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Problems Relating to Propagation on

Coupled Microstrips

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Prepared for

Massachusetts Institute of Technology

Lincoln Laboratory

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Final Report  
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## ABSTRACT

This report presents the results of an investigation on coupled microstrip propagation and related problems. Included is a conformal mapping approach to coupled microstrip propagation in the quasi-static limit. Numerical analysis of the treatment shows that it does not agree with known results in the limit of narrow strip width. A Green's function solution to the problem has been developed and is contained in a separate report. The method is illustrated by applying it to a related problem -- a dielectric rod situated symmetrically between two parallel conducting plates. A perturbation method is considered for the problem of non-reciprocal propagation in coupled microstrip. The method has been adapted so as to make use of numerical results from the Green's function solution of the reciprocal propagation problem. As a preliminary test of the perturbation approach, it is applied to a problem which can be solved exactly.

**Accepted for the Air Force**  
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PROBLEMS RELATING TO PROPAGATION ON  
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PROBLEMS RELATING TO PROPAGATION ON  
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1. Introduction.

The purpose of this report is to summarize the work performed under Subcontract No. 351 during the period September 15, 1967 -- June 15, 1968, on coupled pairs of microstrip transmission lines and, in particular, the problem of non-reciprocal propagation in these structures. Microstrip is a two-conductor transmission line made up of a conductive strip on one surface of a sheet of dielectric material, the other surface being completely coated with conductor to form a ground plane. Introducing a second strip parallel to the first gives rise to coupling and two normal modes of propagation. It has been found with regard to coupled pairs of microstrip that by employing ferrites as the substrate material, nonreciprocal propagation effects can be produced. As yet, however, such effects are only qualitatively understood.

The physical structure of the coupled microstrip lines is shown in Figure 1. The parameters which characterize the microstrip are  $W/H$ ,  $S/H$  and the dielectric constant of the substrate.  $W$  is the strip width,  $H$  the substrate height, and  $S$  the spacing between adjacent edges of the two strips. Calculation of the transmission-line parameters such as propagation velocity and characteristic impedance is complicated by the unsymmetrical geometry of the microstrip cross-section. There are, however, general methods such as conformal mapping and the theory of Green's functions which have been successful in treating structures with similar geometries.

## 2. Conformal Mapping Approach.

A method of solution of the coupled microstrip problem in the "quasi-static" limit, i.e. in the frequency range in which propagation is approximately TEM, was developed by T. G. Bryant and J. A. Weiss at Lincoln Laboratory. The conformal mapping approach which was used involved successive transformations of the original microstrip configuration to a parallel plate configuration suitable for determining the capacitance, characteristic impedance, and velocity of propagation for each of the two normal modes. The transformations for each mode are given in Figures 2 and 3.

In order to adapt this procedure into a form suitable for numerical computation, a single equivalent transformation was determined for each mode. The transformation equation for the odd mode is

$$\omega(z) = \frac{2H}{\pi} \frac{(e^z + 1)^{1/2}}{e^{z_b}} + \frac{H}{\pi} \ln \left[ \frac{(e^z + 1)^{1/2} + 1}{(e^z + 1)^{1/2} - 1} \right] \quad (1)$$

and for the even mode

$$\omega(z) = -\frac{2H}{\pi} \cosh^2\left(\frac{z_b}{2}\right) \tanh \frac{z}{2} + \frac{H}{\pi} z - iH \quad (2)$$

Using these equations the parameters which characterize the microstrip ( $S/H$ ,  $W/H$ ) can be expressed as functions of the parameters which determine the capacitance, etc. for the transformed configurations in Figures 3A and 3C.

Consider the even mode. When  $\omega = S/2$ ,  $z = z_b + i\pi$  and from equation (2),

$$S/H = 2/\pi (\sinh \chi_b + \chi_b) \quad (3a)$$

For  $\omega = S/2 + W/2$ ,  $z = \chi_{c,d} + i\pi$  and (2) becomes

$$W/H = 2/\pi \left[ 2 \sinh^2 \left( \frac{\chi_b}{2} \right) \coth \left( \frac{\chi_{c,d}}{2} \right) + \chi_{c,d} \right] - S/H \quad (3b)$$

In the case of the odd mode when  $\omega = S/2$ ,  $z = \chi_b$  and

$$S/H = \frac{4}{\pi} \frac{(e^{\chi_b+1})^{1/2}}{e^{\chi_b}} + \frac{2}{\pi} \ln \left[ \frac{(e^{\chi_b+1})^{1/2} + 1}{(e^{\chi_b+1})^{1/2} - 1} \right] \quad (4a)$$

When  $\omega = S/2 + W/2$ ,  $z = \chi_{c,d}$  so that

$$W/H = \frac{4}{\pi} \frac{(e^{\chi_{c,d}+1})^{1/2}}{e^{\chi_b}} + \frac{2}{\pi} \ln \left[ \frac{(e^{\chi_{c,d}+1})^{1/2} + 1}{(e^{\chi_{c,d}+1})^{1/2} - 1} \right] \quad (4b)$$

Equations (3b) and (4b) are double-valued for specified S/H and W/H. Since neither equations (3) nor (4) can be solved explicitly for  $\chi_b$ ,  $\chi_c$  and  $\chi_d$ , a program was written to compute these variables by interpolation. Corresponding values of S/H and  $\chi_b$  were computed according to (3a) for the even mode and stored in the program. The desired value of S/H was then read into the program and  $\chi_b$  was obtained. Using the result, corresponding values of W/H and  $\chi_{c,d}$  were determined from (3b) and stored. A value for W/H was then read in and  $\chi_c$  and  $\chi_d$  were obtained. The same procedure was used with the odd mode.

According to the conformal mapping approach described above the strips were divided into proximal and distal halves (see Figure 2A). Fields of the proximal half-strips were characterized by the value of

S/H of the actual structure. Fields for the distal half-strips were taken to be the same as those of the proximal halves in the limit of large spacing (S/H = 200). Values of  $\chi_b$ ,  $\chi_c$  and  $\chi_d$  were determined in each case and were used to get the capacitance for a single coupled strip.

In both even and odd modes the dielectric-air boundary in the transformed configuration is a complicated function of S/H and W/H. It is obtained by letting  $w(z)$  range between 0 and S/2. For the even mode the two equations which determine the boundary are

$$2 \sinh^2\left(\frac{\chi_b}{2}\right) \tan\left(\frac{y}{2}\right) \left[ \frac{1 - \tanh^2 \frac{\chi}{2}}{1 + \tan^2 \frac{y}{2} \tanh^2 \frac{\chi}{2}} \right] + y = \pi \quad (5)$$

$$w(z) = \frac{2H}{\pi} \sinh^2\left(\frac{\chi_b}{2}\right) \tan\left(\frac{\chi}{2}\right) \left[ \frac{1 + \tan^2 \frac{y}{2}}{1 + \tan^2 \frac{y}{2} \tanh^2 \frac{\chi}{2}} \right] + \chi$$

The corresponding equation for the odd mode is

$$w(z) = \frac{2H}{\pi} \frac{(e^{\chi+iy} + 1)^{1/2}}{e^{\chi_b}} + \frac{H}{\pi} \ln \left[ \frac{(e^{\chi+iy} + 1)^{1/2} + 1}{(e^{\chi+iy} + 1)^{1/2} - 1} \right] \quad (6)$$

Rather than attempt to solve these equations or incorporate them into the program, an average dielectric filling was used as a first approximation to determine the capacitance for both the proximal and distal half-strips. For no dielectric the capacitance is given by

$$C'_0 = \left[ \epsilon_0 (\chi_d - \chi_c) / \pi + \epsilon_0 (\chi'_d - \chi'_c) / \pi \right] \quad (7)$$

where the primed quantities refer to solutions for the desired value of W/H with S/H = 200. For partial dielectric filling with dielectric constant  $k$ ,

$$C' = \left\{ \frac{1}{2} \left[ k \epsilon_0 (\chi_d - \chi_c) / \pi + k \epsilon_0 (\chi_d - \chi_b) / \pi + \epsilon_0 (\chi_b - \chi_c) / \pi \right] + \frac{1}{2} \left[ k \epsilon_0 (\chi'_d - \chi'_c) / \pi + k \epsilon_0 (\chi'_d - \chi'_b) / \pi + \epsilon_0 (\chi'_b - \chi'_c) / \pi \right] \right\} \quad (8)$$

The propagation velocity of the medium is determined from the relation

$$v = \frac{c}{\sqrt{\epsilon_{\text{eff}}}} \quad (9)$$

where  $c$  is the velocity of light and  $\epsilon_{\text{eff}} = \frac{C}{C_0}$ . The impedance for each strip is then given by

$$Z = \frac{1}{v C} \quad (10)$$

In the limit of large  $S/H$  spacing there is no coupling and the even and odd modes are degenerate. This limiting case corresponds to the single strip, a problem which was treated by Wheeler<sup>1</sup>. As a test of the correctness of the approach, this limiting case was investigated. Not only should the impedances, etc. for each mode approach the same value when  $W/H$  is specified, but that value should be the same as obtained by Wheeler.

Results indicated that both even and odd mode impedances were identical in this limit, but the values obtained did not correspond to those reported by Wheeler. Agreement with Wheeler's results was not expected because of the arbitrariness of the averaging procedure used to get the capacitance, etc. In order to determine whether or not the solutions for this limiting case could be made to agree with Wheeler's results, impedances were obtained for each mode at various values of  $W/H$  assuming maximum and minimum dielectric filling.

These values are plotted along with Wheeler's curve in Figures 4 and 5. When  $W/H$  is less than 1.0, Wheeler's curve lies outside the region in which agreement is possible. This suggests that some portion of the

field in air had been thrown away. The reason for this difficulty can be understood by considering Figures 6 and 7. It was assumed in the transformations for both modes that each strip could be divided into half-strips which could then be considered separately. This assumption is good only if none of the field lines on the distal half terminate in the region of the proximal half. Such field lines are not accounted for in the final result. Apparently there is an appreciable amount of the field strongly dependent on  $W/H$  for which this is not the case and the loss of this portion of the field in the resulting transformation would account for the discrepancy with Wheeler's results.

### 3. Green's Function Solution.

Due to the inherent difficulties with the conformal mapping approach, a Green's function solution was developed by J. A. Weiss and T. G. Bryant. For a discussion of the theory see "Parameters of Microstrip Transmission Lines and of Coupled Pairs of Microstrip Lines" which is contained in a separate report under the same contract as the report given here.<sup>2</sup> The method is quite general and can be used to determine the field configuration in a number of related problems. As an illustration, it is applied to the problem of a dielectric cylinder situated symmetrically between two infinite conducting plates.

Using a program devised by J. A. Weiss, necessary modifications were made to allow computation of the bound charge distribution on the surface of the dielectric cylinder. A second program was written to determine the potential at desired points between the conducting plates

using the distribution of bound charge. Such points were chosen to be of unit separation and the potentials were printed out in such a way as to allow their decimal points to form a grid. Lines of equipotential could then be constructed to give a graphical picture of the result.

A detailed sketch of the lines of equipotential is given in Figure 8. The potential difference of the plates was taken to be 50 volts and, for convenience, the unit of distance was taken as 1/50th of the plate separation. According to this scale the dielectric rod was chosen to have a diameter of 30 units and the width of the plates was 60 units. With no rod present, the capacitance is 10.6 pf/m. When a cylinder of dielectric constant  $k = 9$  is present, the capacitance as determined from the program is 14.2 pf/m. The effective dielectric constant for the system is 1.34.

The same procedure was also carried out for an analogous problem, that of a cylindrical hole in the presence of dielectric. The capacitance for the system with dielectric constant  $k = 16$  is 148.3 pf/m and the effective dielectric constant was found to be 13.99. The equipotential lines for this problem are shown in Figure 9.

#### 4. Perturbation Method for Non-reciprocal Propagation.

The use of ferrites as the substrate material in microstrip to produce nonreciprocal propagation is a subject of much interest in the design of integrated microwave components but is only poorly understood from a theoretical point of view. As a starting point for the theoretical investigation of this phenomenon a study was made of perturbation method of Suhl and Walker.<sup>3</sup> The usefulness of their formulation is that

longitudinal fields are included in the perturbation equations for transverse components. For completeness a brief review of the theory is included.

A medium whose dielectric and permeability tensors are diagonal, isotropic, and independent of the distance along the z-axis is perturbed such that it is no longer isotropic and the tensors may contain off-diagonal elements. For the unperturbed system  $\mu = \mu_1(x,y)$ ,  $\epsilon = \epsilon_1(x,y)$ . After the perturbation

$$\epsilon(x,y) = \begin{bmatrix} \epsilon_2(x,y) & -i\eta(x,y) & 0 \\ i\eta(x,y) & \epsilon_2(x,y) & 0 \\ 0 & 0 & \epsilon_3(x,y) \end{bmatrix}, \quad \mu(x,y) = \begin{bmatrix} \mu_2(x,y) & -i\kappa(x,y) & 0 \\ i\kappa(x,y) & \mu_2(x,y) & 0 \\ 0 & 0 & \mu_3(x,y) \end{bmatrix} \quad (11)$$

Maxwell's equations for the perturbed system may be written as

$$\begin{aligned} \nabla^* H_z - \frac{\partial H_t^*}{\partial z} - i\omega \epsilon_2 E_t - \omega \eta E_t^* &= 0 \\ \nabla^* E_z - \frac{\partial E_t^*}{\partial z} + i\omega \mu_2 H_t + \omega \kappa H_t^* &= 0 \\ \nabla \cdot H_t^* - i\omega \epsilon_3 E_z &= 0 \\ \nabla \cdot E_t^* + i\omega \mu_3 H_z &= 0 \end{aligned}$$

where  $\nabla^* = (\frac{\partial}{\partial y}, -\frac{\partial}{\partial x})$ ,  $\underline{E}_t^* = (E_y, -E_x)$ ,  $\underline{H}_t^* = (H_y, -H_x)$ ,  $\underline{E}_t$  and  $\underline{H}_t$  may be expanded in terms of the normal modes of the unperturbed system by the relations

$$\underline{E}_t = \sum_n a_n(z) \underline{E}_{tn}(x,y) \quad \underline{H}_t = \sum_n b_n(z) \underline{H}_{tn}(x,y) \quad (12)$$

where  $a_n$  and  $b_n$  are the amplitude functions for the  $n^{\text{th}}$  mode. Using these results

$$\frac{db_n}{dz} + i\beta_n a_n = \frac{i\omega}{\tilde{\Delta}_n} \left[ \int (\epsilon_2 - \epsilon_1) \underline{E}_t \cdot \tilde{\underline{E}}_{tn} ds - i \int \kappa \underline{E}_t^* \cdot \tilde{\underline{E}}_{tn} ds + \int (\mu_3 - \mu_1) H_z \tilde{H}_{zn} ds \right] \quad (13)$$

$$\frac{da_n}{dz} + i\beta_n b_n = \frac{i\omega}{\Delta_n} \left[ \int (\mu_2 - \mu_1) \underline{H}_t \cdot \tilde{\underline{H}}_{tn} ds - i \int \kappa \underline{H}_t^* \cdot \tilde{\underline{H}}_{tn} ds + \int (\epsilon_3 - \epsilon_1) E_z \tilde{E}_{zn} ds \right] \quad (14)$$

where

$$\Delta_n = \int \underline{E}_{tn}^* \cdot \tilde{\underline{H}}_{tn} ds \quad \tilde{\Delta}_n = - \int \underline{H}_{tn}^* \cdot \tilde{\underline{E}}_{tn} ds \quad (15)$$

and the tilde denotes complex conjugation. Equations (13) and (14) are exact but involve the unknown fields  $\underline{E}_t$ ,  $E_z$ ,  $\underline{H}_t$ , and  $H_z$ .

Consider the case in which the perturbation is uniform in  $z$ . In the absence of the perturbation only the  $m^{\text{th}}$  mode is to be present. An unmagnetized ferrite material is introduced into the line. Two cases are to be considered. In the first case the ferrite completely fills the substrate region. In the second case only part of the substrate is ferrite, the rest being dielectric. If the material properties of the ferrite and the original medium are not significantly different, the resulting fields are nearly the same as for the unperturbed system. Denoting the fields in the presence of ferrite by  $\underline{E}_{tm}$  and  $\underline{H}_{tm}$ , we may then set them equal to  $\underline{E}_{tm}$  and  $\underline{H}_{tm}$ . If, on the other hand, the mediums

are significantly different, then  $\underline{E}_{tm_0}$  and  $\underline{H}_{tm_0}$  must be determined by an independent method.

The ferrite is then magnetized along the z-direction and the fields for the  $m^{\text{th}}$  mode become  $a_m(z) E_{tm_0}(x,y)$ ,  $a_m(z) H_{zm_0}(x,y)$ ,  $b_m(z) H_{tm_0}(x,y)$ , and  $b_m(z) E_{zm_0}(x,y)$  where  $a_m(z) = A_m e^{-i\beta z}$   
 $b_m = B_m e^{-i\beta z}$  and  $\beta = \beta_m + \delta\beta$ . Equations (13) and (14) then become

$$i\beta_m A_m - i\beta B_m = \frac{i\omega}{\Delta_m} \left( \int [(\epsilon_2 - \epsilon_1) \underline{E}_{tm_0} \tilde{\underline{E}}_{tm_0} - i\eta \underline{E}_{tm_0}^* \tilde{\underline{E}}_{tm_0} + (\mu_3 - \mu_1) H_{zm_0} \tilde{H}_{zm_0}] ds \right) A_m \equiv iL A_m \quad (16)$$

$$i\beta_m B_m - i\beta A_m = \frac{i\omega}{\Delta_m} \left( \int [(\mu_2 - \mu_1) \underline{H}_{tm_0} \tilde{H}_{tm_0} - i\kappa \underline{H}_{tm_0} \tilde{H}_{tm_0} + (\epsilon_3 - \epsilon_1) E_{zm_0} \tilde{E}_{zm_0}] ds \right) B_m \equiv iM B_m \quad (17)$$

The change in  $\beta$  is determined by adding and then subtracting equations (16) and (17). Requiring that the resulting equations give non-trivial solutions for  $A_m$  and  $B_m$ , one then has that

$$\begin{vmatrix} \delta\beta + L & \delta\beta + M \\ 2\beta - \delta\beta - L & -2\beta + \delta\beta + M \end{vmatrix} = 0$$

Since  $L$ ,  $M$ , and  $\delta\beta$  are small, higher order terms in these variables may be neglected. The result is then

$$\delta\beta = -\frac{1}{2}(L+M) \quad (18)$$

The perturbation equations, as they stand, are not suitable for determining the fields for the microstrip problem since analytic expressions for the fields in this case are not available. However, when the only mode

present in the unperturbed system is TEM the perturbation equations can be conveniently expressed in terms of a potential function  $\phi$ . For this case Maxwell's equations take the form

$$\nabla \times \underline{E}_{TEM} = \frac{i\omega}{c} \mu \underline{H}_{TEM} \quad \nabla \times \underline{H}_{TEM} = -\frac{i\omega}{c} \epsilon \underline{E}_{TEM}$$

and are satisfied if we let

$$\underline{E}_{TEM} = -\nabla_t \phi(x,y) e^{i(\beta z - \omega t)} \quad \underline{H}_{TEM} = \nabla_t \phi^*(x,y) e^{i(\beta z - \omega t)} \quad (19)$$

If one assumes that  $\underline{E}_{tm_0} = \underline{E}_{TEM}$  and  $\underline{H}_{tm_0} = \underline{H}_{TEM}$ , then (16) and (17) become

$$\begin{aligned} i\beta_m A_m - i\beta B_m &= iL A_m \\ i\beta_m B_m - i\beta A_m &= iM B_m \end{aligned} \quad (20)$$

where

$$\begin{aligned} L &= \frac{\omega}{\Delta_{TEM}} \int [(\epsilon_2 - \epsilon_1) \nabla_t \phi \cdot \nabla_t \phi - i\kappa \nabla_t^* \phi \cdot \nabla_t \phi] ds \\ M &= \frac{\omega}{\Delta_{TEM}} \int [(\mu_2 - \mu_1) \nabla_t^* \phi \cdot \nabla_t^* \phi + i\kappa \nabla_t \phi \cdot \nabla_t^* \phi] ds \end{aligned} \quad (21)$$

$$\Delta_{TEM} = - \int \nabla_t^* \phi \cdot \nabla_t^* \phi ds = - \int \nabla_t \phi \cdot \nabla_t \phi ds = \tilde{\Delta}_{TEM} \quad (22)$$

The advantage of the formulation above over equations (16) and (17) is only one of form. However, further modifications may be made so as to allow important simplifications in numerical solutions to the micro-strip problem.

As an example of the method, consider the case of a coaxial cable which is perturbed when part of the medium between the two conductors is altered as in Figure 10. For the unperturbed system

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r} \quad \phi = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{r}\right) \quad a \leq r \leq b$$

The wave and line impedance are

$$Z_w = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \quad Z_L = \frac{1}{2\pi} \ln\left(\frac{b}{a}\right) \sqrt{\frac{\mu_0}{\epsilon_0}} \quad (23)$$

Let the potential remain fixed when the medium specified by  $\theta$  is altered. Furthermore, let  $\epsilon > \epsilon_0$  and  $\mu = \mu_0$  in this region. Then  $M = 0$  in (21).

$$\nabla_t \phi = \frac{\lambda}{2\pi\epsilon_0} \left( -\frac{x}{r^2} \hat{i} - \frac{y}{r^2} \hat{j} \right)$$

$$L = \frac{\omega}{\Delta_{TEM}} \int (\epsilon - \epsilon_0) \nabla_t \phi \cdot \nabla_t \phi \, ds = -\omega (\epsilon - \epsilon_0) \frac{\theta}{2\pi}$$

Therefore

$$\beta = \sqrt{\epsilon_0 \mu_0} \, \omega/c + \frac{\theta}{4\pi} (\epsilon - \epsilon_0) \omega/c \equiv \sqrt{\epsilon_{eff} \mu_0} \, \omega/c$$

$$\epsilon_{eff} = \left( \sqrt{\epsilon_0} + \frac{\theta}{4\pi} \frac{(\epsilon - \epsilon_0)}{\sqrt{\mu_0}} \right)^2$$

From equations (20)

$$A = \frac{\sqrt{\epsilon_0 \mu_0}}{(\sqrt{\epsilon_0 \mu_0} + \theta/4\pi (\epsilon - \epsilon_0))} B$$

The current and potential difference are

$$I = \oint \underline{H} \cdot d\underline{l} = \lambda c B e^{-i\beta z}, \quad \Phi_{ab} = - \int \underline{E} \cdot d\underline{r} = \frac{\lambda}{2\pi\epsilon_0} A e^{-i\beta z} \ln\left(\frac{b}{a}\right)$$

The wave impedance is then

$$Z_w = \sqrt{\frac{\mu_0}{\epsilon_{\text{eff}}}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \left[ \frac{\sqrt{\mu_0 \epsilon_0}}{\sqrt{\mu_0 \epsilon_0 + \theta/4\pi(\epsilon - \epsilon_0)}} \right]$$

and the line impedance is

$$Z_L = \frac{\ln(b/a)}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \left[ \frac{\sqrt{\mu_0 \epsilon_0}}{\sqrt{\mu_0 \epsilon_0 + \theta/4\pi(\epsilon - \epsilon_0)}} \right]$$

In the case where  $\mu > \mu_0$  and  $\epsilon = \epsilon_0$  the same procedure gives the following results

$$\beta = \left[ \sqrt{\epsilon_0 \mu_0} + \theta/4\pi(\mu - \mu_0) \right] \omega/c$$

$$Z_L = \frac{\ln(b/a)}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \left[ \frac{\sqrt{\mu_0 \epsilon_0} + \theta/4\pi(\mu - \mu_0)}{\sqrt{\mu_0 \epsilon_0}} \right]$$

$$Z_w = \sqrt{\frac{\mu_0}{\epsilon_0}} \left[ 1 + \theta/4\pi \frac{(\mu - \mu_0)}{\sqrt{\epsilon_0 \mu_0}} \right]$$

When both  $\mu > \mu_0$  and  $\epsilon > \epsilon_0$  the results are

$$\beta = \left[ \sqrt{\mu_0 \epsilon_0} + \theta/4\pi(\epsilon - \epsilon_0 + \mu - \mu_0) \right] \omega/c$$

$$Z_L = \frac{\ln(b/a)}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \left[ \frac{\sqrt{\mu_0 \epsilon_0} + \theta/2\pi(\mu - \mu_0)}{\sqrt{\mu_0 \epsilon_0} + \theta/4\pi(\epsilon - \epsilon_0 + \mu - \mu_0)} \right]$$

$$Z_w = \sqrt{\frac{\mu_0}{\epsilon_0}} \left[ \frac{\theta/2\pi(\mu - \mu_0) + \sqrt{\mu_0 \epsilon_0}}{\theta/2\pi(\epsilon - \epsilon_0) + \sqrt{\mu_0 \epsilon_0}} \right]^{1/2}$$

It can be seen that the impedances and propagation constants have the correct limiting value when  $\epsilon = \epsilon_0$  and  $\mu = \mu_0$ .

In order to determine L and M in equations (21) for a general problem one must know the potential and its gradient at every point where the field exists. When a functional form for  $\phi$  is unknown but  $\phi$  and  $\text{grad } \phi$  may be determined at certain points, e.g. along the boundaries, it is convenient to have an alternate expression available where the potential need not be considered everywhere. Such an expression can be found using Green's first identity.

The energy of a charge distribution is given by

$$U = \frac{1}{2} \int_V \rho(\underline{r}') \phi(\underline{r}') dV' \quad (24)$$

In a dielectric  $\text{div } \underline{E} = 4\pi(\rho + \rho')$  where primed quantities refer to bound charge. Alternately this relation may be written

$$\nabla^2 \phi = -4\pi(\rho + \rho') \quad (25)$$

Combined with Green's identity

$$\int [\phi \nabla^2 \phi + (\nabla \phi)^2] dV = \oint \phi \nabla \phi \cdot d\underline{s}$$

equations (24) and (25) give the result

$$\frac{1}{8\pi} \int_V \nabla_t \phi \cdot \nabla_t \phi dV' = \frac{1}{8\pi} \oint_S \phi \text{grad } \phi \cdot d\underline{s} + \frac{1}{2} \int_V (\rho + \rho') \phi dV'$$

When only surface charge exists this expression becomes

$$\frac{1}{8\pi} \int \nabla_t \phi \cdot \nabla_t \phi dV' = \frac{1}{8\pi} \oint_S \phi \nabla \phi \cdot d\underline{s} + \frac{1}{2} \int_S (\sigma + \sigma') \phi ds' \quad (26)$$

where  $S'$  includes all surfaces within  $S$  on which charge exists. The usefulness of equation (26) is demonstrated in the following simple example.

Consider a coaxial cable partially filled with dielectric in a cylindrically symmetric fashion as shown in Figure 11. Using Gauss' law we obtain the following expressions for the potential  $\phi$ ,

$$\phi = \frac{k V_0}{k \ln(c/b) + \ln(b/a)} \ln(c/r) \quad b \leq r \leq c$$

$$\phi = \frac{k V_0 \ln(c/b)}{k \ln(c/b) + \ln(b/a)} + \frac{V_0 \ln(b/r)}{k \ln(c/b) + \ln(b/a)} \quad a \leq r \leq b$$

At the interface of the inner conductor and dielectric the bound surface charge is

$$\sigma_1' = \frac{-(k-1)V_0}{4\pi a [k \ln(c/b) + \ln(b/a)]}$$

At the outer boundary of the dielectric

$$\sigma_2' = \frac{(k-1)V_0}{4\pi b [k \ln(c/b) + \ln(b/a)]}$$

There is no volume bound charge density. The free surface charge density on the inner conductor is

$$\sigma = \frac{k V_0}{4\pi a [k \ln(c/b) + \ln(b/a)]}$$

We now desire the energy of the field within the dielectric only. This may be obtained directly from the potential function.

$$\frac{1}{8\pi} \int E^2 dv = \frac{l}{4} \frac{V_0^2}{[k \ln(c/b) + \ln(b/a)]^2} \ln(b/a) \quad (27)$$

where  $l$  is the length of the cable. The same result is obtained by considering a closed surface just inside the outer boundary of the dielectric and using the right-hand side of (26).

$$\begin{aligned} & \frac{1}{8\pi} \oint_S \Phi \text{grad} \Phi \cdot ds + \frac{1}{2} \int_{S_1} (\sigma' + \sigma) \Phi(a) ds' \\ &= -\frac{l}{4} \frac{V_0^2}{[k \ln(c/b) + \ln(b/a)]^2} [k \ln(c/b) + \ln(b/r)] + \frac{l V_0^2}{4[k \ln(c/b) + \ln(b/a)]} \end{aligned}$$

In the limit as  $r \rightarrow b$  this expression is the same as (27). A closed surface just outside the dielectric boundary also gives the same result in the limit as  $r \rightarrow b$ . In this case however one must consider both bound charge densities.

$$\begin{aligned} \frac{1}{8\pi} \int E^2 dv &= \frac{1}{8\pi} \oint_S \Phi \text{grad} \Phi \cdot ds + \frac{1}{2} \int_{S_1} (\sigma + \sigma_1') \Phi(a) ds' \\ &+ \frac{1}{2} \int_{S_2} \sigma_2' \Phi(b) ds' \end{aligned}$$

In addition to verifying equation (26) this problem provides a useful test for the limits of the perturbation method since it can be solved exactly. The perturbation approach is essentially the same as that given in the first example. Assuming that  $\mu = \mu_0$  and  $\epsilon > \epsilon_0$  within the dielectric results of this method are

$$\beta = \left[ \sqrt{\epsilon_0 \mu_0} + \frac{\ln(b/a)}{2 \ln(c/a)} (\epsilon - \epsilon_0) \right] \omega / c$$

$$Z_L = \frac{\ln(c/a)}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \left[ \frac{\sqrt{\mu_0 \epsilon_0}}{\sqrt{\mu_0 \epsilon_0} + \frac{1}{2} \frac{\ln(b/a)}{\ln(c/a)} (\epsilon - \epsilon_0)} \right]$$

$$Z_w = \sqrt{\frac{\mu_0}{\epsilon_0}} \left[ \frac{\sqrt{\mu_0 \epsilon_0}}{\sqrt{\mu_0 \epsilon_0} + \frac{1}{2} \frac{\ln(b/a)}{\ln(c/a)} (\epsilon - \epsilon_0)} \right]$$

The exact solution was not completed although the field components were obtained. They are listed below.

$$E_r = [a_n J_1(k_r^n r) + b_n N_1(k_r^n r)] e^{i(\omega t - \beta z)}$$

$$H_\phi = \frac{\omega \epsilon_n}{c \beta} [a_n J_1(k_r^n r) + b_n N_1(k_r^n r)] e^{i(\omega t - \beta z)}$$

$$E_z = -i/\beta \left[ \frac{1}{r} (a_n J_1(k_r^n r) + b_n N_1(k_r^n r)) \right. \\ \left. + \frac{k_r^n a_n}{2} (J_0(k_r^n r) - J_2(k_r^n r)) + \frac{k_r^n b_n}{2} (N_0(k_r^n r) - N_2(k_r^n r)) \right] e^{i(\omega t - \beta z)}$$

where  $(k_r^n)^2 + \beta^2 = \omega^2/c^2 \mu \epsilon_n$ ,  $n = I, II$ . Six auxiliary equations are needed to determine the constants  $a_n$ ,  $b_n$ , and  $k_r$ . Four of the six equations are obtained from the boundary conditions. From the requirement that the normal component of  $\underline{E}$  be continuous at the boundaries, one has that  $E_z^{II} = 0$  at  $r = a$ ,  $E_z^I = 0$  at  $r = c$ , and  $E_z^I = E_z^{II}$  at  $r = b$ . The remaining boundary condition is obtained by requiring that the normal component of the electric displacement vector,  $\underline{D}$ , be continuous at  $r = b$ . The other two equations which relate the unknown constants are

$$(k_r^I)^2 + \beta^2 = \omega^2/c^2 \mu \epsilon_I$$

$$(k_r^{II})^2 + \beta^2 = \omega^2/c^2 \mu \epsilon_{II}$$

The existence of surface waves on a sheet of dielectric bounded on one side by a conductor was investigated. The resulting fields of this problem closely resemble those in microstrip. In addition, the effect

of surface waves is prominent in the high frequency range. The results are not reported here because this problem is treated in greater detail in the literature.<sup>4</sup>

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## FIGURES

1. Coupled Microstrip
2. Initial transformation -- both modes
3. Final transformations -- both modes
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between Parallel Conducting Plates
10. Partial Dielectric Filling in Coaxial Cable
11. Cylindrically Symmetric Dielectric Filling in Coaxial Cable

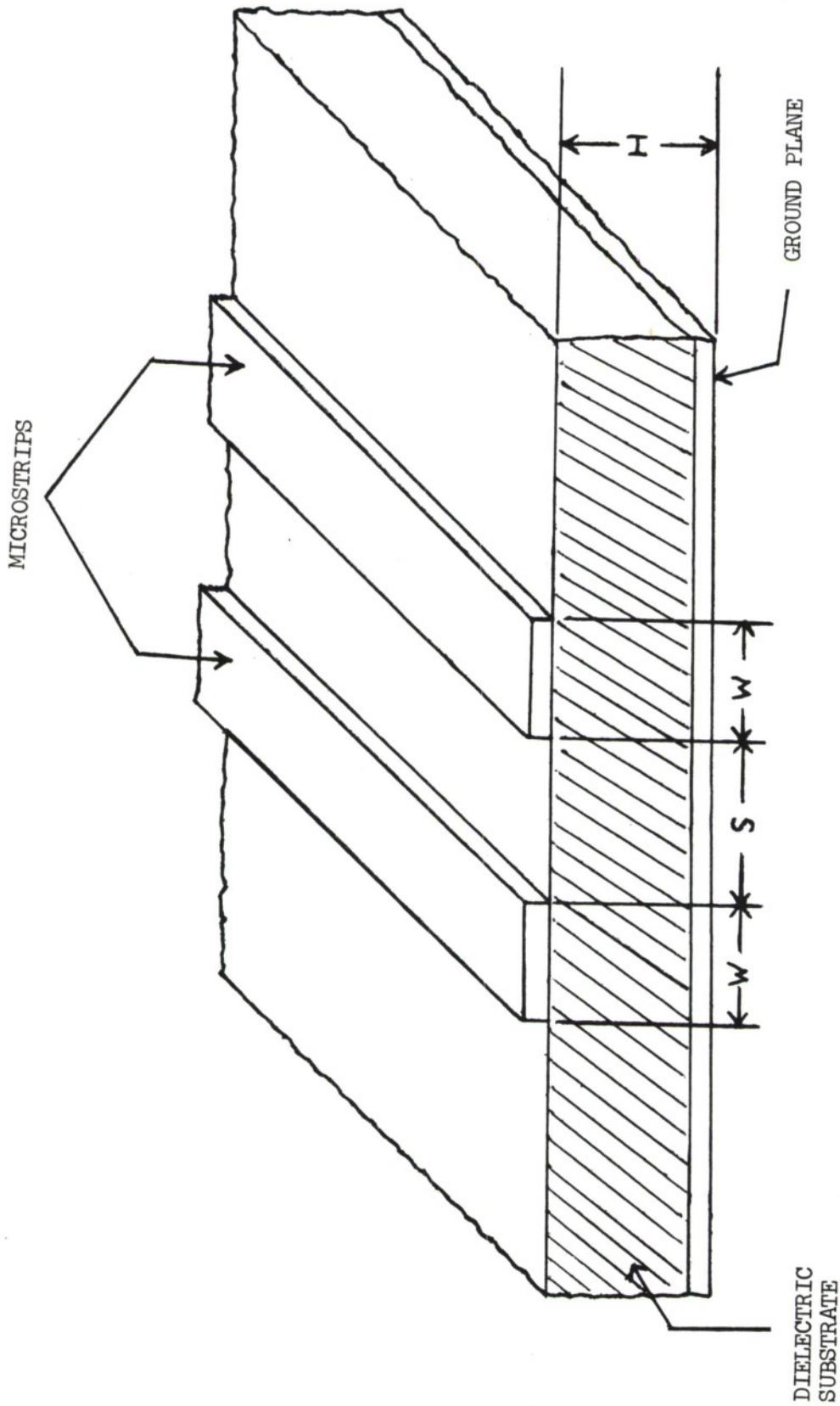


Figure 1. Coupled Microstrip

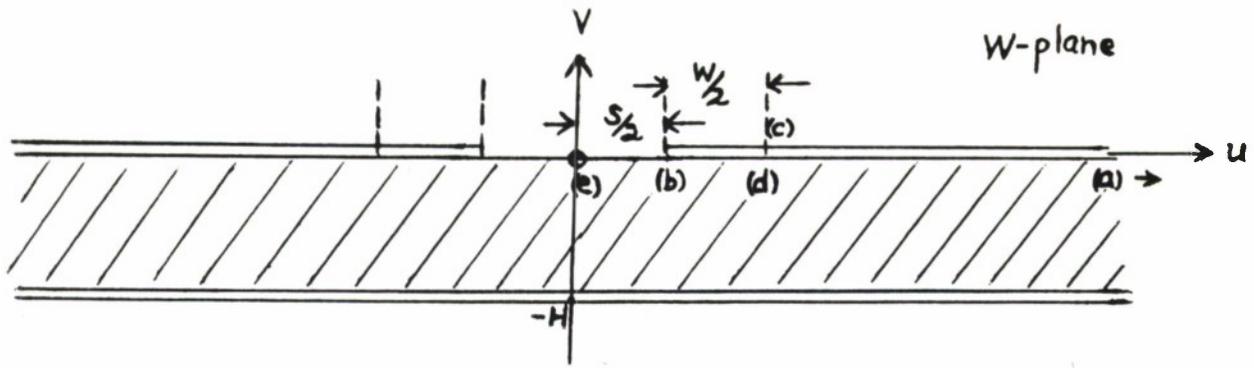


Figure 2A Representation (W-plane) of proximal half-strips as edges of semi-infinite strips

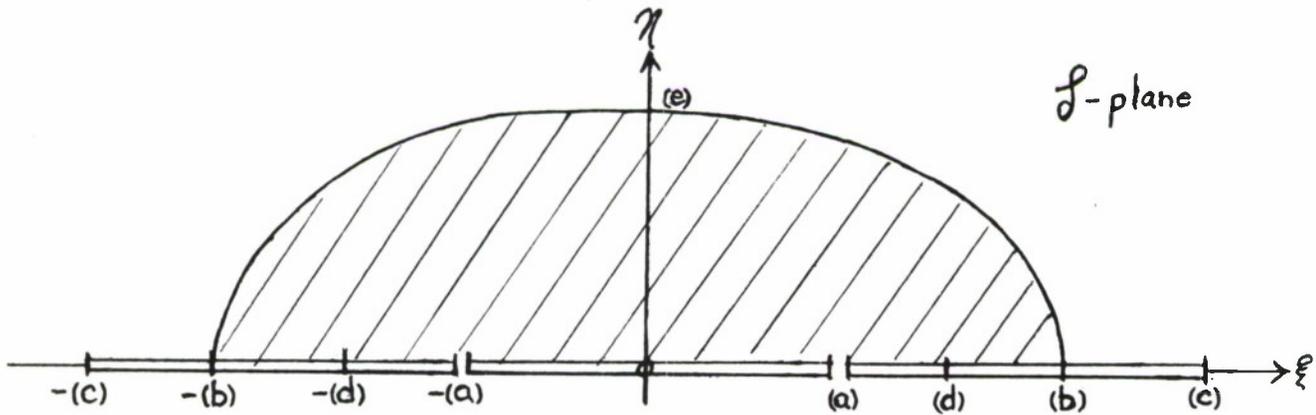


Figure 2B First transformation  $W(z)$  -- both modes

$$W(z) = \frac{2H}{\pi \left[ \left( \frac{f_b}{f_a} \right)^2 - 1 \right]} \left[ \frac{z}{f_a} + \frac{\left( \frac{f_b}{f_a} \right)^2 - 1}{2} \ln \left( \frac{z/f_a + 1}{z/f_a - 1} \right) \right]$$

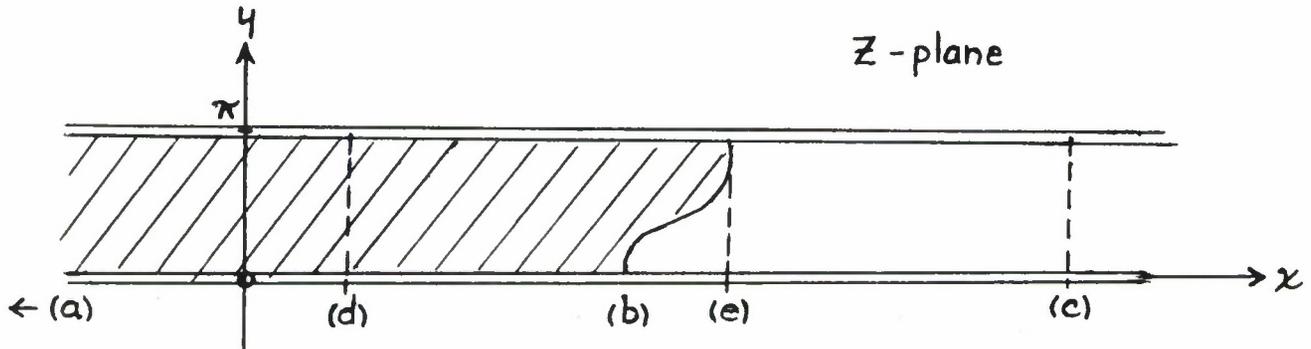


Figure 3A Second transformation  $Z(z)$  -- odd mode.

$$Z(z) = \ln (z^2 - f_a^2)$$

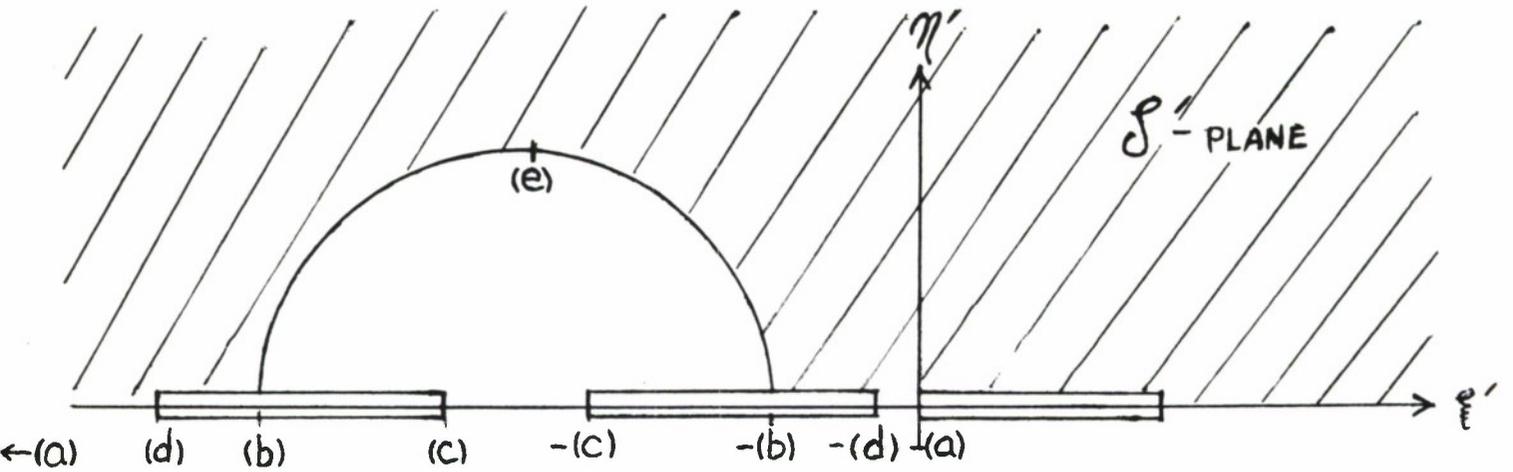


Figure 3B Second transformation  $\zeta'(\zeta)$  -- even mode.

$$\zeta'(\zeta) = \frac{\xi_a + \zeta}{\xi_a - \zeta}$$

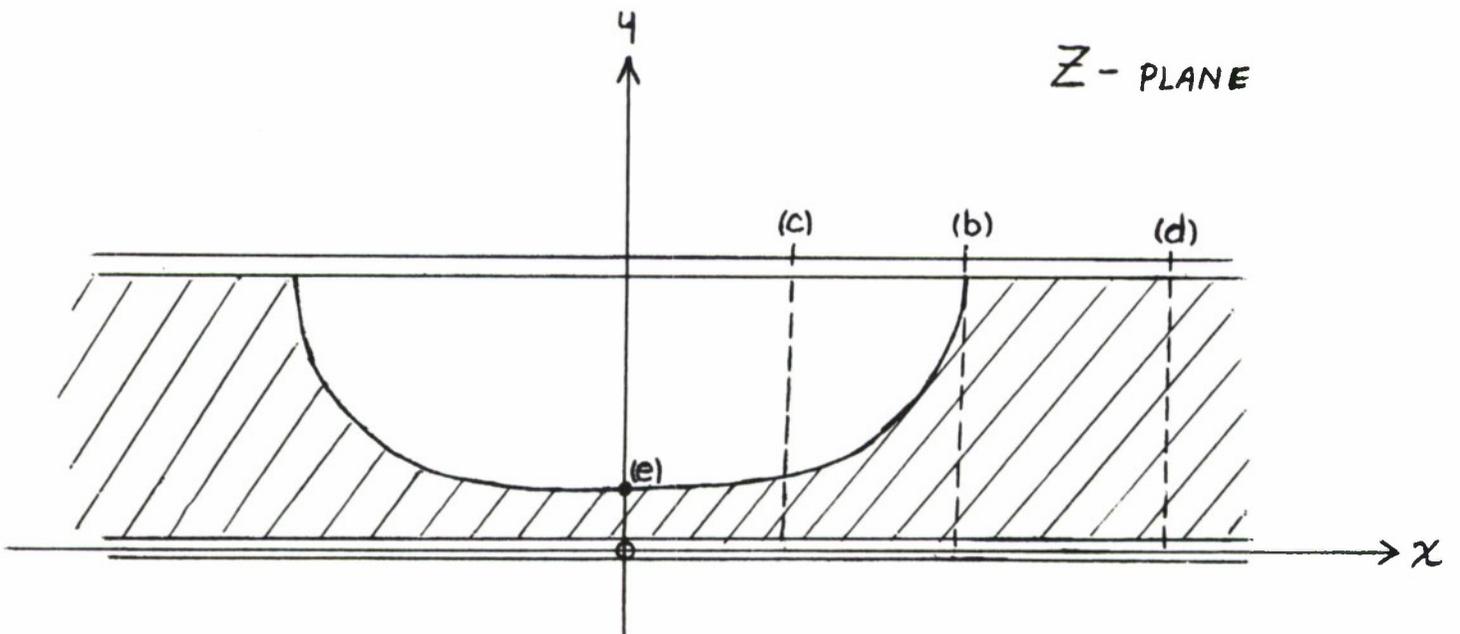
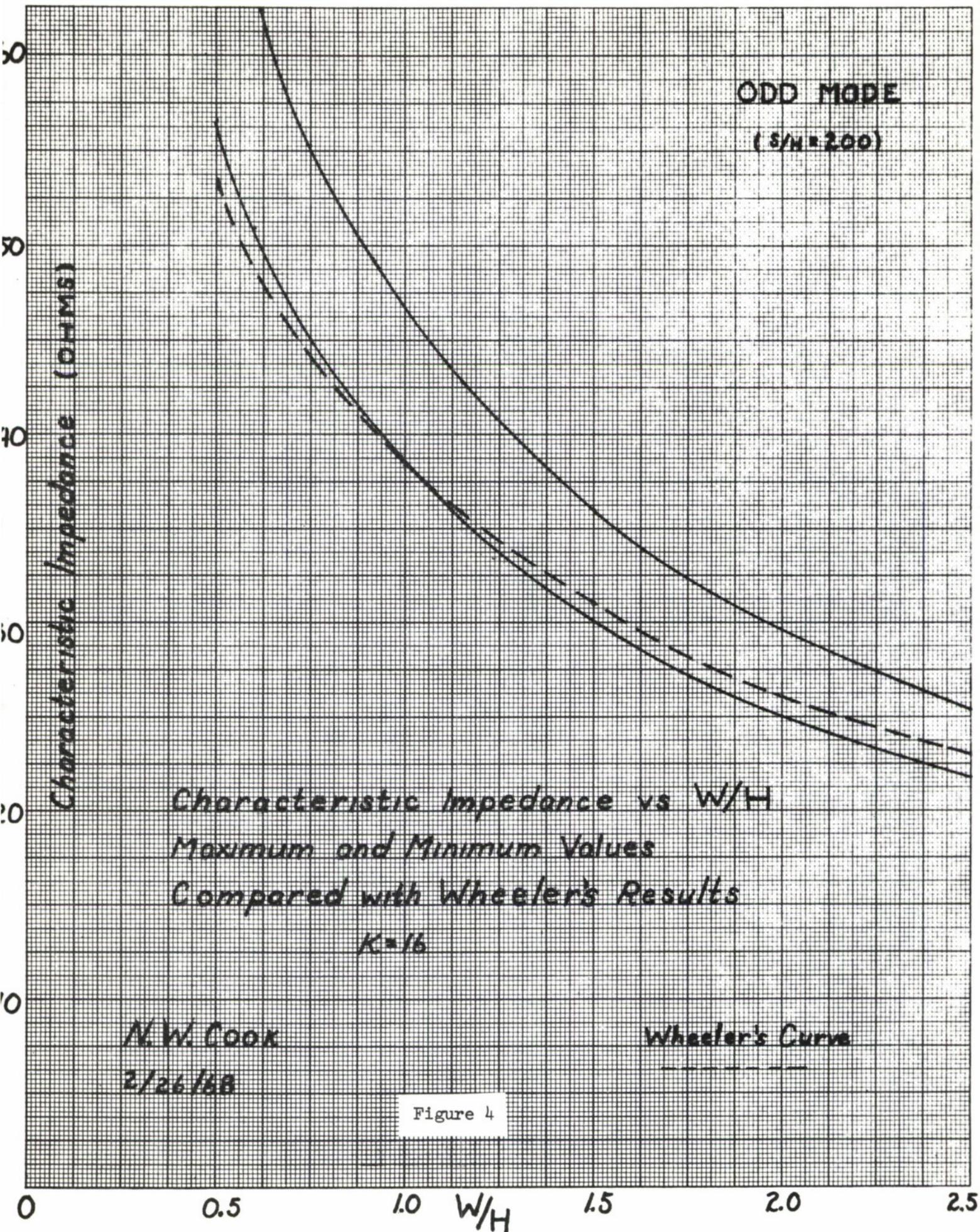
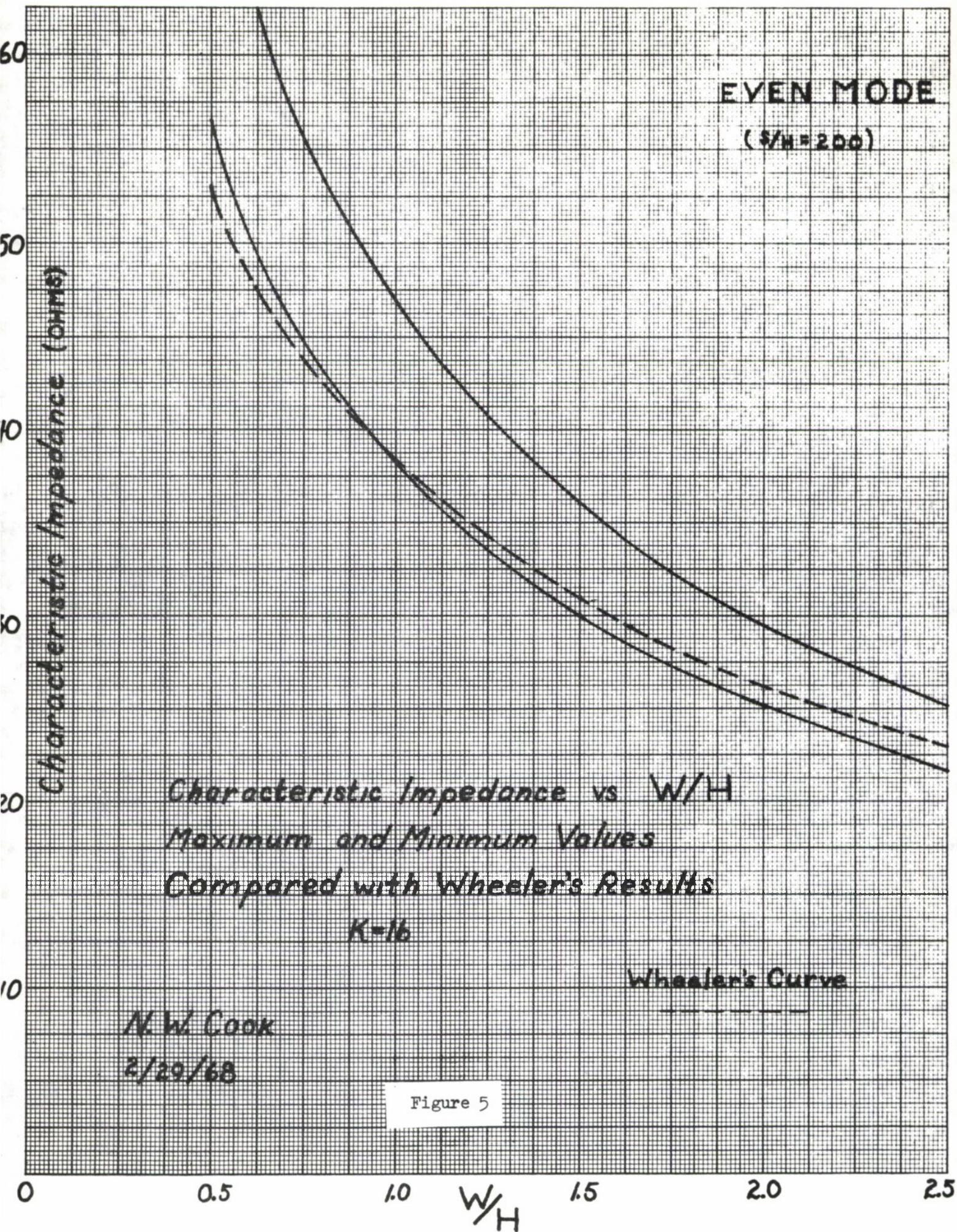


Figure 3C Third transformation  $Z(\zeta')$  -- even mode

$$Z = \ln \zeta'$$





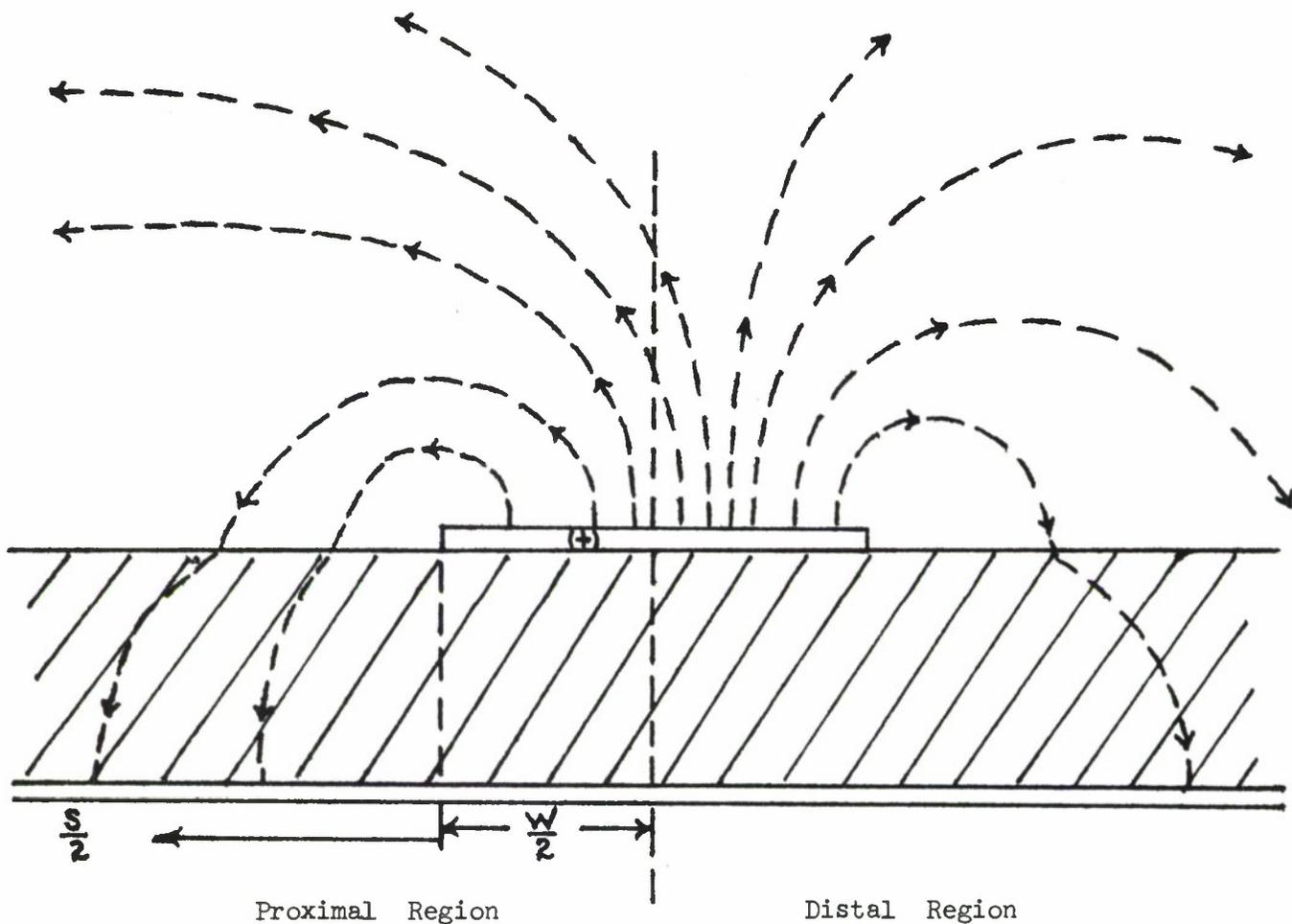


Figure 6 Actual Field of the Coupled Strip ( $S/H = 200$ )

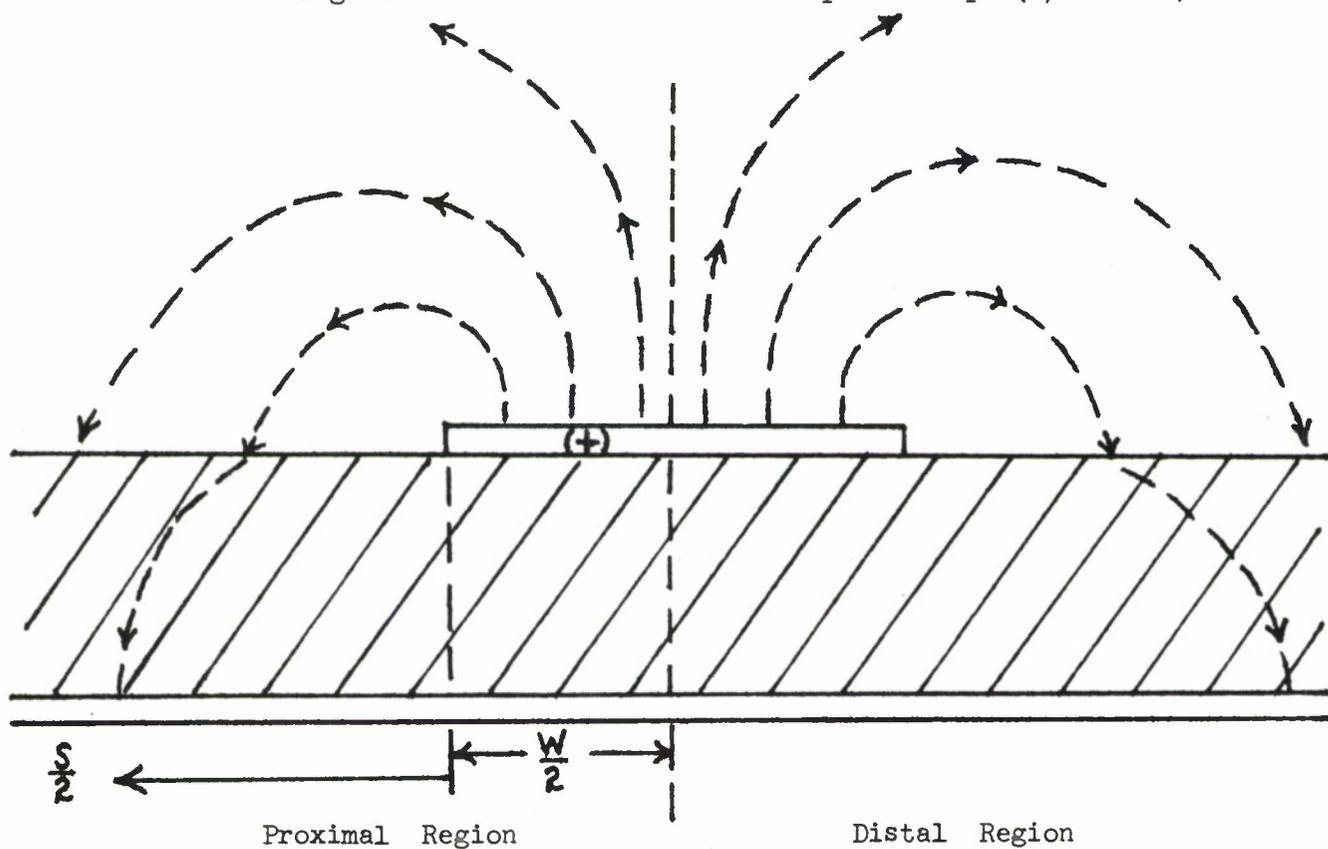


Figure 7 Assumed Field of the Coupled Strip ( $S/H = 200$ )

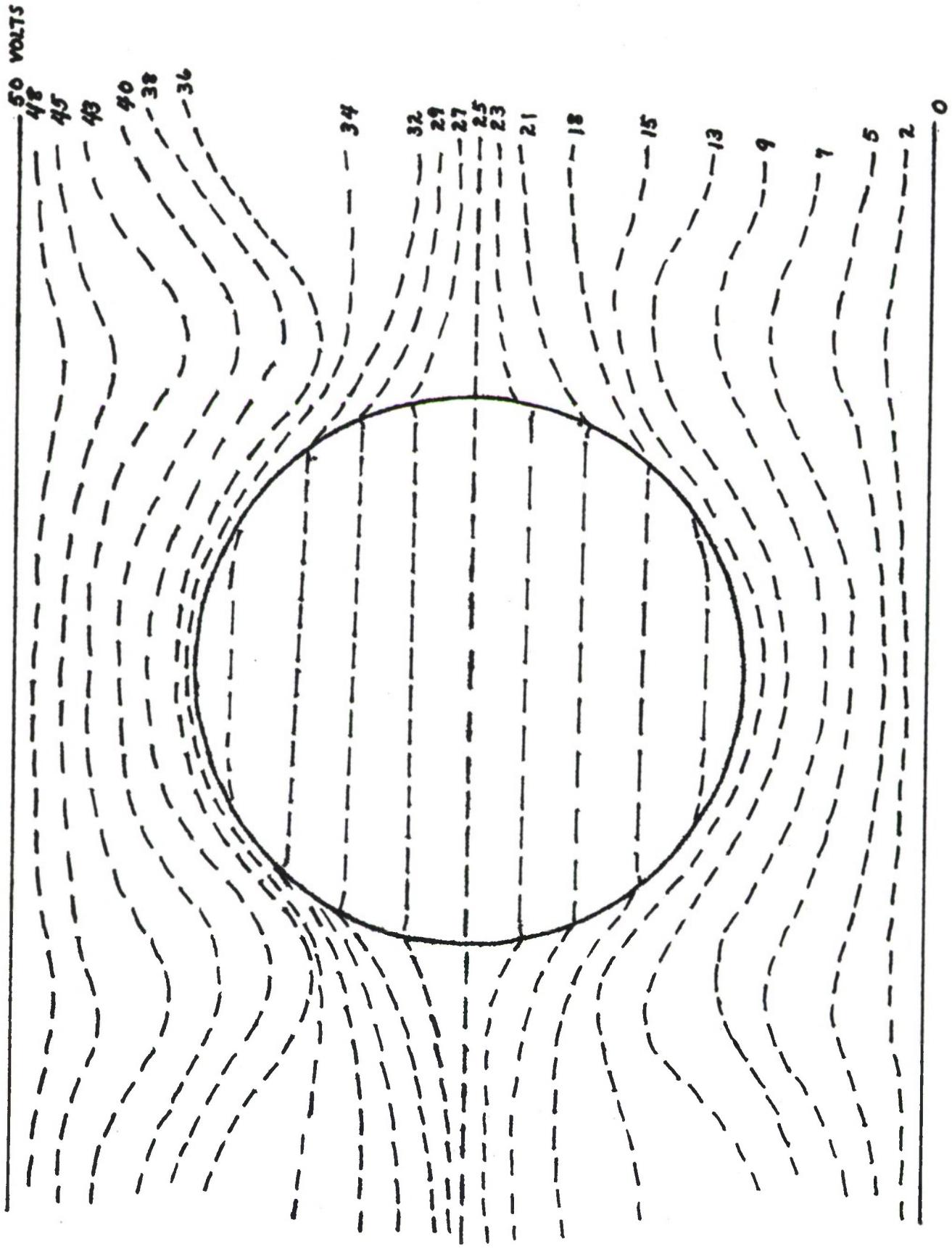


Figure 8 Equipotential Lines for Dielectric Rod ( $k = 9$ ) between Parallel Conducting Plates

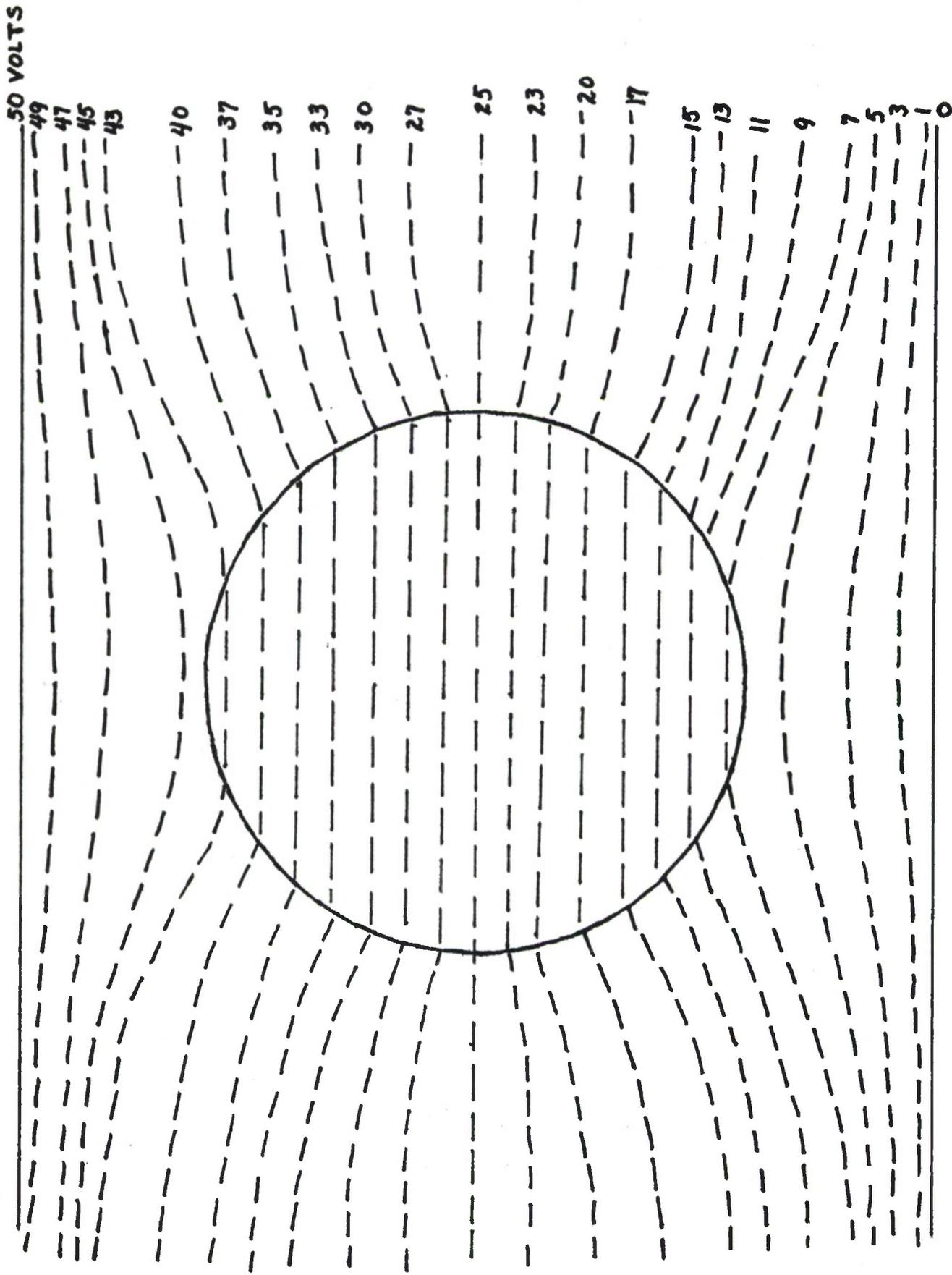


Figure 9 Equipotential Lines for Hollow Rod in Dielectric ( $k = 16$ )  
between Parallel Conducting Plates

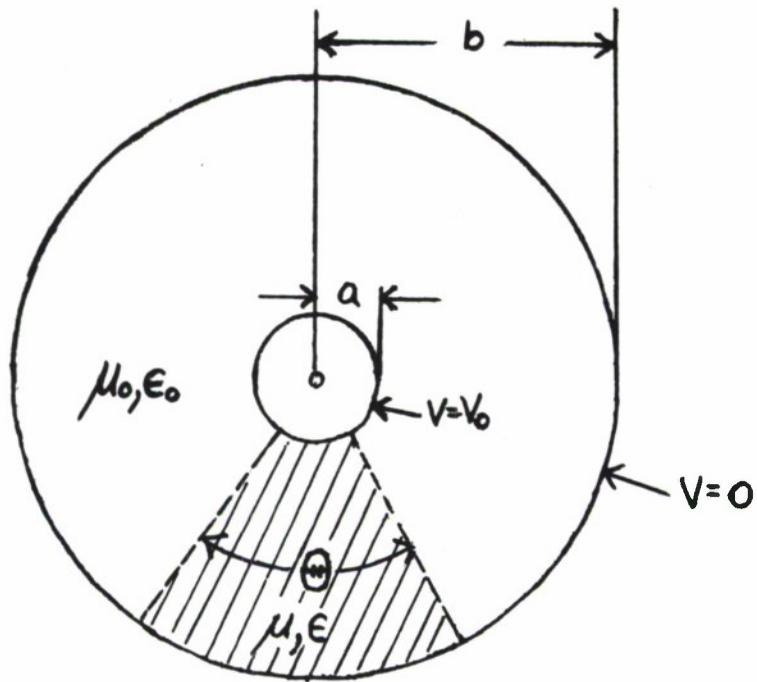


Figure 10 Partial Dielectric Filling in Coaxial Cable

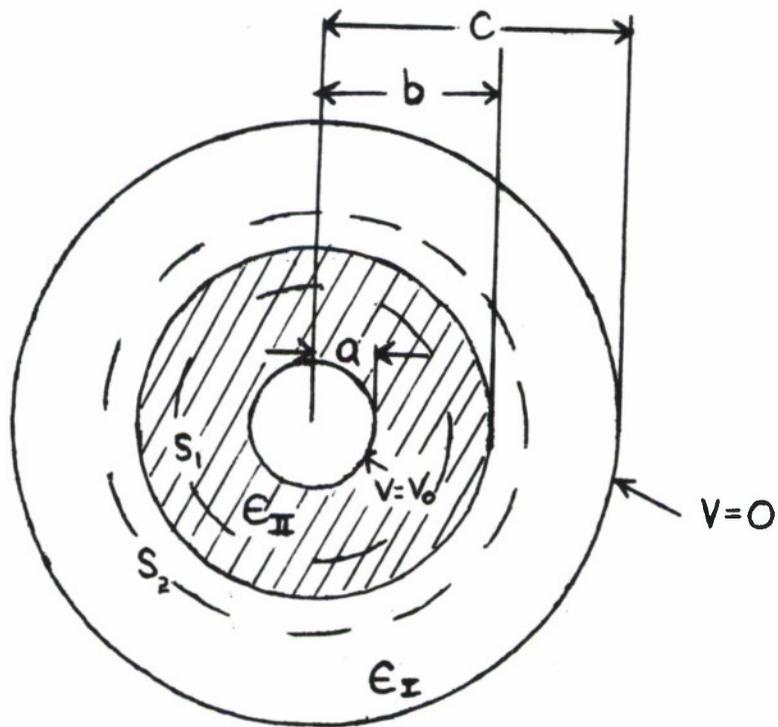


Figure 11 Cylindrically Symmetric Dielectric Filling in Coaxial Cable

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13. ABSTRACT  This report presents the results of an investigation on coupled microstrip propagation and related problems. Included is a conformal mapping approach to coupled microstrip propagation in the quasi-static limit. Numerical analysis of the treatment shows that it does not agree with known results in the limit of narrow strip width. A Green's function solution to the problem has been developed and is contained in a separate report. The method is illustrated by applying it to a related problem -- a dielectric rod situated symmetrically between two parallel conducting plates. A perturbation method is considered for the problem of non-reciprocal propagation in coupled microstrip. The method has been adapted so as to make use of numerical results from the Green's function solution of the reciprocal propagation problem. As a preliminary test of the perturbation approach, it is applied to a problem which can be solved exactly.			
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