000 | 0.7570 | -0.2226 | 0.0276 | -0.1615 |
| Supersonic (curve) | 0.1500 |

WEIGHT OF FUEL | 0.00 LBS
MASS FLOW | 0.00 LBS/SEC
BODY DIAMETER | 11.00 IN
MUZZLE EXIT DIAMETER | -0.00 IN
SPECIFIC IMPULSE | 0.00 SECS

DELAY BEFORE IGNITION | 0.00 SECS
DELAY BEFORE BURNOUT | 0.00 SECS

COMMENTS | TIME | VELOCITY | RANGE | ALTITUDE (FEET) | ALTITUDE (KILMETERS) | CO | MACING | ATTITUDE | YY PKFSS
LABUNCH | 0.0 | 4700. | 0. | 0.000 | 0.235 | 4.297 | 48.090 | 24244.919
| 20.0 | 3765. | 48742. | 48976. | 14.776 | 0.328 | 2.455 | 24.701 | 1484.482
| 40.0 | 2236. | 89292. | 75923. | 23.525 | 0.383 | 2.310 | 27.590 | 253663.653
| 60.0 | 19.7 | 128311. | 90935. | 27.711 | 0.418 | 2.091 | 11.931 | 98.045
| 72.4 | 1915. | 152155. | 93389. | 24.465 | 0.427 | 1.934 | -0.043 | 82.745
| 80.0 | 120. | 166660. | 92473. | 28.166 | 0.425 | 1.934 | -0.254 | 85.384
| 100.0 | 2065. | 204472. | 81350. | 24.798 | 0.403 | 2.132 | -24.909 | 170.615
| 120.0 | 2311. | 241296. | 57990. | 17.676 | 0.374 | 2.3467 | -39.033 | 551.857
| 140.0 | 2329. | 274052. | 24763. | 7.346 | 0.345 | 2.2893 | -50.634 | 2498.681

IMPACT | 154.9 | 1878. | 293227. | -127. | -0.039 | 0.459 | 1.6805 | -57.117 | 521.251

\[
4000 = \frac{581(144)}{c_D \cdot D^2} = \frac{880}{c_D}
\]

\[
c_D = 0.22 \cdot M = 4.4
\]
APPENDIX D

Trajectory Equations for Vehicle under Rocket Thrust with Zero Drag, Constant $g$

Newton's 2nd Law for Directions Parallel and Normal to Trajectory:

Parallel:  \[ F - mg \sin \theta = m \frac{dv}{dt} \]  \( D-1 \)

Normal:  \[ -mg \cos \theta = mV \frac{d\theta}{dt} \]  \( D-2 \)

Eliminate $V$ from $D-1$ and $D-2$ by dividing $D-2$ by $\frac{d\theta}{dt}$ and differentiating with respect to $t$:

\[
m \frac{dv}{dt} = -\frac{d}{dt} \left( \frac{mg \cos \theta}{d\theta/dt} \right) = mg \left\{ \sin \theta + \cos \theta \frac{d^2\theta/dt^2}{(d\theta/dt)^2} \right\} = F - mg \sin \theta
\]

Therefore, on simplifying the above, the differential equation for $\theta(t)$ is obtained:

\[
\frac{d^2\theta}{dt^2} + \left( 2 \tan \theta - \alpha \sec \theta \right) \left( \frac{d\theta}{dt} \right)^2 = 0 \tag{4.1}
\]

Solution for $\alpha = \text{constant}$

Set $\omega = \frac{d\theta}{dt}$ Then Eq. (4.1) can be written

\[
\frac{d\omega}{dt} + \left( 2 \tan \theta - \alpha \sec \theta \right) \omega = 0 \tag{D-3}
\]

This can be integrated directly to give

\[
\frac{\omega_n}{\omega_n-1} = \frac{\cos \alpha \theta_n^{-2} \left( 1 + \sin \theta \right)^\alpha}{\cos \alpha \theta \left( 1 + \sin \theta_{n-1} \right)^\alpha} \tag{D-4}
\]

This is integrated in turn to give, after some simplification:

\[
\int_{\theta_{n-1}}^{\theta_n} \frac{\cos \alpha \theta^2 \theta \ d\theta}{\left( 1 + \sin \theta \right)^\alpha} = \frac{\cos \alpha \theta_{n-1} \Delta \tau_n}{(1 + \sin \theta_{n-1})^\alpha} \tag{4.2}
\]
Solution for $\Delta R_n$ for $\alpha = \text{constant}$

Divide D-1 by $m$ and differentiate with respect to $t$:

$$\frac{d^2v}{dt^2} = -g \cos \theta \frac{d\theta}{dt} = \frac{g^2 \cos^2 \theta}{v}$$  \hspace{1cm} \text{D-5}

using D-2. Again, if D-1 is solved for $g \sin \theta$:

$$g \sin \theta = g \alpha - \frac{dv}{dt}$$  \hspace{1cm} \text{D-6}

or

$$g^2 (1 - \cos^2 \theta) = g^2 \alpha^2 - 2g \alpha \frac{dv}{dt} \left( \frac{dv}{dt} \right)^2$$  \hspace{1cm} \text{D-7}

Equating $g^2 \cos^2 \theta$ from D-5 and D-7, $\theta$ is eliminated:

$$v \frac{d^2v}{dt^2} = -g^2 (\alpha^2 - 1) + 2g \alpha \frac{dv}{dt} \left( \frac{dv}{dt} \right)^2$$

or

$$\frac{d}{dt} \left( V \frac{dv}{dt} \right) - 2g \alpha \frac{dv}{dt} - g^2 (\alpha^2 - 1)$$  \hspace{1cm} \text{D-8}

D-3 can be integrated directly to give:

$$V \left\{ \frac{dv}{dt} - 2g \alpha \right\} = V_{n-1} \left\{ \frac{dv}{dt} \bigg|_{n-1} - 2g \alpha \right\} - g^2 (\alpha^2 - 1) (t - t_{n-1})$$  \hspace{1cm} \text{D-9}

$\frac{dv}{dt}$ can be eliminated, using D-6:

$$V \left\{ \alpha + \sin \theta \right\} = V_{n-1} \left\{ \alpha + \sin \theta_{n-1} \right\} + g (\alpha^2 - 1) (t - t_{n-1})$$

or

$$V = V_{n-1} \left\{ \alpha + \sin \theta_{n-1} \right\} + \frac{g (\alpha^2 - 1) (t - t_{n-1})}{(\alpha + \sin \theta)} = \frac{dh}{dt} \cdot \frac{1}{\sin \theta}$$  \hspace{1cm} \text{D-10}

This gives directly

$$dh = V_{n-1} (\alpha + \sin \theta_{n-1}) \cdot \frac{\sin \theta \, dt}{(\alpha + \sin \theta)} + g (\alpha^2 - 1) \frac{\sin \theta (t - t_{n-1})dt}{(\alpha + \sin \theta)}$$  \hspace{1cm} \text{D-11}

D-11 is made non-dimensional by multiplication with $\frac{2\Delta}{V_{n-1}}$ and then integrated:

$$\Delta R_n = 2(\alpha + \sin \theta_{n-1}) \int_0^{\Delta \tau_n} \frac{\sin \theta \, d\tau}{(\alpha + \sin \theta)} + 2(\alpha^2 - 1) \int_0^{\Delta \tau_n} \frac{\sin \theta \, d\tau}{(\alpha + \sin \theta)}$$

(4.6)
I. Pre-Ignition Climb: \( V_n = 6,000 \) kfps, \( \theta_n = 33.0^\circ \), \( \epsilon_n = 0.03215 \) kfps, \( K = 1.00 \), \( Y = 1.40 \), \( A = 1.478 \) ft²

\( a_{eq} = 2735 \) lbs, \( \frac{Kx}{2a} = \frac{1.00(0.7011428 \times (32.135)}{2735} = 12.15 \) ft²/lb·sec²

II. 1st Stage Burning: \( h_3 = 40.0 \) kft, \( V_{J4} = 9.000 \) kfps, \( \theta_{p4} = \frac{1600}{15} = 106.7 \) lbs/sec, \( C_4 \approx 0.04808/kft, (p/s)_4 = 4.45(10)^{-6} \) lb·sec²/kft

III. 2nd Stage Burning: Start at \( t_4, V_{J4} = 9.320 \) kfps, \( \theta_{p3} = \frac{400}{10} = 40.0 \) lbs/sec, \( \theta_{p3} = 775 \) lbs, \( (p/s)_4 = 0 \)

### I. Pre-Ignition Climb

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**APPENDIX B: TYPICAL TRAFFIC VAY DATA - TACTICAL V SEAD**

**HARP TRAJECTORY PROGRAM - R. W. HICKIE - VE 3**

10 AUG 60  CASE 1046  PAGE 1

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Note: The data includes information such as time, height, range, velocity, and various other parameters for different stages and conditions.
SIMPLIFIED FORMS OF PRELIMINARY TRAJECTORY CALCULATION FOR GUN-LAUNCHED VEHICLES

An approximate numerical method for calculation of trajectories of hypersonic vehicles is described, with particular reference to gun-launched vehicles. Because of the form of the hypersonic drag coefficient, a relatively simple calculation using functions which depend only on altitude is possible. Reasonable accuracy can be obtained even for computations in which very large calculation intervals are used. The method is particularly suitable for preliminary engineering calculations which do not justify a detailed computer solution, and has the added advantage of giving the analytic forms linking input and output. The technique is shown to be applicable to glide trajectories with and without drag and to rocket powered trajectories as well.
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