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J. W. Craig

RLC Realizations of Approximations to Matched Filters for a Rectangular Pulse

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RLC REALIZATIONS OF APPROXIMATIONS TO MATCHED FILTERS FOR A RECTANGULAR PULSE

J. W. CRAIG

Group 62

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ABSTRACT

Several low order rational transfer functions have been found which approximate a matched filter for a rectangular pulse. These transfer functions are optimum in the sense that the ratio of peak-signal to rms-noise at the output is maximized to the extent allowed by the available parameters. Normalized transfer functions are given and their frequency and time response are shown graphically. For the most useful cases, realizations normalized in impedance and pulse duration are given which allow the use of lossy inductors. A short discussion of a bandpass realization is presented. A transfer function with zeros restricted to the $i\omega$-axis is given, along with its realization, which is useful as the lowpass prototype of a bandpass, crystal matched filter.

Accepted for the Air Force
Franklin C. Hudson
Chief, Lincoln Laboratory Office

RLC REALIZATIONS OF APPROXIMATIONS TO MATCHED FILTERS FOR A RECTANGULAR PULSE

I. INTRODUCTION

Matched filters are widely used in communication and radar systems for the optimum detection of a pulse in white noise. The pulse shape is usually rectangular and a passive matched filter is often desirable. Several rational approximations to the transfer function of a matched filter for a rectangular pulse have been found by a computer-aided search and some of these have been synthesized.[1, 2, 3] These approximations are optimum in the sense that for the restrictions placed on the transfer function (number and/or location of poles and zeros) the peak signal to rms noise at the output is maximized. The various results are gathered together here for ready reference.

The optimum pole and zero locations given in sections II B and C and the realization of Fig. 14 have been obtained independently by P. Meyer[4] who, in addition, gives results for optimum approximations for transfer functions with up to 11 poles and 10 zeros.

II. VARIOUS RATIONAL APPROXIMATIONS AND THEIR REALIZATIONS

A. Multiple Order Negative Real Poles

In reference 1 parameters are given for the impulse response of a matched filter approximation whose transfer function has a multiple pole of up to the fourth order on the negative real axis. As one might expect, this restriction makes the approximation inefficient in that for a given complexity (number of poles and zeros) the best approximation is obtained when the poles are allowed to be complex (see sections B and C below). For completeness the parameters for the impulse response of the multiple order pole approximations and their performance compared to the exact matched filter are given in Table 1. T is the pulse duration in seconds,

TABLE 1

n	$\alpha_1 T$	$\alpha_2 T$	$\alpha_3 T^2$	$\alpha_4 T^3$	ρ	$-10 \log \rho$
1	1.2564	—	—	—	0.8145	0.890 db
2	2.8427	10.476	—	—	0.8861	0.526
3	4.5376	−2.8841	34.990	—	0.9149	0.386
4	6.1807	13.830	−73.917	283.75	0.9304	0.313

and ρ is the signal-to-noise ratio out of the approximate filter compared to that out of the exact one.

For an nth order pole, the optimum unit impulse response is

$$\eta_n(t) = \left(1 + \alpha_2 t + \ldots + \alpha_n t^{n-1}\right) e^{-\alpha_1 t} \quad . \tag{1}$$

and the corresponding transfer function (Laplace transform of $\eta_n(t)$) is

$$H_n(s) = \frac{1}{s+\alpha_1} + \frac{\alpha_2}{(s+\alpha_1)^2} + \ldots + \frac{(n-1)!\,\alpha_n}{(s+\alpha_1)^n} \quad . \tag{2}$$

B. <u>Two Complex Poles and One Real Zero</u>

For a transfer function of the form

$$H(s) = \frac{K(s+\mu)}{(s+\alpha)^2 + \beta^2} \quad , \tag{3}$$

where K is an arbitrary gain constant, the optimum parameters for approximating a matched filter are:

$$\mu T = 7.520$$
$$\alpha T = 1.511$$
$$\beta T = 1.826$$

For this approximation $\rho = 0.9140$ (0.390 db) . In performance this is the same as the triple pole approximation above. The corresponding unit impulse response (inverse Laplace transform of $H(s)$) is

$$h\left(\frac{t}{T}\right) = K\left(\cos \beta t + \frac{\mu - \alpha}{\beta} \sin \beta t\right) e^{-\alpha t} \quad . \tag{4}$$

The normalized magnitude of the transfer function, $\left| \frac{H(i\omega)}{H(o)} \right|$, is plotted in Fig. 1. For $K = 1$, the unit impulse response is shown in Fig. 2 and the response to a matched pulse of T seconds duration and amplitude $\frac{1}{T}$ is shown in Fig. 3.

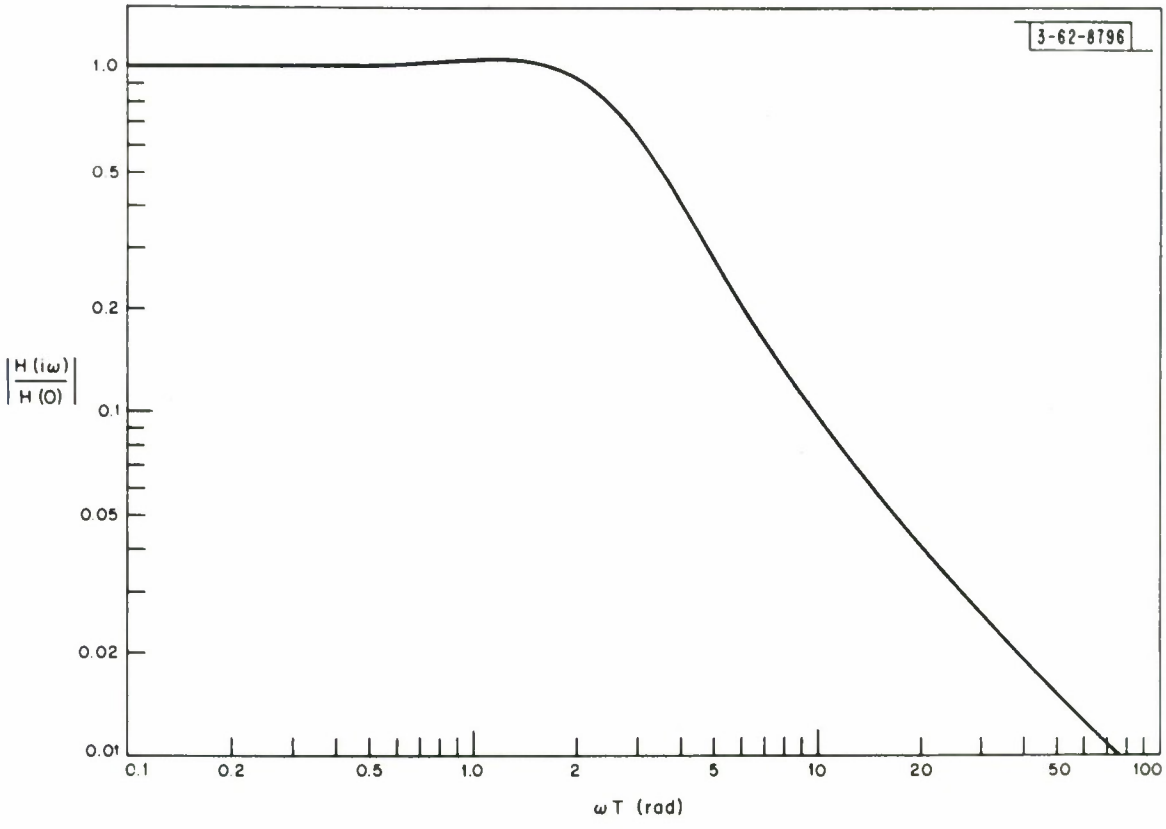


Fig. 1. Normalized magnitude vs. frequency of 2-pole, 1-zero matched filter approximation.

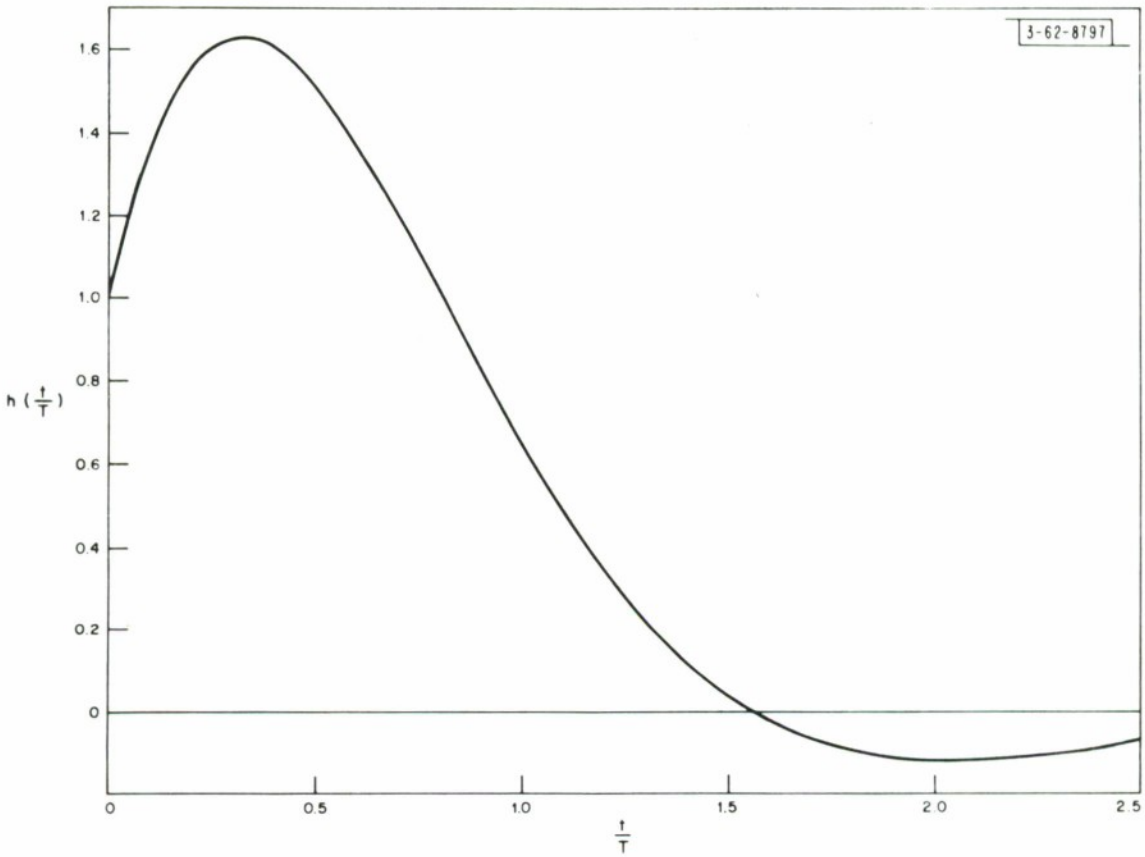


Fig. 2. Unit impulse response of 2-pole, 1-zero matched filter.

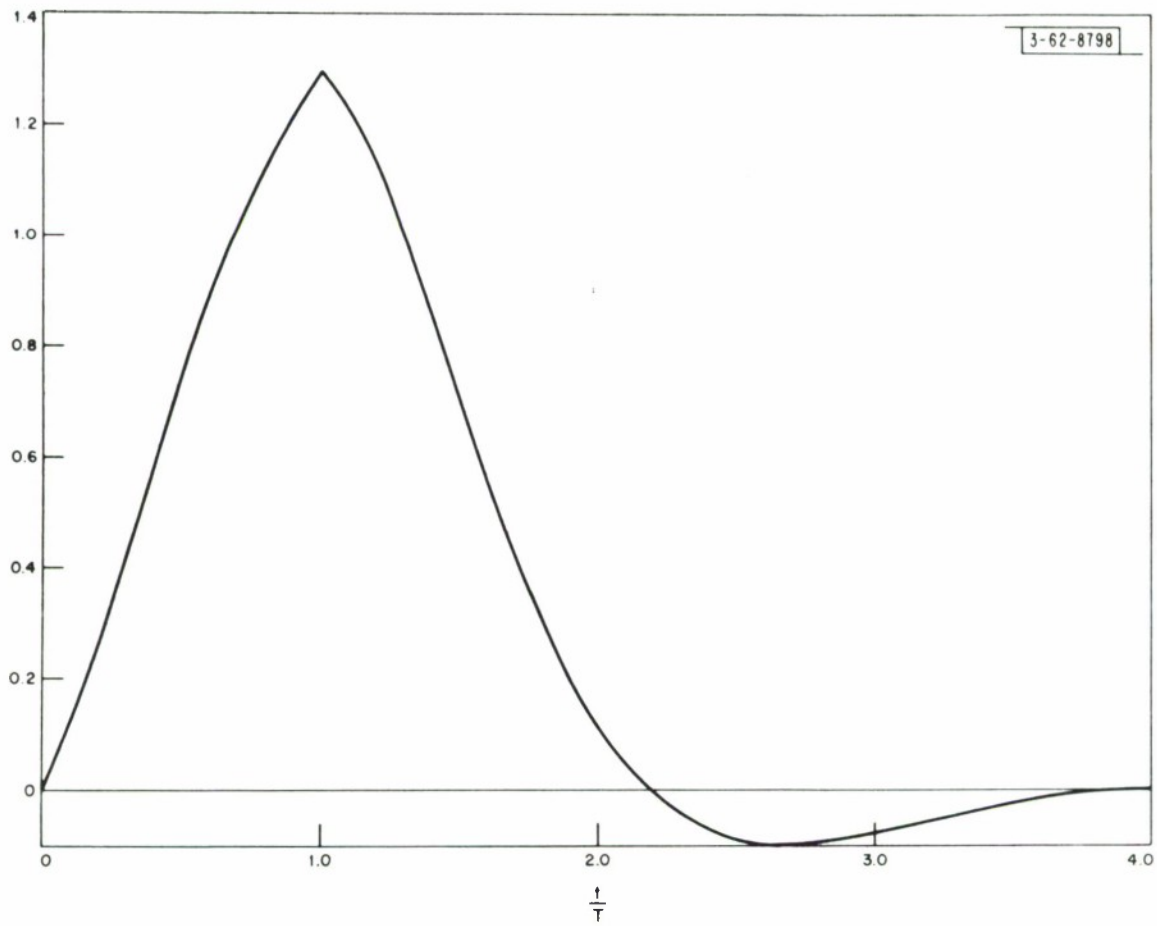


Fig. 3. Response of a 2-pole, 1-zero matched filter to a matched (T second) pulse of amplitude 1/T.

A convenient circuit for realizing this transfer function is shown in Fig. 4.

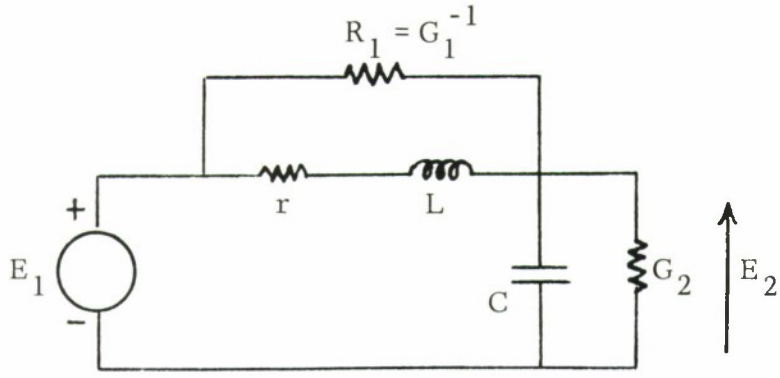


Fig. 4. Circuit for the realization of the 2-pole, 1-zero matched filter approximation.

The equivalent series resistance of the inductor is accounted for by the resistance r. The transfer function of the circuit is

$$H(s) = \frac{E_2(s)}{E_1(s)} = \frac{\left(s + \frac{r+R_1}{L}\right)\left(\frac{1}{CR_1}\right)}{s^2 + s\left(\frac{r}{L} + \frac{G_1+G_2}{C}\right) + \frac{1}{LC} + \left(\frac{r}{L}\right)\left(\frac{G_1+G_2}{C}\right)} . \tag{5}$$

If we use the parameters

$$\sigma_o = \frac{r+R_1}{L} , \quad \sigma_2 = \frac{r}{L} \quad \text{and} \quad \sigma_3 = \frac{G_1+G_2}{C} ,$$

the transfer function has the form

$$H(s) = \frac{(s+\sigma_o)\left(\frac{1}{CR_1}\right)}{s^2 + s(\sigma_2+\sigma_3) + \sigma_2\sigma_3 + \frac{1}{LC}} \tag{6}$$

By equating corresponding polynomial coefficients in Eqs. (3) and (6), we obtain the following three equations for the design.

$$\alpha_o = \sigma_o = \frac{r + R_1}{L} \tag{7}$$

$$2\alpha_1 = \sigma_2 + \sigma_3 \tag{8}$$

$$\alpha_1^2 + \beta_1^2 = \sigma_2\sigma_3 + \frac{1}{LC} \tag{9}$$

These equations do not uniquely specify the component values, but they allow just enough freedom to make the design easy.

A design approach that works well is as follows. Since inductors are the most difficult component to adjust to a specified value, choose an inductance value that is convenient for the bandwidth of the filter being designed and have one wound to this nominal value. Measure its inductance, L, and equivalent series resistance, r, at a frequency approximately equal to $1/T$ Hz. Calculate σ_2 from these measured values of L and r, and then calculate R_1 and σ_3 using Eqs. (7) and (8):

$$R_1 = \sigma_o L - r$$

$$\sigma_3 = 2\alpha_1 - \sigma_2$$

Then calculate C using Eq. (9):

$$C = \left[\left(\alpha_1^2 + \beta_1^2 - \sigma_2\sigma_3 \right) L \right]^{-1}$$

The last component, G_3, is then computed to be

$$G_3 = \sigma_3 C - G_1 \quad .$$

C. Three Pole, Two Zero Transfer Function Approximations

1. Unrestricted Pole and Zero Locations (optimum approximation)

For a transfer function of the form

$$H(s) = K \frac{(s+\mu)^2 + \nu^2}{(s+\sigma)\left[(s+\alpha)^2 + \beta^2\right]} , \tag{10}$$

the optimum parameters for approximating a matched filter are:

$$\mu T = 0.7832$$

$$\nu T = 5.9763$$

$$\sigma T = 2.2464$$

$$\alpha T = 1.4488$$

$$\beta T = 4.1517$$

For this approximation $\rho = 0.9470$ (0.236 db), which is somewhat better than any of the approximations above and the transfer function is not a great deal more complex. Its unit impulse response is

$$h\left(\frac{t}{T}\right) = K\left[2.1181\, e^{-\sigma t} - 1.1231\, e^{-\alpha t} \cos(\beta t - 0.09439)\right] , \tag{11}$$

The normalized magnitude of the transfer function, $\left| \frac{H(i\omega)}{H(o)} \right|$, is given in Fig. 5. For $K = 1$, $h\left(\frac{t}{T}\right)$ is shown in Fig. 6 and the response to a matched pulse of T seconds duration and amplitude $\frac{1}{T}$ is shown in Fig. 7.

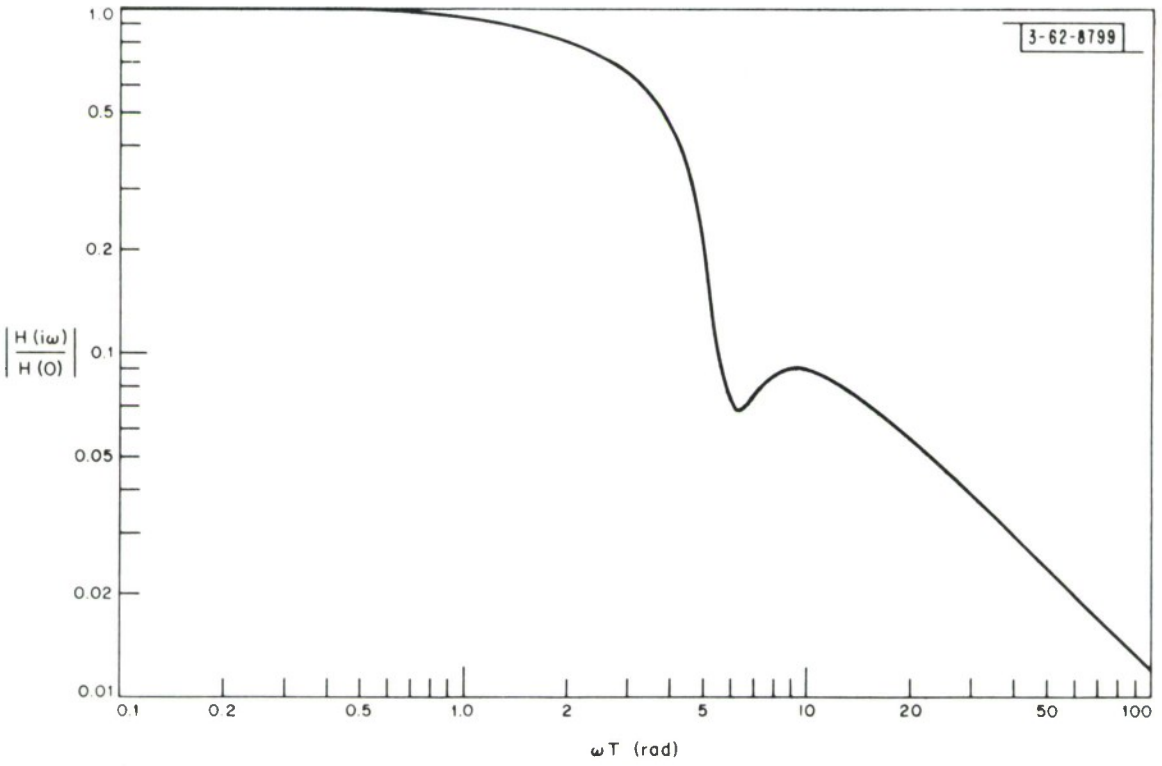


Fig. 5. Normalized magnitude vs. frequency of optimum 3-pole, 2-zero matched filter approximation.

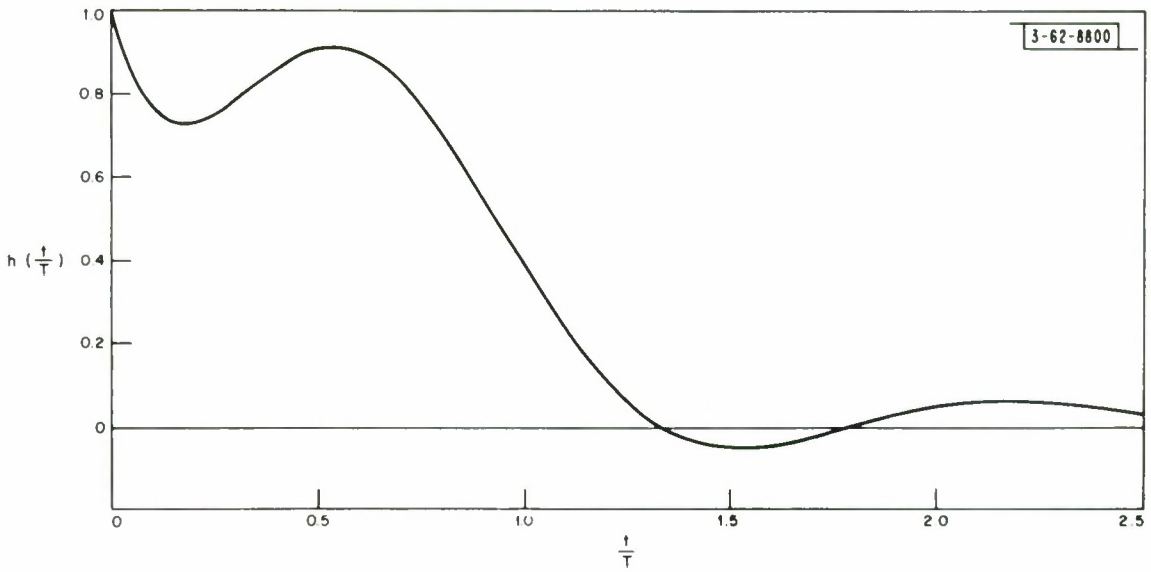


Fig. 6. Unit impulse response of optimum 3-pole, 2-zero matched filter.

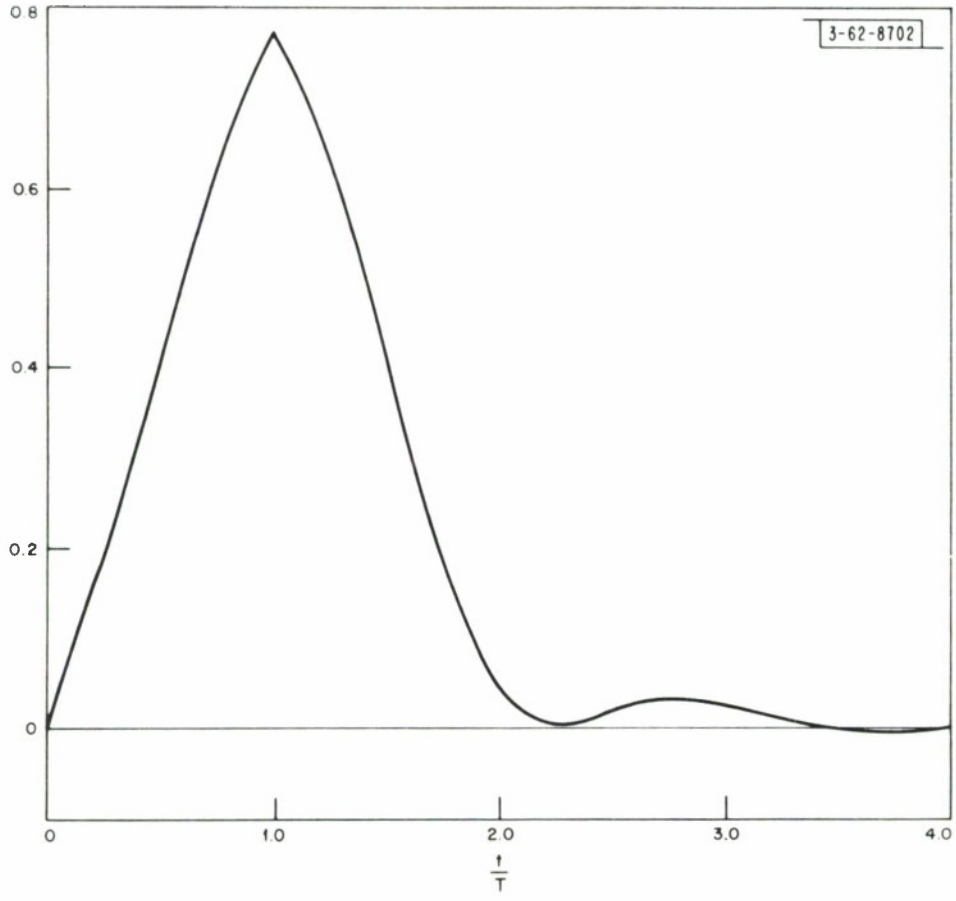


Fig. 7. Response of the optimum 3-pole, 2-zero matched filter to a matched (T second) pulse of amplitude 1/T.

A practical circuit for realizing this transfer function is in Fig. 8.

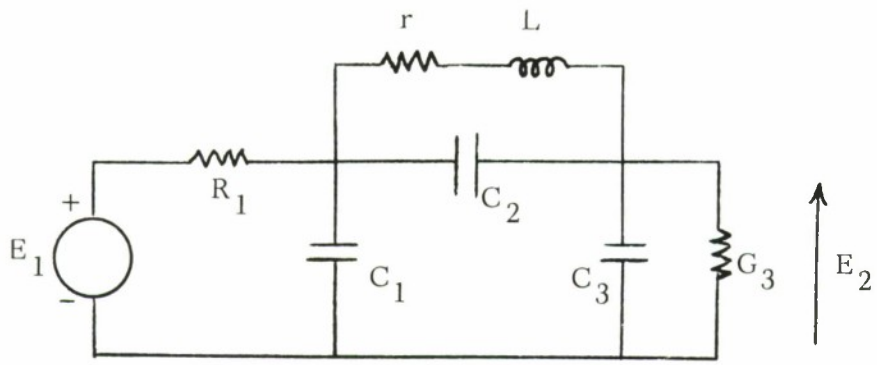


Fig. 8. Circuit for the realization of the optimum 3-pole, 2-zero matched filter approximation.

This realization allows a lossy inductor and it incorporates both source and load resistances. The transfer function of the circuit is

$$H(s) = \frac{E_2(s)}{E_1(s)} = \frac{\frac{G_1C_2}{C_1C_3}\left(s^2 + s\frac{r}{L} + \frac{1}{LC_2}\right)}{\left\{s^3\left(1+\frac{C_2}{C_1}+\frac{C_2}{C_3}\right) + s^2\left[\frac{r}{L}\left(1+\frac{C_2}{C_1}+\frac{C_2}{C_3}\right)+\frac{G_1}{C_1}+\frac{G_3}{C_3}+\frac{C_2(G_1+G_3)}{C_1C_3}\right] + s\left[\frac{r}{L}\left(\frac{G_1}{C_1}+\frac{G_3}{C_3}\right)+\frac{G_1G_3}{C_1C_3}+\frac{rC_2(G_1+G_3)+C_1+C_3}{LC_1C_3}\right] + \frac{G_1+G_3(1+rG_1)}{LC_1C_3}\right\}} \quad (12)$$

The design of this circuit is quite complicated and it is given in Appendix A. The results are that there is a continuum of solutions and ten of these selected from the range of realizable solutions are given in Table II. The value of the source resistance R_1 is normalized to 1 ohm and the load resistance R_3 varies from approximately $0.6R_1$ to infinity. Since both inductance and capacitance are proportional to the pulse duration T (for a given network, the L and C values are inversely proportional to the bandwidth of the network), multiply both the L and C values in the table by T to obtain the component values for a pulse

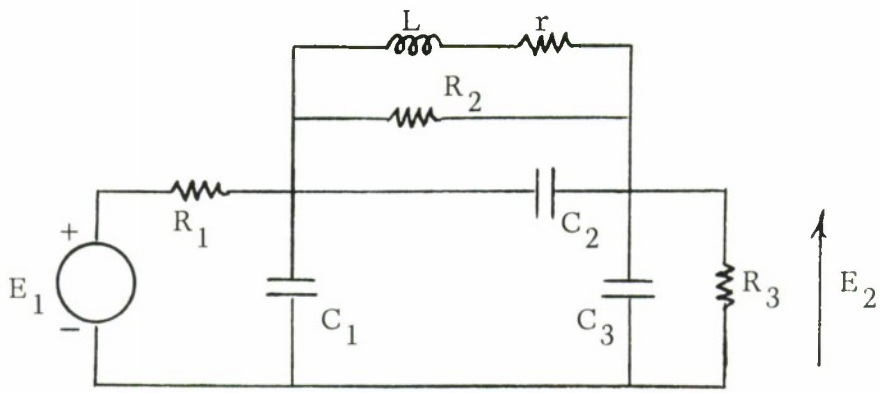
of duration T seconds. Also, the impedance level of a particular design must be increased to obtain practical component values. A particularly convenient way of doing this is to scale the impedance according to the particular inductor that is to be used so that it does not have to be trimmed. If L_o henries is the value of the inductor to be used, multiply the impedance of the circuit by the factor (L_o/LT).

An alternative method of realizing the transfer function of Eq. (10) is to use Darlington's method of synthesis for a network terminated in resistance at both ends.[5,6] In the scheme used here the transfer function is predistorted (all poles and zeros are shifted toward the ω axis by an amount μ) to place the zeros of transmission on the ω axis. This allows lossy reactances and simplifies the synthesis. The work is carried out in Appendix B and the resulting network for a 1 second pulse is given in Fig. 9. The transfer impedance $\frac{E_2(s)}{I_1(s)}$ of this circuit is of the form of Eq. 10 with $K = 0.6743$.

Table II

Set No.	Component Values						
	C_1	C_2	C_3	R_3	L	r	K
1	0.3613	0.2314	0.8584	0.5756	0.1189	0.1863	0.3906
2	0.3177	0.1830	0.5175	1.1524	0.1504	0.2355	0.5769
3	0.2940	0.1684	0.4465	1.537	0.1634	0.2560	0.6579
4	0.2584	0.1532	0.3761	2.478	0.1797	0.2815	0.7881
5	0.2360	0.1467	0.3423	3.698	0.1877	0.2940	0.8857
6	0.2196	0.1433	0.3205	5.585	0.1920	0.3008	0.9698
7	0.2068	0.1416	0.3045	9.132	0.1943	0.3044	1.0463
8	0.1962	0.1408	0.2919	18.68	0.1955	0.3062	1.1175
9	0.1872	0.1406	0.2816	145.5	0.1958	0.3067	1.1849
10	0.1859	0.1406	0.2800	∞	0.1958	0.3067	1.1958

Ten sets of component values for the circuit of Fig. 8, which realize the optimum 3-pole, 2-zero matched filter approximation for a 1 second pulse. $R_1 = 1$ ohm and the other component values are in farads, henries and ohms. K is the gain constant in Eqs. (10) and (11). Observe that both L and C are proportional to the pulse duration so that for a pulse of duration T seconds, multiply both the L and C values by T.



$R_1 = 1\Omega$	$C_1 = 0.5908\,F$	$L = 0.2123\,H$
$R_2 = 9.683$	$C_2 = 0.1318$	$r = 0.1663\Omega$
$R_3 = 1.505$	$C_3 = 0.1627$	

Fig. 9. An alternative realization of the optimum 3-pole, 2-zero matched filter approximation, Eq. (10). The element values are for a 1 second pulse, and the gain factor K = 0.6743.

For lowpass circuits it is almost never necessary to account for the very small losses in the capacitors used in the realization, so for lowpass realizations the circuit of Fig. 8 is preferable to that of Fig. 9. However the latter (before uniform dissipation is added) is useful as a lowpass prototype for the RLC realization of a bandpass matched filter. Fig. B-1 shows the lowpass realization of the predistorted transmission function or the lowpass prototype for a bandpass filter. This circuit is transformed to bandpass and then uniform dissipation is added for the final realization. A particularly convenient form of the standard lowpass to bandpass transformation is given by Saal and Ulbrich in Ref. 7. Applied to the predistorted network, the transformation yields the network in Fig. 10.

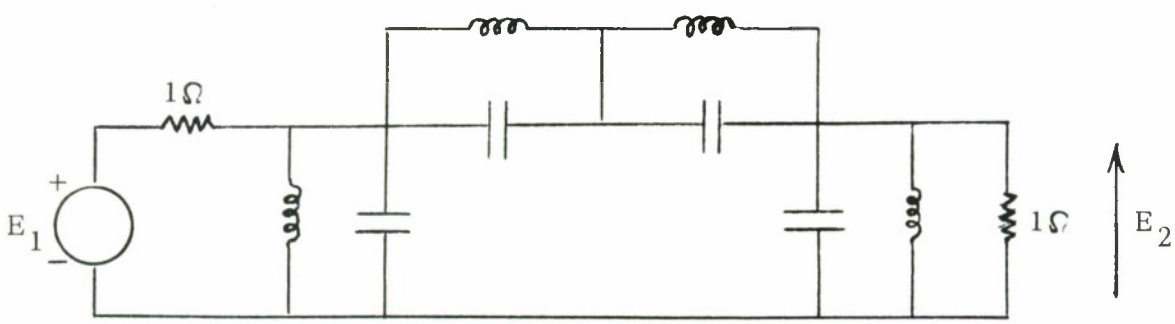


Fig. 10. Form of the circuit for realizing a bandpass matched filter. The reactance values are to be calculated from those of the lowpass prototype through the transformation of Saal and Ulbrich. In the final circuit, the four LC resonant circuits must have equal Q as determined by the predistortion.

The reactance values are calculated from those of the lowpass prototype (Fig. B-1) through the frequency transformation, and the equations are given in the reference. In designing a bandpass matched filter, it is the duration of the impulse response that is of interest so that in the lowpass to bandpass frequency transformation equation,

$$\lambda = \frac{\omega_o}{w}\left(\frac{s}{\omega_o} + \frac{\omega_o}{s}\right) ,$$

ω_o is as usual the center radian frequency, but w must be equal to $4\pi/T$. The Q of each of the four resonant circuits must be equal to $\frac{\omega_1 T}{2\mu_o}$ at its resonant frequency ω_1 rad/sec.

2. Transmission Zeros on the ω Axis

An approximation with the zeros of transmission restricted to the ω axis (imaginary zeros) is required for the realization of a bandpass matched filter using quartz crystal resonators. The lowpass prototype of such a transfer function is

$$H(s) = \frac{K(s^2+\nu^2)}{(s+\sigma)\left[(s+\alpha)^2+\beta^2\right]} \quad . \tag{13}$$

The optimum parameters for approximating a matched filter are:

$$\nu T = 6.122$$
$$\sigma T = 2.097$$
$$\alpha T = 1.296$$
$$\beta T = 4.696$$

For this restricted approximation, $\rho = 0.9408$ (0.265 db) , which is insignificantly worse in performance than the optimum approximation above. Its unit impulse response (inverse Laplace transform of Eq. 13) is

$$h\left(\frac{t}{T}\right) = K\left[1.8453e^{-\sigma t} - 0.9385e^{-\alpha t}\sin(\beta t + 1.1213)\right] \quad . \tag{14}$$

$\left|\frac{H(i\omega)}{H(o)}\right|$, the normalized magnitude of the transfer function is shown in Fig. 11. For K = 1 , the unit impulse response of Eq. (14) is given in Fig. 12, and Fig. 13 shows the response to matched pulse duration T seconds and amplitude $\frac{1}{T}$.

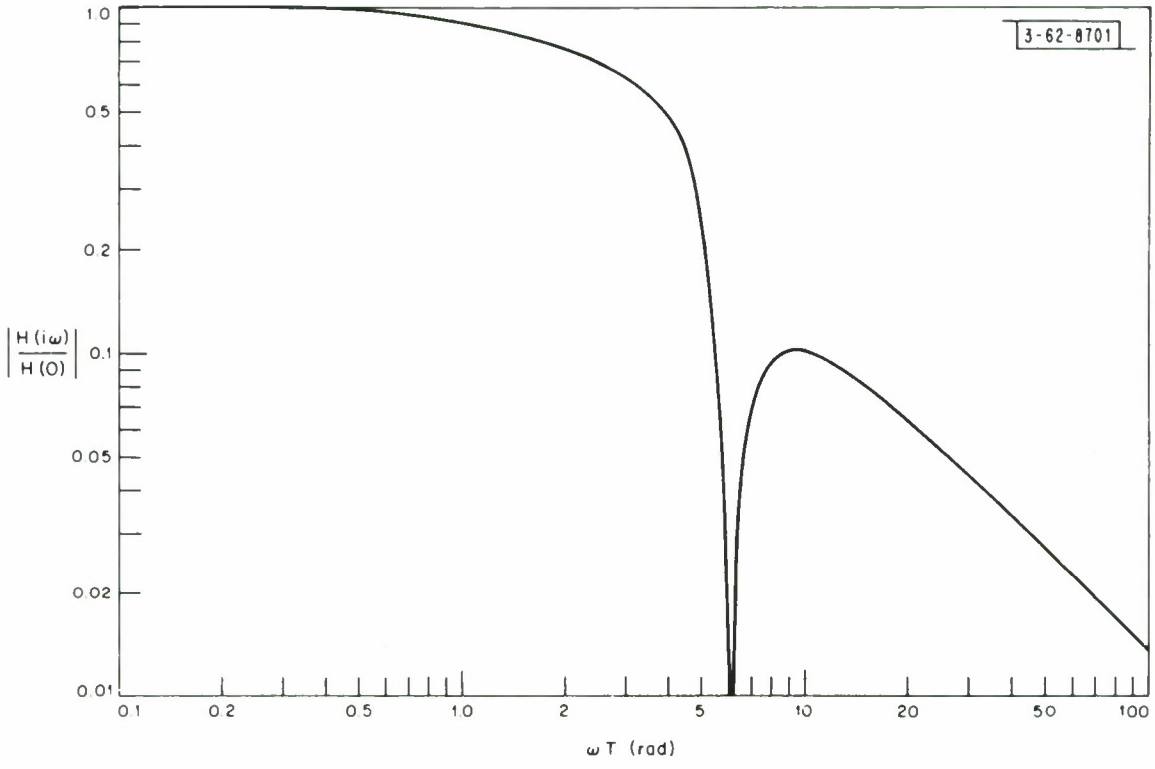


Fig. 11. Normalized magnitude vs. frequency of 3-pole, 2 (imaginary)-zero matched filter approximation.

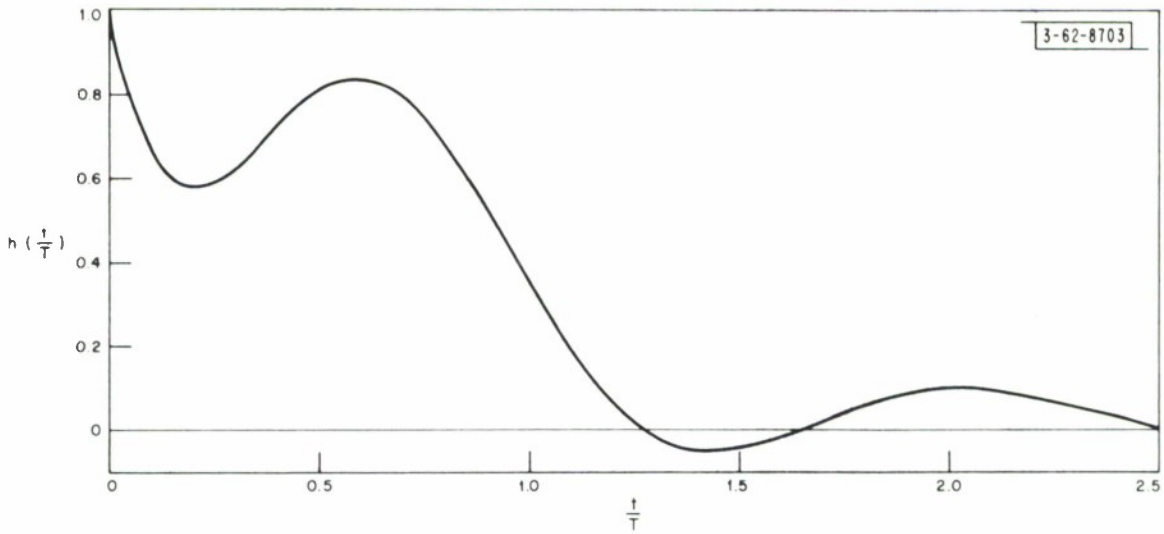


Fig. 12. Unit impulse response of 3-pole, 2 (imaginary)-zero matched filter.

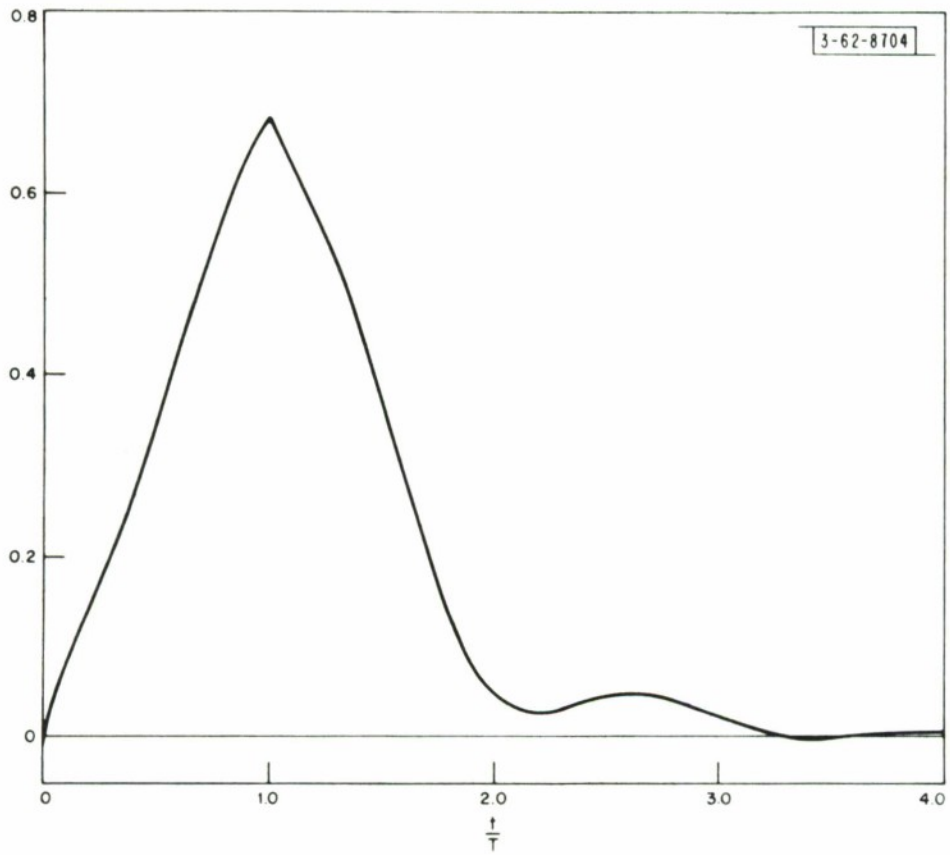


Fig. 13. Response of the 3-pole, 2 (imaginary)-zero matched filter to a matched (T second) pulse of amplitude 1/T.

The transfer function of Eq. (13) is realized for $T = 1$ second by the equally terminated lossless coupling network shown in Fig. 14. The synthesis is carried out in Appendix C. Crystal bandpass matched filters using this circuit as the lowpass prototype have been realized very successfully by Damon Engineering, Inc., and a discussion of their performance is given in Ref. 3.

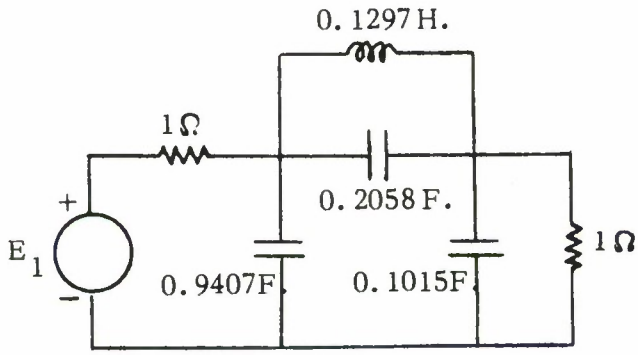


Fig. 14. A realization of the three-pole, two (imaginary)-zero approximation of Eq. (13) for a filter matched to a 1 second pulse. For this circuit the gain constant K equals 0.6639.

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APPENDIX A

A Realization of the Optimum Three-Pole, Two-Zero Matched Filter Approximations

The circuit to realize the transfer function of Eq. (10) is shown in Fig. 8. The transfer function of this circuit (which will not be derived here) is given in Eq. (12). The method used to design the circuit is to equate corresponding coefficients of the numerator and denominator polynomials of the desired and circuit transfer functions (Eqs. (10) and (12)). The resulting set of nonlinear equations is then solved for the circuit element values. A change of variables converts the set of equations to a linear set which greatly simplifies the solution.

The transfer function, Eq. (12), of the network to be designed is of the form:

$$T(s) = \frac{E_2(s)}{E_1(s)} = \frac{K_1(s^2+a_1 s+a_o)}{b_3 s^3+b_2 s^2+b_1 s+b_o} \tag{A-1}$$

By introducing the variables

$$A = \frac{r}{L} \qquad E = \frac{C_2}{C_3}$$

$$B = \frac{1}{LC_2} \qquad F = \frac{G_3}{C_3}$$

$$D = \frac{C_2}{C_1} \qquad H = \frac{G_1}{C_1}$$

the constant factor and the polynomial coefficients are:

$$K_1 = EH \qquad b_o = B(EH + DF) + AFH$$

$$a_o = B \qquad b_1 = A(F + H) + FH + A(EH + DF) + B(D+E)$$

$$a_1 = A \qquad b_2 = A(1 + D + E) + F(1 + D) + H(1 + E)$$

$$b_3 = 1 + D + E$$

Then by equating the corresponding polynomial coefficients in the transfer functions, the following set of equations is obtained.

$$a_1 = 2\mu = A \tag{A-2}$$

$$a_o = \mu^2 + \nu^2 = B \tag{A-3}$$

$$b_2/b_3 = 2\alpha + \sigma = \gamma \tag{A-4}$$

$$b_1/b_3 = 2\alpha\sigma + \alpha^2 + \beta^2 = \delta \tag{A-5}$$

$$b_o/b_3 = \sigma(\alpha^2 + \beta^2) = \rho \tag{A-6}$$

For the constant factor we have $K = K_1/b_3$. The Eqs. (A-2) and (A-3) specify the parameters A and B, so the remaining three equations must be solved for the four other parameters. In these equations the new parameters γ, δ and ρ have been introduced.

By introducing the definitions of the b_k's into Eqs. (A-4), (A-5) and (A-6), they can be put into the following form.

$$(A - \gamma)(D + E) + (F + H) + (DF + EH) = \gamma - A \tag{A-7}$$

$$(B - \delta)(D + E) + FH + A(F + H + DF + EH) = \delta \tag{A-8}$$

$$-\rho(D + E) + AFH + B(DF + EH) = \rho \tag{A-9}$$

In these three equations the variables occur only in certain groupings, and it turns out that a further change of variables produces a set of three linear equations in four unknowns. The new variables are:

$$u = D + E \tag{A-10}$$

$$v = F + H \tag{A-11}$$

$$w = DF + EH \tag{A-12}$$

$$x = FH\ , \tag{A-13}$$

and the new set of equations is

$$(A-\gamma)\,u + v + w = \gamma - A \tag{A-14}$$

$$(B-\delta+A\gamma-A^2)\,u + x = \delta - A(\gamma - A) \tag{A-15}$$

$$-\rho u + Bw + Ax = \rho \tag{A-16}$$

Since there are four unknowns and only three equations, a value for one of the unknowns can be selected and then the values for the other three can be found. A convenient one to select a value for initially is u. The value of x is then found from Eq. (A-15), the value of w from Eq. (A-16), and the value of v from Eq. (A-14). Actually, x, w and v can easily be found as functions of u alone.

The constants in the last set of equations have the values:

$$\begin{aligned}
A &= 2\mu && = 1.56643\\
B &= \mu^2+\nu^2 && = 36.32994\\
\gamma &= 2\alpha+\sigma && = 5.14406\\
\delta &= 2\alpha\sigma+\alpha^2+\beta^2 && = 25.84539\\
\rho &= \sigma(\alpha^2+\beta^2) && = 43.43620\\
\gamma &- A && = 3.57764\\
A(\gamma &- A) && = 5.60411\\
B-\delta &+ A(\gamma - A) && = 16.08866\\
\delta &- A(\gamma - A) && = 20.24129
\end{aligned}$$

As a function of u, the unknowns x, w and v are:

$$\left.\begin{aligned}
x &= 20.24129 - 16.08866u\\
w &= 1.88929u + 0.32287\\
v &= 1.68835u + 3.25477
\end{aligned}\right\} \tag{A-17}$$

For a realizable network all elements must have positive values so the variables v and x must be positive. The first equation of (A-17) thus requires that $u \le 1.2581$. Below we will find a lower limit on u.

Next solve Eqs. (A-10) through (A-13) for D, E, F and H in terms of u, v, w and x. From (A-11) and (A-13) we obtain $v = F + \frac{x}{F}$ or $F^2 - Fv + x = o$, and $H = v - F$. So for F we have

$$F = \frac{v}{2} - \left(\frac{v^2}{4} - x\right)^{\frac{1}{2}}$$

where we have chosen the negative sign for the second term on the right. And for H

$$H = \frac{v}{2} + \left(\frac{v^2}{4} - x\right)^{\frac{1}{2}} .$$

If $v^2 \geq 4x$, F and H will be real and positive (assuming x is positive). By substituting from Eq. (A-17) into this inequality, a lower limit on u is obtained which is $u \geq 0.9031$, so we have $0.9031 \leq u \leq 1.2581$.

Using these results for F and H, we can solve Eqs. (A-10) and (A-12) for D and E. The solutions are

$$D = \frac{1}{2}\left[u - \frac{w - \frac{1}{2}uv}{\left(\frac{v^2}{4} - x\right)^{\frac{1}{2}}}\right]$$

and

$$E = \frac{1}{2}\left[u + \frac{w - \frac{1}{2}uv}{\left(\frac{v^2}{4} - x\right)^{\frac{1}{2}}}\right]$$

Finally, from these results we obtain the component values as follows, normalizing $R_1 = G_1^{-1} = 1\,\text{ohm}$.

$$C_1 = \frac{G_1}{H} = \frac{1}{H}$$

$$C_2 = DC_1 = \frac{D}{H}$$

$$C_3 = \frac{C_2}{E} = \frac{D}{EH}$$

$$R_3 = G_3^{-1} = \frac{1}{FC_3} = \frac{EH}{DF}$$

$$L = \frac{1}{BC_2} = \frac{H}{BD}$$

$$r = AL = \frac{AH}{BD}$$

Ten sets of circuit element values for the range of u noted above are given in Table II.

APPENDIX B

A Realization of the Optimum Three-Pole, Two-Zero Matched Filter Approximation Using Predistortion

The transfer function to be realized is that of Eq. (10) which is

$$H(s) = K \frac{(s+\mu)^2 + \nu^2}{(s+\sigma)\left[(s+\alpha)^2 + \beta^2\right]} \quad . \tag{B-1}$$

The first step is to move the poles and zeros toward the $i\omega$ axis (predistort) by the distance μ, and then form the transmission function of the predistorted network. The result is

$$t(s) = t_o \frac{s^2 + \nu^2}{(s+\sigma-\mu)\left[(s+\alpha-\mu)^2 + \beta^2\right]} = \frac{N_o(s)}{D(s)} \tag{B-2}$$

We will make $|t(i\omega)|$ have its maximum allowable value of unity by making

$$t_o = (\sigma-\mu)\left[(\alpha-\mu)^2 + \beta^2\right]/\nu^2 \quad .$$

The maximum value occurs at $\omega = o$. This transmission function will be realized by a lossless coupling network terminated at both ends in 1 ohm resistors. The transmission function is used to obtain the reflection coefficient $\rho(s)$ at the input terminals of the network from which we calculate its input impedance when its output is terminated in 1 ohm. A lossless network terminated in a 1 ohm resistor is then realized in such a way as to have the required zeros of transmission at $s = \pm i\nu$.

$\rho(s)$ is defined by

$$\rho(s)\,\rho(-s) = 1 - t(s)\,t(-s) = \frac{N_1(s)\,N_1(-s)}{D(s)\,D(-s)} \quad ,$$

from which we find that

$$N_1(s)\,N_1(-s) = (-s^2)\,(s^4 + a_2 s^2 + a_o) \quad ,$$

where

$$a_2 = 2\left[\beta^2 - (\alpha-\mu)^2\right] + (\sigma-\mu)^2 \left\{\frac{\left[\beta^2 + (\alpha-\mu)^2\right]^2}{\nu^4} - 1\right\} ,$$

and

$$a_o = \left[\beta^2+(\alpha-\mu)^2\right]^2 \left[1+\frac{2}{\nu^2}(\sigma-\mu)^2\right] - 2(\sigma-\mu)^2\left[\beta^2 - (\alpha-\mu)^2\right] .$$

The numerical values are

$$a_2 = 31.97149 \qquad \text{and} \qquad a_o = 278.14857 .$$

The roots of $N_1(s)$ are chosen to have negative real parts, so factoring $s^4 + a_2 s^2 + a_o$ and selecting the appropriate factors for $N_1(s)$ yields

$$\rho(s) = \frac{s(s^2+1.17647\,s+16.67779)}{D(s)} .$$

The input impedance of the network is defined as

$$Z_1(s) = \frac{1-\rho(s)}{1+\rho(s)}$$

which, upon substitution of numerical values, is the ratio of polynomials

$$Z_1(s) = \frac{1.61795\,s^2+2.95007\,s+25.86880}{2\,s^3+3.97090\,s^2+36.30565\,s+25.86880} .$$

Realization of this driving point impedance by standard techniques yields the complete network shown in Fig. B-1.

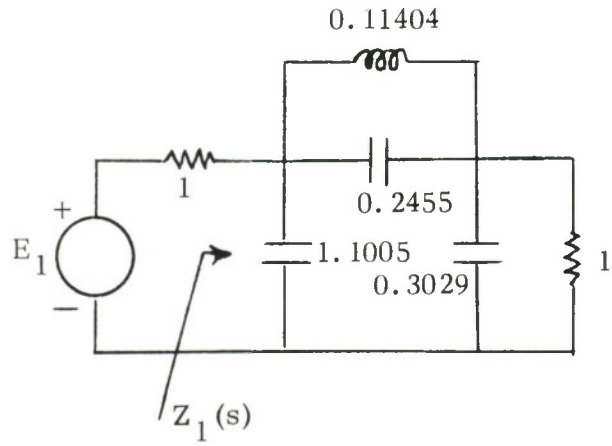

Fig. B-1. Circuit realizing the predistorted transmission function of Eq. (B-2). Units are ohms, farads and henries. The driving point impedance $Z_1(s)$ is indicated by the arrow.

Next add uniform dissipation of the amount μ to all the reactances to move the poles and zeros of the transfer function to the desired locations as specified in Eq. (B-1). The resistance in series with the inductor L is equal to μL ohms and that in parallel with each capacitor C is equal to $(\mu C)^{-1}$ ohms, where $\mu = 0.7832$. The resulting circuit is shown in Fig. B-2. The final form of the network

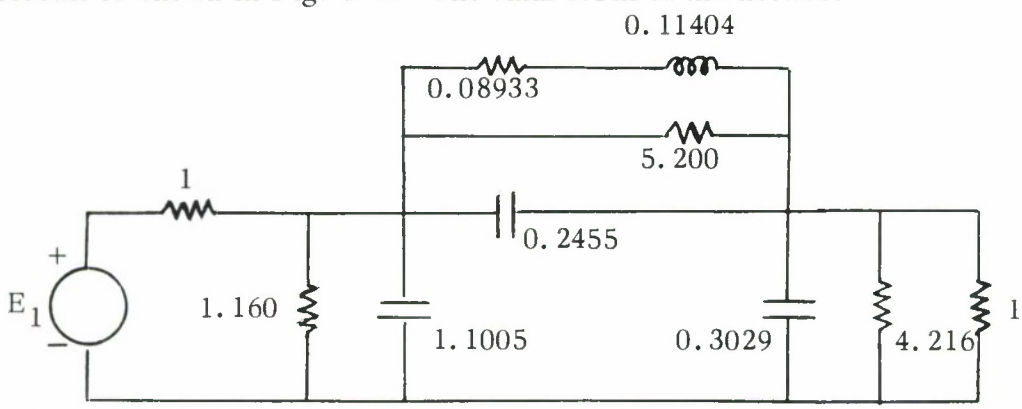

Fig. B-2. Circuit of Fig. B-1 after uniform dissipation has been added to realize the desired transfer function of Eq. (B-1).

is obtained by combining the resistors at each of the input and output terminals and raising the impedance level of the network to obtain a 1 ohm source resistance. The final network is in Fig. 9.

APPENDIX C

A Realization of the Three-Pole, Two (imaginary)-Zero Matched Filter Approximation

The transfer function to be realized is that of Eq. (13) which is

$$H(s) = \frac{K(s^2+\nu^2)}{(s+\sigma)\left[(s+\alpha)^2+\beta^2\right]} \quad . \tag{C-1}$$

The transmission function is chosen to have a maximum magnitude of unity (which occurs at $s = o$), and it is

$$t(s) = \frac{t_o(s^2+\nu^2)}{(s+\sigma)\left[(s+\alpha)^2+\beta^2\right]} \tag{C-2}$$

where

$$t_o = \sigma(\alpha^2+\beta^2)/\nu^2 \quad .$$

The reflection coefficient $\rho(s)$ is defined in terms of $t(s)$ as

$$\rho(s)\,\rho(-s) = 1 - t(s)\,t(-s) = \frac{N_1(s)\,N_1(-s)}{D(s)\,D(-s)} \quad .$$

From this relation we find that

$$N_1(s)\,N_1(-s) = (-s^2)(s^4+a_2 s^2+a_o) \quad ,$$

where

$$a_2 = 2(\beta^2-\alpha^2) + \sigma^2\left[\frac{(\alpha^2+\beta^2)^2}{\nu^4} - 1\right]$$

and

$$a_o = (\alpha^2+\beta^2)^2 \left(1 + 2\frac{\sigma^2}{\nu^2}\right) - 2\sigma^2(\beta^2 - \alpha^2) \quad .$$

The values of these constants are

$$a_2 = 38.11136 \qquad \text{and} \qquad a_o = 516.1973 \quad .$$

$N_1(s)$ is chosen to have zeros with negative real part. Selection of the appropriate factors for $N_1(s)$ yields

$$\rho(s) = \frac{s(s^2+2.70714\,s+22.7200)}{D(s)}$$

The input impedance of the terminated network is then

$$Z_1(s) = \frac{1-\rho(s)}{1+\rho(s)} = \frac{1.98186\,s^2+6.44746\,s+49.56933}{2\,s^3+7.39614\,s^2+5.18875\,s+49.56933}$$

Realization of this driving point impedance by standard techniques yields the network in Fig. C-1.

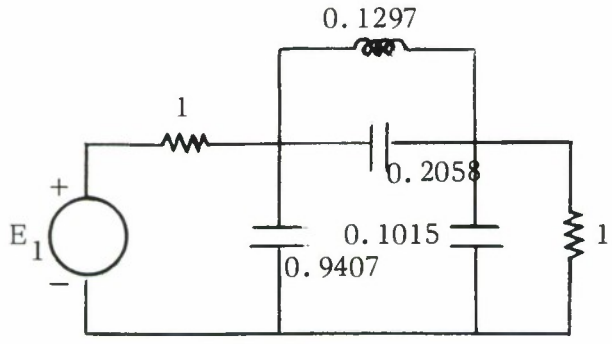

Fig. C-1. Circuit realizing the three-pole, two (imaginary)-zero approximation of a matched filter for a 1 second pulse. Units are ohms, farads and henries.

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13. ABSTRACT

Several low order rational transfer functions have been found which approximate a matched filter for a rectangular pulse. These transfer functions are optimum in the sense that the ratio of peak-signal to rms-noise at the output is maximized to the extent allowed by the available parameters. Normalized transfer functions are given and their frequency and time response are shown graphically. For the most useful cases, realizations normalized in impedance and pulse duration are given which allow the use of lossy inductors. A short discussion of a bandpass realization is presented. A transfer function with zeros restricted to the $i\omega$-axis is given, along with its realization, which is useful as the lowpass prototype of a bandpass, crystal matched filter.

14. KEY WORDS

matched filters
rectangular pulse
transfer functions
pulse detection