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David F. Parker, et al

Lehigh University
Bethlehem, Pennsylvania

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by

D. F. PARKER

and

E. VARLEY

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The Interaction Of Finite Amplitude Deflection And Stretching Waves In Elastic Membranes And Strings.

By D. F. Parker

Department of Theoretical Mechanics,
The University, Nottingham.

and

E. Varley

Center for the Application of Mathematics,
Lehigh University.

SUMMARY.

Disturbances produced by the motion of a driver which is rigidly bonded to the edge of a plate are used to motivate parameter expansion techniques which, when applied to the equations of finite elasticity, generate approximating equations which describe low frequency deflection and stretching waves travelling along stretched elastic plates and rods in the limit when bending forces are negligible compared with membrane forces. The structure of the boundary layer at the driven edge and shock layers, where the low frequency or filament approximations are locally invalid, are also discussed.

The low frequency equations are used to discuss the interaction between progressing finite amplitude deflection and stretch waves in the limit when the stretch rate is small compared with the angular speed of the plate. The disturbance is locally that of a pure deflection simple wave whose amplitude and frequency are modulated by slow variations in the stretch. As the stretch increases the frequency increases while the amplitude decreases. The stretch wave is also modified by deflection of the plate: the speeds of wavelets carrying constant values of stretch are always less than their values in the pure stretch simple wave.

1. Introduction.

This paper is in two parts. In the first part we describe an expansion procedure which, when applied to the equations of finite elasticity, generates equations which govern low frequency, finite amplitude , deflection and stretching disturbances in elastic plates in the membrane limit when, to a first approximation, bending forces are negligible compared with membrane forces. The first term in the expansion gives the classical membrane, or flexible string approximation, (see Cristescu (5) and (6), and Craggs (7)). In the second part, these low frequency equations are used to discuss the interaction between finite amplitude deflection pulses and stretching waves. Detailed analysis is given for plates and is directly applicable (see Parker [10]) to stretched rods in the 'flexible string' limit when bending moments and torques are negligible.

A region is a low frequency region for a deformation if:

(I) the strain rate and angular velocity of the plate at all particles are (negligibly) small compared with the frequency defined by the local sound speed C and plate thickness d;

(II) the plate curvature and stretch gradients at all particles are small compared with d^{-1}.

In section 2 the expansion procedures are motivated by considering the disturbances which are produced when one edge of the plate is rigidly bonded to a driver which deflects and stretches the plate. It is supposed that the angular velocity and stretch rate at the edge of the plate are proportional to $\omega_D d/C$ and $\omega_S d/C$: for time periodic motions these can be taken as the frequencies defined by the variations in direction and magnitude of the applied traction. The dependent and independent variables are then non-dimensionalised in such a way that, in terms of these variables, the governing equations have formal regular asymptotic solutions which are valid as $\omega=\max\{\omega_D,\omega_S\}\to 0$. These satisfy the traction free conditions on the lateral surfaces of the plate, and describe deformations satisfying conditions I and II. The lowest order approximation

agrees with the classical filament, or long wave, approximation and is analogous to the long gravity wave approximation used in hydraulics.

The filament theory cannot satisfy the bonding condition imposed at the edge of the plate since this is inconsistent with the Poisson contraction which occurs in the low frequency region. A boundary layer (called the first diameter by experimentalists (see Bell [1]) occurs in which conditions II are violated. In section 3.2 we introduce boundary layer variables and show how the low frequency expansion must be modified. To a first order inertia effects are negligible and the applied tractions and particle velocities are continuous across the layer.

If the plate is not deflected the low frequency approximation neglects the lateral inertia of the plate compared with its longitudinal inertia. In section 3.1 we use the well known pure stretching simple wave solutions to show how the strain rate and lateral particle speeds can grow until the low frequency approximation becomes invalid. A 'shock layer' forms in which both sets of conditions I and II are violated: in this layer the lateral and longitudinal inertias of the plate are comparable.

In section 3.3 we exhibit solutions of the filament equations which describe pure deflection simple waves; these deflect but do not stretch the plate. Such progressing, non-distorting waves cannot form shocks. The second half of the paper, section 4, emphasises the role played by these waves when the stretch rate is small compared with the angular speed of the plate. Such deformations occur when $\varepsilon=\omega_S/\omega_D \ll 1$.

The interaction between a deflection and stretch wave is illustrated in section 4.1, where discontinuity analysis is used to determine the variation in the angular velocity of the plate at a front which separates the interaction region from a pure stretch precursor wave. If the stretching wave extends (contracts) the plate as it passes the front, the angular speed increases (decreases).

In particular, when the plate is completely unloaded the angular speed tends to zero, although the curvature of the plate increases without bound until conditions II are violated and bending forces become important.

In sections 4.2 and 4.3, motivated by the results of section 4.1 and previous work on pulse propagation by Varley and Cumberbatch (13) and (14), and Varley and Rogers (15), we present expansion techniques, which are valid as $\epsilon \to 0$, to describe conditions in an interaction pulse where the plate is being deflected much more rapidly than it is being stretched. No restriction is placed on the amplitude of the disturbance, and the 'shapes' of the stretch and deflection profiles are quite arbitrary. Even though the expansions only describe conditions at a particle over a limited time, no restriction is placed on the distance the pulse propagates and the modulation of high frequency deflection pulses by slowly varying stretching waves is described in detail. The deformation, at any time, is locally that of a deflection simple wave with amplitude and frequency modulated by the slow variations in stretch. Its amplitude increases as the local stretch decreases and becomes unbounded when the tension is completely released. The stretching wave is itself affected by the mean values of the deflecting disturbance. Wavelets carrying constant values of stretch are always slowed relative to their speeds in the pure stretch wave.

Finally, in section 5, we discuss expansion techniques for deformations in 'shock layers' in which both sets of conditions I and II are violated. In general the speeds of these layers are not those of thermodynamic shocks: dissipation is negligible and the deformation is isentropic. The effect of a shock layer on the low frequency deformation can be obtained by treating the layer as a shock discontinuity across which conservation laws can be applied to derive jump conditions. These laws are identical to those defined by writing the filament equations in appropriate divergence form. If a shock layer separates two low frequency regions with identical deformations at the edge of the layer, conditions in the layer are identical to those in a solitary wave.

2. General formulation.

Our aim is to construct a model which describes the main features of the deformations produced in stretched elastic plates and rods by 'low frequency' deflecting and stretching waves which are controlled by the presence of traction free boundaries. Detailed analysis is presented for plates, and the corresponding results for rods quoted.

The deformation is described in the usual way by explicit relations

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},\tau) \tag{2.1}$$

between the cartesian co-ordinates $\underset{\sim}{x}$ of a particle at time τ and its co-ordinates $\underset{\sim}{X}$ in some reference configuration R where the plate occupies the region $X_1 \geq 0$, $-d \leq X_2 \leq d$, and $-\infty < X_3 < \infty$, and is in equilibrium under uniform hydrostatic pressure*, uniform temperature, and uniform density ρ. We consider adiabatic deformations, which occur when the power generated by the heat flux or other energy sources is negligible compared with the rate at which stresses do work, and restrict our attention to elastic materials which are isotropic and homogeneous with respect to their states in R. For such a material the internal energy is a function of the entropy S and any three independent invariants of the Cauchy-Green strain tensor

$$G = \underset{\sim}{p}^T \underset{\sim}{p}, \tag{2.2}$$

where $\underset{\sim}{p}$, the deformation gradient tensor, has components

$$p_{ij} = \partial x_i / \partial X_j. \tag{2.3}$$

We consider plane-strain deformations for which

$$x_1 = x_1(X_1,X_2,\tau), \quad x_2 = x_2(X_1,X_2,\tau) \text{ and } x_3 = X_3, \tag{2.4}$$

* This is taken as the ambient pressure. In what follows all stresses are measured relative to this hydrostatic stress.

which for $\tau > 0$ are isentropic with $S = S_0$. As the three invariants of $\underset{\sim}{G}$ we take

$$I = \tfrac{1}{2}(\mathrm{tr}.\underset{\sim}{G}-1),\ J = (\det.\underset{\sim}{G})^{\frac{1}{2}},\ \text{and}\ K = 2I+J^2-\tfrac{1}{2}[(\mathrm{tr}.\underset{\sim}{G})^2-\mathrm{tr}.\underset{\sim}{G}^2]; \quad (2.5)$$

in plane strain deformations $K \equiv 0$.

The elastic material is specified by its internal energy $\sigma E(I,J,K:S)$, where σ is a characteristic constant of the material with dimensions of stress, (e.g. Young's modulus at ambient strain), and E is non-dimensional. If we define

$$W(I,J,S_0) = E(I,J,0:S_0) \quad (2.6)$$

then, for plane strain isentropic deformations, (see Green and Adkins ($\underset{\sim}{8}$), p.104), the two dimensional Piola-Kirchoff stress tensor, non-dimensionalised by σ, is

$$\underset{\sim}{T} = (T_{ij}) = (\partial W/\partial p_{ij}) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \partial W/\partial I + \begin{pmatrix} p_{22} & -p_{21} \\ -p_{12} & p_{11} \end{pmatrix} \partial W/\partial J, \quad (2.7)$$

where henceforth the subscripts take the values 1,2.

It is convenient to work with a system of first order equations for $\underset{\sim}{T}$, $\underset{\sim}{p}$ and the particle velocities

$$u_i = \partial x_i/\partial \tau. \quad (2.8)$$

Consequently, to equation (2.7), and the Euler equations

$$\partial T_{ij}/\partial X_j = C^{-2}\ \partial u_i/\partial \tau, \quad (2.9)$$

where $C = \sqrt{\sigma/\rho}$ is a characteristic speed defined by the material, we add the compatibility conditions

$$\partial p_{ij}/\partial\tau = \partial u_i/\partial X_j, \text{ and } \partial p_{ij}/\partial X_k = \partial p_{ik}/\partial X_j \qquad (2.10)$$

which, together with suitable initial conditions, imply (2.3) and (2.8). During the deformation the surfaces of the plate $X_2=\pm d$ are traction free so that (2.7), (2.9) and (2.10) are to be solved subject to the conditions

$$T_{22} = T_{12} = 0, \quad \text{on } X_2 = \pm d. \qquad (2.11)$$

The disturbance in the plate is produced by the forced motion of its edge $X_1=0$. For definiteness we take this edge to be rigidly bonded to a moving plate so that

$$p_{22}-1 = p_{12} = 0, \quad \text{on } X_1 = 0. \qquad (2.12)$$

This driver transmits a total force per unit breadth to the plate with varying magnitude $\tilde{F}(\tau)$ and direction $\tilde{\theta}(\tau)$. The variation of $\tilde{F}(\tau)$ stretches the plate, and the variation in $\tilde{\theta}(\tau)$ deflects it. We suppose that the time variations of $\tilde{F}$ and $\tilde{\theta}$ introduce two characteristic frequencies $\tilde{\omega}_S$ and $\tilde{\omega}_D$ so that

$$\tilde{F}'(\tau) = 2d\sigma\tilde{\omega}_S F'(\tilde{\omega}_S\tau), \quad \text{and} \quad \tilde{\theta}'(\tau) = \tilde{\omega}_D\theta'(\tilde{\omega}_D\tau), \qquad (2.13)$$

where F and θ are bounded functions of their arguments with piecewise continuous bounded derivatives. In the special case of time periodic variations in applied force $\tilde{\omega}_S$ and $\tilde{\omega}_D$ can be taken as the frequencies defined by the variations of $\tilde{F}$ and $\tilde{\theta}$.

In order to make the statement 'low frequency wave' precise we work with non-dimensional independent variables

$$t = \tilde{\omega}\tau, \quad X = \tilde{\omega}X/C, \quad \text{and} \quad Y = X_2/d \qquad (2.14)$$

where $\tilde{\omega}=\max.\{\tilde{\omega}_S,\tilde{\omega}_D\}$. As non-dimensional dependent variables we take $\underset{\sim}{T}$, $\underset{\sim}{p}$ and

$$(v_1, v_2) = C^{-1}(u_1, u_2). \tag{2.15}$$

In terms of these variables conditions (2.7) are unchanged; (2.9) are replaced by

$$\partial T_{12}/\partial Y = \omega\left(\partial V_1/\partial t - \partial T_{11}/\partial X\right) \tag{2.16}$$

and

$$\partial T_{22}/\partial Y = \omega\left(\partial V_2/\partial t - \partial T_{21}/\partial X\right), \tag{2.17}$$

where

$$\omega = \tilde{\omega}d/C, \tag{2.18}$$

and (2.10) are replaced by

$$\partial p_{11}/\partial t - \partial v_1/\partial X = \partial p_{21}/\partial t - \partial v_2/\partial X = 0, \tag{2.19}$$

and

$$\partial p_{11}/\partial Y = \omega\,\partial p_{12}/\partial X, \quad \partial p_{21}/\partial Y = \omega\,\partial p_{22}/\partial X. \tag{2.20}$$

The parameter ω occurring in (2.16)-(2.20) is the ratio of the maximum frequency of the traction imposed at the edge of the plate to the characteristic frequency C/d defined by the material and the stress free boundaries. Alternatively, it is the ratio of plate thickness to a typical wavelength of the disturbance.

Here we are interested in low frequency, long wave, disturbances for which $\omega \ll 1$. More precisely, a region is a low frequency region if as $\omega \to 0$ the functions $\underset{\sim}{T}(X,Y,t:\omega)$, $\underset{\sim}{p}(X,Y,t:\omega)$ and $\underset{\sim}{v}(X,Y,t:\omega)$ have limiting values, satisfying the stress free conditions (2.11). These functions have bounded derivatives which need not be identically zero.

If $\overset{\circ}{f}$ denotes the limiting value of any variable f as $\omega \to o$ then (2.11), (2.16) and (2.17) imply that

$$\overset{\circ}{T}_{12} \equiv \overset{\circ}{T}_{22} \equiv 0, \tag{2.21}$$

so that, by (2.7), the components of $\overset{\circ}{\underset{\sim}{p}}$ are implicitly related by the conditions

$$\overset{\circ}{p}_{11}\overset{\circ}{p}_{12} + \overset{\circ}{p}_{22}\overset{\circ}{p}_{21} = 0 \tag{2.22}$$

and

$$\overset{\circ}{p}_{22}\,\partial W/\partial I + \overset{\circ}{p}_{11}\,\partial W/\partial J = 0. \tag{2.23}$$

(2.19), (2.20), (2.22) and (2.23) then imply that $\partial\overset{\circ}{\underset{\sim}{p}}/\partial Y = 0$ so that

$$\overset{\circ}{\underset{\sim}{p}} = \overset{\circ}{\underset{\sim}{p}}(X,t). \tag{2.24}$$

Conditions (2.22) and (2.23) imply that in the limit of low frequency disturbances the families of fibres X = constant, Y = constant, which are mutually orthogonal in R, remain orthogonal and in directions of principal stretch. Moreover, like the free surfaces $Y = \pm 1$, any surface Y = constant is stress free. If (λ,μ) denote the limiting principal stretches and $(\overset{\circ}{T}_1,\overset{\circ}{T}_2)$ the principal Piola-Kirchoff stresses then, in terms of

$$G(\lambda_1,\lambda_2:S_0) = W(\tfrac{1}{2}(\lambda_1^2 + \lambda_2^2),\lambda_1\lambda_2:S_0), \tag{2.25}$$

λ and μ are implicitly related by the condition

$$\overset{\circ}{T}_2 = \frac{\partial G}{\partial \lambda_2}(\lambda,\mu:S_0) = 0, \tag{2.26}$$

while

$$\overset{\circ}{T}_1 = \frac{\partial G}{\partial \lambda_1}(\lambda,\mu:S_0), = T(\lambda) \text{ say}, \tag{2.27}$$

where the dependence of T on S_0 is not stated. Relation (2.27), with $\mu(\lambda)$ given by (2.26), is exactly the relation between force per unit undeformed area (engineering stress) and extension for simple static stretching of the plate with ambient temperature Θ adjusted so that

$$\Theta = -\sigma\frac{\partial W}{\partial S}(I,J,0;S_0). \tag{2.28}$$

3.1 Asymptotic expansion for the low frequency region.

In the low frequency region $\underset{\sim}{p}$, $\underset{\sim}{T}$ and $\underset{\sim}{v}$ admit a formal asymptotic expansion

$$\underset{\sim}{p} = \overset{o}{\underset{\sim}{p}}(X,t) + \sum_{n=1}^{N} \omega^n \overset{n}{\underset{\sim}{p}}(X,Y,t),$$

$$\underset{\sim}{T} = \overset{o}{\underset{\sim}{T}}(X,t) + \sum_{n=1}^{N} \omega^n \overset{n}{\underset{\sim}{T}}(X,Y,t), \qquad (3.1)$$

and

$$\underset{\sim}{v} = \overset{o}{\underset{\sim}{v}}(X,t) + \sum_{n=1}^{N} \omega^n \overset{n}{\underset{\sim}{v}}(X,Y,t).$$

When (3.1) is inserted, (2.16)-(2.20) imply that

$$\partial \overset{o}{v}_1/\partial X - \partial \overset{o}{p}_{11}/\partial t = \partial \overset{o}{v}_2/\partial X - \partial \overset{o}{p}_{21}/\partial t = 0, \qquad (3.2)$$

and

$$\partial \overset{1}{T}_{12}/\partial Y = \partial \overset{o}{v}_1/\partial t - \partial \overset{o}{T}_{11}/\partial X$$

$$\partial \overset{1}{T}_{22}/\partial Y = \partial \overset{o}{v}_2/\partial t - \partial \overset{o}{T}_{21}/\partial X. \qquad (3.3)$$

Since the right-hand sides of (3.3) are independent of Y, and since $\overset{1}{T}_{12} \equiv \overset{1}{T}_{22} \equiv 0$ on $Y = \pm 1$,

$$\overset{1}{T}_{12} \equiv \overset{1}{T}_{22} \equiv 0, \qquad (3.4)$$

and

$$\partial \overset{o}{v}_1/\partial t - \partial \overset{o}{T}_{11}/\partial X = \partial \overset{o}{v}_2/\partial t - \overset{o}{T}_{21}/\partial X = 0. \qquad (3.5)$$

Equations (3.2) and (3.5), together with (2.7), (2.22) and (2.23), provide a complete set of equations for the zeroth order variables. They are the low frequency membrane equations which are usually derived by postulating (2.21) and assuming $\underset{\sim}{p}$ is independent of Y, (see Craggs (7)). These conditions are more conveniently written if we introduce as basic dependent variables the angle ψ between the fibres Y = constant and the x_1-axis, and (v,w) the components of particle velocity along and normal to these fibres. In terms of these variables, the stretch λ, and the tension $T(\lambda)$,

$$\begin{aligned} T'(\lambda)\partial\lambda/\partial X &= \partial v/\partial t - w\partial\psi/\partial t, \\ \partial\lambda/\partial t &= \partial v/\partial X - w\partial\psi/\partial X, \end{aligned} \qquad (3.7)$$

and

$$\begin{aligned} T(\lambda)\partial\psi/\partial X &= \partial w/\partial t + v\partial\psi/\partial t, \\ \lambda\partial\psi/\partial t &= \partial w/\partial X + v\partial\psi/\partial X. \end{aligned} \qquad (3.8)$$

Equations (3.7) and (3.8) are the equations governing the plane vibrations of an extensible string with $T(\lambda)$ representing the tension in the string. They also govern the one dimensional motion produced when an infinite half-space of <u>incompressible</u> elastic material is sheared at X=0 by a force which varies in strength and direction with time (see Chu (2), and Collins (3)): then λ represents the shear strain, $T(\lambda)$ the shear stress, ψ the direction of shear, and (v,w) the components of velocity parallel and normal to the shear direction.

We restrict attention to situations when the plate is in tension $(T>0)$'; when $T<0$ bending forces are important. When $T>0$ equations (3.7) and (3.8) are totally hyperbolic with characteristic Lagrangian speeds

$$dX/_{dt} = \pm\ [T'(\lambda)]^{\frac{1}{2}}, = \pm\ C_S(\lambda) \text{ say}, \tag{3.9}$$

and

$$dX/_{dt} = \pm\ [T(\lambda)/_{\lambda}]^{\frac{1}{2}}, = \pm\ C_D(\lambda) \text{ say}. \tag{3.10}$$

If equations (3.7) and (3.8) are formally linearized about $\lambda=\lambda_0$, and $v=w=\psi=0$, the pair (3.7), which describes the stretching of the plate, uncouple from the pair (3.8), which describes its deflection. (λ,v) and (ψ,w) then satisfy the linear wave equations with constant Lagrangian speeds $C_S(\lambda_0)$ and $C_D(\lambda_0)$. In particular, the linear theory predicts that any disturbance entering a uniform region which is prestrained by a tension $T(\lambda_0)$ is generated by two progressing, undistorted, noninteracting waves: a longitudinal wave composed of wavelets, each moving with the same Lagrangian speed $C_S(\lambda_0)$, at which (λ,v) are constant, and a transverse wave with wavelets moving with the same Lagrangian speed $C_D(\lambda_0)$ at which (ψ,w) are constant. In terms of the driving conditions (2.13)

$$T = T(\lambda_0) + F(\omega_S[t - C_S^{-1}X]), \text{ and } \psi = \theta(\omega_D[t - C_D^{-1}X]), \tag{3.11}$$

where

$$\omega_S = \tilde{\omega}_S/\tilde{\omega}, \text{ and } \omega_D = \tilde{\omega}_D/\tilde{\omega}. \tag{3.12}$$

The only known exact solutions to the non-linear equations (3.7) and (3.8) which describe disturbances moving into an undisturbed region are the simple wave solutions. These are of two dinstinct

types. Stretching (longitudinal) simple waves for which (λ, v) vary while (ψ, w) are constant, so the plate is not deflected; and deflection (flexural) simple waves for which (ψ, w) and v vary while λ is constant, so the plate is not stretched. We might suspect that these also provide a good 'local' description of deformations for which $\tilde{\omega}_S/\tilde{\omega}_D >> 1$ and $\tilde{\omega}_S/\tilde{\omega}_D << 1$. In sections 3.1 and 3.3 we describe these simple waves and in section 4 compare these with asymptotic solutions to (3.7) and (3.8) as $\omega_S/\omega_D \to 0$.

3.2 Stretching simple wave.

The stretching simple wave is governed by the non-linear equations

$$T'(\lambda)\partial\lambda/\partial X - \partial v/\partial t = \partial\lambda/\partial t - \partial v/\partial X = 0, \tag{3.13}$$

which are discussed in detail by Courant and Friedrichs (4). For such waves (λ, v) are constant on wavelets which travel with constant but distinct Lagrangian speeds. If $\beta(X,t)$ = constant is the wavelet which left $X = 0$ at $t = \beta(>0)$ then

$$\lambda = \Lambda(\beta) \tag{3.14}$$

is arbitrary;

$$X = U_S(\beta)(t-\beta), \tag{3.15}$$

where

$$U_S(\beta) = C_S(\Lambda), \tag{3.16}$$

and

$$v = v(\lambda) = -\int^{\lambda} C_S(r)dr, = V(\beta) \text{ say.} \tag{3.17}$$

The roles of these simple waves are well understood. Any disturbance produced by a driver motion for which $\theta' \equiv 0$ is generated by a stretching simple wave with $\Lambda(\beta)$ determined from the condition

$$T(\lambda) = T(\lambda_0) + F(\omega_S\beta). \tag{3.18}$$

If $\theta' \equiv 0$, but only strains satisfying

$$T'(\lambda) > T/_{\lambda}, \quad \text{so } C_S(\lambda) > C_D(\lambda), \tag{3.19}$$

are produced, then, at any time, the disturbed region can be separated into two distinct regions: a region neighbouring $X = 0$ in which ψ, λ, v and w may all vary, and preceding this a simple wave region in which (ψ, w) are constant while (λ, v) vary according to (3.14)-(3.17) for some $\Lambda(\beta)$.

In the simple wave region the displacement field is given by

$$\omega x_1/d = \Lambda(\beta)X + V(\beta)t + \int^{\beta} r\Lambda'(r)U_S(r)dr, \quad \text{and } x_2/d = \mu(\Lambda)\, Y \tag{3.20}$$

where $\beta(X,t)$ is given implicitly by (3.15). In particular the equations of the free surfaces $Y = \pm 1$ are given by

$$x_2/d = \pm\, \mu(\Lambda). \tag{3.21}$$

Conditions (3.20) imply that to a first order in the low frequency region

$$\frac{v_2}{v_1} = \frac{\mu'(\lambda)}{v(\lambda)} Y \omega \frac{\partial \lambda}{\partial t}$$

so that in regions where $\partial\lambda/\partial t$ is bounded $v_2/v_1 = 0\ (\omega)$ as $\omega \to 0$. However, as is well known, (3.14) and (3.15) predict that either all compressive or all expansive wavelets begin to coalesce and $\partial\lambda/\partial t$ may become unbounded. In section 5 we show that wavelets are in fact prevented from coalescing by the formation of a 'shock layer' in which the low frequency expansion (3.1) is not valid. In this layer $\partial\lambda/\partial t = O(1)$ as $\omega \to 0$ and the lateral and longitudinal inertias of the plate are of the same order.

In section 4 we discuss how a 'rapid' long wavelength deflection disturbance modifies a stretching simple wave of much lower frequency. Such disturbances are produced at X=0 when F(t) varies slowly compared with $\theta(t)$. The front separating this disturbance and the pure stretch simple wave is necessarily a characteristic wavelet moving with Lagrangian speed

$$C_D(\lambda), = U_D(\beta) \text{ say.} \tag{3.23}$$

If $\bar{\alpha}(X,t)$ = constant is any such wavelet moving in a direction of increasing X then (3.14), (3.15) and (3.23) imply that $\bar{\alpha}$ can be chosen so that

$$X = X_0(\beta)-\bar{\alpha}X_1(\beta),$$

and (3.24)

$$t = t_0(\beta)-\bar{\alpha}t_1(\beta),$$

where

$$X_0-U_S(t_0-\beta) = X_1- U_S t_1 = 0, \tag{3.25}$$

and (t_0,t_1) satisfy the ordinary differential equations

$$(U_S-U_D)t_0' + U_S't_0-(\beta U_S)' = (U_S-U_D)t_1' + U_S't_1 = 0, \tag{3.26}$$

subject to

$$t_0(0) = 0, \quad t_1(0) = 1. \tag{3.27}$$

3.3 First diameter effect

In addition to the shock layer region, where the lateral and longitudinal inertias of the plate are of the same order, the low frequency expansion is, in general, invalid in some vicinity of the driven edge X=0. There the imposed conditions are not, in general, compatible with (2.22) and (2.23). For example the pure stretch simple wave solutions, given by (3.12)-(3.18), satisfy the boundary conditions (2.13) with $\theta' \equiv 0$, but produce varying strain fields which do not satisfy the bonding condition (2.12) which requires that $p_{22}=1$ at X=0. In fact a boundary layer in which $X=O(\omega)$ occurs where the stress and strain gradients are larger than those in the low frequency region by a factor $O(\omega^{-1})$. In this boundary layer, which is called the first diameter by experimentalists (see Bell (1)) the deformation depends upon the precise details of the loading distribution on X=0.

In the boundary layer region the variations of the dependent variables in the X_1 and X_2 directions balance. If we introduce the boundary layer variable

$$\bar{X} = \omega^{-1}X = X_1/d \tag{3.28}$$

equations (2.16), (2.17) and (2.20) become

$$\partial T_{11}/\partial\bar{X} + \partial T_{12}/\partial Y = \omega\,\partial v_1/\partial t,$$

$$\partial T_{21}/\partial\bar{X} + \partial T_{22}/\partial Y = \omega\,\partial v_2/\partial t \tag{3.29}$$

and

$$\partial p_{11}/\partial Y - \partial p_{12}/\partial\bar{X} = \partial p_{21}/\partial Y - \partial p_{22}/\partial\bar{X} = 0,$$

where $\underset{\sim}{T} = \underset{\sim}{T}(\underset{\sim}{p})$ is given by (2.7). Equations (2.19) yield

$$\partial v_1/\partial \bar{X} = \omega^{\partial p_{11}/\partial t}, \quad \text{and} \quad \partial v_2/\partial \bar{X} = \omega^{\partial p_{21}/\partial t}. \tag{3.30}$$

To the zeroth order inertia effects are negligible and the only time dependence is introduced by the boundary conditions at X=0 and the usual matching conditions (see Van Dyke (13)) which require that as $\bar{X}\to\infty$

$$\underset{\sim}{v} \to \overset{0}{\underset{\sim}{v}}(0,t) + \sum_{n=1}^{N} \omega^n \overset{n}{\underset{\sim}{v}}(0,Y,t)$$

and (3.31)

$$\underset{\sim}{p} \to \overset{0}{\underset{\sim}{p}}(0,t) + \sum_{n=1}^{N} \omega^n \overset{n}{\underset{\sim}{p}}(0,Y,t).$$

Equations (3.30) and (3.31) imply that to the zeroth order $\underset{\sim}{v}=\overset{0}{\underset{\sim}{v}}(0,t)$ throughout the layer, while (3.29) imply that the stresses satisfy the equilibrium equations

$$\partial T_{11}/\partial \bar{X} + \partial T_{12}/\partial Y = \partial T_{21}/\partial \bar{X} + \partial T_{22}/\partial Y = 0. \tag{3.32}$$

The boundary layer deforms like a semi-infinite plate rigidly bonded at $\bar{X}=0$ under a load $2d\sigma T(\lambda)\,[\cos\psi, \sin\psi]$ applied at $\bar{X}=\infty$. This static elastic deformation has not been quantitatively determined. However, because of the divergence form of (3.29) the total load at time t is transmitted across this layer. Consequently, this layer does not affect either the velocity or load transmitted to the low frequency region, and conditions (2.13), or equivalent conditions on the velocity field, can be specified as boundary data on X=0 for equations (3.7) and (3.8).

3.4 Deflection simple wave.

The deflection simple wave has not, to our knowledge, been discussed previously. λ does not vary, so the membrane is not stretched, while ψ, w, and v are constant on wavelets $\bar{\alpha}(X,t)=$ constant which move with the same Lagrangian speed $C_D(\lambda)$. Since $U_S' \equiv U_D' \equiv 0$, (3.24) - (3.27) integrate to give

$$t = \left(1-{}^{U_D}/_{U_S}\right)\beta^{-1} - \bar{\alpha}, \quad X = U_D\left(1-{}^{U_D}/_{U_S}\right)\beta^{-1} - U_S\bar{\alpha}, \tag{3.33}$$

or, eliminating β,

$$U_D t - X = (U_S - U_D)\bar{\alpha}. \tag{3.34}$$

The deflection angle

$$\psi = \Psi(\bar{\alpha}) \tag{3.35}$$

is arbitrary, while

$$\begin{aligned} v + (\lambda T) &= q\cos\overline{\psi-\gamma} \text{ and} \\ w &= -q\sin\overline{\psi-\gamma} \end{aligned} \tag{3.36}$$

where q and γ are constant Riemann invariants. Conditions (3.35) imply that to a first order the velocity field is given by

$$v_1 = -(\lambda T)^{\frac{1}{2}}\cos\psi + q\cos\gamma, \text{ and } v_2 = -(\lambda T)^{\frac{1}{2}}\sin\psi + q\sin\gamma, \tag{3.37}$$

which produces a displacement field

$$\omega x_1/d = tq\cos\gamma - (\lambda T)^{\frac{1}{2}}\int^{\bar{\alpha}}\cos\Psi(r)dr, \quad \text{and } \omega({}^{x_2}/d - \mu(\lambda)\gamma) = tq\sin\gamma$$
$$-(\lambda T)^{\frac{1}{2}}\int^{\bar{\alpha}}\sin\Psi(r)dr. \tag{3.38}$$

If the simple wave is moving into a uniform region where $\psi=0$ then

$$q = (\lambda T)^{\frac{1}{2}}, \quad \text{and } \gamma = 0 \tag{3.39}$$

so that conditions (3.36) become

$$v_1 = (\lambda T)^{\frac{1}{2}}(1-\cos\psi), \quad \text{and} \quad v_2 = -(\lambda T)^{\frac{1}{2}}\sin\psi. \tag{3.40}$$

Deflection simple waves can occur adjacent to a uniform region which is prestrained so that $C_D > C_S$. They completely define the disturbance in a membrane, or string, which is produced by changing the direction but not the magnitude of the applied traction. Then $F' \equiv 0$ and

$$\Psi(\bar{\alpha}) = \theta[(U_S/U_D - 1)\bar{\alpha}]$$

We now show how the statements (3.33) - (3.40), which are exact when λ is constant, may be modified to provide a good description of disturbances in which λ, q and γ are 'slowly' varying compared with ψ. Such distrubances are produced when $\tilde{\omega}_S/\tilde{\omega}_D \ll 1$.

4.1 Modulation of finite amplitude deflection pulses by slow stretching waves.

The deformations described in sections 3.2 and 3.4 correspond to exact solutions of equations (3.7) and (3.8). They are generated by special motions of the edge X=0. Although the pure stretch waves are produced quite naturally by simple rectilinear motions of the driver, the pure deflection waves are produced only by driving motions which rotate but do not stretch the membrane at X=0. These edge motions produce no variation of stretch at other values of X and are somewhat artificial. If, however, we consider those driver motions for which F(t) varies slowly compared with $\theta(t)$ ($\tilde{\omega}_S/\tilde{\omega}_D \ll 1$) and which produce progressing disturbances such that the stretch λ, at some station $X=X_o$ at time $t=t_o$ is varying 'slowly' compared with ψ, it is not unreasonable to expect that in some interval around (X_o,t_o) the disturbance is 'approximately' described by a simple wave solution (3.35) and (3.36) with appropriately chosen Riemann invariants λ, q and γ. The problem then is to construct a procedure for 'enveloping' these local simple waves to obtain a global statement for conditions in a progressing wave in which λ, q and γ vary slowly compared with ψ. Of course we cannot expect any one simple procedure to apply for all F(t) and $\theta(t)$ at all (X,t). To illustrate these procedures we investigate conditions in an interaction pulse which is preceded by an arbitrary pure stretch simple wave described in section 3.2.

We construct formal asymptotic solutions to (3.7) and (3.8) as $\varepsilon = \omega_S/\omega_D \to 0$ which describe how a pure deflection wave is modulated by slow variations of the stretch λ. Even though the solutions are valid at a given X only for a finite time interval after the passage of the interaction front $\alpha=0$ of (3.24), no restriction is imposed on the distance this pulse propagates. Conditions in the pulse are related to $\theta(t)$ and to conditions in the pure stretch wave which precedes it: at X=0 the pulse lasts only while λ changes by $O(\varepsilon)$ although θ can change by any amount.

The interaction is best described in terms of new dependent and independent variables. As dependent slow 'Riemann' variables we take q, γ and

$$a = \lambda C_D = (\lambda T)^{\frac{1}{2}} ,$$

and as our fast variable

$$\phi = \psi - \gamma . \tag{4.1}$$

These are related to v and w as in (3.36), where the slow variables are Riemann invariants. As independent variables we take characteristic parameters (α,η) where the $\alpha(X,t)$ = constant and $\eta(X,t)$ = constant wavelets propagate with local speed a, or Lagrangian speed C_D, in directions of increasing and decreasing X. The characteristic parameters are chosen so that $\alpha>0$ in the interaction pulse and $\alpha<0$ in the precursor simple wave region, while $\eta \equiv t$ on the interaction front $\alpha=0$. In terms of the variables q, γ, a, ψ and

$$\Omega = \frac{\partial t}{\partial \alpha}\Big/\frac{\partial t}{\partial \eta}, \text{ and the material function* } \sinh H(a) = 1 - \frac{d}{d}\Big\{\frac{n\lambda}{na} = \frac{C_S^2 - C_D^2}{C_S^2 + C_D^2} \tag{4.2}$$

equations (3.7) and (3.8) are replaced by

$$\frac{\partial a}{\partial \alpha} = \Omega \operatorname{cosec} H \left\{ 2\frac{\partial a}{\partial \eta} - \cos\phi \frac{\partial q}{\partial \eta} - q \sin\phi \frac{\partial \gamma}{\partial \eta} \right\}, \tag{4.3}$$

$$\frac{\partial q}{\partial \alpha} = \Omega \sinh H \cos\phi \frac{\partial a}{\partial \eta} , \tag{4.4}$$

$$\frac{\partial \gamma}{\partial \alpha} = \Omega \sinh H \sin\phi \; q^{-1} \frac{\partial a}{\partial \eta} , \tag{4.5}$$

and

$$2a\frac{\partial \phi}{\partial \eta} + (2a - q\cos\phi)\frac{\partial \gamma}{\partial \eta} + \sin\phi \frac{\partial q}{\partial \eta} = 0, \tag{4.6}$$

* This relation is readily established by using (4.1) and (3.9).

while the definitions of (α,η) imply that

$$\frac{\partial X}{\partial \eta} = C_D \frac{\partial t}{\partial \eta}, \text{ and } \frac{\partial X}{\partial \alpha} = -C_B \frac{\partial t}{\partial \alpha}. \tag{4.7}$$

If $t_\alpha = \frac{\partial t}{\partial \alpha}$ and $t_\eta = \frac{\partial t}{\partial \eta}$ are regarded as dependent variables then they are related by the compatibility conditions

$$\frac{\partial t_\eta}{\partial \alpha} = \frac{\partial t_\alpha}{\partial \eta}, \tag{4.8}$$

while elimination of X from (4.7) implies that

$$2a\frac{\partial t_\alpha}{\partial \eta} + \sin H\left(\frac{\partial a}{\partial \eta}t_\alpha + \frac{\partial a}{\partial \alpha}t_\eta\right) = 0. \tag{4.9}$$

In what follows we take (4.3) - (4.6), (4.8), (4.9) and

$$t_\alpha = \Omega t_\eta \tag{4.10}$$

as a complete set of equations for $\phi, \underline{a}, q, \gamma, t_\alpha, t_\eta$ and Ω. These are to be solved for $\alpha>0$ subject to the condition that the front $\alpha=0$ borders an arbitrary stretching wave which implies that on $\alpha=0$

$$t_\eta - 1 = \gamma = \phi = 0, \tag{4.11}$$

while by (3.17) and (3.36)

$$q_o(\eta) = q(0,\eta), \text{ and } a_o(\eta) = a(0,\eta) \tag{4.12}$$

vary so that

$$q_o' = [1 + \cos H(a_o)]a_o', \text{ where } \frac{\pi}{2} < H < \frac{3\pi}{2} \tag{4.13}$$

Note that the only material function which occurs in the complete set of equations (4.3) - (4.6) and (4.8) - (4.10) is $\sinh(a)$. In the important practical case when $2C_D^2/C_S^2$ can be neglected compared with unity* we can take $H=\pi/2$ in these equations which then are identical for all materials. Of course, even in this limit, the determination of $X(\alpha,\eta)$ from (4.7), and the placing of the characteristics in (X,t) space involves the material function $\lambda(a)$. However, at constant X conditions (4.7) imply that η varies with α so that

$$\left.\frac{\partial\eta}{\partial\alpha}\right|_X = \Omega(\alpha,\eta). \tag{4.14}$$

Accordingly, the variations with t, at constant X, of the variables in (4.3) - (4.6) and (4.8) - (4.10) can be determined if one point on the (η,α) curve defined by (4.14) is known; this could, for example, correspond to the arrival time η of the front $\alpha=0$ at X.

4.2 Acceleration front.

The expansion procedure used to determine conditions in the pulse is best motivated by considering conditions at the front $\alpha=0$ whose trajectory is given by (3.24) with $\bar{\alpha}=0$ and $[X_o(\beta),t_o(\beta)]$ determined by conditions in the precursor wave by (3.25) - (3.27). At the front the dependent variables in (4.3) - (4.6) and (4.8) - (4.10) are continuous, and so, according to (4.3) - (4.6), are $\frac{\partial a}{\partial \alpha}$, $\frac{\partial q}{\partial \alpha}$ and $\frac{\partial \gamma}{\partial \alpha}$ which are given by (4.3) - (4.6) and (4.13) as

$$\frac{\partial a}{\partial \alpha} = \Omega_o \tan\tfrac{1}{2}H(a_o)a_o', \quad \frac{\partial a}{\partial \alpha} = \Omega_o \sin H(a_o)a_o', \quad \text{and } \frac{\partial \gamma}{\partial \alpha} = 0 \qquad (4.15)$$

where, by (4.9), (4.11) and (4.13), $\Omega_o(\eta)=\Omega(0,\eta)=t_\alpha(0,\eta)$ varies so that

$$\Omega_o \exp\left[\sqrt{2}\int^{a_o} r^{-1}\sin\tfrac{1}{2}H(r)\sin\tfrac{1}{2}(H(r)+\tfrac{\pi}{2})dr\right] = \text{constant}, = \varepsilon \text{ say.} \qquad (4.16)$$

Note that Ω_o, regarded as a function of $\underline{a}$, is proportional to t_1, which satisfies (3.26). ϕ is the only dependent variable which may have a discontinuous first derivative at the front. If (4.6) is differentiated with respect to α, and the results (4.15) are used, then an ordinary differential equation for $\frac{\partial \phi}{\partial \alpha}(0,\eta) = \frac{\partial \psi}{\partial \alpha}(0,\eta)$ is obtained which integrates to give

$$\frac{\partial \phi}{\partial \alpha}(0,\eta)\exp\left[\int^{a_o} r^{-1}\cos^2\tfrac{1}{2}H(r)dr\right] = \text{constant}, = k \text{ say.} \qquad (4.17)$$

In the special case when $H \equiv \frac{\pi}{2}$, (4.16) and (4.17) simplify to give

$$a_o\Omega_o = \varepsilon, \quad \text{and} \quad a_o^{\frac{1}{2}}\frac{\partial \phi}{\partial \alpha}(0,\eta) = k. \qquad (4.18)$$

The results (4.16) - (4.18) can be used to compute and compare the values of $\frac{\partial \psi}{\partial t}(t,X)$ and $\frac{\partial a}{\partial t}(t,X)$ at the passage of the front. These are given in terms of $a_o(\eta)$ as

$$\frac{\partial\psi}{\partial t} = (2\Omega_o)^{-1}\frac{\partial\phi}{\partial\alpha}(0,\eta), \text{ and } \frac{\partial a}{\partial t} = \tfrac{1}{2}(1+\tan\tfrac{1}{2}H(a_o))a_o' , \qquad (4.19)$$

which when $H = \frac{\pi}{2}$ can be simplified to give

$$\frac{\partial\psi}{\partial t} = \tfrac{1}{2}k\epsilon^{-1}a_o^{\frac{1}{2}} \quad \text{and} \quad \frac{\partial a}{\partial t} = a_o' . \qquad (4.20)$$

Conditions (4.19) and (4.20) imply that if the stretching wave extends (contracts) the plate as it passes the front, so that a_o increases (decreases), the $|\frac{\partial\psi}{\partial t}|$ increases (decreases). In particular $|\frac{\partial\psi}{\partial t}| \to 0$ as the plate is unloaded ($a_o \to 0$).
However, since at the front

$$\frac{\partial\psi}{\partial X} = -\lambda(a_o)a_o^{-1}\frac{\partial\psi}{\partial t} , = -\tfrac{1}{2}k\epsilon^{-1}\lambda(a_o)a_o^{-\frac{1}{2}} \text{when } H \equiv \frac{\pi}{2} , \qquad (4.21)$$

the low frequency theory predicts that as $a_o \to 0$ the curvature of the plate become unbounded. Again this is prevented by the formation of a layer, the 'bending layer', in which the low frequency membrane approximation is invalid and bending forces are important. In this layer the stretch varies across the plate and as $\omega \to 0$ the ratio of plate curvature in the layer to plate curvature outside is $O(\omega^{-1})$.

4.3 First order solutions.

Conditions (4.19) and (4.20) show that at the front, at times when $(a_o^{\frac{1}{2}})'$ is bounded, $\frac{\partial\psi}{\partial t}\Big/\frac{\partial a}{\partial t}$ can be made arbitrarily large by taking ε/k sufficiently small. Motivated by this result, and by previous work on finite amplitude pulses by Varley and Cumberbatch (14), (15), Varley and Rogers (16), and Seymour and Varley (11), we seek asymptotic solutions to (4.3) - (4.6) and (4.8) - (4.10), which satisfy (4.11) - (4.13) for arbitrary $a_o(\eta)$, which are of the form

$$a-a_o(\eta) = \sum_1^N \varepsilon^n a_n(\alpha,\eta), \quad q-q_o(\eta) = \sum_1^N \varepsilon^n q_n(\alpha,\eta),$$

$$\gamma = \sum_1^N \varepsilon^n \gamma_n(\alpha,\eta), \quad t-\eta = \sum_1^N \varepsilon^n t_n(\alpha,\eta), \quad \Omega = \sum_1^N \varepsilon^n \Omega_n(\alpha,\eta) \tag{4.22}$$

and $\phi = \phi_o(\alpha,\eta) + \sum_1^N \varepsilon^n \phi_n(\alpha,\eta)$.

These solutions describe deformations in which $\underline{a}$ varies slowly compared with ψ. This is best seen by noting that at constant X

$$\frac{\partial a}{\partial t}\Big/\frac{\partial \phi}{\partial t} = \left(\frac{\partial a}{\partial \alpha} + \Omega\frac{\partial a}{\partial \eta}\right)\Big/\left(\frac{\partial \phi}{\partial \alpha} + \Omega\frac{\partial \phi}{\partial \eta}\right), \tag{4.23}$$

which, together with (4.3) and (4.22), implies that $\frac{\partial a}{\partial t}\Big/\frac{\partial \psi}{\partial t} = O(\varepsilon)$ as $\varepsilon\to 0$, (except, perhaps, at isolated points).

If (4.22) is inserted in (4.6) this integrates to give

$$\tan\tfrac{1}{2}\psi_o = \tan\tfrac{1}{2}\phi_o = A(a_o)G(\alpha), \tag{4.24}$$

where the signal function $G(\alpha)$ is arbitrary, and the amplitude modulation function

$$\begin{aligned} A(a_o) &= \exp\left[-\int^{a_o} r^{-1}\cos^2\tfrac{1}{2}H(r)dr\right], \\ &= a_o^{-\frac{1}{2}} \text{ when } H \equiv \tfrac{\pi}{2}. \end{aligned} \tag{4.25}$$

Note that the zeroth order approximation (4.24) gives the <u>exact</u> variation of $\frac{\partial\phi}{\partial\alpha}(0,\eta)$ for all a_o, when $K=2G'(0)$. If now (4.22) is inserted in (4.9), and (4.3) and (4.14) are used, an ordinary differential equation is obtained for the variation of $\Omega_1=\frac{\partial t}{\partial\alpha}1$ on the α=constant wavelets: this integrates to give

$$\Omega_1=\exp[-\int^{a_o}\sqrt{2}r^{-1}\{\sin\tfrac{1}{2}H(r)\sin\tfrac{1}{2}(H(r)+\tfrac{\pi}{2})+\sqrt{2}\cos^2\tfrac{1}{2}H(r).A^2(r)G^2(\alpha)[1+A^2(r)G^2(\alpha)]^{-1}\}dr],$$

$$= a_o^{-1}[1+a_o^{-1}G^2(\alpha)] \quad = \quad a_o^{-1}\sec^2\tfrac{1}{2}\phi_o \text{ when } H \equiv \tfrac{\pi}{2}. \qquad (4.26)$$

In terms of $\Omega_1(\eta,\alpha)$, to a first order,

$$\epsilon^{-1}(t-\eta) = \int_o^\alpha \Omega_1(\eta,s)ds,$$

$$= a_o^{-1}[\alpha+a_o^{-1}\int_o^\alpha G^2(s)ds] \text{ when } H \equiv \tfrac{\pi}{2}, \qquad (4.27)$$

while, according to (4.7),

$$\lambda(a_o)(X(0,\eta)-X) = a_o(t-\eta),$$

where

$$X(0,\eta) = \int_o^\eta \frac{a_o(r)dr}{\lambda(a_o(r))}. \qquad (4.28)$$

Note that η and $X(0,\eta)$ are related to conditions in the precursor wave on $\alpha=\bar{\alpha}=0$ by (3.24) so that

$$\eta = t_o(\beta), \quad \text{and} \quad X(0,\eta) = X_o(\beta). \qquad (4.29)$$

The statements (4.24) - (4.28) provide a first approximation for the variation of $\psi(X,t)$ in terms of the <u>arbitrary</u> functions $a_o(\eta)$ and $G(\alpha)$. If (4.24) is inserted in (3.36) and (3.37), and if it is noted that to a first approximation $\gamma\equiv 0$, these then provide a first approximation for (v,w) and (v_1,v_2). In the special case when $a_o'(\eta)\equiv 0$ this

first 'approximation' is an <u>exact</u> solution and describes the pure deflection waves of section (3.3). Note, however, that if $a_o'(\eta) \not\equiv 0$ the expansion (4.22) is only appropriate for a <u>pulse</u>. For at a particle X the assumed expansion for <u>a</u>, together with (4.14) which shows that $\frac{\partial \eta}{\partial \alpha}\Big|_X = O(\varepsilon)$, implies that the expansion only describes conditions over a time interval after the passage of the front in which <u>a</u> changes by $O(\varepsilon)$. The variation of ψ in this time interval is, however, unrestricted. Because the expansion (4.22) only describes conditions at constant X over a limited time it cannot, in general, be used to relate $a_o(\eta)$ and $G(\alpha)$ to the driving conditions (2.13) at X=0. In a future paper Parker, using techniques analogous to those described in recent work of Whitham (17) and Luke (9), will discuss high frequency expansion procedures which allow a finite time variation at constant X.

If the plate is only **stretched** so that $C_D/C_s < 1$, which corresponds to $\frac{\pi}{2} < H < \pi$, and so that $a_o^{-2} a_o'$ is bounded, the expansion (4.22) is valid for <u>all</u> <u>t</u> in some region behind the front $\alpha=0$, and (4.24)-(4.28) describe how a bending pulse is modulated by the passage of loading and unloading waves shich vary a_o. Condition (4.24) implies that as the plate is loaded (unloaded) at an α=constant wavelet, which occurs when a increases (decreases), $|\tan\frac{1}{2}\psi|$ decreases (increases) while, to a first order, the magnitude of the angular velocity of the plate at a particle

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= (2\varepsilon\Omega_1)^{-1}\frac{\partial \phi}{\partial \alpha}, \\ &= \varepsilon^{-1} a_o^{\frac{5}{2}}[a + G^2(\alpha)]^{-2} G'(\alpha) \text{ when } H \equiv \frac{\pi}{2}, \end{aligned} \tag{4.30}$$

increases (decreases). In particular, as the plate is unloaded ($a_o \to 0$), (4.24) and (4.30) predict that $\psi \to \pm\pi$ and that $\frac{\partial \psi}{\partial t} \to 0$. However, since in this limit the filament theory predicts that $\frac{\partial \psi}{\partial X}$ and the curvature of the plate are unbounded it is locally invalid. A bending layer forms in which the radius of curvature and thickness of the plate are of the same order as $\omega \to 0$.

So far we have considered how a deflection pulse is modulated by a stretching wave, and have shown that to a first order current conditions at a wavelet α are determined by conditions at α at any one previous time, and by current conditions in the stretching wave at the front $\alpha=0$. To a first approximatio the deflection also interacts with and changes the mode of propagation of the stretching wave. This interaction is illustrated by comparing the Lagrangian speed, $\frac{dX}{dt}|_{\underline{a}}$, of wavelets carrying constant values of $\underline{a}$ with the Lagrangian speed C_S of the β=constant characteristic wavelets: in the precursor wave, and whenever $\phi=0$, these are equal. For since

$$\frac{dX}{dt}\Big|_{\underline{a}} = C_D\left(\frac{dn}{d\alpha}\Big|_{\underline{a}} - \Omega\right)\Big/\left(\frac{dn}{d\alpha}\Big|_{\underline{a}} + \Omega\right), \qquad (4.31)$$

(4.3), (4.13) and (4.22) imply that to a first approximation

$$C_S^{-1}\frac{dX}{dt}\Big|_{\underline{a}} = 1-2\sin^2\tfrac{1}{2}\phi\tan\tfrac{1}{2}H(1+\tan\tfrac{1}{2}H)^{-1}(\sin^2\tfrac{1}{2}\phi+\tfrac{1}{2}\tan\tfrac{1}{2}H[1-\tan\tfrac{1}{2}H]^{-1} \qquad (4.32)$$

$$= 0 \text{ when } H = \frac{\pi}{2}, \qquad (\phi\neq 0),$$

$$= 1 \text{ when } H = \pi.$$

According to (4.32), if $C_D/C_S<1$ in the bending pulse, the wavelets carrying constant values of $\underline{a}$ are slowed relative to the β=constant wavelets though $C_D^{-1}\frac{dX}{dt}|_{\underline{a}}>1$. When $C_D/C_S=1$ the $\underline{a}$=constant wavelets, the β=constant wavelets and the α=constant wavelets move with the same Lagrangian speed C_D.

4.3 Higher Order approximations

Further terms in the expansions (4.22) can be obtained by applying a simple iteration procedure to (4.3)-(4.5) and (4.8)-(4.10). We insert the expressions (4.3) and (4.10) for $\frac{\partial \underline{a}}{\partial \alpha}$ and t_α into (4.9) and regard this as an ordinary differential equation (I) for Ω on the α=constant wavelets with coefficients depending on $\underline{a}, q, \gamma, \phi, t_\eta$ and their η-derivatives. The nth order approximation for Ω then satisfies the equation obtained by replacing the coefficients in (I) by their $(n-1)^{th}$ approximations. The nth order approximations for $\underline{a}, q, \gamma$ and t_η are obtained by replacing Ω by its nth order approximation, and its coefficient by its $(n-1)^{th}$ order approximation, and its coefficient by its $(n-1)^{th}$ order approximation, in the right hand sides of (4.3)-(4.5) and (4.8): the resulting equations are then integrated from the front $\alpha=0$ along the η=constant wavelets. Finally, the nth order approximation for ϕ satisfies the ordinary differential equation obtained by replacing the coefficients in (4.6) by their $(n-1)^{th}$ approximations. We note, but do not further discuss, that according to (4.3) this iteration scheme, and the formal expansions (4.22), must be modified if $H\to\pi$ which occurs when $C_D \to C_S$.

As an illustration, which is physically important and also avoids messy algebra, we apply the iteration scheme when $H=\frac{\pi}{2}$ to calculate the first approximation for the 'slow' variables $\underline{a}, q$ and γ in the pulse. These approximations, together with the results already established, give the first approximations for the time variations of all dependent variables at a particle X as the pulse passes. In particular, they give the first approximation to the strain rate $\partial\lambda/\partial t$. We obtain, to a first approximation,

$$\begin{aligned} a &= a_o + a_o^{-1} a_o' \{\alpha + 3a_o^{-1}\int_o^\alpha G^2(s)ds\}, \\ q &= a_o + a_o^{-1} a_o' \{\alpha - a_o^{-1}\int_o^\alpha G^2(s)ds\}, \end{aligned} \tag{4.33}$$

and

$$\gamma = 2\varepsilon a_o^{-\frac{5}{2}} a_o' \int_o^\alpha G(s)ds.$$

The results (4.32) indicate that our formal iteration scheme can only be valid in a pulse; that is for $|\varepsilon a_0' \alpha| \ll 1$. Note that whereas, to a first approximation, the variation in the fast variable ψ at an α=constant wavelet is independent of conditions at all precursor wavelets, in any <u>finite amplitude</u> disturbance the slow variables are influenced by conditions at all precursor wavelets.

5. Shock layer

In section 3.1 we pointed out that according to the low frequency or long wave theory a stretching simple wave can develop limit surfaces, which are envelopes of β-wavelets, on which strain gradients and strain rates become unbounded. However, shortly before this occurs the lateral velocities become comparable with the longitudinal velocities so that the underlying assumption of that theory is violated. In analogy with the laminar bore of hydraulics a 'shock layer' is generated in which the strains and velocities vary rapidly. This layer is driven forward to seperate two low frequency regions. By introducing a new expansion procedure to describe regions of rapidly varying strain, we can obtain the equations governing the structure of such a layer and can, in general, relate its speed of propagation to the jump in strain across it.

In a layer where the strains vary rapidly (compared with their values in the low frequency region) the weighted scales (2.14) are not appropriate: the right hand sides of (2.16), (2.17) and (2.20) are not vanishingly small as $\omega \to 0$. If the shock layer is centred about the shock trajectory $X=S(t)$ then in terms of the shock layer variables Y, t and

$$\tilde{X} = \omega^{-1}(X-S(t)) \tag{5.1}$$

equations (2.16), (2.17), (2.19) and (2.20) become

$$\begin{aligned} \frac{\partial}{\partial \tilde{X}}(T_{11}+S'(t)v_1) + \frac{\partial T_{12}}{\partial Y} &= \omega\frac{\partial v_1}{\partial t} \\ \frac{\partial}{\partial \tilde{X}}(T_{21}+S'(t)v_2) + \frac{\partial T_{22}}{\partial Y} &= \omega\frac{\partial v_2}{\partial t} \end{aligned} \tag{5.2}$$

$$\begin{aligned} \frac{\partial}{\partial \tilde{X}}(v_1+S'(t)p_{11}) &= \omega\frac{\partial p_{11}}{\partial t} \\ \frac{\partial}{\partial \tilde{X}}(v_2+S'(t)p_{21}) &= \omega\frac{\partial p_{21}}{\partial t} \end{aligned} \tag{5.3}$$

and

$$\frac{\partial p_{12}}{\partial \tilde{X}} - \frac{\partial p_{11}}{\partial Y} = \frac{\partial p_{22}}{\partial \tilde{X}} - \frac{\partial p_{21}}{\partial Y} = 0. \qquad (5.4)$$

In addition to (5.2) - (5.4), the stresses are related to the deformation gradients by (2.7) and satisfy the stress free boundary conditions $T_{22} = T_{12} = 0$ on $Y = \pm 1$. As $\tilde{X} \to \infty(-\infty)$ the solutions to (5.2)-(5.4) tend to the solutions of the low frequency equations as $X \to S(t)$ from $X > S(t)$ ($< S(t)$), see Van Dyke (1964)).

Conditions (5.3), together with the boundary data at $\tilde{X} = \pm\infty$, imply that to the zeroth order in the shock layer

$$v_1 + S'p_{11} = g_1(t), \quad \text{and} \quad v_2 + S'p_{21} = g_2(t) \qquad (5.5)$$

so that at any time $v_1 + S'p_{11}$ and $v_2 + S'p_{21}$ are continuous across the layer. To this order, (5.5) imply that the strain field satisfies the compatibility conditons (5.4) and, from (5.2), the equations

$$\frac{\partial}{\partial \tilde{X}}(T_{11} - S'^2 p_{11}) + \frac{\partial T_{12}}{\partial Y} = \frac{\partial}{\partial \tilde{X}}(T_{21} - S'^2 p_{21}) + \frac{\partial T_{22}}{\partial Y} = 0, \qquad (5.6)$$

which are identical to the equilibrium equations of finite elasticity with a body force $-S'^2(\frac{\partial p_{11}}{\partial \tilde{X}}, \frac{\partial p_{21}}{\partial \tilde{X}})$. If the divergence theorem is applied to (5.6), and if the boundary conditions at the stress free surfaces $Y = \pm 1$ and those at $\tilde{X} = \pm\infty$ are used, we obtain the result that throughout the layer

$$\int_{-1}^{1} [T_{11}(\tilde{X},r,t) - S'^2 p_{11}(\tilde{X},r,t)]dr = f_1(t),$$

and (5.7)

$$\int_{-1}^{1} [T_{21}(\tilde{X},r,t) - S'^2 p_{21}(\tilde{X},r,t)]dr = f_2(t).$$

If $[v_1]$, $[v_2]$, ... denote the differences in values of v_1, v_2,... at $\tilde{X}=\infty$ and $\tilde{X}=-\infty$, and if it is remembered that these differences are independent of Y, (5.5) and (5.7) imply that

$$[v_1] + S'[p_{11}] = [v_2] + S'[p_{21}] = 0, \qquad (5.8)$$

and that

$$[T_{11}] + S'[v_1] = [T_{21}] + S'[v_2] = 0. \qquad (5.9)$$

Conditions (5.8) and (5.9) are trivially satisfied, with no restriction on S', if (5.6) have solutions which are symmetric about $\tilde{X}=0$. Such solutions describe <u>solitary waves</u>. After the passage of a solitary wave conditions at a particle return to those which existed prior to its arrival. Note that if the speed of such a wave is constant, solutions to (5.4) and (5.6) satisfy (5.2)-(5.4) exactly.

When the passage of a shock layer changes conditions at a particle, (5.8), (5.9) and (3.6) relate conditions in the low frequency disturbance which occurs immediately after the passage of the shock layer to S' and conditions in the low frequency disturbance which existed immediately before its arrival. The shock layer is a narrow region in which the low frequency approximation is locally invalid. The propagation speed, given in terms of the jumps in physical variables by (5.8) and (5.9), is not that of a thermodynamic shock; in the layer dissipation forces remain negligible and the entropy is unchanged.*

If we are not interested in the structure of the shock layer but only in how it affects the deformation in the low frequency region, we may regard this shock layer as a discontinuity surface

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* The deformation in the shock layer may, however, produce velocity gradients which are large enough for dissipation to be important. Then thermodynamic shocks may be formed or, in the language of hydraulics, the laminar bore may degenerate into a breaking bore.

across which the jump conditions (5.8) and (5.9) hold. Then (5.9) state that the change in force balances the change in momentum, while (5.8) state that the position of a particle is unchanged by the passage of the shock. These jump conditions are identical with those defined by the conservation equations (3.2) and (3.3) which govern the low frequency deformation. This result was anticipated by Whitham (18) in his work on magnetohydrodynamic waves. He also noted that the use of the jump conditions defined by the conservation laws (3.2) and (3.3) can only be justified by either having a model for the shock structure, or using sound physical arguments. Equations (3.2) and (3.3) can be rewritten in many alternative ways as conservation laws and these would define jump conditions which, in general, would be incorrect. If, for example, t and the Eulerian co-ordinate $\underset{\sim}{x}$ of a particle are used as independent variables and the equations are written in the usual form as conservation laws the associated jump conditions are incorrect. (In hydraulics the hydraulic jump conditions are derived from sound physical arguments and not from the conservation-type differential equations governing the flow, (see Stoker (12)).

There are two possible ways of satisfying the jump conditions (5.8) and (5.9). The longitudinal shock, across which (ψ, w) are continuous but (λ, v) are discontinuous, for which

$$S' = \sqrt{\frac{[T]}{[\lambda]}} \, . \tag{5.10}$$

Such a shock can be generated from an initially shockless pure stretch simple wave. The transverse shock, across which λ is continuous but (v, w, ψ) are discontinuous, which moves with characteristic speed

$$S' = \sqrt{\frac{T}{\lambda}} \, . \tag{5.11}$$

This shock cannot be generated from an initially shockless pure bending wave and must be regarded as the mathematical limit of a non-distorting simple wave.

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13. ABSTRACT

Disturbances produced by the motion of a driver which is rigidly bonded to the edge of a plate are used to motivate parameter expansion techniques which, when applied to the equations of finite elasticity, generate approximating equations which describe low frequency deflection and stretching waves travelling along stretched elastic plates and rods in the limit when bending forces are negligible compared with membrane forces. The structure of the boundary layer at the driven edge and shock layers, where the low frequency or filament approximations are locally invalid, are also discussed.

The low frequency equations are used to discuss the interaction between progressing finite amplitude deflection and stretch waves in the limit when the stretch rate is small compared with the angular speed of the plate. The disturbance is locally that of a pure deflection simple wave whose amplitude and frequency are modulated by slow variations in the stretch. As the stretch increases the frequency increases while the amplitude decreases. The stretch wave is also modified by deflection of the plate: the speeds of wavelets carrying constant values of stretch are always less than their values in the pure stretch simple wave.

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