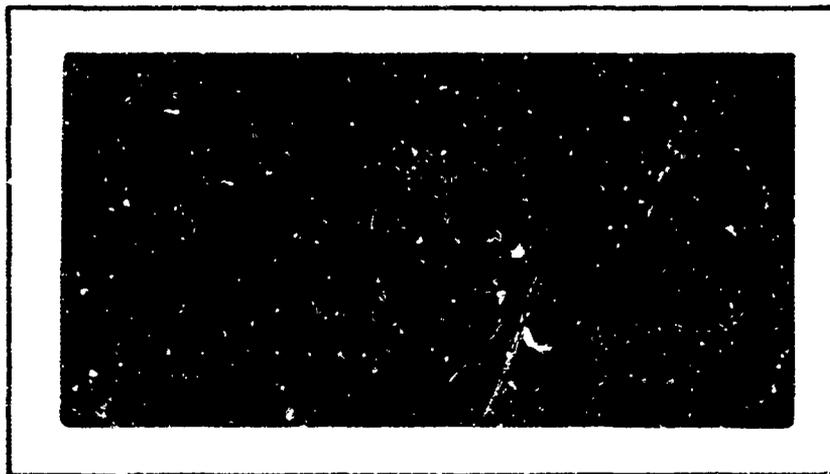


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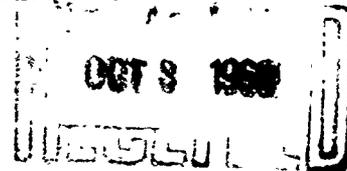


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APPLICATIONS OF EXTREME VALUE THEORY
IN THE RELIABILITY ANALYSIS
OF NON-ELECTRONIC COMPONENTS

THESIS

Presented to the Faculty of the School of Engineering of
the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

By

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Preface

This thesis is the presentation of the results of an intensive literature search to discover the applications of extreme value theory in the reliability analysis of non-electronic components.

In an attempt to discover as many examples as possible, over four hundred abstracts were reviewed. Of these, approximately one hundred were selected for further investigation. Final selection was made of forty-four references which were applicable to this study.

It is assumed that the reader is familiar with statistical theory and basic concepts of reliability analysis. For the reader who is unfamiliar with extreme value theory, a discussion of this theory is found in the text to the extent which I feel is necessary to understand the applications presented.

I wish to thank my thesis advisor, Professor Albert H. Moore for providing me with an interesting thesis topic and for his helpful guidance and advice.

Finally, I wish to thank my wife for being patient and for understanding a student-husband.

Cletus B. Kuhla

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ABSTRACT

Manufacturers of non-electronic components and systems are required to accurately determine the reliability of their products in order to meet the demands of government weapon system contracts, safety programs and commercial product warranties. In an effort to establish simple but accurate techniques for determining reliability factors of mechanical components, increased use of statistical theory is being made in analyzing component failure data.

This thesis illustrates the application of extreme value theory in the reliability analysis of mechanical systems and components. The basic theory of extreme values is presented and the exact and asymptotic forms of the extreme value distributions are developed. Applications of the extreme value distributions are presented in example problems.

The Type I extreme value distribution is applicable to the analysis of corrosive pitting of aluminum and the analysis of maximum loads. The Type III extreme value distribution is useful in the failure analysis of step motors, automobile door lock mechanisms, corrosion resistance of magnesium, automobile structural components and electromagnetic relays.

APPLICATIONS OF EXTREME VALUE THEORY
IN THE RELIABILITY ANALYSIS
OF NON-ELECTRONIC COMPONENTS

I. Introduction

In the past decade, the term reliability has begun to appear with greater frequency in all areas of industry. Although manufacturers were always concerned with producing an operational product, not too much emphasis was placed on the reliability of the product. The prevailing attitude seemed to be that if a product worked it was a reliable product. The present day concept of reliability is quite different from merely having a product that operates. Reliability may be defined as the probability that a system, subsystem, component or part will perform as specified for a given period of time under stated conditions.

The first major effort to produce reliable products was initiated when the electronics industry sought a solution to the unreliability problem of electronics equipment in the early 1950's. The crash program undertaken by the government and aerospace industry to create the ballistic missile weapon system during the mid-1950's resulted in even greater demands for high reliability in products. The early satellite programs and the manned

space project further stressed the need for developing usable techniques for determining the reliability of components and whole systems.

Because of the continued pressure of military reliability prediction and demonstration requirements and the increased demands made on commercial manufacturers in the form of warranties, a simplified method of accurately predicting the reliability of mechanical components and systems is constantly being sought. The natural trend is to attempt to incorporate methods which have been used in the analysis of electronic components. The use of statistical distributions and statistical analysis has been proven to be quite accurate in the determination of reliability factors for electronic components and systems. This fact is realized when an evaluation is made of the methods available for reliability prediction. Practically every reliability analysis or prediction technique which is in popular use by the government and aerospace industry is designed for electronic systems.

Unfortunately, the characteristics of mechanical and electro-mechanical components are greatly different

from those of electronic components which are conducive to the use of probability theory and statistical analysis. Electronic components are highly standardized, mass produced, relatively low cost and fail primarily because of manufacturing defects. In contrast, mechanical components are usually high cost, of special design rather than mass produced, and non-standardized. The typical mechanical component is designed for long life and endurance, therefore, the usual life testing which is performed with vacuum tubes, transistors, resistors and other electronic components would be costly and time consuming when performed on mechanical parts.

Despite the inherent difficulties, the ability to predict the useful life characteristics of mechanical components and systems has been improving over the years. This trend has been the result of advances made in a few but important areas. First of all, the physics of failure and breaking strengths for mechanical parts are more precisely known than those of electronic parts. Secondly, the advances made by the study of fatigue strength has greatly improved mechanical reliability. Among the by-products of fatigue studies of materials,

especially metals, are good design practices for mechanical structures such as the use of fillets and rounds, gradual cross-sections, better surface finishes and heat treatments without which many industries of today, such as the aircraft industry, would not have been possible (Ref. 28).

Despite the many advances made by industry, the determination of accurate reliability measurements for mechanical and electro-mechanical devices is still a difficult task. Systems requirements are becoming more severe and design complexity is increasing. Greater and more binding demands are being made on the engineer to specify and produce accurate reliability figures for mechanical systems. Contracts no longer specify "goals" and "best efforts", terms in common use a few years ago, but instead require firm guarantees of both reliability and maintainability (Ref. 39).

In order to meet these demands, the current trends in the methods of mechanical reliability analysis has been an increased emphasis on the probabilistic approach to the design of mechanical systems, especially structures. Such an approach is necessary to aid in understanding the

variations associated with design, materials and application factors. The gathering and statistical analysis of field failure data is continuing to increase. Analysis of this nature results in greater knowledge of mechanical reliability as a function of application, environment and design factors. Also, the statistical techniques used in these analysis procedures are being extended (Ref. 42).

The purpose of this thesis then, is to present the results of an extensive literature research which was undertaken to uncover examples showing the applications of statistical theory, in particular, extreme value theory, in the reliability analysis of mechanical or electro-mechanical components and systems. The initial step in the literature survey was an investigation of the Technical Abstracts Bulletin, Reliability Abstracts and Technical Reviews and the Quality Control and Applied Statistics Abstracts. Publications found in these sources led to further applicable articles. The results of this survey is presented in the following order.

First, a discussion of the theory of extremes including the exact distribution and asymptotic distribution of extremes is presented. Next, some commonly

used statistical methods of evaluating test data are presented. Following this section, is an example problem utilizing the exact distribution of the smallest extreme. The next section is used to further discuss the asymptotic distribution of extremes and two problems are presented which show the application of the Type I asymptotic distribution in mechanical reliability analysis. The Type III asymptotic distribution is then presented in the following section with five examples of applying this distribution in reliability analysis of mechanical components. Finally, the results are summarized and conclusions are made.

II. Extreme Value Theory

One of the main problem areas of interest to mathematicians was how to cope with values which lie well away from the central values of a set of data. The concern over outlying values in astronomical data led to the development of a solution in the theory of errors. In the early 1900's, statisticians became interested in sampling distribution, the estimation of distribution parameters and estimation errors. Included in this work were studies in the sampling distribution of the range and the largest normal value of the normal distribution which indicated that the largest value and the range are random variables possessing their own distribution.

Some of the earliest applications of extreme value statistics were in the field of human life statistics, radioactive emission and strength of materials. Further advances in the theory of order statistics and parameter estimation techniques has led to the verification of key extreme - value distribution theories. The most important factor in the development of present day extreme value theory has been the new knowledge of exponential type distributions and order statistics.

These two advancements in theory increase the possible means of data analysis by offering new distributions and characteristic properties from which to choose in formulating hypothesis (Ref. 31).

An extreme value is an ordered sample value. If a sample of size N is collected and the sample values are arranged in increasing or decreasing values the sample is an ordered sample. The values of the sample are subscripted x_i with i indicating the order. If the sample is ordered from lowest to highest value, x_1 is the smallest extreme and x_N is the largest extreme.

Suppose we are interested in finding the probability density function of the M^{th} ordered extreme in a sample of size N . To find the probability density function, $f(x_M)$, let us consider the following argument: divide the real axis into three parts, one going from $-\infty$ to x_M , a second going from x_M to x_M+h , where h is a positive constant and the third from x_M+h to $+\infty$. If the common probability density of the random variables x_i is given as $f(x)$, the probability that $M-1$ of the sample fall into the first interval, one value falls into the second interval and $N-M$ fall into the third interval is

$$\frac{N!}{(M-1)!(N-M)!} \left[\int_{-\infty}^{x_M} f(x) dx \right]^{M-1} \left[\int_{x_M}^{x_M+h} f(x) dx \right] \left[\int_{x_M+h}^{+\infty} f(x) dx \right]^{N-M}$$

Using the law of the mean we have

$$\int_{x_M}^{x_M+h} f(x) dx = f(\xi) h \quad x_M \leq \xi \leq x_M+h$$

and if we let $h \rightarrow 0$, the probability density function of x_M , the random variable whose value is the M^{th} ordered statistic becomes

$$\phi_M(x) = \frac{N!}{(M-1)!(N-M)!} \left[\int_{-\infty}^x f(x) dx \right]^{M-1} f(x) \left[\int_x^{+\infty} f(x) dx \right]^{N-M} \quad (2.1)$$

for $-\infty < x < +\infty$ (Ref. 14)

Another form of this formula is

$$\phi_M(x) = \frac{N!}{(M-1)!(N-M)!} [F(x)]^{M-1} f(x) [1-F(x)]^{N-M} \quad (2.2)$$

If in a sample of N independent observations from a population whose density is $f(x)$, we are interested in the probability density function of the smallest ordered statistic or smallest extreme, it is found by substituting the value $M=1$ into equation (2.2). The resulting density function for the smallest extreme is

$$\phi_1(x) = N f(x) [1 - F(x)]^{N-1} \quad (2.3)$$

In the same manner, the probability density function of the largest order statistic can be found by substituting in equation (2.2). Therefore, the density function of the largest extreme is

$$\phi_N(x) = N f(x) [F(x)]^{N-1} \quad (2.4)$$

It is now possible to determine the probability distribution function for the smallest and largest extreme.

The probability that X is the smallest among N independent observations is defined as $\Phi_1(x)$. This distribution function can be derived by integration of equation (2.3) over the limits of $-\infty$ to x . Thus,

$$\Phi_1(x) = 1 - [1 - F(x)]^N \quad (2.5)$$

In a similar manner, the probability distribution function of the largest extreme can be found by integrating equation (2.4) over the limits of $-\infty$ to x . Thus, the probability $\Phi_N(x)$, that X is the largest among N independent observations is

$$\Phi_N(x) = [F(x)]^N \quad (2.6)$$

The equations previously derived, that is, equations (2.3), (2.4), (2.5) and (2.6) are also presented by Gumbel (Ref. 19) The names given to these equations are the exact distributions of extremes.

If the initial distributions were known, equations (2.3), (2.4), (2.5) and (2.6) could be utilized in deriving the probability density and distribution functions of the extremes. In most instances, direct calculation would lead to complex integrals that could be approximated only by long and tedious numerical methods.

Because of the difficulty involved when applying the exact distribution of extremes, most reliability analysis performed makes use of the asymptotic extreme value distribution. The derivation of the three asymptotic distributions will not be presented here. It is felt that it is sufficient to present the results of the derivation which appears in Ref. 19. The three asymptotic distribution functions are

Type I

$$\text{Largest Value} \\ \mathbb{Q}_N(x) = \exp \left[-e^{-\frac{(x-\mu)}{b_N}} \right] \quad (2.7)$$

$$b_N > 0$$

Smallest Value

$$\Phi_1(x) = 1 - \exp \left[-e^{\left(\frac{x-\mu}{b_1}\right)} \right] \quad (2.8)$$

$b_1 > 0$

Type II

Largest Value

$$\Phi_N(x) = \exp \left[- \left(\frac{x-\varepsilon}{V_N-\varepsilon} \right)^{K_N} \right] \quad (2.9)$$

$K_N > 0 \quad x \geq \varepsilon \quad V_N > \varepsilon \geq 0$

Smallest Value

$$\Phi_1(x) = 1 - \exp \left[- \left(\frac{W-x}{W-V_1} \right)^{K_1} \right] \quad (2.10)$$

$K_1 > 0 \quad x \leq W \quad V_1 < W$

Type III

Largest Value

$$\Phi_N(x) = \exp \left[- \left(\frac{W-x}{W-V_N} \right)^{K_N} \right] \quad (2.11)$$

$x \geq W \quad V_N < W \quad K_N > 0$

Smallest Value

$$\Phi_1(x) = 1 - \exp \left[- \left(\frac{x-\varepsilon}{V_1-\varepsilon} \right)^{K_1} \right] \quad (2.12)$$

$x \geq \varepsilon \quad K_1 > 0 \quad V_1 > \varepsilon \geq 0$

The Type I extreme value distribution is also known as the exponential, double exponential and Gumbel. Type I extreme value distribution. This asymptotic distribution is derived on the assumption that the underlying or initial variant is of the exponential type. By exponential type it is meant that the probability distribution

function $F(x)$ converges toward unity with increasing x at least as fast as an exponential function. The prototype is the exponential distribution function itself while the Normal, Log-Normal, Logistic and Chi-square distribution are all members of this type. The Type II extreme value distribution is referred to as the Cauchy type because it was derived on the assumption of an underlying Cauchy distribution as the initial distribution. The Type III distribution is referred to as the limited or bounded type distribution because it was developed as the asymptotic distribution of initial distributions which were limited or bounded on the right for largest values and on the left for smallest values. Another common name for the Type III extreme value distribution is the Weibull distribution because it was first used by Weibull in the analysis of breaking strengths of metals (Ref. 19).

When using the theory of extreme values in explaining or analyzing extreme values observed under a given set of conditions and to make predictions of the extremes which may be expected when the same or equivalent set of conditions exist the following conditions must be adhered to:

1. It is necessary to deal with statistical variants.
2. The initial distribution from which the extremes have been drawn and its parameters must remain constant from one sample to another or that the changes that have occurred or will occur are determined and eliminated.
3. The observed data must be independent (Ref. 20).

In regards to the asymptotic theory of extremes, an important relationship exists between the Type I asymptotic distribution and the Type III asymptotic distribution. This relationship is such that the Type III distribution can be obtained from the Type I by a logarithmic transformation of the variate. Conversely, the Type I distribution is reached from the Type III by a linear transformation of the variate and a limiting process on one of the parameters (Ref. 19).

The most popular use made by this relationship is the following. If X is a random variable having the Weibull distribution (Type III) with location parameter equal to 0 then $Z = \ln X$ is a random variable which has the Gumbel Type I extreme value distribution of smallest extremes. The relationship between the parameters of

these two distributions as a result of this transformation are

Gumbel (Type I)		Weibull (Type III)
location parameter μ	→	scale $\gamma = e^{\mu}$
scale parameter b	→	shape $\beta = \frac{1}{b}$

Another important concept in the study of extreme value theory is that of "return period". The "return period" is defined as $T(x)$ with

$$T(x) = \frac{1}{1 - F(x)}$$

The definition of "return period" is that it is the number of observations or sample size such that on the average there is just one observation equalling or exceeding the value x (Ref. 29). This concept of return period is useful in the application of extreme value theory in reliability analysis. A use of the return period will be made in one of the extreme value theory applications presented later in this paper.

In the process of researching the literature for uses of the extreme value distributions it was noted that no applications of the Type II asymptotic distribution could be found. The reasons why the Type II distribution is not used in reliability analysis is because

the variate X is in the negative domain for this distribution (Ref. 28). The domain of the variate for the Type II distribution prevents any logical analysis of failure data because the negative value would indicate failure before the component is put into use.

This concludes the presentation of the theory of extreme values. It is not intended that this presentation would be all inclusive. Rather, it is presented to assist the reader in understanding the application of extreme value theory in reliability analysis which are presented in the remaining portion of this thesis. For a more detailed presentation of the theory of extremes, the reader is referred to the publications of Gumbel (Ref. 19 and 20).

III. Methods of Determining A Probability Distribution Based On Life Test Data

There are several methods available for determining the underlying distribution from a set of data. The importance of accurately determining the underlying distribution can be seen when we consider that we are attempting, on the basis of a sample, to establish as close a measure as possible of the parent population and the various statistics of the parent population which are used in determining the reliability of a component. Thus, if a given distribution is arbitrarily chosen for a set of data and in reality, the data has a different distribution, the statistics computed for the data can be misleading. Two methods for determining the underlying distribution will be discussed. Both of these methods were found to be most frequently used in the example applications appearing in the literature. Other methods will be mentioned at the end of the section.

Estimation of Parameters

In reference 19, Gumbel presents the equations for determining the parameters of the Type I asymptotic distribution. The cumulative distribution function for the Type I largest and smallest extreme are

Largest Extreme

$$D_N(x) = e^{-e^{-y}} \quad (3.1)$$

Smallest Extreme

$$D_1(x) = 1 - e^{-e^y} \quad (3.2)$$

where

$$y = \frac{x - \mu}{b} \quad (3.3)$$

The value y is the so called reduced variate, while μ and b are parameters of the distribution. The quantity μ is the mode or location parameter while b is the scale parameter. In order to explicitly define the equation of the probability distribution function, it is sufficient to determine the values of the parameters.

Using the method of moments, the following are estimators for μ and b as presented in Ref. 19.

$$\hat{b} = \frac{\sqrt{6}}{\pi} S_x \quad (3.4)$$

$$\hat{\mu} = \bar{x} - \nu b \quad \text{for the largest extreme} \quad (3.5)$$

$$\hat{\mu} = \bar{x} + \nu b \quad \text{for the smallest extreme} \quad (3.6)$$

where $\nu = .5772$ (Euler's constant)

$\bar{x} = \sum_{i=1}^N x_i / N$ is the sample mean

$S_x = \left[\frac{\sum_{i=1}^N x_i - \bar{x}}{N-1} \right]^{1/2}$ is the sample standard deviation

Through some algebraic manipulation and substitution of equation (3.4) into equations (3.5) and (3.6) the results are

$$\hat{\mu} = \bar{X} - .4502 S_x \quad (\text{Largest Extreme}) \quad (3.7)$$

$$\hat{\mu} = \bar{X} + .4502 S_x \quad (\text{Smallest Extreme}) \quad (3.8)$$

The above equations can be utilized to estimate the parameters of the Type I (Largest or Smallest) extreme value distribution. Our hypothesis as the correct choice of the distribution selected for describing the underlying distribution of the data will be rejected or accepted on the basis of a test. Two widely used tests are the Chi-Square and the Kolmogorov-Smirnov goodness-of-fit tests. When the number of samples is small it is best to use the Kolmogorov-Smirnov test because the Chi-Square test loses power due to grouping.

For the purpose of evaluating data utilized in the examples presented in the following sections, a computer program was written in Fortran language to estimate the parameters using equations (3.4), (3.7) and (3.8). A

Kolomogorov-Smirnov test was also incorporated in the computer program in order to determine whether or not the sample data fit the Type I extreme value distribution.

Graphical Solution

A very convenient method of evaluating sample data is to plot the data on an appropriate graph paper. Graph papers have been designed for both the Gumbel Type I distribution and the Weibull or Type III distribution. The design of the probability graph paper is such that data sampled from the governing distribution would plot as a straight line on the applicable graph paper. The use of a best-fitted straight line drawn through the plotted data and the plotting of curves reflecting a desired confidence limit can be used to determine if the data fits the selected distribution. In cases where the plotted data is found to fit a selected distribution, the parameters of the distribution can be estimated by using the appropriate scales on the graph paper. The methods of using the Type I extreme value probability paper (Gumbel Type I) can be found in Refs. 30 and 31. For the methods of using Weibull probability paper, the reader is referred to references 27 and 30.

Other Methods

The increased availability of computers for use by mathematicians, statisticians and engineers in the past seven years has greatly increased the methods of evaluating test data. Methods of evaluating test data to determine the underlying distribution and estimating the parameters of the distribution which were formerly regarded as too difficult or impossible to apply are becoming commonplace methods because of increased use of computers. A computer program written by Gross (Ref. 16) uses a regression technique to determine estimates for the shape, scale and location parameters of the Weibull distribution. Greater application of order statistics has provided new methods of estimating the parameters of the Type I and Type III extreme value distributions. Among the new estimation techniques for the Type I extreme value distribution are the "best linear invariant estimators" proposed by Mann (Ref. 34) and the nearly best linear unbiased estimators presented by Hassanein (Ref. 22). Linear estimators with polynomial coefficients were presented by Downton (Ref. 3). In Ref. 24, Harter and Moore present methods of determining the maximum-likelihood estimates for the parameters of the

Type I asymptotic distribution from doubly censored samples.

Mann (Ref. 35) presented tables for obtaining the best linear invariant estimates of the parameters of the Weibull distribution. In Ref. 23, Harter and Moore present a method of finding the maximum likelihood estimates for the three parameters of the Weibull distribution from complete and censored samples. Other methods for estimating the parameters of the Weibull distribution are presented by White (Ref. 43 and 44), Menon (Ref. 32) and Leiberman (Ref. 30).

The purpose of this section is to familiarize the reader with the methods available for analyzing test data. It is felt that a step by step procedure for the application of each method is not necessary and is not within the scope of this thesis. The use of some of the previously mentioned methods will be made in the following sections which present the applications of extreme value theory in reliability analysis.

IV. Application of the Exact Distribution
of the Smallest Extreme

This particular application makes use of the weakest-link theory. Under this theory of failure, each component is treated as consisting of many sub-components which make up the component itself in the same manner as links form a chain. Then this characteristic life pattern of the component (chain) is equivalent to the characteristic life pattern of the sub-component (link). Assuming that the life length of all N sub-components are independently and identically distributed with a probability density function $f(x)$ and cumulative distribution function $F(x)$, the life of the composite component would be distributed according to the smallest order statistic or smallest extreme; thus,

$$\overline{Q}_1(x) = 1 - [1 - F(x)]^N \quad (4.1)$$

and

$$Q_1(x) = Nf(x) [1 - F(x)]^{N-1} \quad (4.2)$$

The reliability function $R(x)$ is defined as $1 - \overline{Q}_1(x)$ thus,

$$R_1(x) = [1 - F(x)]^N \quad (4.3)$$

The example is a hypothetical case but is a direct application of the theory applied by Epstein (Ref. 8) in the analysis of paper capacitor failures and the theoretical example presented by Lloyd and Lipow (Ref. 33). Although the following problem is not based on an actual situation, the theory can be applied in many areas, such as failures due to corrosive pitting, failures as a result of propagation of cracks due to thermal fatigue, vibration or stress and also failures resulting from flaws in a component.

The problem is stated as follows:

A structural member, containing a constant cross-section area element one-half inch thick, is subjected to a vibratory stress environment. The element is fabricated from steel stock containing a large number of surface defects which have a high probability of developing into cracks in the vibratory environment. A microscopic examination reveals that the number of these defects in the parent stock is such that a piece with an area equal to the critical surface area of the element will contain approximately 668 defects on the average.

It is also determined that the defects in the parent stock may be represented with an exponential distribution, with a defect mean depth of 0.02 inches. Computations of the stress-strength relationship in the element indicates that it will fail (thereby failing the structural member) when cracked half way through. Vibratory tests, approximating the environmental stress, show a crack growth rate, once started, which is proportional to the time of exposure. The constant of proportionality is of the form of an acceleration, and is equal to 1.5×10^{-14} in./sec.². Compute the period of operation which will degrade the reliability to 95.1%.

Given:

$$D = \frac{1}{2} \text{ in.}$$

$$Z_i = \text{initial depth of } i^{\text{th}} \text{ pit}$$

$$N = 668 \text{ (number of defects)}$$

$$\mu = .02 \text{ in/defect} \quad \lambda = 50 \text{ defect/in}$$

$$\text{Constant of proportionality } C = 1.5 \times 10^{-14} \text{ in/sec}^2$$

The density function of defects is

$$f(z_i) = \lambda e^{-\lambda z}$$

A failure occurs when crack propagates to depth $D/2$

Solution:

It is necessary to determine a new distribution function for the distribution of defects. The original distribution was exponential with the range of 0 to $+\infty$ for the variate Z . The new distribution will be truncated at D (thickness of component) because the allowable range of the variate is restricted to 0 to D . Therefore, the truncated density function is

$$f(z_i) = \frac{\lambda e^{-\lambda z}}{1 - e^{-\lambda D}} \quad (4.4)$$

Let τ_i = time of penetration of the i^{th} crack

$$\tau_i = \frac{D/2 - Z_i}{K} \quad (4.5)$$

where $k = ct^2$
 $= 1.5 \times 10^{-14} t$

Reliability of the component $R(t)$ is defined as:

$$\begin{aligned} R(t) &= P(\tau_{i_{\text{MIN}}} > t) = \prod_{i=1}^N [P(\tau_i > t)] \\ &= \prod_{i=1}^N [1 - P(\tau_i < t)] \\ &= \prod_{i=1}^N [1 - F(t)] \\ R(t) &= [1 - F(t)]^N \quad (4.6) \end{aligned}$$

This equation is the same as equation (4.3).

Equation (4.6) can be approximated as follows

$$R(t) \approx e^{-NF(t)} \quad (4.7)$$

The basis for this approximation can be found in Ref. 33 and 37.

Using equation (4.5)

$$\begin{aligned} P_r(\tau_i < t) &= P_r\left(\frac{D/2 - Z_i}{k} < t\right) \\ &= P_r(D/2 - Z_i < kt) \\ &= P_r(Z_i > D/2 - ct^2) \\ \therefore F(t) &= P_r(Z_i > D/2 - ct^2) \end{aligned}$$

The probability density function for the distribution of Z_i is equation (4.4).

$$\begin{aligned} F(t) &= P_r(Z_i > D/2 - ct^2) = \int_{D/2 - ct^2}^D f(z_i) dz \\ &= \frac{e^{-\lambda(D/2 - ct^2)} - e^{-\lambda D}}{1 - e^{-\lambda D}} \end{aligned}$$

Multiplying through by $\frac{e^{\lambda D}}{e^{\lambda D}}$ we obtain

$$F(t) = \frac{e^{\lambda(D/2 + ct^2)} - 1}{e^{\lambda D} - 1} \quad (4.8)$$

Substituting equation (4.8) into equation (4.7) we obtain

$$R(t) = e^{-N \left(\frac{e^{\lambda(D/2 + ct^2)} - 1}{e^{\lambda D} - 1} \right)} \quad (4.9)$$

Because the value of λ is large (50) in the above equation the magnitude of e raised to the λ power is very large; therefore, the following approximation is reasonable

$$R(t) = e^{-N \left(\frac{e^{\lambda(D/2 + ct^2)}}{e^{\lambda D}} \right)} \quad (4.10)$$

We now solve for the operating time t which will degrade the reliability of the structure to 95.1%.

Using equation (4.10)

$$.951 = e^{-N \left[\frac{e^{\lambda(D/2 + ct^2)}}{e^{\lambda D}} \right]}$$

$$\ln(.951) = -N \left(e^{\lambda(D/2 + ct^2)} \cdot e^{-\lambda D} \right)$$

$$t = 2 \times 10^6 \text{ seconds}$$

$$= 556 \text{ hours}$$

Thus, the operating time required to degrade the reliability of the structure to 95.1% is 556 hours.

V. Application of Type I Asymptotic
Extreme Value Distribution

Corrosion Pitting of Aluminum

The effect that a corrosive environment has on a metal structure has always been of great importance to design engineers because of the possibility of failure in the form of holes caused by corrosive pitting. A large amount of money is spent each year in repairing damage caused by failure of underground storage tanks, water supply pipes and other metal structures which fail as a result of the corrosive pitting of the metal. The study of corrosive pitting of metal is made difficult by the lack of a suitable variable which can be measured quantitatively and treated mathematically. One of the most commonly used measures is the maximum pit depth developed on metal samples exposed to a corrosive environment for a fixed period of time.

Aziz (Ref. 1) utilized the method of measuring the maximum pit depth found on aluminum coupons which were exposed to tap water for various lengths of time.

For purposes of the experiment, coupons measuring 5" X 2" were manufactured from various aluminum alloys. Strings, each containing ten of these coupons, were

immersed in a 300 gallon tank containing tap water. At the end of a specified time period, the sample coupons were removed from the water and the maximum pit depth was measured and recorded. The recorded maximum pit depth for Alcan 3S-0 aluminum appears in Table 5-1. This data is found in reference 1.

TABLE 5-1

MAXIMUM PIT DEPTH (MICRONS)

<u>Rank</u>	<u>Two Weeks</u>	<u>One Month</u>	<u>Two Month</u>	<u>Four Month</u>	<u>Six Month</u>	<u>One Year</u>
1	330	570	600	620	640	700
2	460	620	670	620	650	700
3	500	640	770	670	670	750
4	500	640	790	680	700	770
5	530	700	790	720	720	780
6	540	740	830	780	730	810
7	560	780	860	780	750	820
8	560	810	930	800	770	830
9	580	840	1030	830	780	830
10		910		920	850	930

The data presented in Table 5-1 was plotted on Type I extreme value probability paper, Figures 5-1 and 5-2, to determine if the sample data can be fitted to the extreme value distribution. For purposes of plotting the data on probability paper, plotting positions were used versus the ranked data. The median rank plotting position is defined as $\frac{M}{N+1}$ where M is the assigned rank and N

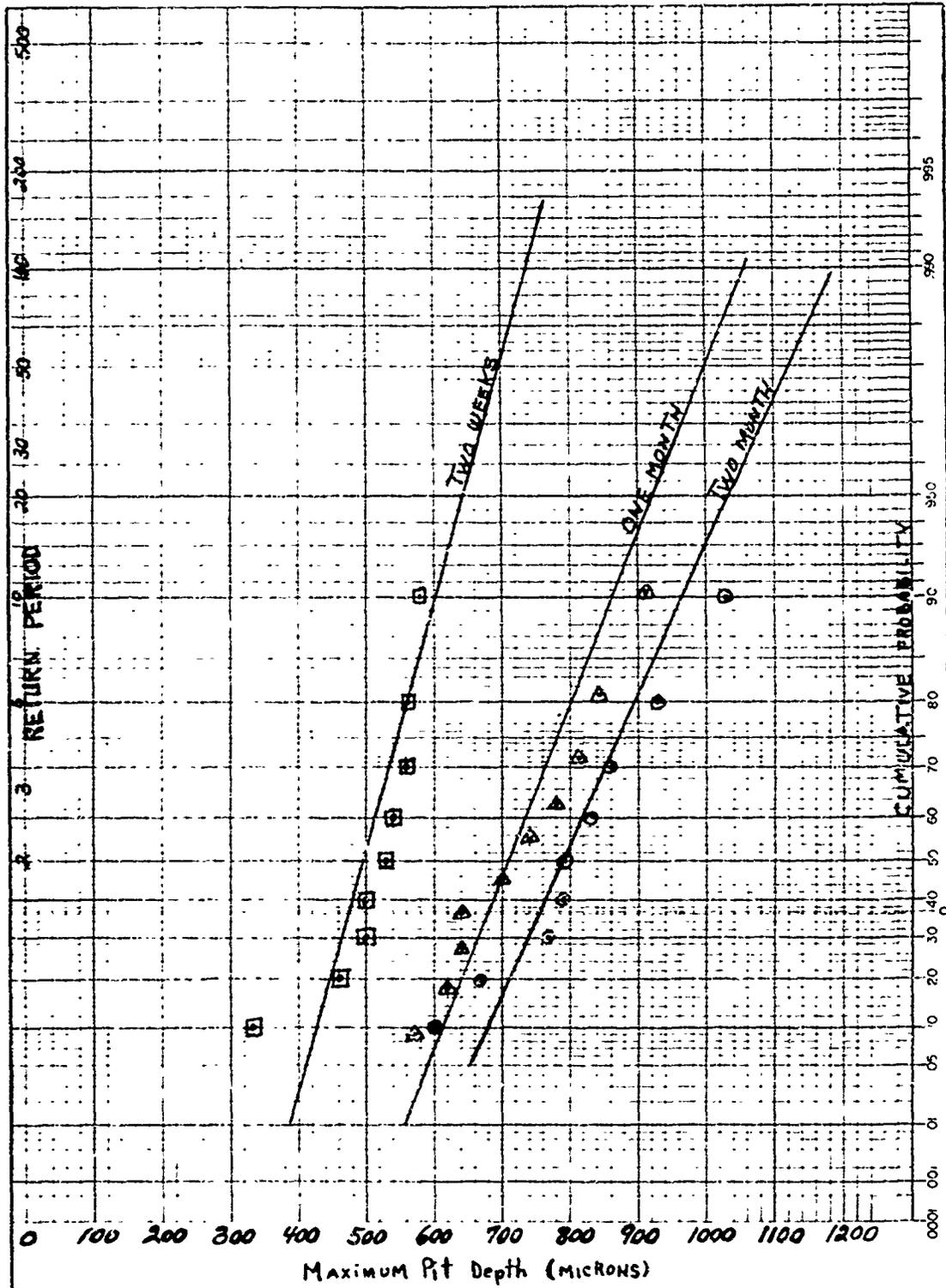
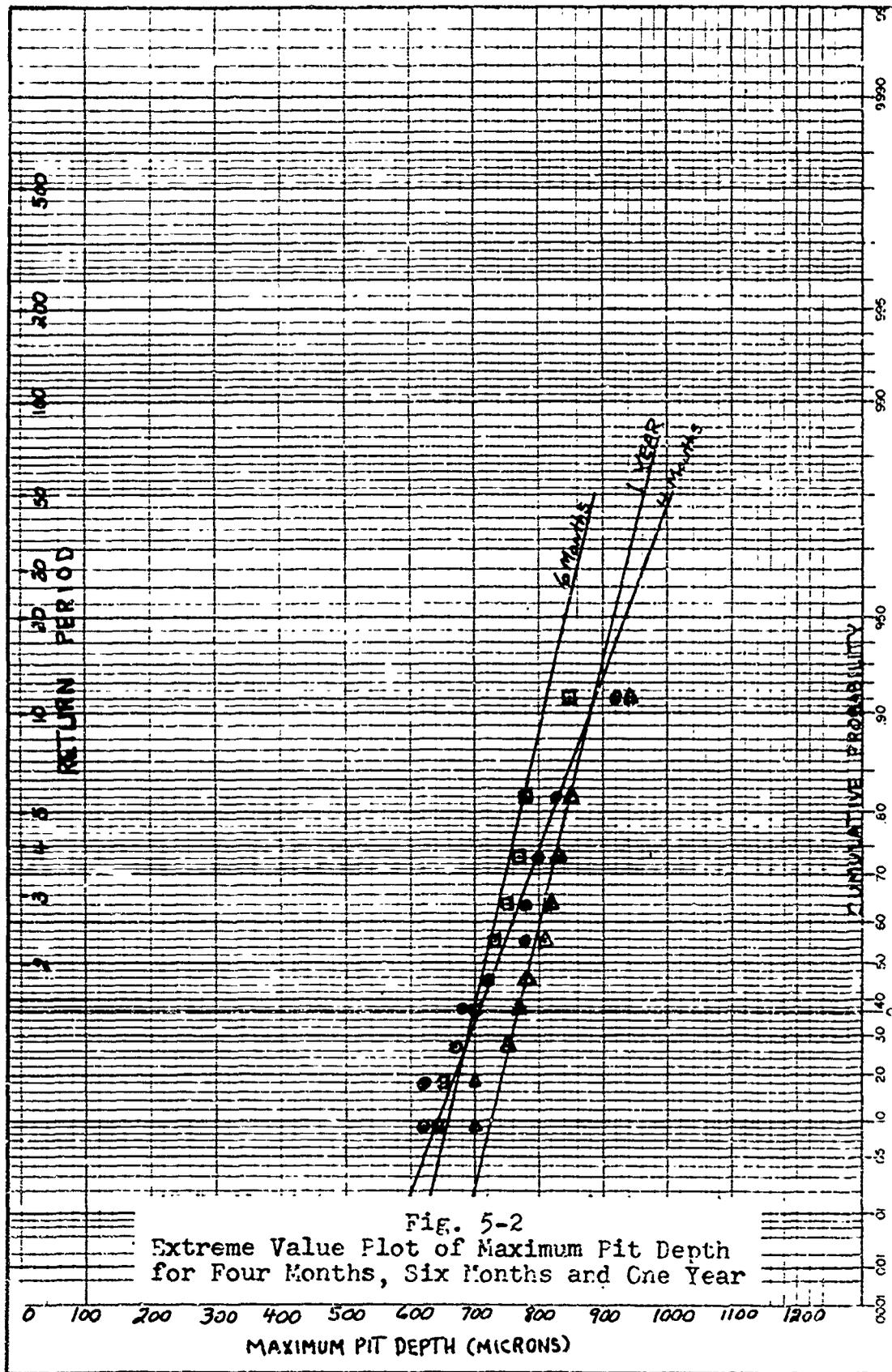


Fig. 5-1
 Extreme Value Plot of Maximum Pit Depth
 for Two Weeks, One Month and Two Months



is the sample size. Reasons for using median rank plotting positions can be found in references 18 and 19.

Using graphical methods to estimate the parameters from the sample data, the results are presented in Table 5-2. The Fortran program, mentioned in section III was

TABLE 5-2

ESTIMATES OF DISTRIBUTION PARAMETERS
FOR MAXIMUM PIT DEPTH DATA

	<u>Graphical Solution</u>			<u>Method of Moments</u>	
	μ	b	Mean	μ	b
2 weeks	475	59.6	510	474.4	55.86
1 month	680	77.9	725	678.2	81.04
2 month	755	95.4	810	753.3	94.33
4 month	705	69.5	745	700.5	71.87
6 month	700	48.4	728	698.3	47.96
1 year	765	52.0	795	762.7	50.75

utilized to estimate the parameters from the sample data and also perform a Kolomogorov-Smirnov goodness-of-fit test at the 95% confidence level. The results of the computer analysis are also presented in Table 5-2. A comparison of the results shows that there is not a significant difference in the accuracy of the graphical

solution and the computer solution. The results of the goodness-of-fit indicates that the data can be assumed to be a sample from the Type I largest extreme value distribution.

Because the corrosive pitting of aluminum can be accurately represented by the Type I extreme value distribution, the use of extreme value theory can lead to some interesting results. In this application, the natural experimental unit was a coupon of fixed area which was exposed to a corrosive environment and from which the deepest pit developed was recorded. Thus, the return period is the number of coupons that on the average, must be exposed in order to obtain a pit depth greater than the observed pit depth. The return period indicates that the deepest pit observed is a function of the area exposed to the corrosive environment (Ref. 1).

The value of the return period can be read directly from the plot on the probability paper. Thus, from the one month data of Figure 5-1, it can be determined that at least 100 coupons must, on the average, be exposed in order to have a pit develop to at least 1060 microns; whereas on the ten coupons exposed the deepest observed pit was 910 microns. This can also be stated in another

way: on 100 square inches, the deepest pit that will be observed, on the average is 910 microns, whereas on 2000 square inches, the deepest observed pit, on the average will be 1060 microns.

Reliability Applications

Suppose we have a container manufactured from Alcan 3S-0 aluminum and the container is to be used to store tap water for a one month period. The inner surface area of the container is 100 square inches and the walls measure 900 microns in thickness. We wish to determine the reliability of this container for the one month period. In this case, reliability is defined as the probability that a pit less than the thickness of the container walls will develop. The reliability value can be determined directly from the one month plotted data of Figure 5-1 or using the following equation

$$R(x) = e^{-e^{-\left(\frac{900-678.2}{81.04}\right)}}$$

Using either method for solution, the reliability of the container is found to be 93.5% for the one month period.

This example shows that extreme value theory can readily be applied to the analysis of corrosive pitting. In cases where weight and cost of the storage container

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This example shows that extreme value theory can readily be applied to the analysis of corrosive pitting. In cases where weight and cost of the storage container

are of major interest, such as the fuel or oxidizer tank of a ballistic missile, it is believed that the use of extreme value theory in the analysis of the effect of a corrosive environment can possibly lead to a more optimum design.

The successful application of extreme value theory in the analysis of corrosive pitting is also presented by Eldredge (Ref. 4) and by Finley and Toncre (Ref. 11).

Load Analysis

The problem of designing a structure to adequately meet the maximum load expected to be experienced by the structure in the intended environment has always proved to be a major dilemma to design engineers. When problems of proper load design arose in the design phase of a project, it was a common practice for the engineer to base his calculations on average values and then multiply his answer by an arbitrary number called the "safety factor". In many cases, the safety factor represented nothing but a vague feeling of danger involved in the specification (Ref. 18).

A structure or component experiences a load failure when the load applied equals the strength of structure. This interaction of applied load and structure strength can be evaluated in two ways. In the first case, the strength of the structure remains constant but the load applied is variable. When the applied load equals or exceeds the strength of the structure, a failure will occur. In the second case, the strength of the structure changes over a period of time which in effect is that the largest allowable load changes over time. As can readily be seen, both cases deal with the extremes encountered and not with the average value. Therefore, it is logical that the theory of extremes would be quite applicable in analyzing design problems of this nature.

The following is an example of the application of extreme value theory in load analysis. An impactograph, mounted on the skid of a missile shipping container, was used to record the largest shock acceleration encountered by a missile on a road trip between Schenectady, New York and White Sands, New Mexico. The total distance was about 2500 miles, over a variety of paved roads. The resulting recorded data is presented in Table 5-3.

(Ref. 26).

TABLE 5-3
 MAXIMUM VERTICAL ACCELERATION

<u>Missile Serial</u>	<u>Vertical Acceleration (g's)</u>	<u>Trip</u>
A5	2.4	N.Y. - N.M.
	4.5	N.M. - N.Y.
A6	6.0	N.Y. - N.M.
	8.0	N.M. - N.Y.
A7	10.0	N.Y. - N.M.
	3.0	N.M. - N.Y.
A8	2.5	N.Y. - N.M.
B1	4.0	N.Y. - N.M.
	5.0	N.M. - N.Y.
B2	7.0	N.Y. - N.M.
	7.5	N.M. - N.Y.
B3	4.6	N.Y. - N.M.
	5.0	N.M. - N.Y.
B4	3.0	N.Y. - N.M.
	5.0	N.M. - N.Y.

A plot of this data on extreme value probability paper, Figure 5-3, indicates that the data can be represented by the Type I largest extreme value distribution. Using the computer program to estimate the parameters from the sample data and to perform a Kolomogorov-Smirnov test, the scale parameter estimate

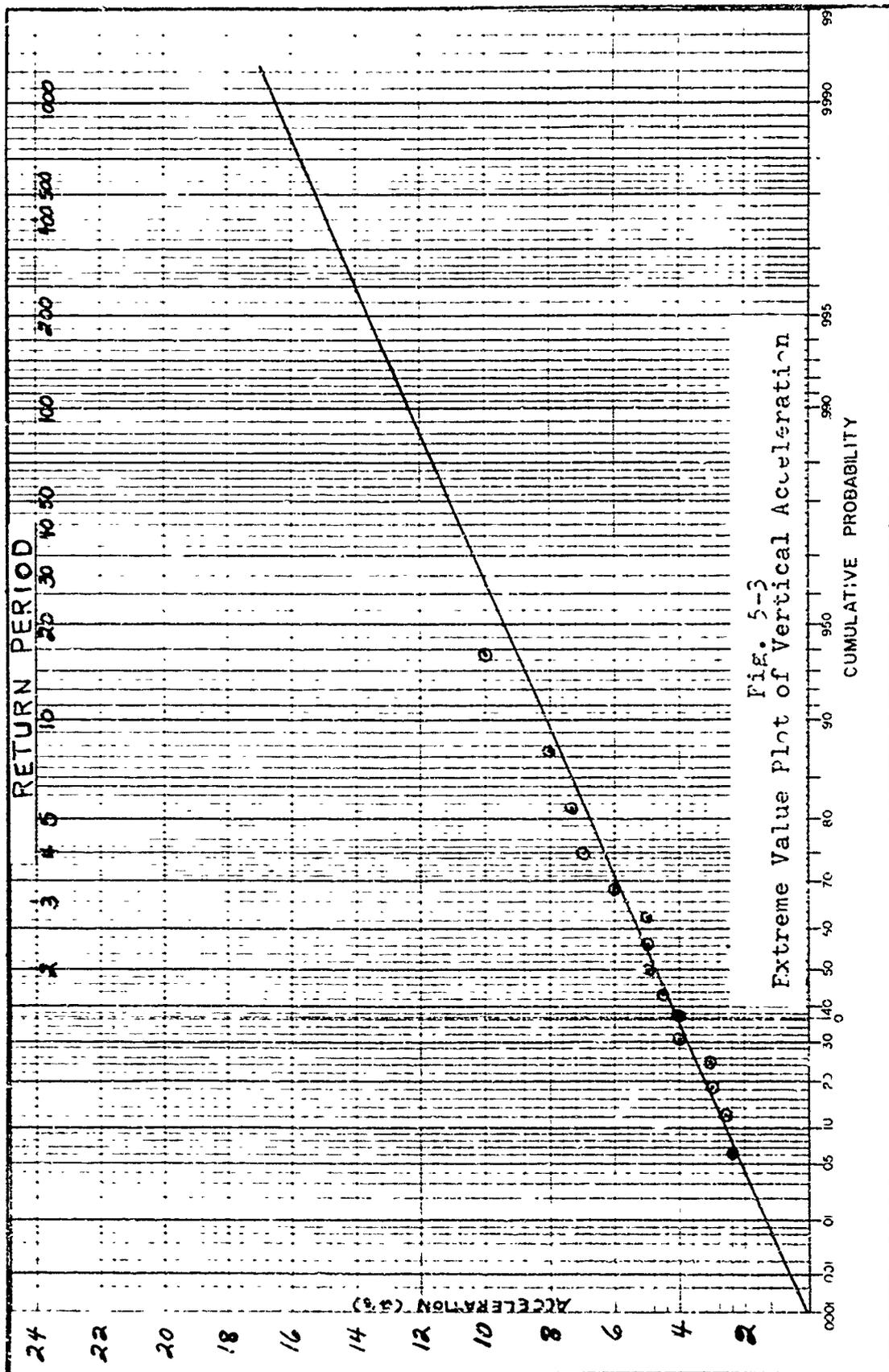


Fig. 5-3
Extreme Value Plot of Vertical Acceleration

is 1.77 and the location parameter is 4.105. The results of the goodness-of-fit test at the 95% confidence level indicates that we can accept the hypothesis that the sample was drawn from a Type I largest extreme value distribution.

If the usual method of applying the three sigma rule as a safety factor was applied in this case, the estimated "largest load" would be 11.4 g. This implies that a load of 11.4 g would occur once in 1,000 trips or an equivalent of 2,500,000 road miles (Ref. 26).

Since it was shown that the data can be represented by the Type I extreme value distribution, the use of extreme value theory should lead to more accurate results. As can readily be seen, the basic unit for the "return period" is the number of trips required which, on the average, would result in a load acceleration equal to or greater than the given value. From Figure 5-3, it can be seen that a value of 11.4 g corresponds to a return period of 60 additional trips or 150,000 miles. It can also be seen that the "expected largest value" in 1,000 trips is 16.5 g which is over three times the mean value and which approaches 1.5 times the value of the three sigma rule. Comparison of the results obtained

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using the theory of extremes rather than the three sigma rule indicates the error involved when the wrong assumption is made concerning the governing distribution. By using the three sigma rule, a normal distribution was assumed as the underlying distribution when in reality, the data was best modeled by the Type I largest extreme value distribution.

VI. Type III Asymptotic Extreme Value Distribution

The cumulative distribution functions for the Type III largest and smallest extreme value were presented in Section II equations (2.11) and (2.12). Although both distributions are referred to as Type III, only the smallest extreme distribution will be presented in this section. It has been discovered that the Type III smallest extreme value distribution has greater application in the engineering field than does the largest extreme value distribution.

The Type III distribution of smallest extremes was first applied by W. Weibull to analyze data resulting from failures caused by contact stresses. Because Weibull was the first to successfully apply the Type III smallest extreme value distribution in engineering studies, the distribution is more familiarly known as the Weibull distribution.

The more commonly defined form of the Weibull probability density function is

$$f(x) = \frac{\beta}{\eta} \left(\frac{x-y}{\eta} \right)^{\beta-1} e^{-\left(\frac{x-y}{\eta} \right)^{\beta}} \quad (6.1)$$

where; β = shape parameter or Weibull slope

π = scale parameter

γ = location parameter

The general shape of the Weibull density function is determined by the value of β , the shape parameter. A plot of the Weibull density function for various values of β with $\pi = 1$ and $\gamma = 0$ is presented in Figure 6-1. It should be noted that when $\beta = 1$, the Weibull distribution specializes to the exponential distribution. When $\beta = 2$, the resulting distribution is the Rayleigh distribution.

The Weibull cumulative distribution function, which is derived from equation (6.1) by integration is

$$\begin{aligned}
 F(x) &= \int_0^x \frac{\beta}{\pi} \left(\frac{x-\gamma}{\pi}\right)^{\beta-1} e^{-\left(\frac{x-\gamma}{\pi}\right)^\beta} dx \\
 &= 1 - e^{-\left(\frac{x-\gamma}{\pi}\right)^\beta}
 \end{aligned} \tag{6.2}$$

The reliability function $R(x) = 1 - F(x)$ is

$$R(x) = e^{-\left(\frac{x-\gamma}{\pi}\right)^\beta} \tag{6.3}$$

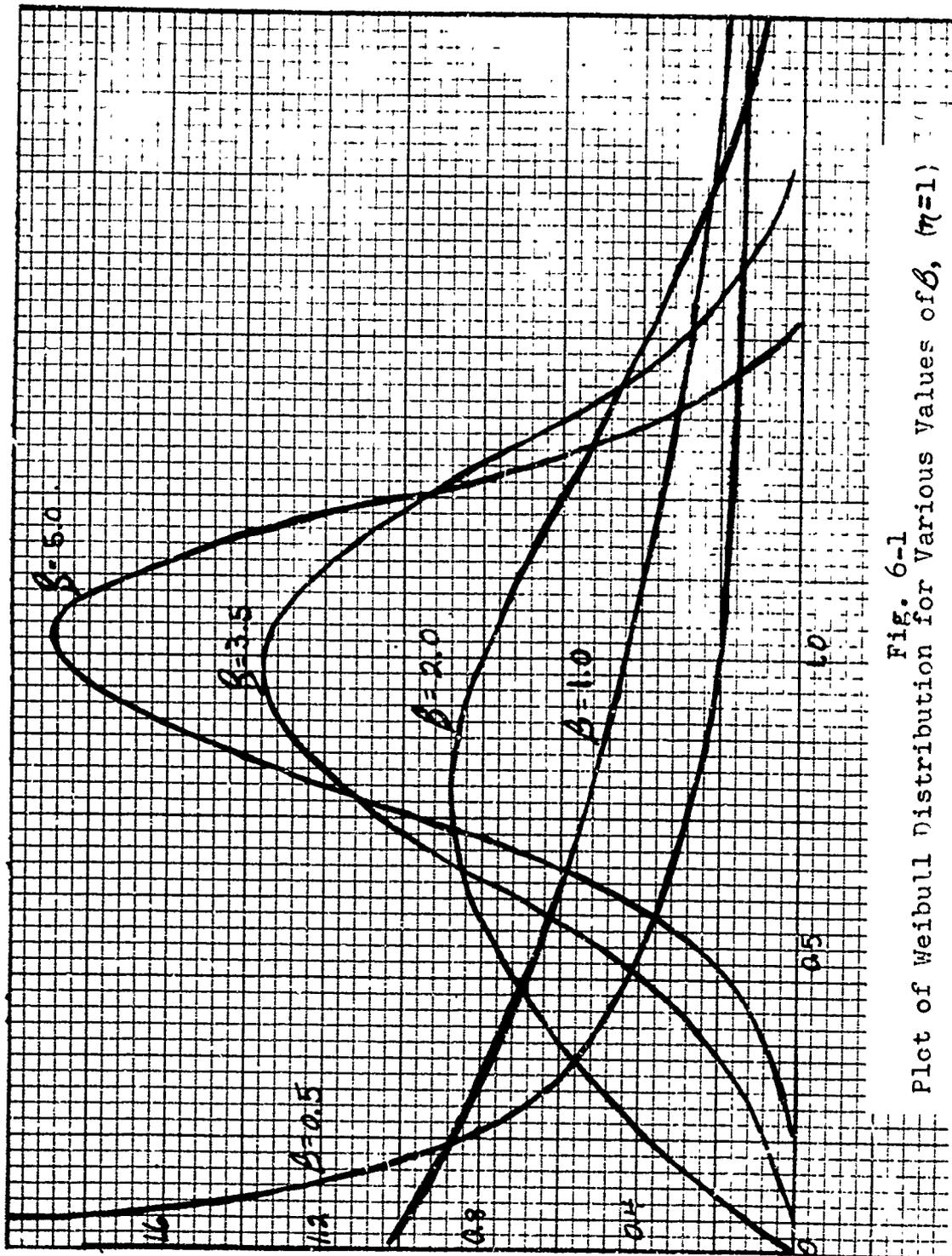


Fig. 6-1
Plot of Weibull Distribution for Various Values of β , ($\eta=1$)

Analysis of Failures of Door Lock Mechanism Assembly

This particular application of the Weibull distribution was presented by Forgione (Ref. 13). The object of the test was to determine the life characteristic of a door lock assembly which was used on an automobile. The purpose of the door lock mechanism was to secure the door in a closed position. Other requirements were that the door must open and close smoothly and, in the locked condition, withstand specified static loads so as to prevent the door from opening.

For purposes of the test, a sample of twelve mechanisms were placed on test, using a special test fixture which accurately simulated the impact loads incurred when a door is slammed shut. The twelve door lock mechanisms were placed on simultaneous test and the numbers of opening and closing cycles required until failure were recorded.

In order to analyze the data on probability paper, use of median rank tables was made to assign a cumulative percent failure value to each failure. As was pointed out by Forgione, the purpose for using median rank is that when failure history of the entire population is not known, a statistical estimate is made for the rank

of each failure. The median rank, which has a 50% probability of being either too high or too low, is the best estimate of the actual failure rank.

The recorded data and the corresponding value of median rank is presented in Table 6-1. As is reflected in Table 6-1, the test was concluded after recording nine failures with 3 mechanisms removed from test after 122,218 cycles without failure. The three mechanisms removed are referred to as suspended items. Suspended items must be taken into account when selecting the median rank values for the failed items, therefore, the median rank values in Table 6-1 were obtained from a median rank table for a sample size of 12.

A plot of the data on Weibull probability paper is presented on Figure 6-2. Included on the plot, are the 90% confidence bands to aid in determining if the data is a representative sample from the Weibull distribution. The resulting straight line plot of the data is an indication that the underlying failure distribution of the population is Weibull with cumulative distribution function given as

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}} \quad (6.4)$$

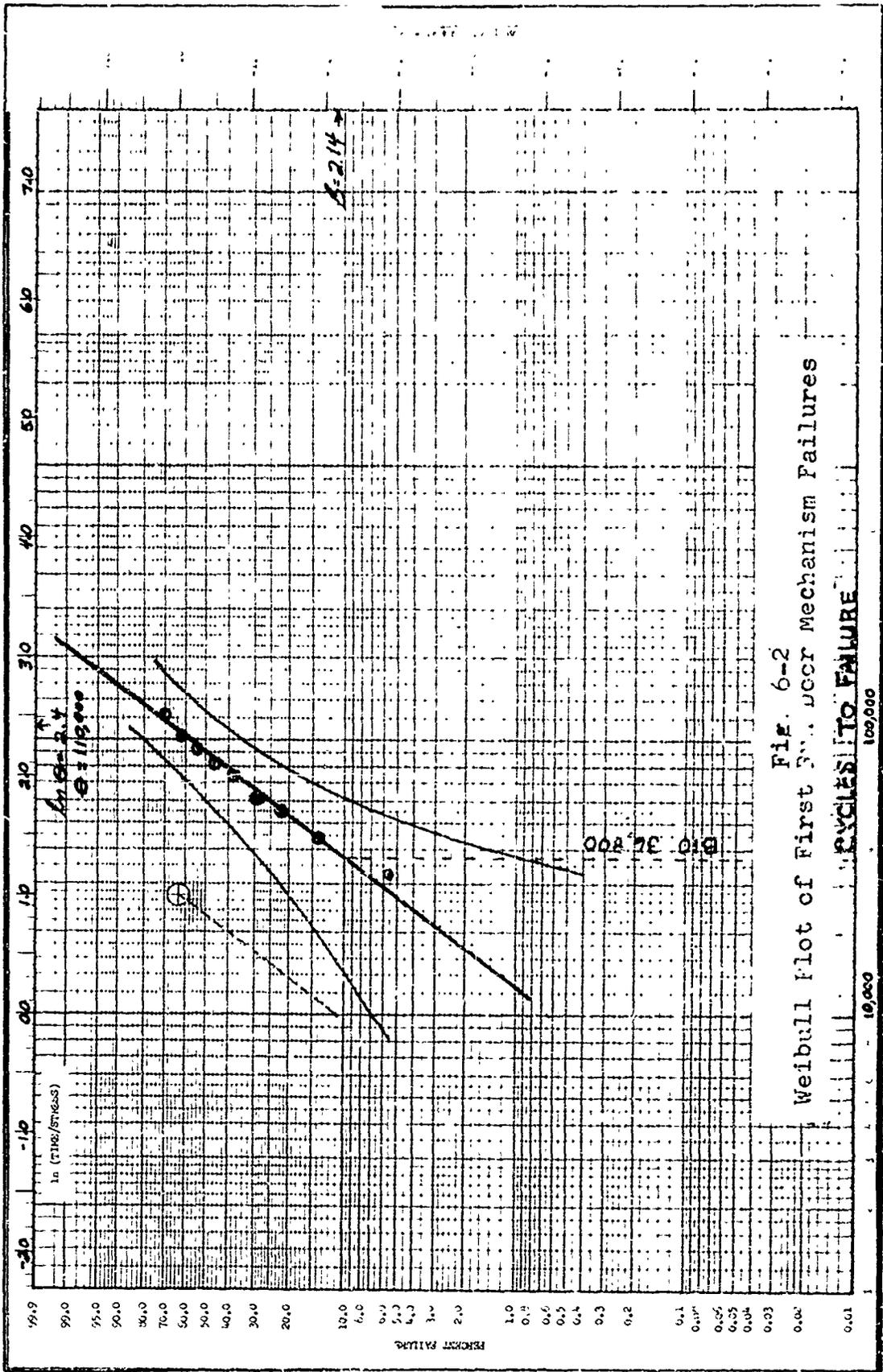


Fig. 6-2
Weibull Plot of First P.M. Occr Mechanism Failures

TABLE 6-1

DOOR LOCK MECHANISM FAILURE DATA

Item Number	Failure Number	Cycles to Failure	Median Rank
1	1	32,680	0.056
2	2	44,560	0.137
3	3	55,442	0.218
4	4	61,074	0.298
5	5	73,998	0.379
6	6	81,468	0.460
7	7	92,900	0.540
8	8	100,690	0.621
9	9	122,218	0.701
10	Suspension	122,218	
11	Suspension	122,218	
12	Suspension	122,218	

where β is the shape parameter

θ the characteristic life or the number of cycles at which 63.2% of the items have failed

This distribution is the same as that given by equation

(6.2) except that the location parameter ν is 0 .

One of the objectives of the test was to determine the BIO life of the component. The BIO life is defined as the life of the component at 90% reliability. The value of the BIO life can be read directly from the graph as 36,800 door slam cycles. Another method of determining the BIO life would be to use the graphical method or computer method to determine the estimates of the parameters and use equation (6.3) to solve for the value of x corresponding to R (.90). Using graphical methods and the method presented in Ref. 23 to estimate parameters and equation (6.3) to determine the BIO life, the results are presented in Table 6-2.

TABLE 6-2

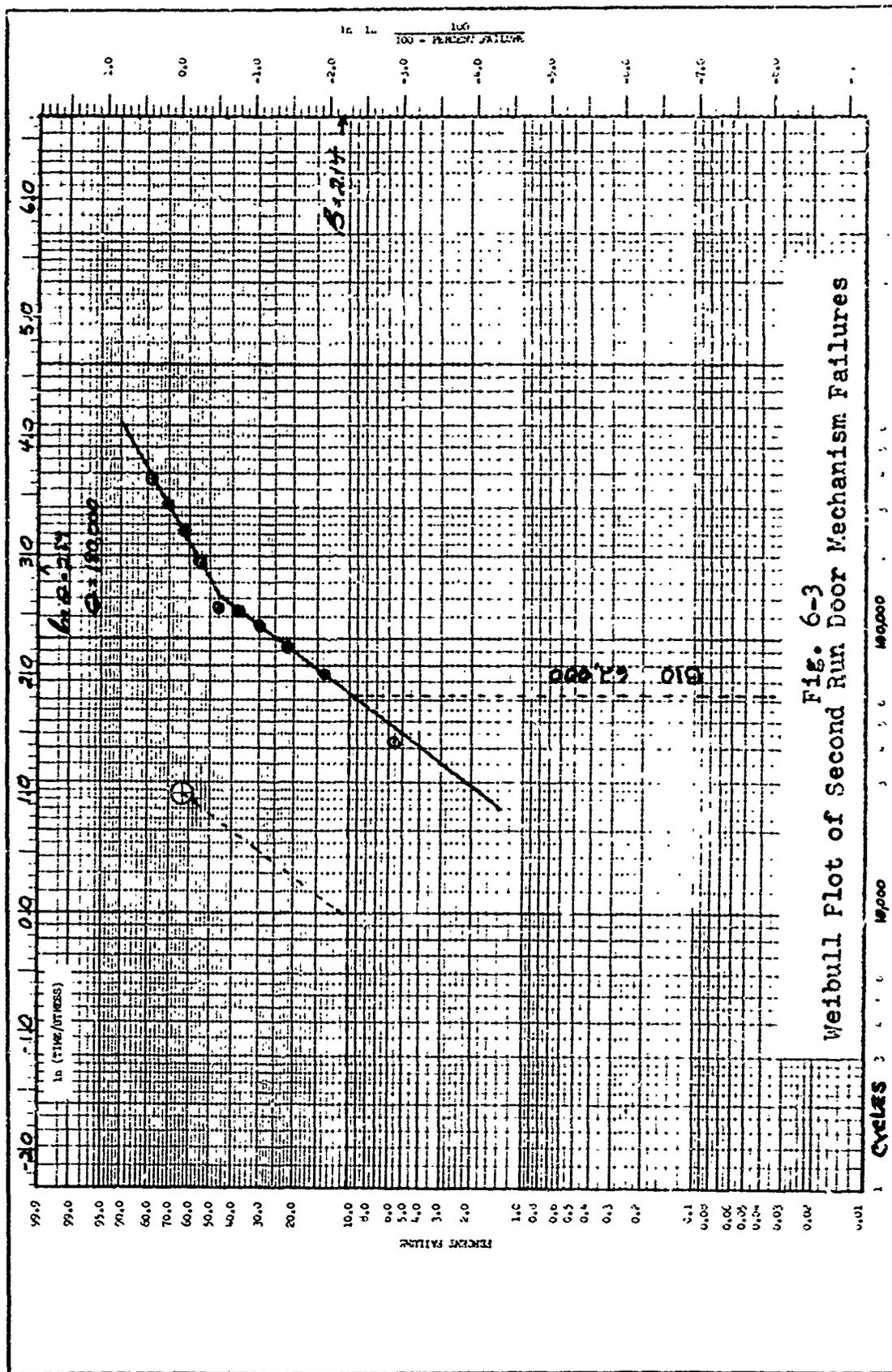
BIO LIFE AND PARAMETER ESTIMATES OF
DOOR LOCK MECHANISM FAILURE DATA

<u>Graphical</u>	<u>Max. Likelihood</u>
$\beta = 2.14$	2.37
$\theta = 110,000$	111,854
$\gamma = 0$	0
BIO = 36,800	43,253

The BIO life determined from the first set of sample components was below that required for the component. An evaluation of the failed parts indicated that all nine failures were caused by fatigue failure of an actuating link. Redesign of this component was performed and another twelve locking mechanisms were placed on test. The resulting data is shown plotted on Weibull paper in Figure 6-3. The BIO life of the redesigned mechanisms was determined to be 62,000 cycles. Statistical tests performed by the author indicated an 88% confidence that the BIO life of the redesigned item was at least equal to or greater than the original mechanism. (Ref. 13).

The change of slope of the data plotted in Figure 6-3 indicates a mixed population. As was pointed out by the author, an evaluation of the cause of failure indicated that the change of slope resulted from failures of an actuating spring. This could be expected since the first failures were link failures having one distribution and another component would have an altogether different distribution of failure.

This example shows the flexibility of the Weibull distribution in that it was used to compare two designs of a component and different values of the shape

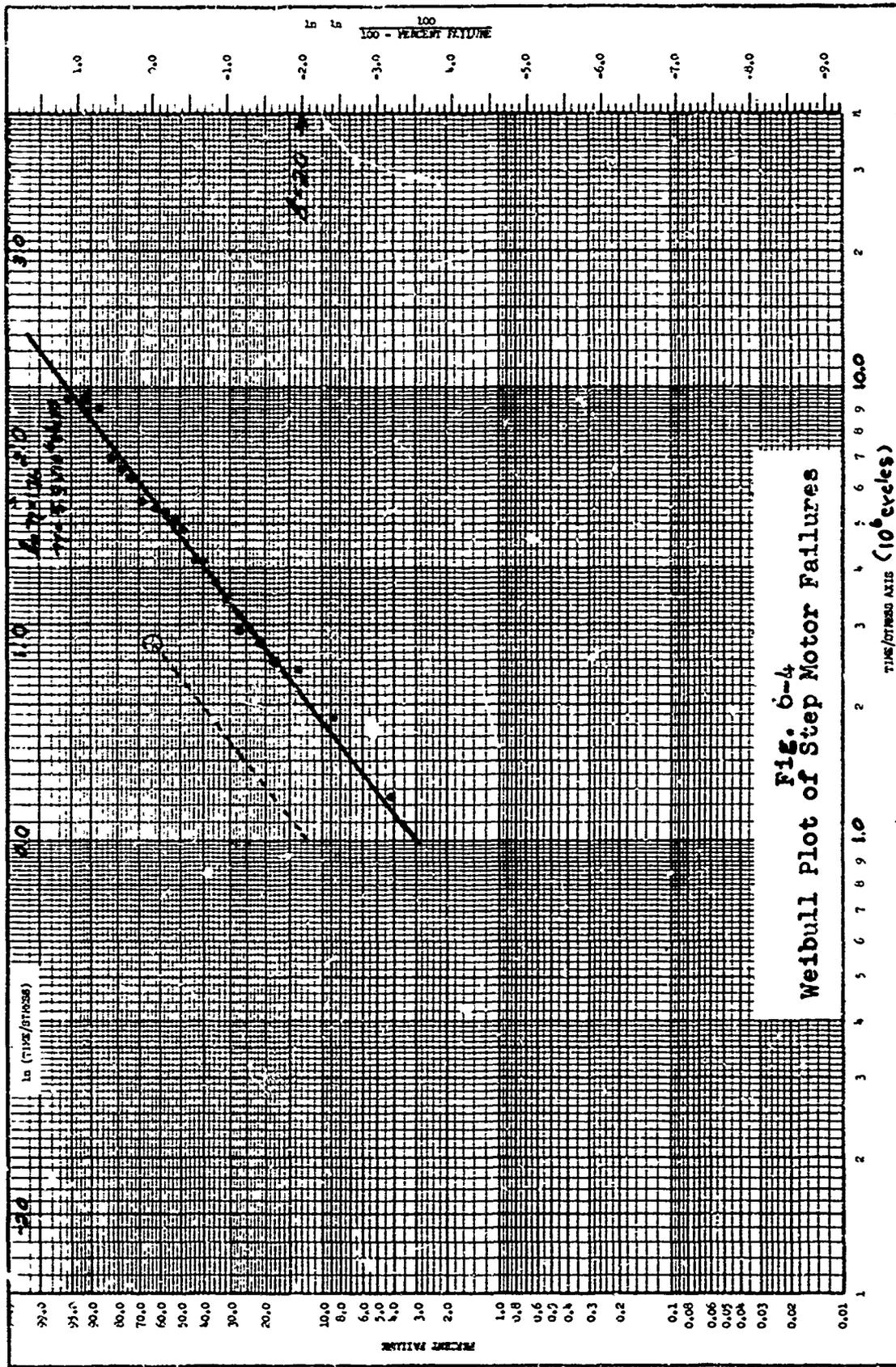


parameter which indicated a failure of a different component.

Analysis of Step-Motors

In many engineering design applications, the various life characteristics of a component are used in selecting a required component for use in a larger system. Life characteristics such as BIO life, characteristic life, and mean time between failure (MTBF) are used in various industries as a measuring device in comparing one component with another. In this example, the Weibull distribution is used to determine the MTBF of step motors from failure data. The methods and data presented in this example were utilized by Webb (Ref. 40).

In the article presented by Webb, failure data resulting from the test of twenty-three motors were plotted on Exponential, Normal and Weibull paper in an effort to determine the ~~underlying~~ distribution. The data plotted on Weibull paper resulted in a nearly straight line indicating that the data is representative of the Weibull distribution. A plot of the data on Weibull paper is presented in Figure 6-4.



The equation used to determine the MTBF for the Weibull distribution is given as

$$MTBF = \eta \Gamma \left(1 + \frac{1}{\beta} \right) \quad (6.5)$$

In this equation η is the scale parameter or characteristic life, β is the shape parameter and Γ indicates the gamma function. Values of the $\Gamma(N)$ can be found in any standard math table. Using graphical methods, $\beta = 2.0$ and $\eta = 5.8 \times 10^6$. Solving equation (6.5) results in a value of 5.15×10^6 steps for MTBF. In this particular application, the failures were measured in steps, hence, the mean steps before failure is used instead of MTBF.

Corrosion Resistance of Magnesium Alloy Plates

The effects of a corrosive environment have always been important in selecting the correct metal in the design of a component or structure which is to be used in that environment. Corrosive action results in a depletion of the metal with a resultant loss of structural strength thereby increasing the possibility of failure. This particular analysis of the effects of corrosion was presented by Berrettoni (Ref. 2).

The items being tested are magnesium alloy plates approximately two square inches in area and with a thickness of one-tenth of an inch. The specification made on the plates was that the corrosion weight loss shall not exceed .120 milligrams per square centimeter per day (MSCD) with an AQL of 1% when the plates were immersed in an inhibited aqueous solution of $MgBr_2$ for a seven day period.

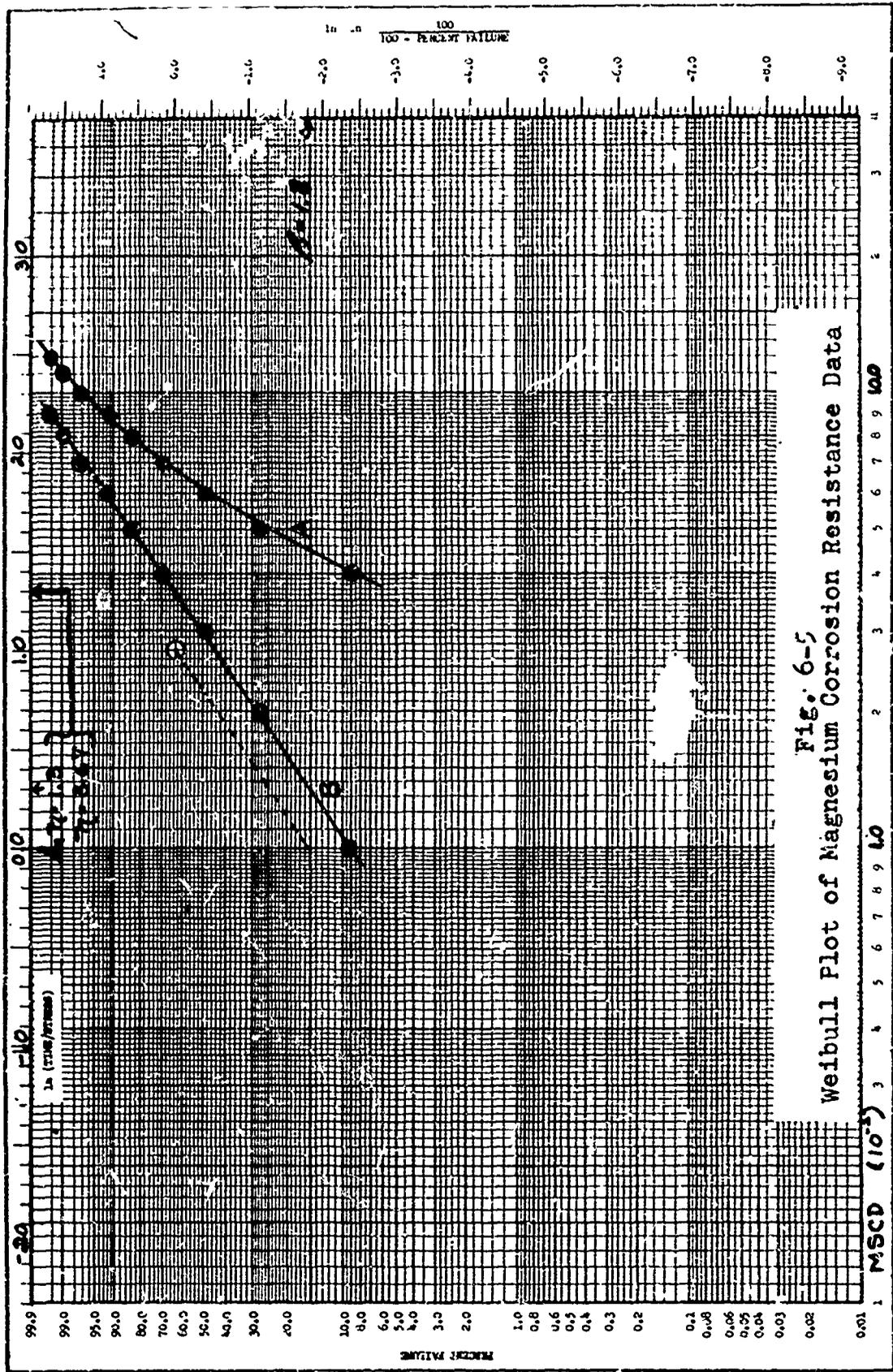
The resulting data of the test performed on two hundred and ten plates is shown in Table 6-3 (Ref. 2). An examination of the test data indicates that the specifications were satisfied because the sample portion exceeding .120 MSCD was only .48% which was less than the limit of 1%.

In an attempt to determine what distribution function characterized the corrosion variation, the data was plotted on Weibull paper. A plot of this data appears in Figure 6-5. Two curves are shown in the figure, curve A is a plot of the original data and curve B is the resulting plot after using graphical techniques for estimating the location parameter. The graphical estimates of the Weibull parameters are $\beta = 1.8$, $\gamma = 3$ and $\tau = 3.67$. Using these estimates for the

TABLE 6-3

CORROSION DATA OF MAGNESIUM ALLOY PLATES

MSCD (10^{-2})	Percent Frequency	Percent Less Than
3	9.05	0
4	19.05	9.05
5	20.48	28.10
6	20.48	48.58
7	14.76	69.06
8	7.14	83.82
9	5.71	90.96
10	1.90	96.67
11	0.95	98.57
12	0.48	99.52
13	0	100.00



parameters, the distribution function for the corrosion variation is

$$F(x) = 1 - e^{-\left(\frac{x-3}{3.67}\right)^{1.8}}$$

Electromagnetic Relay Life Characteristics

This particular application of the Weibull distribution has been discussed by Fontana (Ref. 12). In this reference is described a test program performed with a newly developed relay to determine the functional relationships between the relay life expectancy and the operating parameters of load current, ambient temperature and operating frequency.

A group of 150 new relays were divided into fifteen test samples of ten relays each. Each test sample was submitted to life test under varying combinations of operating stress levels. The range of test levels for the three operating parameters were:

Contact current (I) amp: 5.5 to 14.5

Ambient Temperature (°C): 0 to 150

Operating Frequency
(cycles/min): 5 to 60

In order to provide a mathematical model which showed the relationship between the life characteristics of the relay as a function of the three operating parameters (current, temperature and frequency) a regression equation was formulated based on the results of the fifteen test runs. The equation presented in the reference is:

$$Y = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 \\ + b_{11} x_1^2 + b_{22} x_2^2 + b_{33} x_3^2 \\ + b_{12} x_1 x_2 + b_{13} x_1 x_3 + b_{23} x_2 x_3$$

The values of the b's are the regression coefficients and the x_1 , x_2 and x_3 are coded values of current, temperature and frequency used on each test run. The relationship between the coded values and the operating parameters are:

$$\text{Current (I)} = 3 x_1 + 10$$

$$\text{Temperature (}^\circ\text{C)} = 50 x_2 + 75$$

$$\text{Frequency (CPM)} = 18.5 x_3 + 32.5$$

The values of the x's were given as + 1.5 to indicate the maximum parameter value and -1.5 to indicate the minimum parameter value. As an example the first test run had values of x_1 , x_2 and x_3 equal to -1 which

corresponds to value of 7 amps, 25°C and 14 cycles per minute.

The fifteen groups of ten relays each were placed on life test and the resulting times of failure in terms of cycles were plotted on Weibull graph paper. The data for the first run is shown plotted on Weibull paper in Figure 6-6. Graphical estimates of the shape and scale parameter were made for each test run. The parameter estimates were used as the value of γ in the regression equation to determine a set of regression coefficients for each test run. An optimum set of regression coefficients were obtained which could be used in the regression equation to estimate the shape and location parameter for various levels of operating parameters as reflected by x_1 , x_2 and x_3 . The regression coefficients are given in Table 6-4.

To give an example of using the results of the experiment suppose we were interested in estimating the reliability of the relay after 250 hours of operation with a current of 5 amps, frequency of 20 cpm and ambient temperature of 85°C as operating parameters. Using the regression equation to estimate the shape parameter and scale parameter the results are $\beta = 2.65$ and $\ln \alpha = 6.3$.

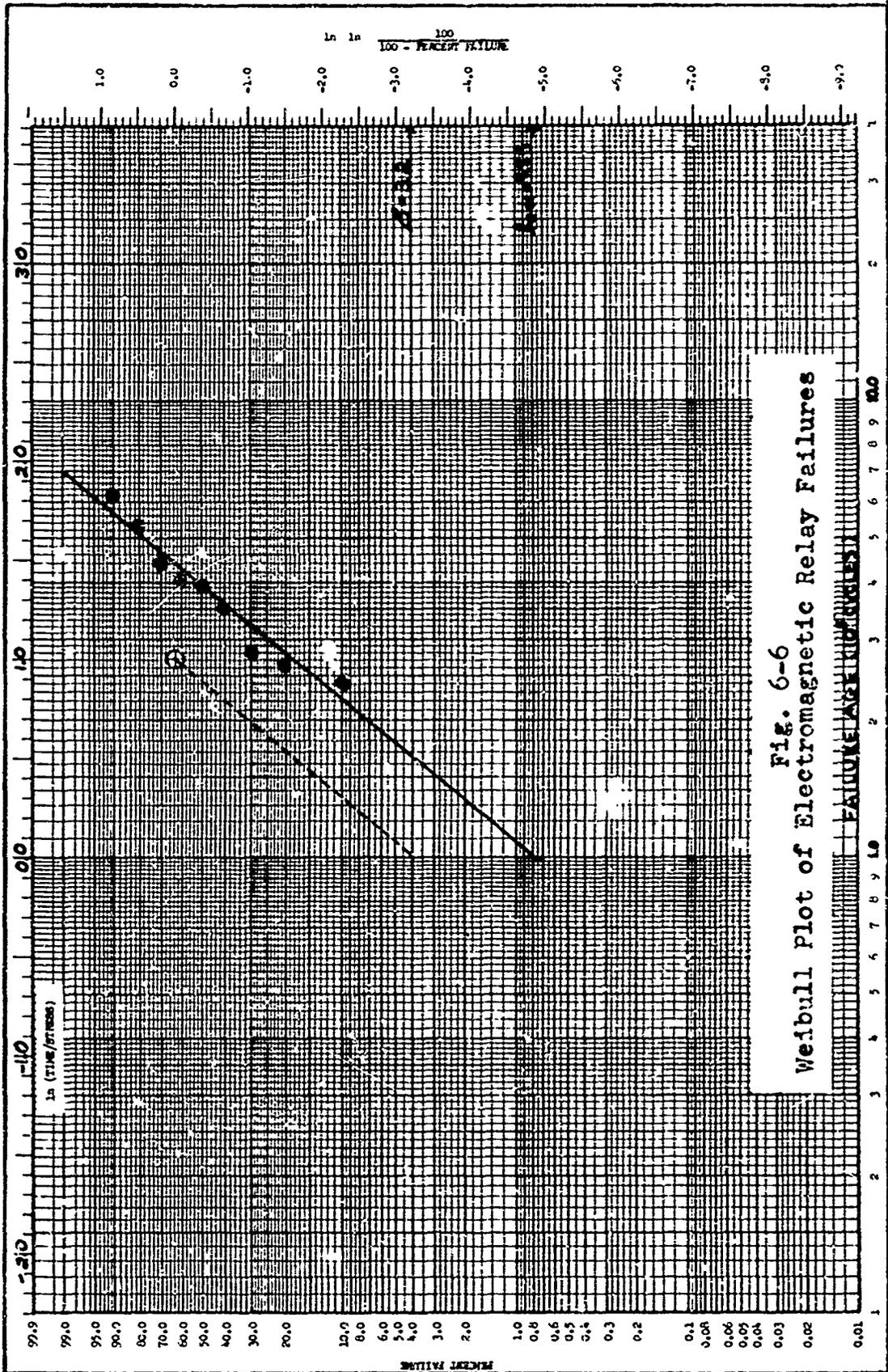


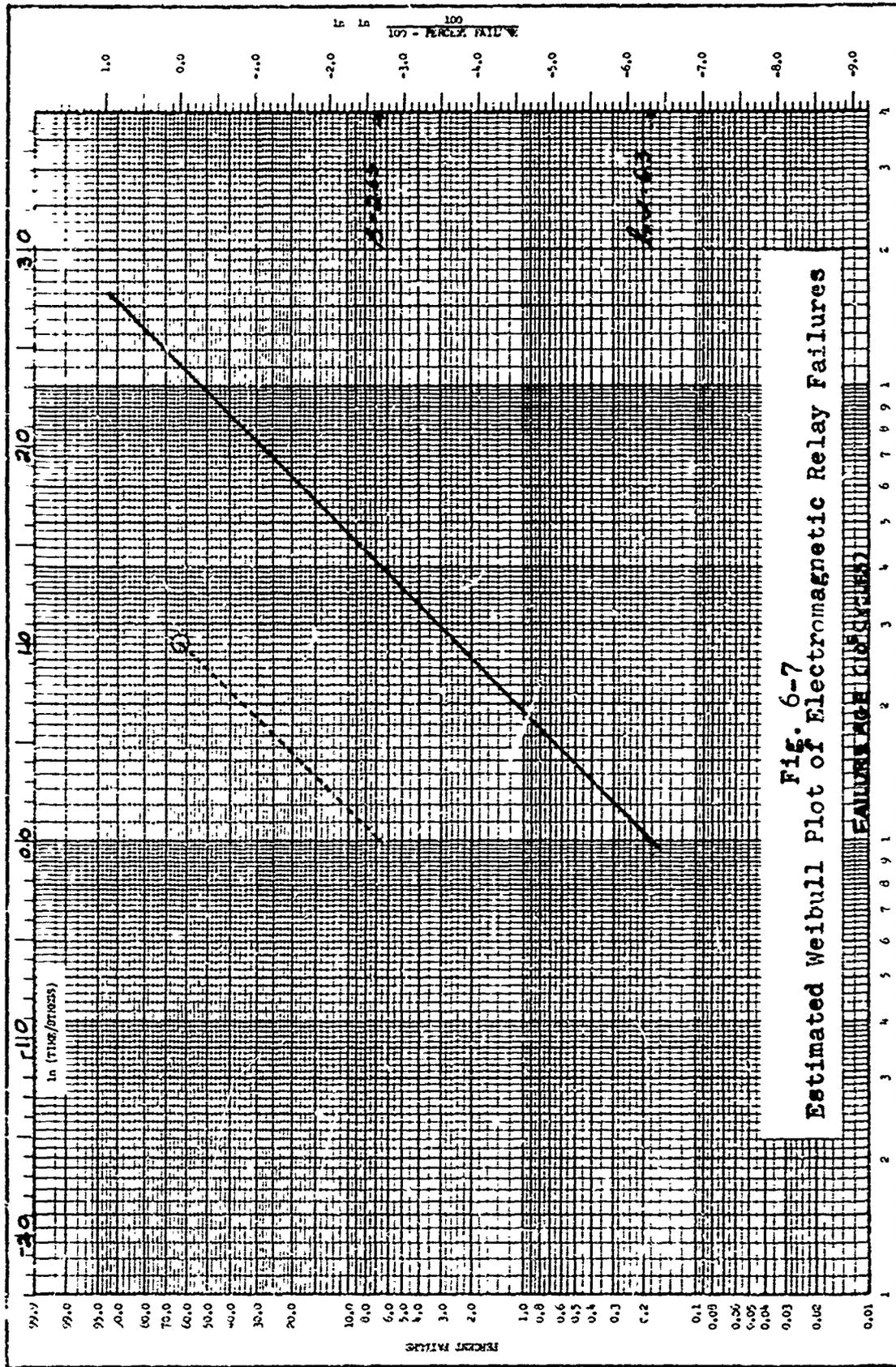
Fig. 6-6
Weibull Plot of Electromagnetic Relay Failures

TABLE 6-4
REGRESSION COEFFICIENTS FOR ELECTROMAGNETIC RELAYS

	Coefficients for Estimating	Coefficients for Estimating
b_0	4.03	3.45
b_1	-1.61	+0.07
b_2	+0.09	-0.41
b_3	-0.12	-0.03
b_{11}	-0.33	-0.38
b_{22}	-0.54	-0.01
b_{33}	-0.60	-0.39
b_{12}	+0.63	+0.14
b_{13}	-0.19	+0.31
b_{23}	-0.04	-0.14

An estimated Weibull plot using the estimated parameters appears in Figure 6-7. The reliability for 250 hours of operation (3×10^5 cycles) can be read from the Weibull graph as 97%.

This experiment by Fontana (Ref. 12) shows the important application of the Weibull distribution and regression techniques in determining the life characteristics of an electromechanical component. The method



enables the determination of reliability values of the component which were not attainable by conventional life testing.

Analysis of Automobile Component Failure Data Using Weibull Distribution and Type I Smallest Extreme Value Distribution

This example of the use of the Weibull distribution will show the relationship between the Weibull distribution and the Type I extreme value distribution. It is further intended to show the application of this relationship in the reliability analysis of components exhibiting a failure pattern which can be modeled with the two-parameter Weibull distribution.

The relationship that exists between the Type I and Type III extreme value distribution is: If x is a random variable having a two parameter Weibull distribution with shape parameter β and scale parameter η , then the random variable $Z = \ln x$ has the following distribution function:

$$F(z) = 1 - \exp \left[-e^{\beta(z - \ln \eta)} \right]$$

This equation is the same as that given for the Type I smallest extreme (Equation 2.8, page 12). The parameters of the Type I smallest extreme value distribution given

in terms of the Weibull parameters are, scale parameter $b = \frac{1}{\beta}$ and location parameter $\mu = \ln \eta$.

The importance of the relationship that exists between these distribution is that methods which are used to estimate the parameters of the Type I distribution from sample data can be used to estimate the parameters of the Weibull distribution data by making the necessary transformation.

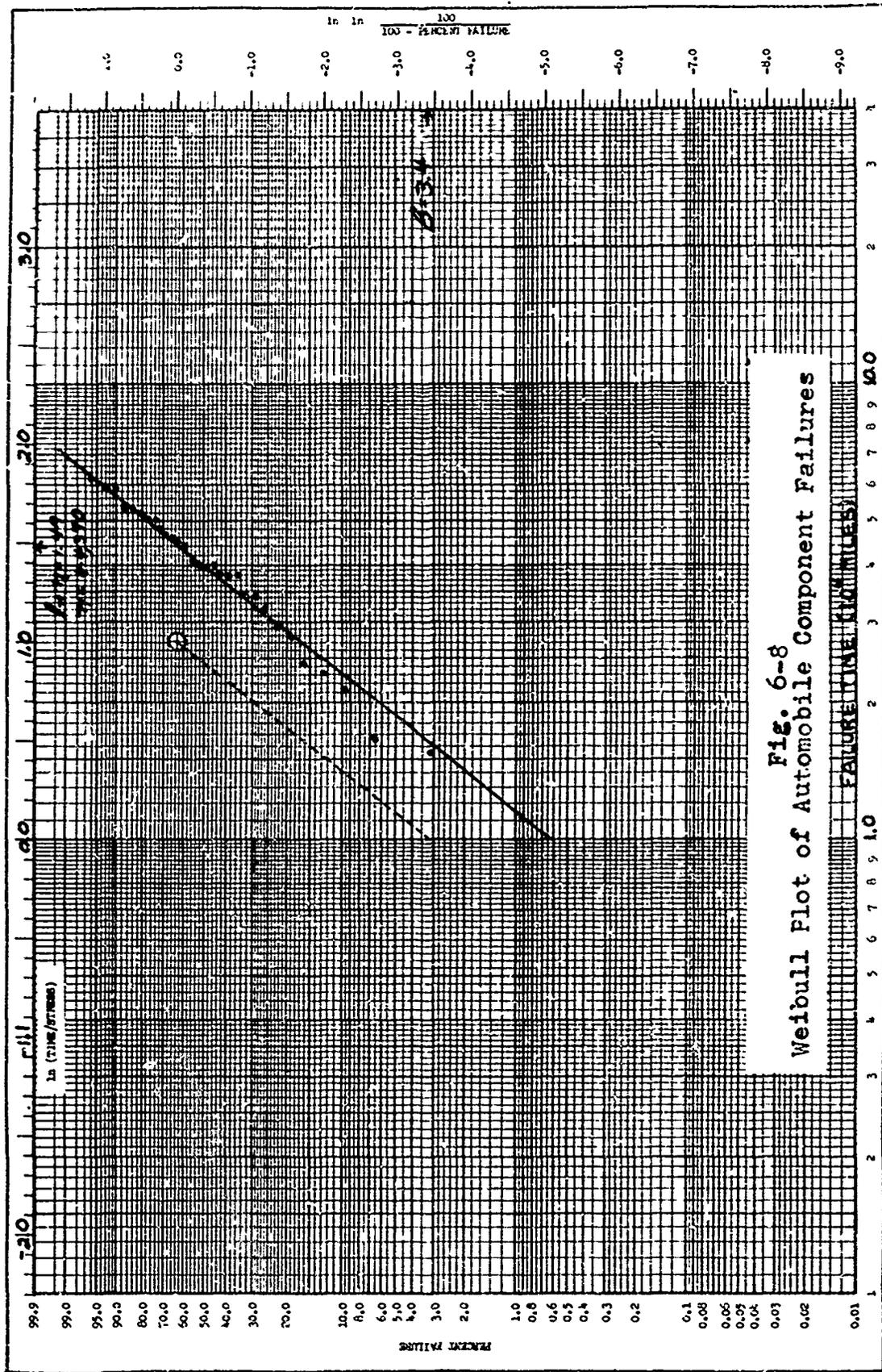
In an effort to show how the previously discussed relationships can be applied to the reliability analysis of mechanical components, use will be made of data presented in Ref. 43. The data was recorded on thirty mechanical components which were installed on test automobiles. The data in terms of miles to failure for all thirty components are presented in Table 6-5.

As with most data that results for the life test of components that exhibit a wear-out type failure, a plot of the data on Weibull probability paper can be used as a method of determining the underlying failure distribution. The plot of the data (Figure 6-8) indicates that the underlying distribution appears to be the two-parameter Weibull. A further test of the data using the methods presented by Gross (Ref. 16) resulted in accepting

TABLE 6-5

FAILURE DATA FOR AUTOMOBILE COMPONENTS

	<u>Ordered Data</u> (miles)	<u>Natural Logarithms</u>
1	15517.4	9.64972
2	16723.5	9.72457
3	21249.9	9.96411
4	23013.3	10.0438
5	24567.6	10.1092
6	27760.4	10.2314
7	29228.0	10.2829
8	32445.7	10.3873
9	34006.0	10.4343
10	34424.4	10.4465
11	38057.6	10.5469
12	38110.1	10.5482
13	38525.0	10.5591
14	39803.3	10.5917
15	40120.9	10.5997
16	40781.9	10.6160
17	41182.9	10.6258
18	44445.5	10.7020
19	45845.4	10.7330
20	46147.9	10.7396
21	46301.0	10.7429
22	48513.2	10.7896
23	50745.2	10.8346
24	51440.7	10.8482
25	52471.1	10.8680
26	53415.1	10.8858
27	53572.4	10.8888
28	58794.9	10.9818
29	60233.1	11.0060
30	62370.5	11.0408



the hypothesis that the underlying distribution is the two-parameter Weibull with $\beta = 3.37$ and $\eta = 44,320$.

The acceptance is based on the Kolomogorov-Smirnov "goodness-of-fit" test at the 95% confidence level.

Maximum-likelihood estimators of the Weibull parameters based on the sample data were obtained using the method presented by Harter and Moore (Ref. 23).

The resulting estimates of the Weibull parameters using graphical solution, regression techniques (Ref. 19) and maximum-likelihood estimators (Ref. 23) are presented in Table 6-6.

TABLE 6-6
WEIBULL PARAMETER ESTIMATES

	Graphical Techniques	Regression Techniques	Max.-Likelihood Techniques
β	3.4	3.371	3.725
η	44,370	44,319.5	44,760.59

An alternate method for determining the failure distribution from the sample data would be to plot the value $Z = \ln X$ from Table 6-5 on Type I extreme value probability paper. A plot of the data appears in Figure 6-9. The resulting straight line fit of the data indicates that the variate Z has a Type I smallest

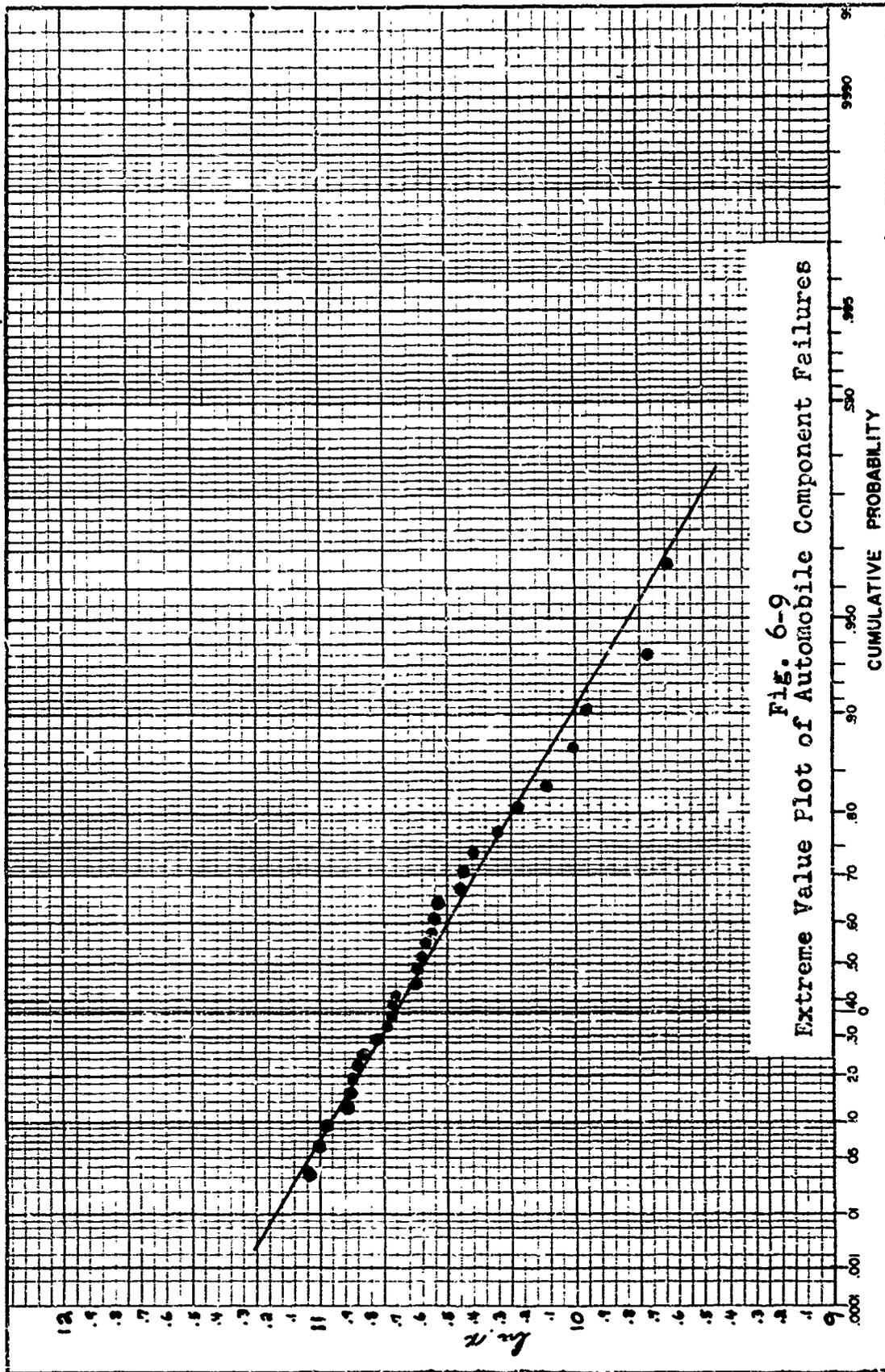


Fig. 6-9
Extreme Value Plot of Automobile Component Failures

extreme value distribution and hence the X variate has a two-parameter Weibull. Acceptance of the hypothesis that the underlying distribution of the Z variate is the Type I smallest extreme was based on a Kolomogorov-Smirnov goodness-of-fit test with a 95% confidence level. Using graphical methods and the method of moments to estimate the parameters of the Z variate distribution the results are shown in Table 6-7. Estimates for the Weibull parameters were made by using the equations $\beta = \frac{1}{b}$ and $\eta = e^{\mu}$. These results are also shown in Table 6-7.

TABLE 6-7

EXTREME VALUE TYPE I PARAMETER ESTIMATES

<u>Graphical</u>	<u>Method of Moments</u>
$b = .295$.2885
$\mu = 10.70$	10.7139
$\beta = 3.39$	3.466
$\eta = 44,440$	44,933

Reliability Application

Suppose we are interested in determining the reliability of the components for 30,000 miles. Since

we have two distributions which can be used as a model to determine the answer, a solution will be given using each model. The reliability value can be read directly from the probability graph paper or by using the parameter estimates and solving the applicable reliability equation for the distribution. The results using both methods are:

Type I Extreme Value Distribution

Graphical: $R(10.308) = 77\%$

Solving
$$R(z) = \exp\left[-e^{\left(\frac{10.308 - 10.7139}{.2885}\right)}\right]$$

$$= 77.8\%$$

Weibull Distribution

Graphical: $R(30,000) = 76\%$

Solving
$$R(z) = e^{-\left(\frac{30,000}{44,320}\right)^{3.371}}$$

$$= 76.4\%$$

As can be seen from this example, either distribution can be used as a model to analyze wear-out type failure.

If other than a graphical estimate of the parameters is desired, transformation of the data from Weibull

distribution to the Type I smallest extreme distribution enables the use of the method of moments to estimate the parameters. The advantage of using the transformed data is that the distribution of the variate depends linearly on the mean and standard deviation of the transformed variate (Ref. 43).

VII. Summary and Conclusions

Summary

The past decade has seen an increased demand being placed on manufacturers to specify accurate reliability estimates for mechanical systems and components. Industry has been required to establish reliability programs to meet the demands of government contracts, product guaranties and safety requirements.

Reliability analysis techniques have been firmly established in the electronics industry. Unfortunately, the nature of mechanical components has prevented successful application of these same techniques in the reliability analysis of mechanical systems. In an attempt to provide better estimates of mechanical reliability, greater use is being made of statistical theory to analyze failure characteristics.

This thesis presents the results of a literature search to determine the extent to which statistical theory, in particular extreme value theory is employed in mechanical reliability analysis. The extreme value distributions and theory are presented in sufficient depth to enable the interested reader to understand their applications in the examples presented in the text.

It is felt that extreme value theory would prove to be quite useful in establishing reliability estimates of mechanical components because of two reasons. First, increased success in applying effective quality control techniques has eliminated or controlled the interaction of a large number of product variables. Control or elimination of these variables result in products which possess a skewed failure distribution. Second, in many applications, especially design, the critical variable is the extreme rather than the average value.

In the examples presented, actual failure data was used in applying the analysis techniques. For the Type I extreme value distribution, two examples were presented. A total of five examples were presented for the Weibull or Type III extreme value distribution. Graphical analysis as well as techniques which are based on order statistics were used to estimate the parameters of both extreme value distributions from the failure data.

It must be remembered that with most statistical techniques, the objective is to predict the life characteristics of an entire population on the basis of a sample of the population. Any predictions made are based on the assumption that failure patterns

determined from the data are representative of the parent population.

Conclusions

The Type I extreme value distribution has been found useful in the analysis of corrosive pitting of metals and the study of loads. The use of this distribution and extreme value theory has allowed the determination of the effects of corrosion which had been impossible to quantify by other methods. When applied to the study of loads, the Type I distribution provided more accurate results than the commonly applied three sigma rule.

The Weibull distribution has been found to apply to the reliability analysis of electromagnetic relays and step motors, fatigue failures of automobile components, the comparison of quality of door locking mechanisms and the corrosion resistance of magnesium alloy plates. The Weibull distribution is particularly useful in the analysis of "wear-out" type failures and has been found to apply to a wide range of mechanical components.

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