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NUMERICAL INTEGRATION OF BAND-LIMITED SIGNALS – WITH APPLICATIONS TO COMPUTER IMPLEMENTED SIGNAL PROCESSING

DECEMBER 1967

John T. Lynch

Prepared for

DEVELOPMENT ENGINEERING DIVISION
DIRECTORATE OF PLANNING AND TECHNOLOGY
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Project 7150
Prepared by
THE MITRE CORPORATION
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FOREWORD

The work reported in this document was performed by The MITRE Corporation, Bedford, Massachusetts, for the Directorate of Planning and Technology, Electronic Systems Division, of the Air Force Systems Command under Contract AF 19(628)-5165.

REVIEW AND APPROVAL

Publication of this technical report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

Anthony P. Trunfio

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ABSTRACT

In many computer-implemented signal-processing systems, it is necessary to perform numerical integrations. If the function to be integrated is a band-limited time signal represented by a set of samples, serious distortion can occur if the wrong integration technique is used. In particular, if the integration is used to calculate a Fourier transform, large error sidelobes will be introduced if Simpson's rule or other higher-order rules are used. It is shown that the rectangular rule introduces no distortion and gives a result which is consistent with the number of samples.

ACKNOWLEDGMENT

The work presented in this paper has been guided by Ron Haggarty. Gerry O'Leary deserves special mention because we worked so closely on all of the material. Ron's and Gerry's ideas so completely permeate the work that individual references have not been possible. The programming and mathematical backup work of Paul Gleason is greatly appreciated.

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SECTION I

INTRODUCTION

The digital computer offers the possibility of increased signal processor performance by eliminating the inaccuracies inherent in an analog operation, and by making it relatively easy to implement complicated, flexible or physically unrealizable techniques.

Often a signal processor is required to correlate two signals or compute a Fourier transform of a band-limited signal. These two operations are performed by numerical integrations of data which are quantized in amplitude and time (or frequency). It is shown in this paper that the "better" numerical integration techniques can give the worst results for band-limited signals. Specifically, the rectangular integration rule is superior for these applications because it does not introduce distortions such as error sidelobes in the Fourier transform example.

The use of the digital computer as a signal processor is becoming more popular in many fields of endeavor such as seismic array analysis or electrocardiogram interpretation. Within the MITRE Corporation there are three radar systems which have independently caused the author to consider the numerical integration problem. These include the ARPA Signal Processor (ASP) System, the oblique HF Sounder Experiment (run with the cooperation of Stanford Electronics Labs), and the MITRE Six Inch Radar experimental program.

SECTION II

NUMERICAL INTEGRATION

A common linear operation that is performed on sampled data is the integral

$$Y = \int_{T_B}^{T_A} X(t)\, dt \qquad (1)$$

where only samples of $X(t)$ are known; i.e.,

$$x_i = X(t)\Big|_{t = i\tau} \qquad \text{for } i = 0 \; .... \; N-1$$

and

$$T_A = k\tau$$

$$T_B = j\tau$$

$$k-j = N = \text{number of intervals}$$

$$\tau = \text{sample spacing}$$

If $X(t)$ is known to be a polynomial of degree less than $N-1$, the integral can be computed exactly because the samples represent the function without approximation.* Usually this condition is not

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*Note: The reader is referred to the treatment of Lagrange Interpolation in Linear Algebra by Hoffman and Kuntz.

assumed by the integration rules. The function $X(t)$ is often assumed to be represented by a polynomial of much smaller degree over a small portion of its argument (t). For example, Simpson's rule fits a second order curve to every two adjacent intervals (i.e., 3 points define each second order polynomial). These rules result in the following formula if the number of intervals is even:

$$Y = \left[X_o + 4X_1 + 2X_2 + 4X_3 \; \ldots\ldots + 2X_{N-3} + 4X_{N-2} + X_{N-1} \right] \frac{\tau}{3} \qquad (2)$$

where $\quad N$ = odd integer = number of samples

$$X_i = X(t) \Big|_{t = i\tau}$$

Using a different order fit to each set of points results in a different integration rule. In general, the higher order the fit and the larger the integer N, the lower will be the maximum error. There are many good texts on numerical analysis which cover this broad subject in considerable detail and which derive the accuracy bounds for each of the rules.

Time functions, however, have different properties than polynomial functions and therefore may have different integration rules as shown in the following discussion. If $X(t)$ is a time function, band-limited to W cps, then there is a very simple interpretation of the integral. By the sampling theorem $X(t)$ can be written

$$X(t) = \sum_{i=-\infty}^{+\infty} X_i \frac{\sin \pi W(t - i\tau)}{\pi W(t - i\tau)} \tag{3}$$

where $\tau \leq \frac{1}{2W}$

The X_i are the values of $X(t)$ at $t = i\tau$.

Substituting the expression of $X(t)$ of Equation 3 into the integral in Equation 1

$$Y = \sum_{i=-\infty}^{+\infty} X_i \int_{T_B}^{T_A} \frac{\sin \pi W(t - i\tau)}{\pi W(t - i\tau)} \, dt \tag{4}$$

$$Y = \sum_{i=-\infty}^{+\infty} X_i \, g_i$$

Equation 4 implies that in order to integrate a band-limited signal over definite limits, the samples of the signal must be known for all time. The weight given each sample x_i is the integral of a portion of the $\frac{\sin x}{x}$ function. If $i\tau$ lies outside the interval $T_A - T_B$ the weights become very small. In fact,

$$\lim_{i \to \infty} g_i = 0$$

If i_i lies in the interval $T_A - T_B$, the integral specifying g_i is approximately $1/T$ because most of the area under the $\frac{\sin x}{x}$ curve is in the main-lobe region.

As the limits of integration become $\pm\infty$ the coefficients g_i become equal, and have value $1/T$. This is true because the area under each of the displaced $\frac{\sin x}{x}$ functions is independent of i.

Therefore if

$$Y = \int_{-\infty}^{\infty} x(t)dt$$

then

$$Y = \frac{1}{T} \sum_{-\infty}^{\infty} X_i$$

In practice $X(t)$ may be a pulsed band-limited signal. Although in theory it is not possible for a signal to be both time-limited and bandlimited, it is possible to have most of the energy in a signal confined to a time-frequency band. For this type of signal, samples of $X(t)$ for $i > N$ will be less than some arbitrary small value and therefore

$$Y \sim \frac{1}{T} \sum_{i=1}^{N} X_i \tag{5}$$

In numerical analysis, Equation 5 is referred to as a "rectangular" integration rule.

The discussion thus far has included only the integration of band-limited time signals. The duality of time and frequency guarantees that it would have been equally valid to discuss the integral of time-limited frequency functions. An example of such an integration is given in the next section on Fourier transforms, and indicates the type of errors which can occur using sampling techniques.

Equation 5 is the major result of this paper. It states that the best integration rule to use for band-limited functions is an equally weighted sum of the samples; namely the so-called rectangular integration rule.

SECTION III

FOURIER TRANSFORM TECHNIQUES

An application of the previous discussion is the Fourier transform of a frequency function. Assume that $N+1$ samples of a "band-limited" "time-limited" frequency function are known. The time signal is given by the following integration:

$$f(t) = \int_{-\frac{W}{2}}^{+\frac{W}{2}} F(f)\, e^{j2\pi ft}\, df \tag{6}$$

where $F(f)\Big|_{f=k\Delta f} = F_k$ are known .

Using the rectangular integration rule and applying the sampling theorem to represent $f(t)$ the integral becomes

$$f(t)\Big|_{t=\frac{i}{2W}} = f_i = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} F_k\, e^{j2\pi \frac{\Delta f}{2W} ik} \tag{7}$$

The computed time function represented by f_i is only equal to the desired time function $f(t)$ for a range of time less than $1/\Delta f$. This is true because evaluating $F(f)$ only at intervals of Δf

causes a periodicity in $f(t)$ with period $1/\Delta f$ as seen from the following argument.

The phase of the complex samples F_k can only be measured modulo 2π, therefore the samples can be multiplied by $e^{j2\pi}$ without changing them.

$$F'_k = F_k e^{\frac{j2\pi f \ell}{\Delta f}} \tag{8}$$

where ℓ is any integer.

The exponential term corresponds to a delay $\ell/\Delta f$, which implies that the computed time signal, <u>$f(t)$</u>, <u>will be periodic</u>. Therefore,

$$\begin{aligned} f(t) &= f(t - \ell T) \\ f_i &= f_{i+\ell N} \end{aligned} \tag{9}$$

where $T = 1/\Delta f$

$\frac{W}{\Delta f} = N+1$

Besides limiting the values of i for which the f_i represent $f(t)$ the periodicity caused by sampling $F(f)$ also introduces an error into f_i. The following example illustrates this point. The transform of a set of equal amplitude zero phase samples, (i.e., $F_k = 1$ for $|k| \le \frac{N-1}{2}$) is the following periodic function.

$$f(t) = \frac{\sin \pi W t}{\sin \frac{\pi W t}{N+1}} = \sum_{\ell=-\infty}^{+\infty} \frac{\sin \pi W (t - \frac{\ell}{W})}{\pi W (t - \frac{\ell}{W})} \tag{10}$$

If F_k represents the samples of a constant zero phase band-limited frequency characteristic, then the desired transform $f_d(t)$ is one of the terms in the last sum. Namely, the desired function is one period of $f(t)$

$$f_d(t) = \frac{\sin \pi W t}{\pi W t} \tag{11}$$

If N is large enough then

$$f_d(t) \approx f(t) \qquad \text{for } -\frac{T}{2} < t < \frac{T}{2} \tag{12}$$

The error in Equation 12 can be approximated by considering the periodicity in $f(t)$. In the simplest approximation, the error in Equation 12 is $+ f_d(t - \frac{N}{W}) + f_d(t) \approx f(t)$. The term $f_d(t - \frac{N}{W})$ should be small for $t \approx 0$, therefore

$$f_d\left(\frac{N}{W}\right) \approx \frac{\sin \pi N}{\pi N}$$

If N is 100 the error will be -50 db with respect to the peak of the desired signal. If sidelobe weighting is used the error is more difficult to determine. Numerical calculations for specific examples give an indication of the magnitude of the errors. Figures 1, 2, 3* give the time signal for $N = 11$, 21 and 101 sample points respectively, with 40 db Taylor weighting $(\bar{n} = 6)$. An analytic bound for the error is difficult to derive because the Taylor-weighted signal is difficult to characterize between $t = \bar{n}/W$ and its asymtotic value of $\frac{\sin x}{x}$. To characterize the distortion for other types of signals one may invoke the sampling theorem, which states that any band-limited time signal is composed of a sum of $\frac{\sin \pi Wt}{\pi Wt}$ functions, each separated by $1/W$ seconds. To bound the error in general does not seem possible; rather it is hoped that the above discussion will help the reader understand the distortion mechanism.

The discussion of the Fourier transform has thus far been limited to the rectangular-rule integration which was shown to be optimum in the earlier section. The Fourier integral is an easy vehicle for demonstrating the type of errors which result when higher-order integration rules are applied to band-limited signals.

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*Note: The reader must observe that the bandwidth for a rectangular case is $BW = N\Delta f$ while for Simpson's rule the bandwidth is $BW = (N-1)\Delta f$.

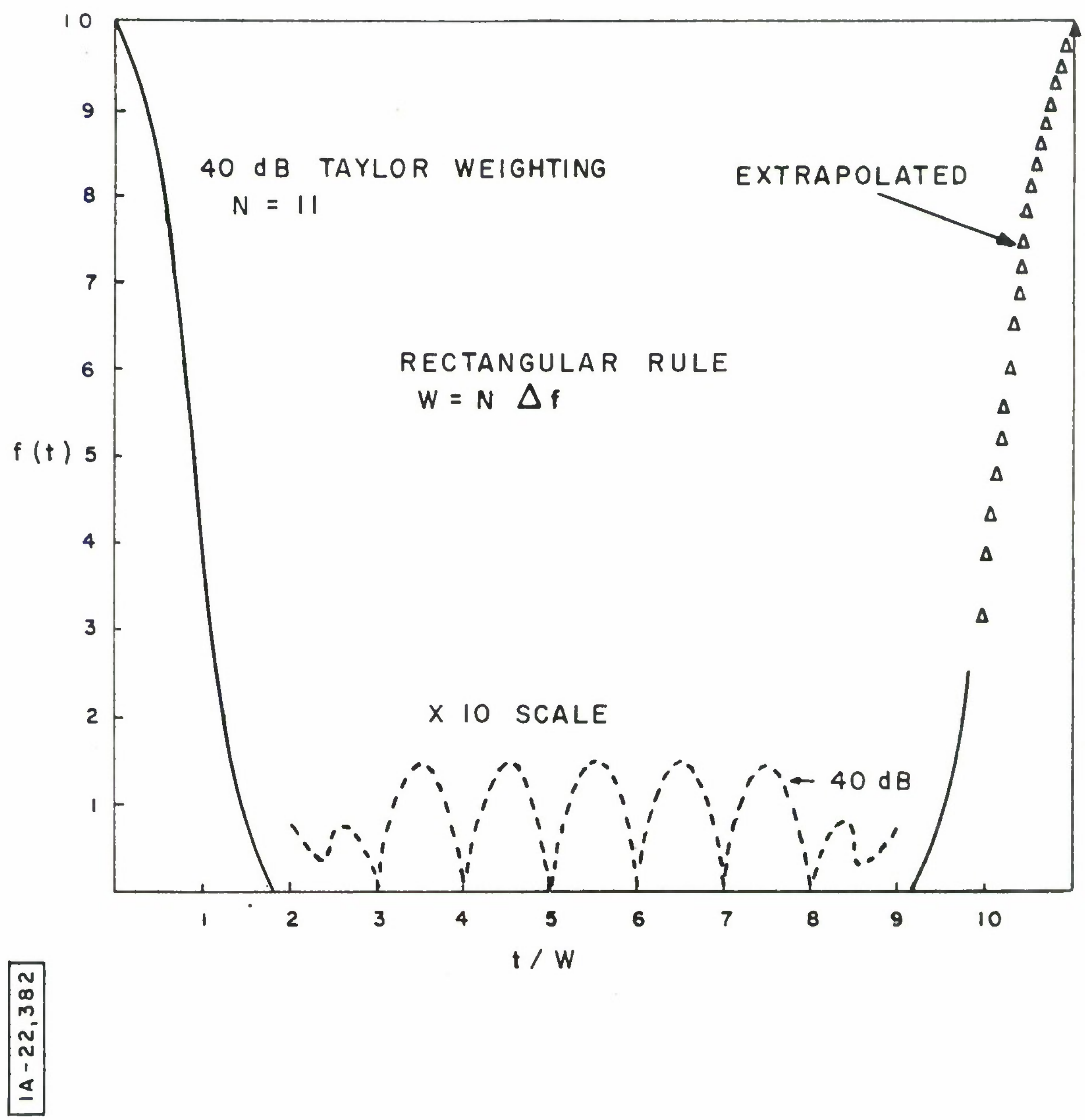

Figure 1 TRANSFORMED TAYLOR WEIGHTING FUNCTION

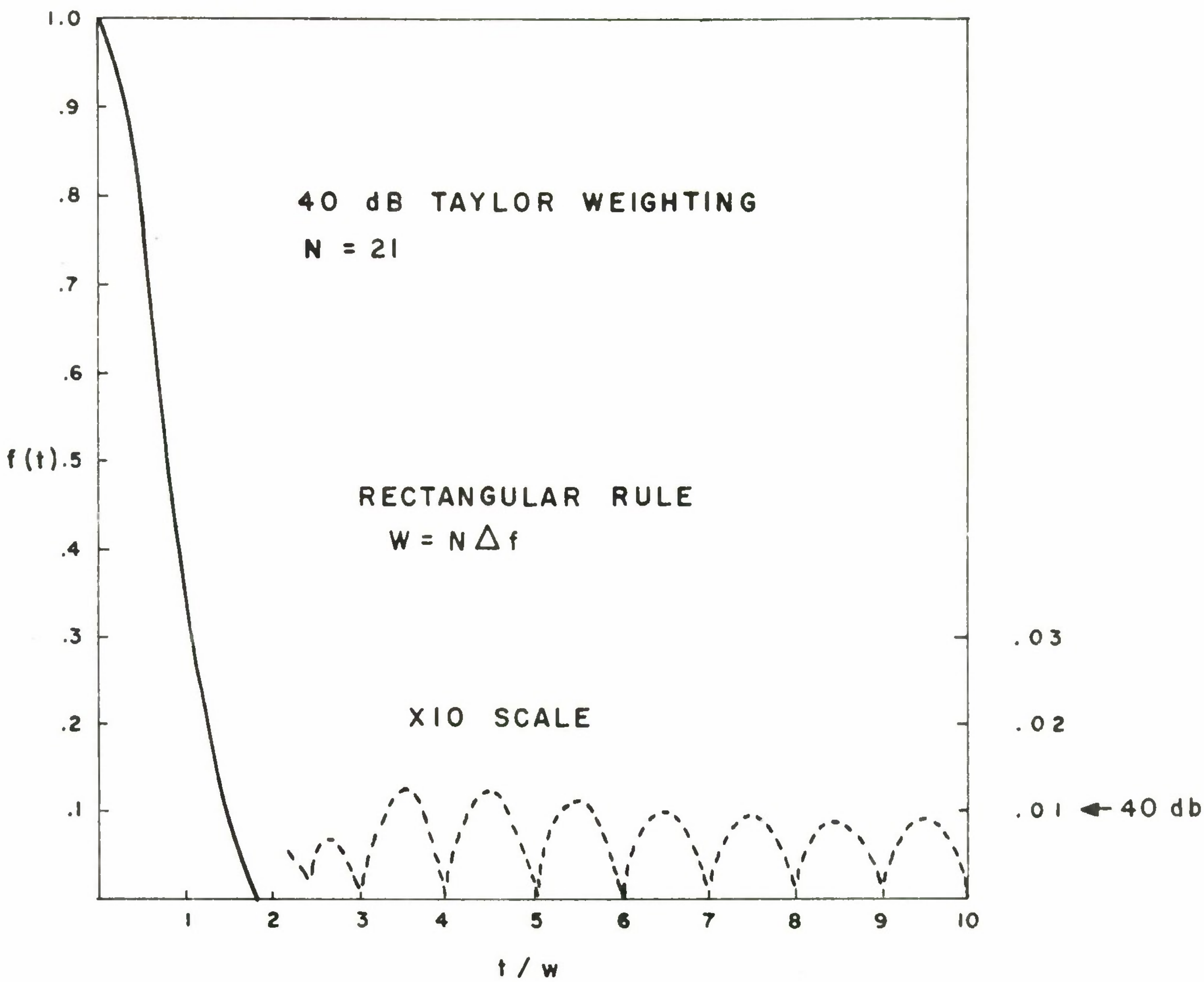

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Figure 2 TRANSFORMED TAYLOR WEIGHTING FUNCTION

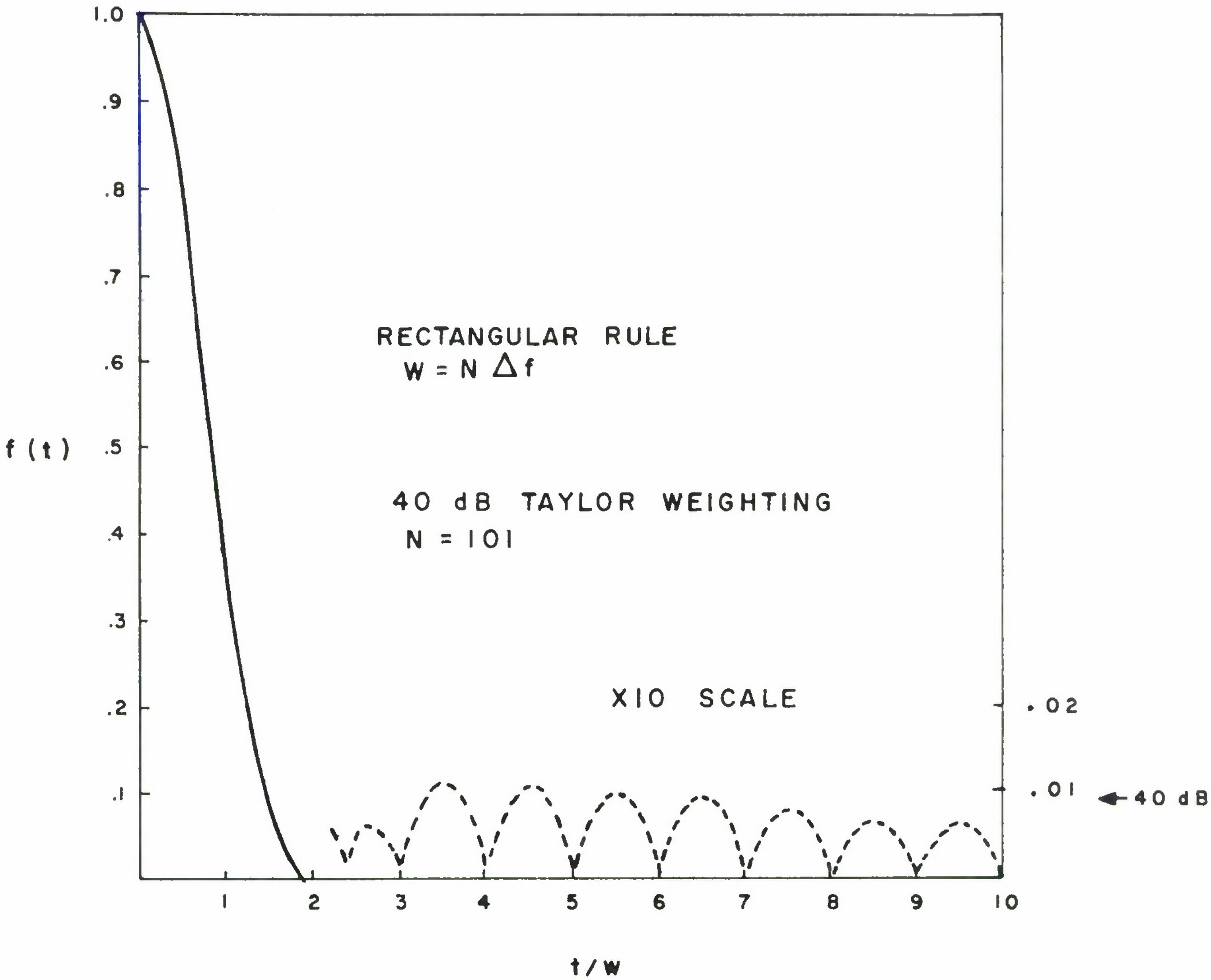

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Figure 3 TRANSFORMED TAYLOR WEIGHTING FUNCTION

The following discussion is a direct application of Fourier theory. Sampling a function $[G(F)]$ in the frequency domain is equivalent to multiplying it by a set of impulses $[X(f)]$ spaced by Δf in frequency $[G(f) \cdot X(f)]$. An infinite set of impulses in the frequency domain, spaced by Δf , transforms to an infinite set of impulses in the time domain, spaced by $1/\Delta f$ in time $[X(t)]$. The sampled function is represented in the time domain as the convolution of the original time function and the impulses spaced by $1/\Delta f$ in time $[g(t) \otimes X(t)]$. If $1/\Delta f$ is larger than the duration in time of the signal, then a simple periodicity will be introduced into $g(t)$. If $1/\Delta f$ is smaller than the duration in time of the signal, then distortion is introduced since $g(t)$ repeats and overlaps itself.

Application of this Fourier theory to sampling and numerical integration gives the following general rule.

> "The transform of a function $F(f)$ is the desired signal, $f(t)$, convolved with the Fourier transform of the integration rule."

The trapezoidal rule indicated in Equation 13 merely weights the first and last sample. It, therefore, will tend to broaden the main lobe of a $\frac{\sin x}{x}$ function slightly and to reduce its sidelobes slightly.

$$y = \left[\frac{1}{2} X_o + X_1 + X_2 ---- X_{N-2} + \frac{1}{2} X_{N-1}\right] \tau \qquad (13)$$

Simpson's rule, indicated in Equation 2, weights every other sample by a factor of 2. Thus, this rule has a strong Fourier component. The transform of an infinite set of equal samples (impulses) is a periodic $\frac{\sin x}{x}$ function. This corresponds to the rectangular rule as shown in Figure 4A. The transform of a set of equal samples alternately weighted as in Simpson's rule has an additional -10 db component, as shown in Figure 4C. Simpson's rule is approximately the sum of two sets of weights, the rectangular weights of Figure 4A and the alternating plus and minus weights of Figure 4B. The transform of a sampled band-limited function which is computed using Simpson's integration rule will be the convolution of the desired transform with the function in Figure 4C. If N is made large enough, however, it is possible to cause the -10 db error sidelobe to be far away from the main lobe and, therefore, avoid significant errors in the near-in region of observation.

The Fourier integral perhaps is a worst-case example since the Fourier components of the rule are so readily visible. The Fourier integral plays such an important part in computer-implemented signal processing that these results have a very practical significance. To emphasize the type of errors which can result, a simple delay-line network is analyzed. Figure 5A shows three delay lines, each with a different gain. The impulse response of this linear network is found by passing a sinusoid, $e^{j2\pi ft}$, through the network and sampling the output. The frequency is varied to give the frequency characteristic, IB-22,383

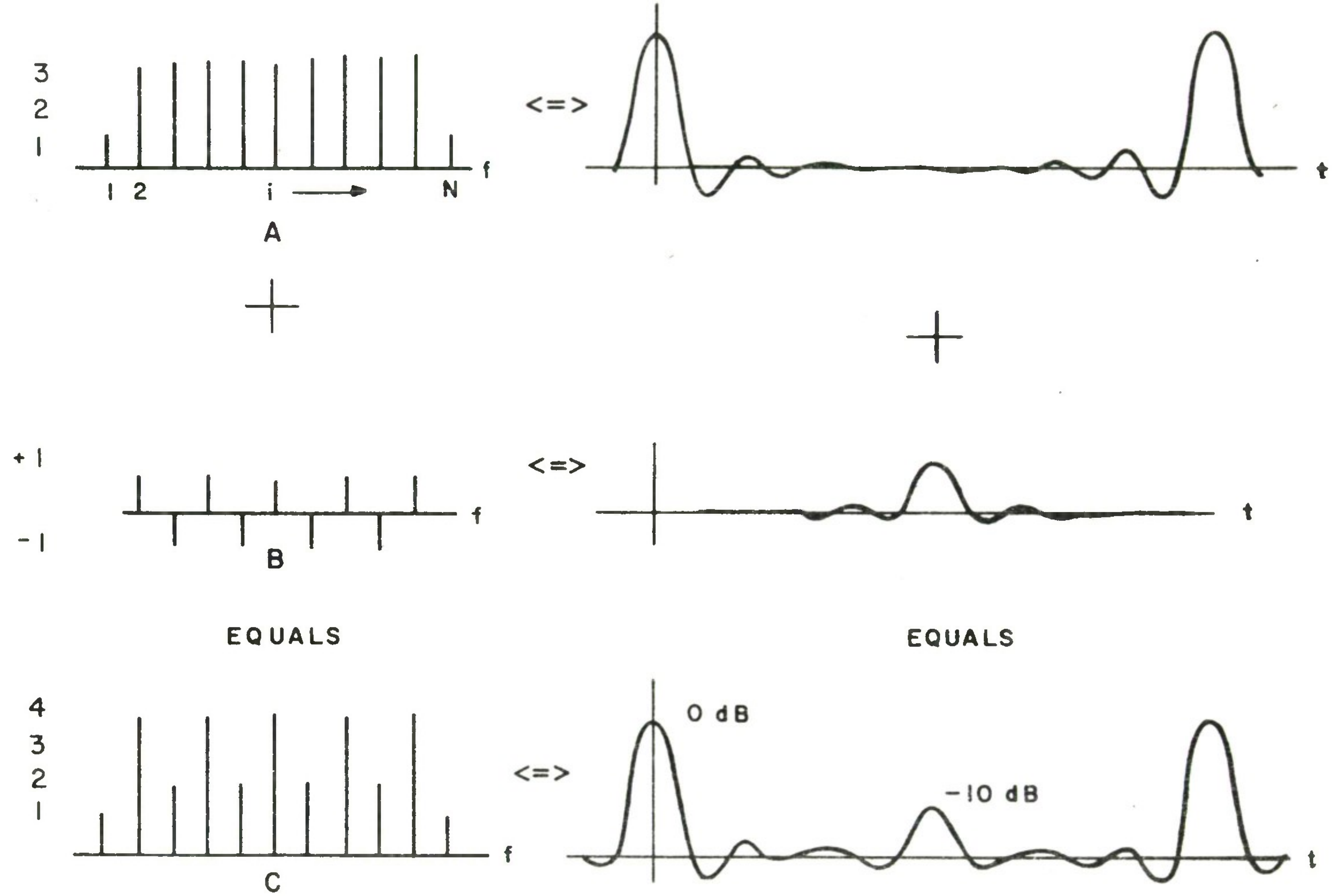

Figure 4 FOURIER INTERPRETATION OF SIMPSON'S INTEGRATION RULE

$F(f)$, which is then Fourier transformed. Figure 5B gives the resultant impulse response if a rectangular rule is used. Note that N is large enough to guarantee that aliasing will not occur. That is, a portion of the time function represents the desired response. Simpson's rule can also be used to compute the Fourier transform as shown in Figure 5C. The confusion caused by the error sidelobes is apparent from the figure.

These predictions were tested by performing a transform on a digital computer. Figures 1, 2, 3 give the transform of a zero-phase, Taylor weighted spectrum for $N = 11, 21, 101$ samples respectively using the rectangular rule. Figures 6, 7, and 8 give the same transform using Simpson's rule. In addition to pointing out the obvious periodicities and the -10 db error sidelobe, these figures give an indication of the increase in sidelobes if N is made too small.

The use of higher-order integration rules will also introduce error sidelobes. The magnitude and position of these sidelobes can be found by simply finding the Fourier components of the rule.

Again the observation is made that N can be made large enough to guarantee good results in the limited region of interest. In a signal processing system, however, where the number of samples is often constrained, the rectangular rule will require fewer samples than higher-order integration rules.

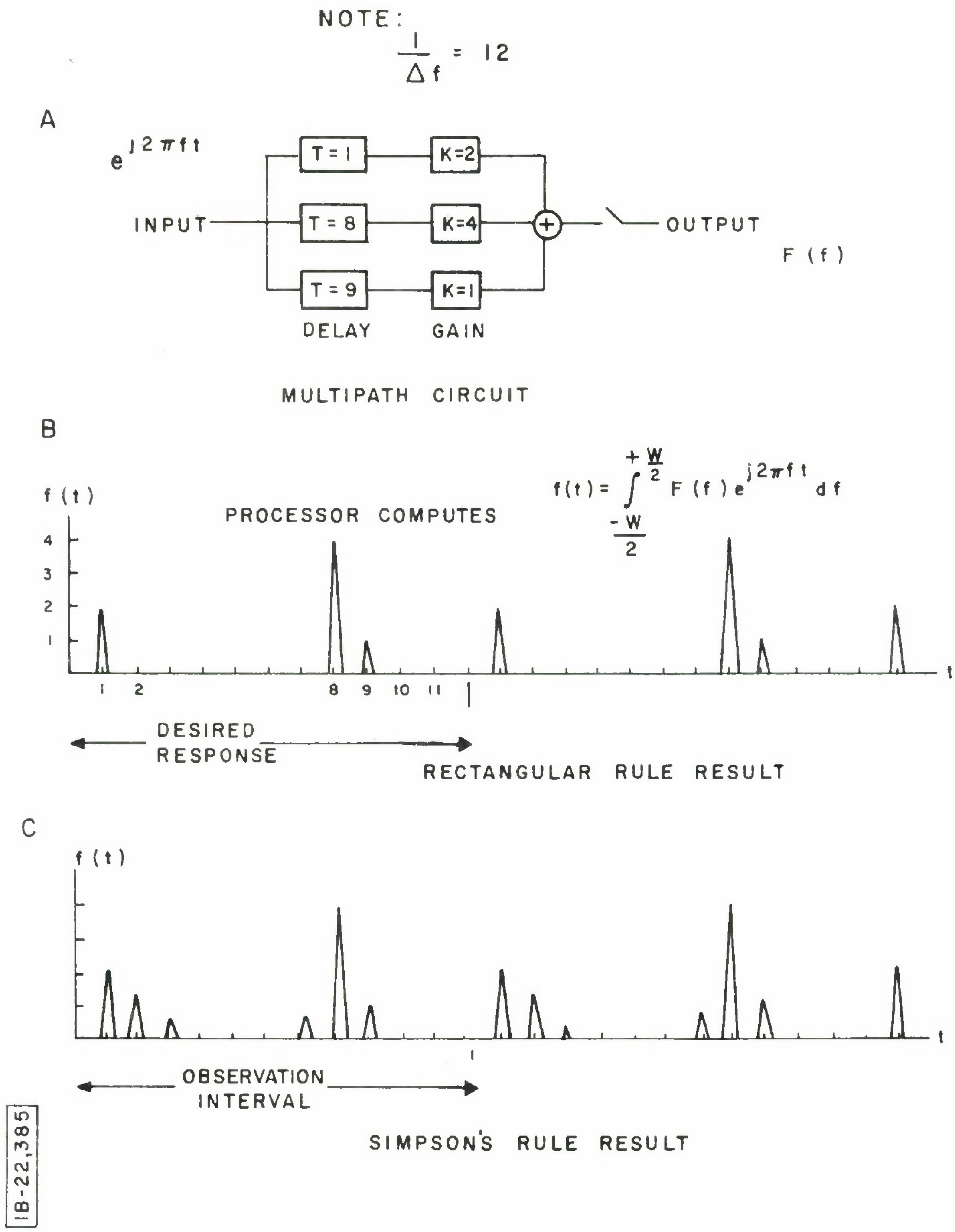

Figure 5 POSSIBLE DISTORTION FROM SIMPSON'S RULE

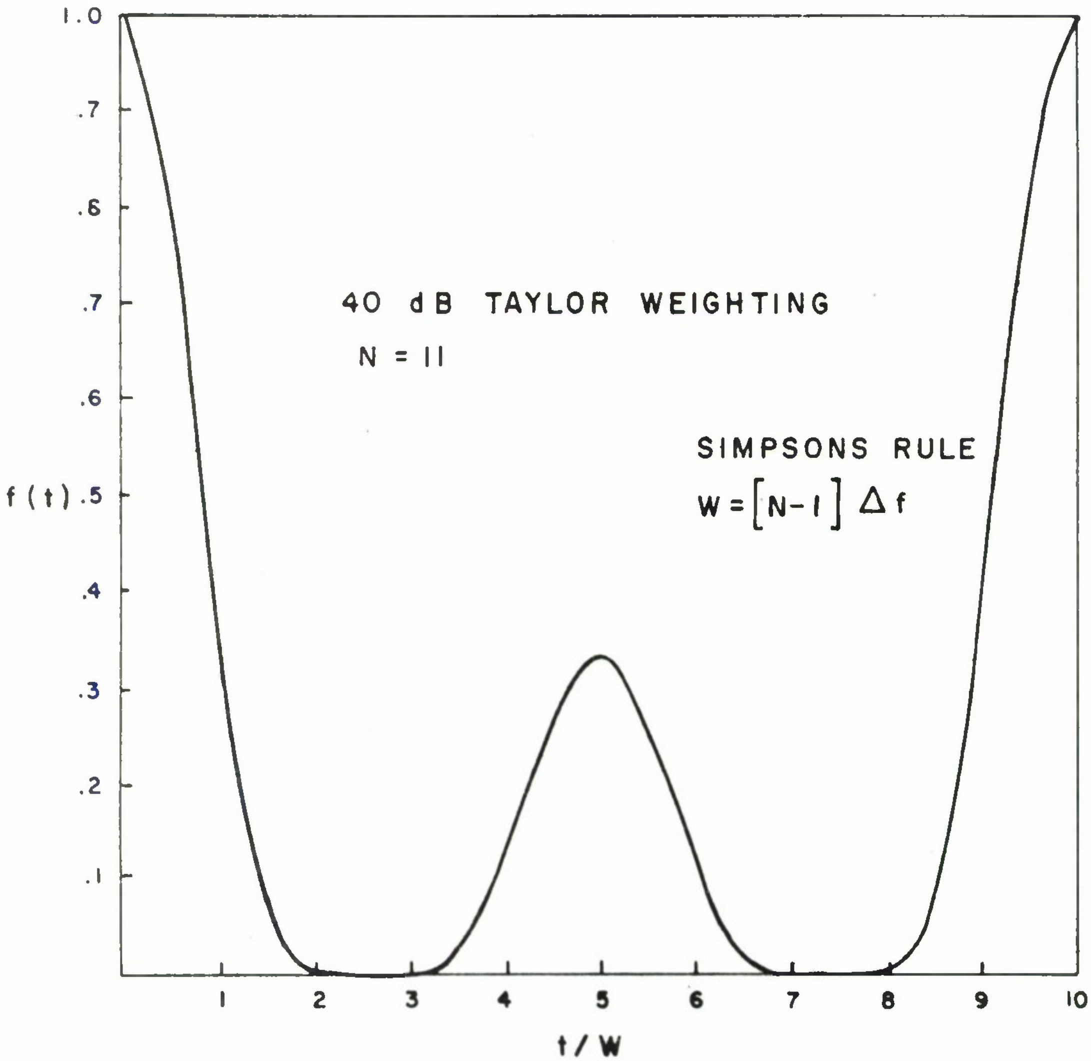

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Figure 6 TRANSFORMED TAYLOR WEIGHTING FUNCTION

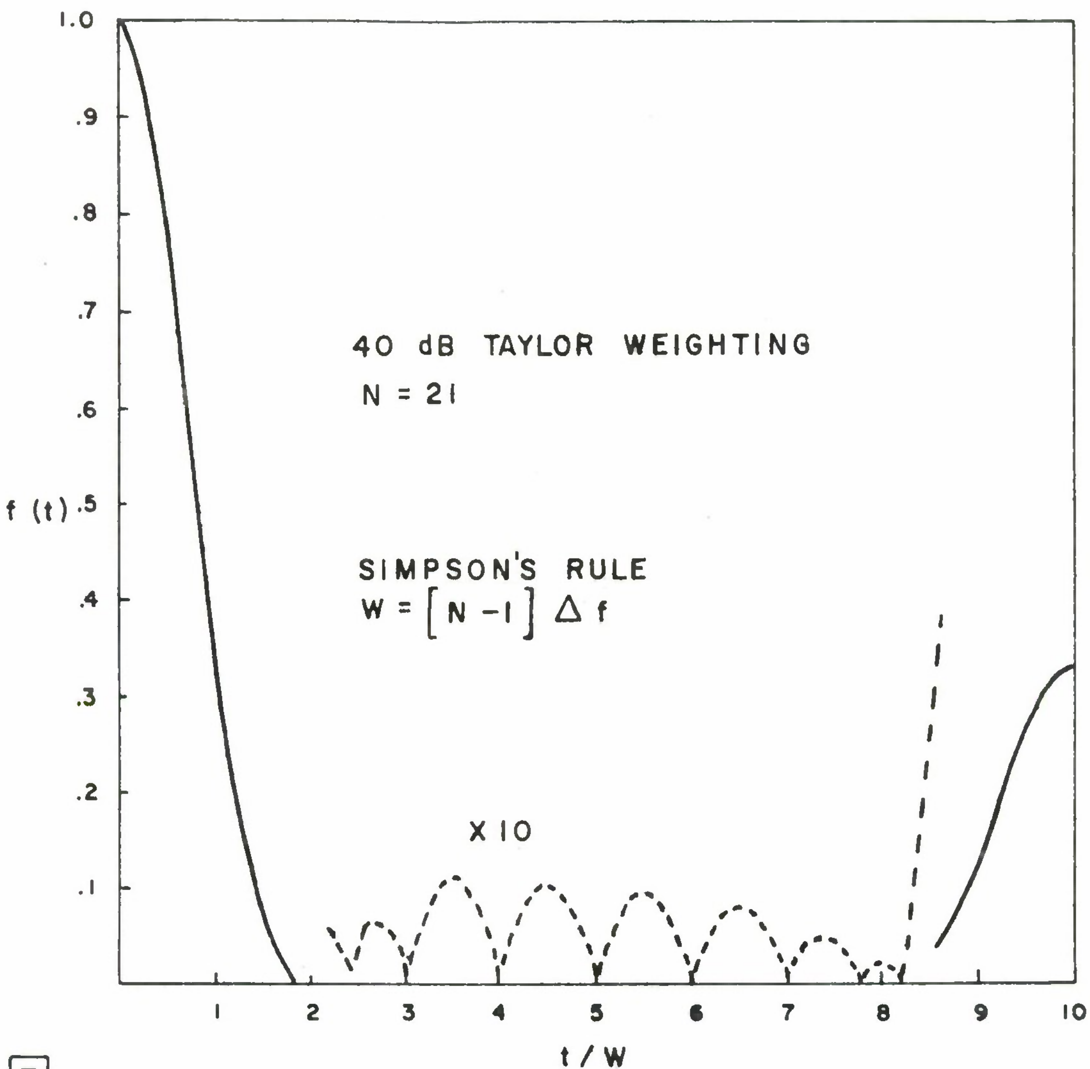

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Figure 7 TRANSFORMED TAYLOR WEIGHTING FUNCTION

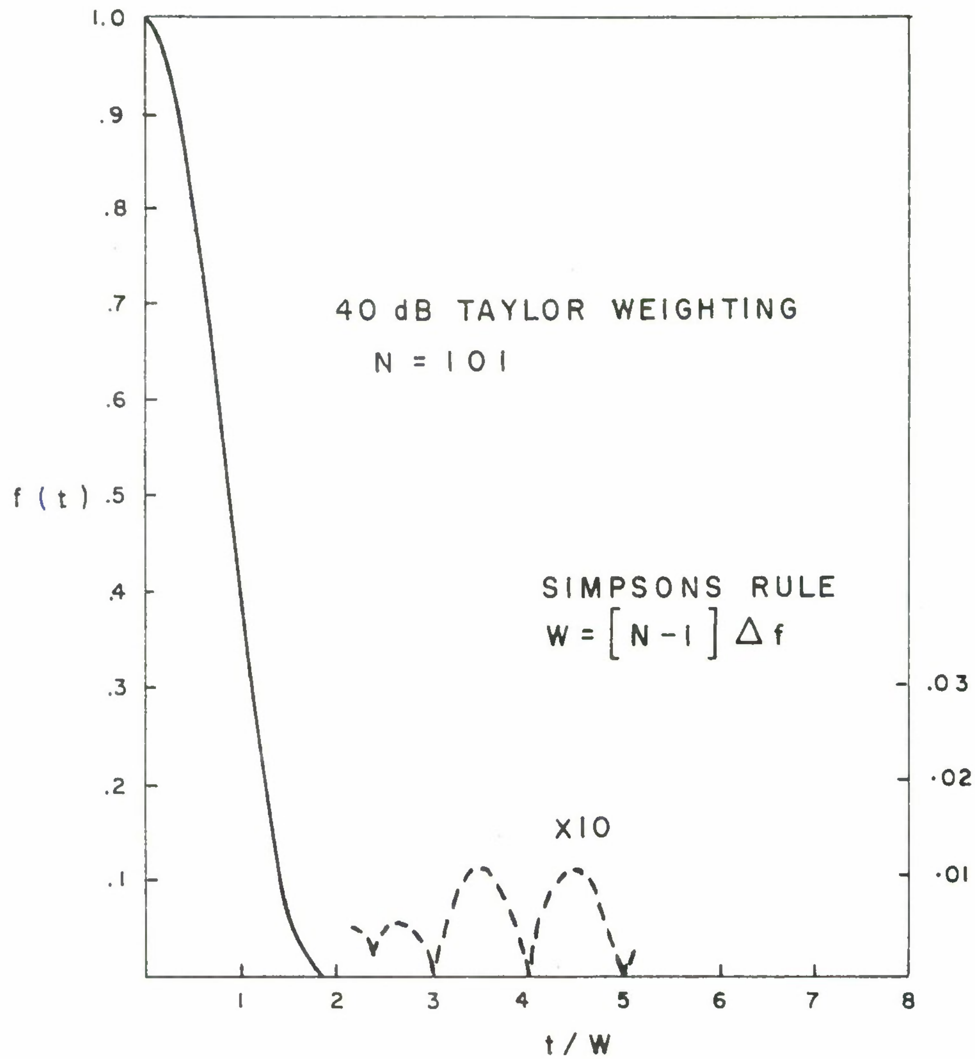

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Figure 8. TRANSFORMED TAYLOR WEIGHTING FUNCTION

SECTION IV

SUMMARY

The integration of band-limited signals which are accurately represented by a set of samples is a weighted sum of the samples. If the signal is a pulse so that most of its energy is confined in time and frequency, the integral is simply an unweighted sum of the samples. This algorithm corresponds to the rectangular integration rule described in numerical analysis.

The above results were applied to the Fourier integral. It was shown that Simpson's rule and other higher-order rules would introduce error sidelobes if a flat spectrum was transformed. The rectangular rule does not introduce this type of distortion. When using the rectangular rule, it is necessary to consider the value of N and the type of signal to be transformed. An unweighted flat spectrum may require $N = 100$ to achieve small errors, while a 40 db weighted spectrum may require an N as small as 25 to achieve small error.

In a signal-processing system the number of samples N of a waveform is usually desired to be as small as possible in order to minimize analog-to-digital converter costs, computation time, etc. The results of this paper apply most readily to signal-processor problems where savings in the number of samples can be realized or where serious sidelobe problems can be avoided.

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13. ABSTRACT

In many computer-implemented signal-processing systems, it is necessary to perform numerical integrations. If the function to be integrated is a band-limited time signal represented by a set of samples, serious distortion can occur if the wrong integration technique is used. In particular, if the integration is used to calculate a Fourier transform, large error sidelobes will be introduced if Simpson's rule or other higher-order rules are used. It is shown that the rectangular rule introduces no distortion and gives a result which is consistent with the number of samples.

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