MULTIPLE COMPARISONS WITH A CONTROL 
FOR MULTIPLY-CLASSIFIED VARIANCES 
OF NORMAL POPULATIONS 

by 
Robert E. Bechhofer 

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MULTIPLE COMPARISONS WITH A CONTROL

FOR MULTIPLY-CLASSIFIED VARIANCES OF NORMAL POPULATIONS*

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and

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Abstract

Dunnett has prepared tables for making multiple comparisons with a control for single-factor (or multi-factor) experiments involving means of normal populations. In the present paper the author uses the multiplicative model for variances which he introduced in an earlier paper, as the basis for making multiple comparisons with a control for multi-factor experiments involving variances of normal populations. A limited set of tables for use with this procedure is included.

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1. Introduction

In [1] it was shown how to design experiments for ranking multiply-classified variances of normal populations. For such experiments there will, of course, be situations in which ranking of the "effects" is not the objective of the experimenter, but rather he would like to estimate these effects. The purpose of the present note is to show how the theory developed in [1] provides a direct method of making certain joint confidence interval estimates of the effects. We shall use the notation of [1], and in particular that of Section 2.

2. Percentage points for joint interval estimation

For fixed $a \geq 2$, $b \geq 1$, $n \geq 1$, and for $i = 2, 3, ..., a$ we define the $a-1$ chance variables

$$G_n^{(i),b} = \prod_{j=1}^{b} \chi_n^2(i,j) / \chi_n^2(1,j),$$

where the $\chi_n^2(i,j)$ ($1 \leq i \leq a; 1 \leq j \leq b$) are $a \cdot b$ independent chi-square variates, each based on $n$ d.f. (Thus for fixed $i$, $G_n^{(i),b}$ is the product of $b$ independent central $F$-statistics, each based on $(n,n)$ d.f.; for $i_1 \neq i_2$, $G_n^{(i_1),b}$ and $G_n^{(i_2),b}$ are correlated.) For $0 < \lambda < 1$ let $G_n^{a,b}(1-\lambda)$ be defined by the equation

$$P(G_n^{a,b}(1-\lambda) < G_n^{(i),b} (i = 2, 3, ..., a)) = 1-\lambda.$$

We now show how the $G_n^{a,b}(1-\lambda)$ can be used to make certain types of inferences concerning the effects $\alpha_i$ defined by (2.1) of [1] when (2.2) holds. As in Section 3.3 of [1], we let $s_{ij}^2$ denote the sample variance based on $n$ d.f. associated with the $i^{th}$ "level" of Factor A and the $j^{th}$ "level" of Factor B.
3. Confidence statements

3.1 Multiple comparisons with a control

Suppose that the first level of Factor A is a "control" level, and that the remaining are "test" levels, i.e., that $a_1$ is the effect associated with the control level, and that $a_2, a_3, \ldots, a_n$ are the effects associated with the test levels. Then it may be desired to make a joint confidence statement concerning the effect-ratios $a_i/a_1$ ($i=2,3,\ldots,n$). Clearly the following statement holds with confidence coefficient $1-\lambda$:

$$
\begin{align*}
\frac{a_i}{a_1} &< \frac{1}{\sum_{j=1}^{n} b_{ij} (1-\lambda)} \left( \frac{b \sum_{j=1}^{n} s_{ij}^2}{n} \right)^{1/b} \\
&\text{(i=2,3,\ldots,n)}
\end{align*}
$$

This is sometimes referred to as a (one-sided) multiple comparison with a control, and is the analogue for multiply-classified variances of [2] (wherein $b=1$) for means. (See also [3] (wherein $b=1$) for the corresponding two-sided comparisons for means.)

3.2 Two-sided estimates when $a=2$

When $a=2$ and $(0 \leq \lambda_1, \lambda_2; \lambda_1 + \lambda_2 < 1)$, the following statement holds with confidence coefficient $1 - \lambda_1 - \lambda_2$:

$$
\begin{align*}
\left[ 1 - \frac{b \sum_{j=1}^{n} s_{ij}^2}{n} \right]^{1/b} &< \frac{a_2}{a_1} < \left[ 1 - \frac{b \sum_{j=1}^{n} s_{ij}^2}{n} \right]^{1/b} \\
&\text{C}_{n}^2 b (\lambda_2) \sum_{j=1}^{n} s_{ij}^2 \\
\end{align*}
$$

This is a two-sided interval estimate of $a_2/a_1$. 

-2-
4. Evaluation of the constants $G_n^{a,b}(\lambda)$

Comparison of (2) with (4.1) and (6.2) of [1] shows that

$$P_A(n|a,b; \theta^*_A) = 1-\lambda$$

(5a)

$$(1/\theta^*_A)^b = G_n^{a,b}(1-\lambda).$$

(5b)

Thus, for fixed $a \geq 2, b \geq 1, n \geq 1$ and specified $\lambda$, one can determine $\theta^*_A$ uniquely from (5a); this value of $\theta^*_A$ substituted in (5b) then yields $G_n^{a,b}(1-\lambda)$. Clearly, for $a=2$ we have

$$1/G_n^{2,b}(1-\lambda) = G_n^{2,b}(\lambda) = (\theta^*_A)^b.$$

We have used these relationships and the formulae of Section 4.1 of [1] to determine exact values of $G_n^{2,b}(\lambda)$ for $\lambda = 0.25, 0.15, 0.10, 0.05, 0.025, 0.01, 0.005, 0.001, 0.0005$; Table I gives the $G$-values for $b=2$ and $n=1(1)6,8$ while Table II gives the $G$-values for $b=3$ and $n=2,4$. These percentage points are correct to the number of significant figures which are tabulated. The tables can be regarded as extensions of the tables [4]
Table I
Table of exact upper percentage points, $G_{n}^{2,2}(\lambda)$, of the product of two independent central F-statistics, each based on $(n,n)$ degrees of freedom

\[ P(G_{n}^{2,2}(\lambda) < G_{n}^{2}) = \lambda \]

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.25</th>
<th>0.15</th>
<th>0.10</th>
<th>0.05</th>
<th>0.025</th>
<th>0.01</th>
<th>0.005</th>
<th>0.001</th>
<th>0.0005</th>
</tr>
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<tr>
<td>1</td>
<td>15.081</td>
<td>72.626</td>
<td>226.91</td>
<td>1405.4</td>
<td>7917.0</td>
<td>71.272</td>
<td>35.983x10^3</td>
<td>13.983x10^3</td>
<td>65.707x10^3</td>
</tr>
<tr>
<td>2</td>
<td>5.1157</td>
<td>12.746</td>
<td>24.259</td>
<td>66.115</td>
<td>166.94</td>
<td>529.35</td>
<td>1224.2</td>
<td>7987.8</td>
<td>17.547</td>
</tr>
<tr>
<td>3</td>
<td>3.5038</td>
<td>6.9963</td>
<td>11.321</td>
<td>23.697</td>
<td>46.344</td>
<td>105.42</td>
<td>189.98</td>
<td>696.74</td>
<td>1195.2</td>
</tr>
<tr>
<td>4</td>
<td>2.8633</td>
<td>5.0940</td>
<td>7.5656</td>
<td>13.905</td>
<td>23.969</td>
<td>46.361</td>
<td>74.034</td>
<td>205.43</td>
<td>312.57</td>
</tr>
<tr>
<td>5</td>
<td>2.5166</td>
<td>4.1627</td>
<td>5.8860</td>
<td>9.9441</td>
<td>15.878</td>
<td>27.873</td>
<td>41.433</td>
<td>97.585</td>
<td>138.41</td>
</tr>
<tr>
<td>8</td>
<td>2.0310</td>
<td>2.9817</td>
<td>3.8773</td>
<td>5.7520</td>
<td>8.1480</td>
<td>12.322</td>
<td>16.433</td>
<td>30.312</td>
<td>38.767</td>
</tr>
</tbody>
</table>

Table II
Table of exact upper percentage points, $G_{n}^{2,3}(\lambda)$, of the product of three independent central F-statistics, each based on $(n,n)$ degrees of freedom

\[ P(G_{n}^{2,3}(\lambda) < G_{n}^{2}) = \lambda \]

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<th>0.10</th>
<th>0.05</th>
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<th>0.005</th>
<th>0.001</th>
<th>0.0005</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>7.6601</td>
<td>23.570</td>
<td>51.557</td>
<td>171.46</td>
<td>510.39</td>
<td>1939.4</td>
<td>5039.3</td>
<td>41.118</td>
<td>97.971</td>
</tr>
<tr>
<td>4</td>
<td>3.6712</td>
<td>7.4496</td>
<td>12.111</td>
<td>25.224</td>
<td>48.422</td>
<td>105.71</td>
<td>182.76</td>
<td>592.34</td>
<td>954.27</td>
</tr>
</tbody>
</table>
when the d.f. \( (v_1, v_2) \) of those tables are such that \( v_1 = v_2 = n \).

(Thus the diagonal entries in the tables [4] are the percentage points \( C^2_{n,1}(\lambda) \).)

Additional exact values of \( C_{n}^{a,b}(\lambda) \) for \( a > 1, b, n \) small, and \( \lambda \) close to zero must await the evaluation of the integral (4.4) of [1] to obtain formulae like those of Section 4.1 of [1].

For \( a=2 \), Tables I and II can also be used for size and power calculations associated with the test \( H_0: \alpha_1 = \alpha_2 \) vs. a one-sided alternative \( H_1: \alpha_1 > \alpha_2 \) (say).


**REPORT TITLE**

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**AUTHORS**

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**ABSTRACT**

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<tbody>
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<td>Mathematical statistics</td>
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<tr>
<td>Multiple comparisons with a control</td>
</tr>
<tr>
<td>Variances</td>
</tr>
<tr>
<td>Factorial experiments</td>
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<td>F-test of homogeneity of variances</td>
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<td>Confidence intervals</td>
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