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OCTOBER 1967

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COMPOUND SIMPLE GAMES, III:  
ON COMMITTEES

Lloyd S. Shapley

PREPARED FOR:  
UNITED STATES AIR FORCE PROJECT RAND

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-iii-

### PREFACE

The mathematical research presented in this Memorandum deals with the structural properties of a certain basic type of multiperson game. Though closely related in subject matter, this Memorandum may be read independently of the two preceding ones in the series (Refs. [5] and [6]), which treated the same class of games from the standpoint of a particular solution concept. The expected applications of this work lie in the direction of organization theory.

A portion of the present research was carried out during the author's consultantship for the Western Management Science Institute at the UCLA Graduate School of Business Administration, and some of the results were presented at a conference entitled "Modern Methodology: New Methods of Thought and Procedure," held at the California Institute of Technology in May 1967.

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ABSTRACT

This is an investigation of the structural properties of those multi-person games, called "simple", in which every coalition either can win outright or is completely defeated. The central idea is the concept of a "committee", which may be characterized roughly as a set of players whose internal politics are independent of the rest of the game. The possible relationships between different committees in the same game are explored: co-existing committees may be disjoint and independent, or one committee may contain another; but only under special circumstances can two committees overlap without inclusion.

The principal result is to establish that every simple game can be decomposed into a hierarchy of "prime" games (i. e. , committee-free games), in which the player-positions are filled either by individual players or by other prime games or sums or products thereof, and that this decomposition is essentially unique.

CONTENTS

PREFACE.....	iii
ABSTRACT.....	v
INTRODUCTION.....	1
Section	
1. SIMPLE GAMES.....	3
2. COMMITTEES.....	6
3. MINIMAL SETS AND DUMMIES.....	8
4. THE SUBSTITUTION PROPERTY.....	12
5. SUBCOMMITTEES.....	17
6. CONTRACTIONS.....	19
7. COMPOUND SIMPLE GAMES.....	23
8. SUMS AND PRODUCTS.....	26
9. OVERLAPPING COMMITTEES.....	29
10. THE UNIQUE PRIME DECOMPOSITION.....	35
REFERENCES.....	40

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## COMPOUND SIMPLE GAMES, III: ON COMMITTEES

### INTRODUCTION

In this paper we offer the reader an opportunity to inspect, at close hand, the substance and methodology of a special corner of descriptive game theory. The "simple games" that populate this area are finite, combinatorial structures that are not only amusing to mathematicians but can serve as abstract representations of voting systems or other group-decision procedures. As such, they have found applications in political science and organization theory, as well as in certain branches of pure mathematics.

The substantive theme of the paper will be the analysis of the structural role played by "committees." The methodological theme will be the intensive use of the language and logic of Boolean algebra and elementary set theory—almost to the exclusion of any other mathematical apparatus. In this respect, the theory of simple games provides a striking example of the trend away from the techniques of classical analysis, as mathematical theories and mathematical model building invade ever-wider areas of the nonphysical sciences.

This paper is not intended as a survey of the theory of simple games, except incidentally. Rather, it works toward a specific goal, a "unique factorization" theorem that describes how a simple game may be decomposed into a hierarchical arrangement of committees,

subcommittees, and individual agents. Since this is a new theorem, we give it a fully rigorous treatment, and almost half of the following text is taken up with the sometimes complicated (but entirely elementary!) proofs that are required to "keep us honest." These proofs may be skipped on first reading without loss of continuity.

The study of simple games was initiated by von Neumann and Morgenstern in their epochal book on game theory, first published in 1944.\* Since then many authors have made many contributions not only to the descriptive theory but to various solution theories and to several domains of application.\*\*

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\*Reference [8] at the end of this Memorandum.

\*\*The author's expository article [4] includes a bibliography complete up to 1961. For more recent work, see e. g., [1], [2], [3], [7].

## 1. SIMPLE GAMES

The theory of simple games is primarily oriented toward problems of organization and control, rather than payoff and strategy. Its point of departure is the primitive notion of "winning coalition." It makes no attempt to treat situations where the costs of winning are significant or partial victories are possible, or where the fruits of victory are not freely divisible within a winning coalition. A simple game may be thought of as an idealized power structure, a voting system, a legislature, or indeed any constituted procedure for arriving at group decisions.

If one wishes to relate simple games to the game-theory models more commonly found in economics or operations research, with their explicit strategies and their numerical payoff functions, one may imagine that there is a lump sum of money to be won, and that the strategic opportunities open to the players are such that it takes coordinated action by the members of some winning coalition in order to capture and divide the prize. To relate simple games to the numerical characteristic-function form introduced by von Neumann and Morgenstern (see [8]), it suffices to assume that the characteristic function takes on only two values: "1" for the winning coalitions and "0" for the others.

The notation  $\Gamma(N, W)$  will be used to denote a simple game. Here  $N$  is a finite set, the players; and  $W$  is a collection of subsets

of  $N$ , the winning coalitions. We lay down three conditions on  $W$ :

- (1a)  $N \in W$ ;
- (1b)  $O \notin W$ ; and
- (1c) If  $S \subseteq T \subseteq N$  and  $S \in W$ , then  $T \in W$ .

The first two merely suppress certain extreme cases, for technical reasons. The third condition, however, expresses a fundamental monotonicity property, inherent in the notion of "winning": any coalition containing a winning coalition must also win. A sort of converse would also be a natural condition:

- (1d) If  $S \cap T = O$  and  $S \in W$ , then  $T \notin W$ ,

i. e., any coalition completely disjoint from a winning coalition must lose. But we do not impose this condition a priori. Games that satisfy (1d) are called proper; all others, improper. Though rarely found in application, improper games play an important role in the structural theory, somewhat analogous to that of imaginary numbers in algebra.

Our notation for simple games is deceptively concise. The double abstraction " $W$ " (a set of sets) embodies in a single symbol a possibly intricate web of voting rules. We have made exhaustive counts of the different simple-game structures that are possible on small sets of players. Excluding games with dummies (see Section 3 below) and games that are merely permutations of games already

counted, we found the following:

No. of players .....	1	2	3	4	5	6
No. of games .....	1	2	5	20	179	?

In the face of this explosive growth, we can hardly expect to explore the possibilities in an effective way without the aid of patterns and symmetries and special classes of games having "nice" properties. It would be helpful to know how to detect and exploit substructures within a game (i. e., "committees") that allow it to be decomposed into smaller games. That this happens often enough to be worthwhile is shown by the following data:

No. of players .....	1	2	3	4	5	6
No. of decomposables ....	0	2	4	14	78	1210
No. of "primes" .....	1	0	1	6	101	?

In this paper we shall achieve a complete analysis of the structural possibilities for decomposition. However, the "detection" problem remains in an unsatisfactory state; an efficient algorithm is sorely needed for finding committees in larger games.

## 2. COMMITTEES

In analyzing simple games for decomposability, the fundamental idea is to identify certain subsets of individuals, called "committees", that can be treated en bloc. Then we can separate the internal politics of the committee from the external politics of the rest of the game. In determining whether a given large coalition can win, in a game where there are committees, we do not have to ask which particular committeemen are party to the coalition, but only whether they have the controlling voices in their respective committees. Thus, an ancillary notion of "winning" comes into play, defined inside each committee. In fact, a committee is a simple game in its own right, embedded in the larger game.

Formalizing these ideas, we define a committee of the simple game  $\Gamma(N, W)$  to be another simple game  $\Gamma(C, W_C)$ , with  $O \subset C \subseteq N$ , which is related to the first as follows: for every coalition  $S \subseteq N$  such that:

$$(2a) \quad S \cup C \in W \text{ and } S - C \notin W,$$

we have:

$$(2b) \quad S \in W \text{ if and only if } S \cap C \in W_C.$$

Condition (2a) expresses that the participation of members of the committee  $C$  is crucial to the success of  $S$ . Condition (2b) expresses that the effect of their participation is entirely determined by the rules of the committee game  $\Gamma(C, W_C)$ .

By extension, the word "committee" will also be used for the set C, whenever a game  $\Gamma(C, W_C)$  exists that is a committee in the sense just defined. (No confusion should result from this; see the corollary at the end of the next section.)

A game always has certain extreme committees, namely the "committee of the whole" and the committees consisting of single individuals. By a proper committee we shall mean one of intermediate size. Obviously, only proper committees can lead to significant decomposition of the game structure.

### 3. MINIMAL SETS AND DUMMIES

If  $W$  is any collection of subsets of  $N$ , we shall denote by  $W^m$  the collection of minimal elements of  $W$ —i. e., those sets in  $W$  that have no proper subsets in  $W$ . If  $W$  is known to be monotonic (condition (1c)), then  $W$  can easily be reconstructed from a knowledge of  $W^m$ . Accordingly, we shall often use just the minimal winning coalitions to identify a particular game.

The abbreviation  $W^m(C)$  will be much used in the sequel, standing for the collection of all coalitions in  $W^m$  that "meet"  $C$ , i. e., that have a nonempty intersection with  $C$ .  $W^m(C)$  should not be confused with  $W_C^m$ , the set of minimal elements of  $W_C$ .

A player belonging to no minimal winning coalition is called a dummy, since he never makes any difference to the status of a coalition. Any set of dummy players is vacuously a committee by our definition, since the hypothesis (2a) is never met. Such an all-dummy committee is called inessential; all others essential. The internal rules of an inessential committee are quite arbitrary, being irrelevant to the game as a whole.

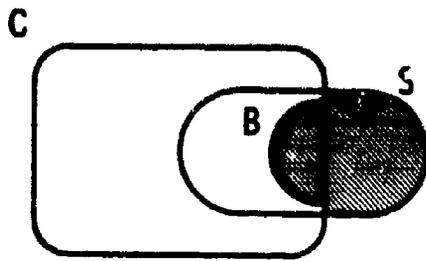
A player who is a dummy in the committee game is obviously a dummy in the full game too. Conversely, a dummy in the full game is a dummy in any essential committee to which he may happen to belong. If a dummy is dropped from, or added to, any committee, the resulting set remains a committee.

Our first theorem gives the relation between the minimal winning coalitions of a committee and the minimal winning coalitions of the full game.

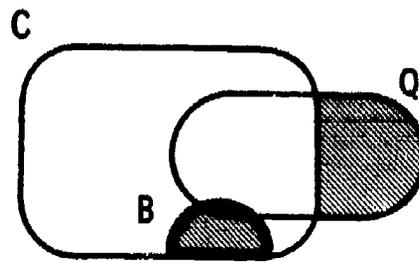
**THEOREM 1.** Let  $\Gamma(C, W_C)$  be a committee of  $\Gamma(N, W)$ . Then for every  $S \in W^m(C)$  there exists  $B \in W_C^m$  such that  $S \cap C = B$ . Conversely, for every  $B \in W_C^m$  there exists  $S \in W^m$  such that  $S \cap C = B$ , unless the committee is inessential.

Proof. (a) Given  $S \in W^m(C)$ , we have  $S \cup C \in W$  by (1c) and  $S - C \notin W$  by the minimality of  $S$ . Hence (2a) applies, and  $S \cap C \in W_C$  by (2b). Hence there exists  $B \subseteq S \cap C$  with  $B \in W_C^m$ . Let  $T = (S - C) \cup B$  (see the diagram); then (2a) holds for  $T$ , and  $T \in W$  by (2b). But  $T \subseteq S$ ; hence  $T = S$  by the minimality of  $S$ . Hence  $S \cap C = T \cap C = B$ , as required.

(b) Unless  $C$  is inessential there is a nondummy in  $C$ , and hence a set  $Q \in W^m(C)$ . We have  $Q \cup C \in W$  by (1c) and  $Q - C \notin W$  by the minimality of  $Q$ . Given  $B \in W_C^m$ , define  $R = (Q - C) \cup B$  (see diagram). Then  $R \cup C = Q \cup C$  and  $R - C = Q - C$ ; hence (2a) holds for  $R$ , and  $R \in W$  by (2b). Hence there exists  $S \subseteq R$  with  $S \in W^m$ . By (1c) we see that  $S \cup C \in W$  and also  $S - C \notin W$ ; hence  $S \cap C \in W_C$  by (2a), (2b). But  $S \cap C \subseteq R \cap C = B$ ; hence  $S \cap C = B$  by the minimality of  $B$ . This completes the proof.



T shaded  
(a)



R shaded  
(b)

Diagrams for Theorem 1

Theorem 1 reveals that except for the inessential case the elements of  $W_C^m$  are precisely the intersections of the elements of  $W^m(C)$  with  $C$ . Thus the following corollary holds, justifying our double use of the term "committee":

COROLLARY. Each committee set  $C$ ,  
unless it consists entirely of dummies,  
corresponds to a unique committee game  
 $\Gamma(C, W_C)$ .

#### 4. THE SUBSTITUTION PROPERTY

The connection between  $W^m(C)$  and  $W_C^m$  proves to be even closer than Theorem 1 would indicate. Indeed, the next theorem asserts that we may take any element of  $W_C^m$  and adjoin to it any set of the form  $S-C$  with  $S \in W^m(C)$ , and the result will be an element of  $W^m$ .

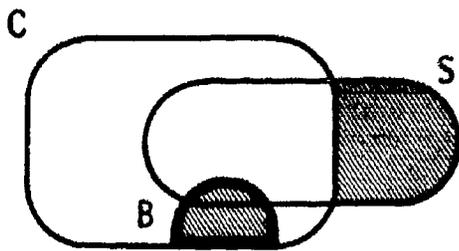
**THEOREM 2.** Let  $\Gamma(C, W_C)$  be a committee of  $\Gamma(N, W)$ . Then

$$(4a) \quad W^m(C) = \{ BU(S-C) \mid B \in W_C^m \text{ and } S \in W^m(C) \} .$$

Proof. Theorem 1 tells us at once that " $\subseteq$ " holds in (4a); it remains to show that " $\supseteq$ " holds in order to establish equality. In other words, we must consider an arbitrary  $B \in W_C^m$  and an arbitrary  $S \in W^m(C)$  and show that  $BU(S-C) \in W^m(C)$ . We shall do this in two steps.

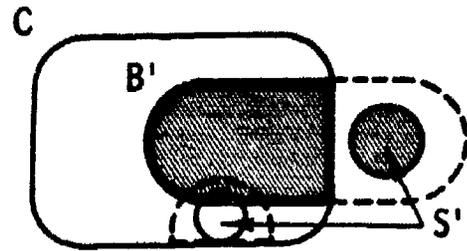
(a) Write  $T$  for  $BU(S-C)$ . (See the diagram.) We have  $T \cup C \in W$  by (1c); also  $T-C \not\in W$  by the minimality of  $S$ . Hence (2a) holds for  $T$ , and we have  $T \in W$  if and only if  $T \cap C \in W_C$ . But  $T \cap C = B \in W_C^m$ ; hence  $T \in W$ .

(b) To show that  $T$  is minimal, let  $S' \subseteq T$  with  $S' \in W^m$ . Then we must have  $S' \in W^m(C)$ , since  $S'-C \subseteq T-C \not\in W$ . Write  $B'$  for  $S' \cap C$  (see the diagram); by Theorem 1 we have  $B' \in W_C^m$ . We can therefore



T shaded

(a)



T' shaded

(b)

Diagrams for Theorem 2

repeat the argument of (a), with primed letters, and obtain  $T' \in W$ , where  $T' = B' \cup (S' - C)$ . But  $T' \subseteq S$ ; hence  $T' = S$  by the minimality of  $S$ . Thus we have

$$S' - C = T' - C = S - C = T - C.$$

Also, we have  $S' \cap C \in W_C^m$  by Theorem 1. But  $S' \cap C \subseteq B$ ; hence

$$S' \cap C = B = T \cap C,$$

by the minimality of  $B$ . The two displayed equations established that  $S' = T$ , and hence that  $T \in W^m$ . Finally, since  $T \supseteq B \supseteq O$ , we have  $T \in W^m(C)$ . This completes the proof.

Theorem 2 enables us to "substitute" the portion of any  $S \in W^m$  that lies within a committee for the portion of any other  $S' \in W^m$  that lies within that committee (provided that both portions are nonempty), with the assurance that the resulting coalition is also minimal winning. According to the next theorem, this doesn't work for any set that is not a committee. In other words, the substitution property is a sufficient as well as a necessary condition for committeehood.

**THEOREM 3.** Let  $\Gamma(N, W)$  be a simple game, and let  $C \subseteq N$ . Then  $C$  is a committee if and only if, for every  $S, S' \in W^m(C)$ ,

$$(4b) \quad (S \cap C) \cup (S' - C) \in W^m.$$

Proof. If  $C$  is a committee, the result is immediate from Theorem 2. Conversely, suppose that (4b) holds for all  $S, S' \in W^m(C)$ . Possibly  $W^m(C)$  is empty; in that case the players in  $C$  are all dummies, and  $C$  is a committee (inessential). Otherwise, let  $Q$  be a fixed element of  $W^m(C)$  and define the collection  $W_C$  to consist of all sets  $B \subseteq C$  such that  $B \cup (Q - C) \in W$ . Clearly  $\Gamma(C, W_C)$  is a simple game, in the sense of (1a), (1b), (1c); we must verify that it is a committee of  $\Gamma(N, W)$ .

Referring to the definition of committee, we see that we must show that " $T \in W$ " is equivalent to " $T \cap C \in W_C$ ", for every  $T$  with  $T \cup C \in W$  and  $T - C \notin W$ . Thus, suppose  $T \in W$ . Then we can find  $S \in W^m$  with  $S \subseteq T$ .  $S$  must meet  $C$ , since  $S - C \subseteq T - C \notin W$ . Applying (4b), we substitute  $S$  for  $Q$  in  $C$  and obtain  $(S \cap C) \cup (Q - C) \in W^m$ . Hence  $(T \cap C) \cup (Q - C) \in W$ . Hence, by the definition of  $W_C$ , we have  $T \cap C \in W_C$ .

For the other direction, suppose  $T \cap C \in W_C$ . Then  $(T \cap C) \cup (Q - C) \in W$ , and we can find  $S \in W^m$  with  $S \subseteq (T \cap C) \cup (Q - C)$ . Since  $Q$  is minimal,  $S$  must meet  $C$ . Similarly, since  $T \cup C \in W$ , we can find  $S' \in W^m$  such that  $S' \subseteq T \cup C$ ; and since  $S' - C \subseteq T - C \notin W$ ,  $S'$  must meet  $C$ . Applying the substitution (4a), we obtain  $(S \cap C) \cup (S' - C) \in W^m$ . But

$$\begin{aligned} (S \cap C) \cup (S' - C) &\supseteq [(T \cap C) \cup (Q - C)] \cap C \cup [(T \cup C) - C] \\ &= (T \cap C) \cup (T - C) \\ &= T. \end{aligned}$$

Hence  $T \in W$ . This completes the proof.

Note that Theorem 3 makes no mention of the committee game  
 $\Gamma(C, W_C)$ . This eliminates much of the clumsiness involved in  
testing committeehood directly from the definition.

5. SUBCOMMITTEES

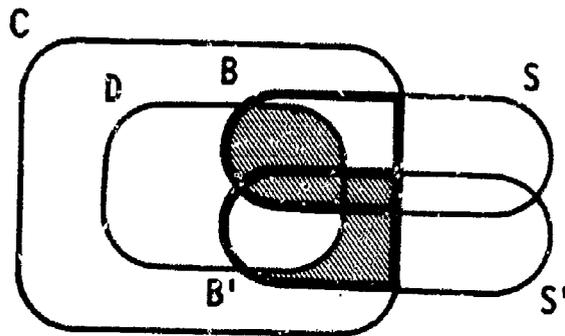
**THEOREM 4.** Let  $C$  be an essential committee of  $\Gamma(N, W)$  with winning sets  $W_C$ , and let  $D \subseteq C \subseteq N$ . Then  $D$  is a committee of  $\Gamma(N, W)$  if and only if  $D$  is a committee of  $\Gamma(C, W_C)$ .

Proof. (a) Suppose  $D$  is a committee of  $\Gamma(N, W)$ . Take any  $B, B' \in W_C^m$  with  $B \cap D \neq \emptyset, B' \cap D \neq \emptyset$ . By Theorem 1 there are  $S, S' \in W^m$  with  $S \cap C = B, S' \cap C = B'$  and we have  $S, S' \in W^m(D)$ . Let  $T = (S \cap D) \cup (S' - D)$ . By Theorem 3 we have  $T \in W^m$ . By Theorem 1 we have  $T \cap C \in W_C^m$ . But  $T \cap C = (B \cap D) \cup (B' - D)$ . (See the diagram.) By Theorem 3,  $D$  is a committee of  $\Gamma(C, W_C)$ .

(b) Conversely, suppose  $D$  is a committee of  $(C, W_C)$ . Take any  $S, S' \in W^m(D)$  and define  $B = S \cap C, B' = S' \cap C$ . By Theorem 1,  $B \in W_C^m$  and  $B' \in W_C^m$ . By Theorem 3,  $(B \cap D) \cup (B' - D) \in W_C^m$ . By Theorem 1, there is  $T \in W^m(C)$  with  $T \cap C = (B \cap D) \cup (B' - D)$ . By Theorem 3, we have  $(T \cap C) \cup (S' - C) \in W^m$ . But

$$\begin{aligned} (T \cap C) \cup (S' - C) &= (B \cap D) \cup (B' - D) \cup (S' - C) \\ &= [(S \cap C) \cap D] \cup [(S' \cap C) - D] \cup (S' - C) \\ &= (S \cap D) \cup (S' - D) \end{aligned}$$

(same diagram illustrates). A final application of Theorem 3 shows that  $D$  is a committee of  $\Gamma(N, W)$ .



TNC shaded

Diagram for Theorem 4

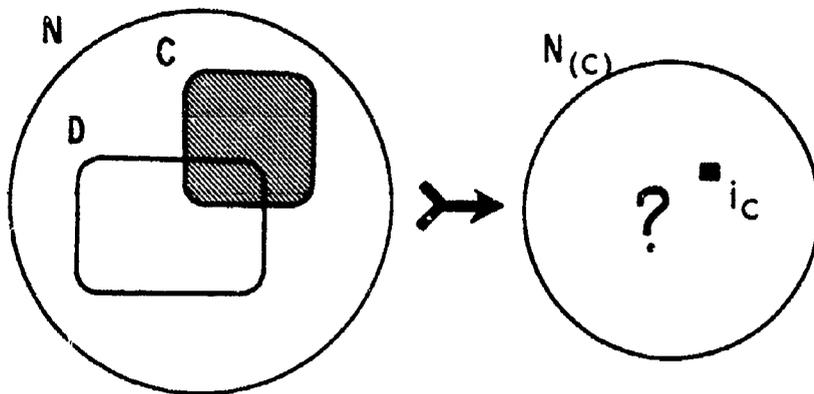
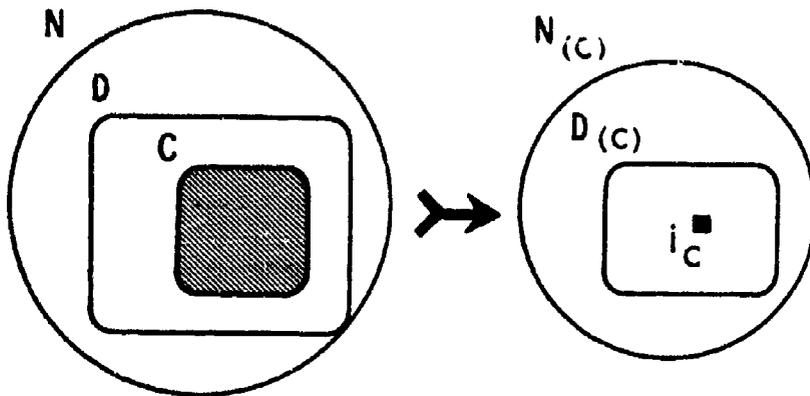
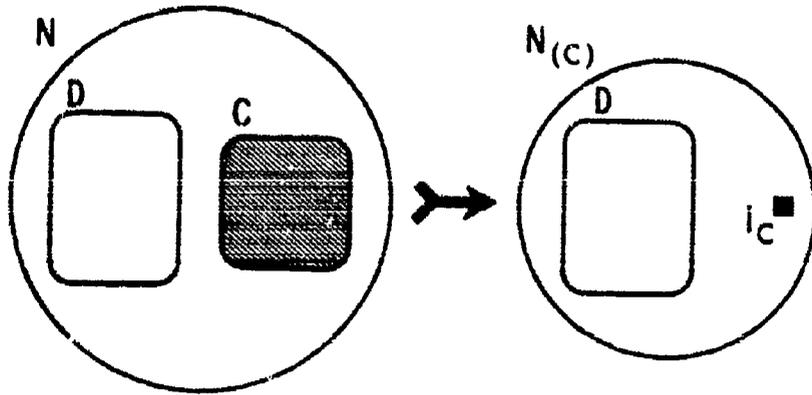
## 6. CONTRACTIONS

By the "contraction" of a game on one of its committees, we shall mean the game obtained by coalescing the committee into a single player. More specifically: (a) the members of the committee are dropped from the game; (b) a single new player is introduced; (c) coalitions containing the new player win if and only if, with that player replaced by the whole committee, they won in the original game; and (d) coalitions not containing the new player win if and only if they won in the original game.

The next two theorems describe a contraction's effect on committees that either are disjoint from or include the committee on which the contraction takes place. (See the figure.) We omit the simple proofs.

**THEOREM 5.** Let  $C$  be a committee of  $\Gamma(N, W)$  and let  $D \subseteq N - C$ . Then  $D$  is a committee of  $\Gamma(N, W)$  if and only if  $D$  is a committee of the contraction of  $\Gamma(N, W)$  on  $C$ .

**THEOREM 6.** Let  $C$  be a committee of  $\Gamma(N, W)$  and let  $C \subseteq D \subseteq N$ . Let  $D_{(C)}$  denote the set  $(D - C) \cup \{i_C\}$ , where  $i_C$  is the new player introduced by contracting on  $C$ . Then  $D$  is a committee of  $\Gamma(N, W)$  if and only if  $D_{(C)}$  is a committee of the contraction of  $\Gamma(N, W)$  on  $C$ .



Diagrams for Theorems 5 and 6

If  $D \subset C$ , then of course  $D$  would disappear in any contraction on  $C$ , and we can infer nothing about  $D$ 's committeehood. There remains the case where  $D$  and  $C$  "overlap", i. e., where the sets  $C \cap D$ ,  $C - D$ , and  $D - C$  are all three nonempty. This is a more complex situation, since contraction on  $C$  now violates the integrity of  $D$ —some of the players in  $D$  are eliminated, others remain. It is not clear intuitively how the committeehood of  $D$  is related, if at all, to the committeehood (in the contracted game) of either  $D - C$  or  $(D - C) \cup \{i_C\}$ , the two most likely candidates for comparison.

It might be hoped that the problem does not really arise, i. e., that the committees of a simple game can never overlap (at least if we ignore dummy players). Then the committees, ordered by inclusion, would form a tree-like hierarchy, and we could determine the "prime" game played by any committee by contracting on all of its proper subcommittees. With no overlapping present, Theorems 5 and 6 would assure that the result would be independent of the order in which the contractions were carried out.

Unfortunately, overlapping does occur, without the aid of dummies, and we have a real problem.\* A unique decomposition based solely on the contraction principle is not attainable, since we would sometimes be forced to decide whether to contract on  $C$  and

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\* For example, in the game  $\Gamma(N, \{N\})$  every subset of  $N$  (except  $\emptyset$ ) is a committee.

spoil D or to contract on D and spoil C. Some further structural concepts must be introduced before we can cope with this difficulty.

## 7. COMPOUND SIMPLE GAMES

A new notational device will be useful at this point. Let  $H_1, \dots, H_m$  be simple games having disjoint sets of players, and let  $K$  be another simple game having exactly  $m$  players, indexed by the numbers from 1 to  $m$ . Then we shall let the symbol

$$K[H_1, \dots, H_m]$$

represent the compound game, defined by taking as players all the players of the component games  $H_i$ , and by taking as winning coalitions all sets that include winning contingents from enough of the components to spell out, by their indices, a winning coalition of the quotient game  $K$ .

Repeating this definition more formally, let  $M = \{1, \dots, m\}$ ; let  $\{N_i \mid i \in M\}$  be a collection of disjoint sets with union  $N$ ; let  $H_i = \Gamma(N_i, W_i)$ , all  $i \in M$ ; and let  $K = \Gamma(M, U)$ . Then  $K[H_1, \dots, H_m]$  is a compound representation of the game  $\Gamma(N, W)$ , where  $W$  is defined as the collection of all  $S \subseteq N$  such that  $\{i \mid S \cap N_i \in W_i\} \in U$ .

If the quotient is a one-person game, or if all of the components are one-person games, then "compounding" is a trivial operation. The compound is the same as the lone component in the first case, and the same as the quotient in the second case. We say that a game is decomposable only if it possesses a nontrivial compound representation. We say that a game is prime if and only if it is not decomposable.

What is the connection between compounds and committees? Clearly, each component in a compound representation is a committee of the compound game. Conversely, any committee of a simple game can be made a component of some compound representation of the game. In fact, we can use the contraction on the committee for a quotient, and let all the other components be one-person games. Of course, if we start with a trivial committee (one-player or all-player), then we get the trivial compound representations just described. Thus, the decomposable games are exactly those that possess proper committees.

In the case of a one-person component, we shall often write the name of the player rather than the name of the game in the compound-representation symbol. This will give us, in particular, a way of displaying the names of the players in any game, thus:

$$G[ p_1, \dots, p_m ].$$

Subcommittees can be displayed by using compound representations as components. For example, the symbol

$$K[ H_1 [ G_1, G_2, p, q ], H_2, H_3, r, s ]$$

reveals that  $G_1$  and  $G_2$  are committees of  $H_1 [ G_1, G_2, p, q ]$ , which is in turn a committee of the full game.

For a further example, let us consider the following five-person compound simple game:

$$M_3[p, M_3[q, r, s], t].$$

Here "M<sub>3</sub>" denotes the simple majority game on three players.

The reader will verify that there are exactly seven minimal winning coalitions, namely:

$$pqr, pqs, prs, pt, qrt, qst, rst.$$

This is of course a decomposable game, since  $qrs$  is a proper committee. But might there be other proper committees, not revealed by the given compound representation? The game  $M_3$  is prime, but how can we be sure that there are not other compound representations, in this and similar situations, that distribute the players into components in a completely different way?

This is the question that motivated the present study. Compound simple games were introduced many years ago, and their solutions have been extensively studied.\* But the question of the uniqueness of a compound representation with prime quotients has been elusive, and can apparently be resolved only by going to the more fundamental notion of "committee" that we have introduced in this paper.

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\*References will be found in the papers cited at the end of this Memorandum.

### 8. SUMS AND PRODUCTS

As our notation indicates, a quotient is essentially a function. Since both the arguments and the values are games, however, it can also be regarded as an operation, acting on games in much the same way as Boolean operations act on sets, or as logical operations act on truth values. Two extreme cases, among the possible quotients, play a special part in the theory, and it will be convenient to represent them as operations rather than functions. The first, denoted by  $\oplus$ , corresponds to quotients having the maximum possible number of winning coalitions; the second, denoted by  $\otimes$ , corresponds to quotients having the least possible number of winning coalitions.

Specifically, we define the sum of  $m$  games:

$$H_1 \oplus \dots \oplus H_m, \quad m \geq 2,$$

to be the compound game  $K[H_1, \dots, H_m]$  where  $K = \Gamma(M, \{S \subseteq M \mid S \neq \emptyset\})$ . That is, a coalition wins in a sum of games whenever it contains a winning contingent from at least one of them.

In similar fashion, we define the product of  $m$  games:

$$H_1 \otimes \dots \otimes H_m, \quad m \geq 2,$$

to be the compound game  $L[H_1, \dots, H_m]$  where  $L = \Gamma(M, \{\emptyset\})$ . That is, a coalition wins in a product of games only if it contains winning contingents from all of them.

These operations are obviously associative and commutative, provided that we properly identify the players after any re-ordering. A distributive law relating them cannot even be stated, however, so long as we require the player sets of different components in a compound to be disjoint. But there is a duality that can be developed between  $\oplus$  and  $\otimes$ , analogous to the Boolean duality between  $\cup$  and  $\cap$  and the logical duality between "or" and "and".

It is worth noting that one-person games are not trivial building blocks in the formation of sums and products. In fact, there is an interesting class of games that is generated by repeated applications of the operations  $\oplus$  and  $\otimes$ , in which all the components are one-person games.

A sum is distinguished by the fact that its minimal winning coalitions are precisely the minimal winning coalitions of its components. No minimal winning coalition meets more than one component.\* Conversely, any game whose minimal winning coalitions "fit" within a partition in this manner is decomposable as a sum. It is easy to see that such a game has a unique representation as a sum of games that are not themselves sums—we merely take the finest partition that does not split any minimal winning coalition.

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\* It follows that a sum is always an improper game.

A similar unique decomposition holds for products. (This holds from the duality between the two operations, mentioned above, or can be proved directly.) Since no game can be both a sum and a product, there is even a unique decomposition of any game into a polynomial expression in  $\oplus$  and  $\otimes$ , the components of which are indecomposable with respect to both operations.

For later reference, we note that there is a simple test that tells whether a given committee gives rise to a product decomposition of its parent game, that is, whether  $\Gamma(N, W)$  has a compound representation of the form  $\Gamma(C, W_C) \otimes H$  for some  $H$ , where  $C$  is the given committee. In fact, such a decomposition exists if and only if  $W^m(C) = W^m$  and there is a nondummy in  $N-C$ . The proof is a simple application of Theorem 2.\* A committee that passes this test will be called a factoring committee. If  $C$  is a factoring committee, then obviously so is its complement  $N-C$ .

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\*For sums, the corresponding condition is  $W^m(C) = W_C^m$  and a nondummy outside  $C$ ; or, more succinctly,  $W^m \supset W_C^m$  (note the strict inclusion).

### 9. OVERLAPPING COMMITTEES

The seeming digression of the last two sections has equipped us to deal with the problem of overlapping committees. In fact, the next theorem reveals that when committees overlap it is either due to the presence of "floating" dummy players, who can safely be disregarded, or it is due to the associativity of the operations  $\oplus$  and  $\otimes$ , since when three or more components are added or multiplied together the partial sums or products will involve overlapping sets of players. Except for these cases, committees do not overlap.

We shall say that  $C$  and  $D$  "overlap essentially" if each of the sets  $C \cap D$ ,  $C - D$ , and  $D - C$  contains a nondummy.

**THEOREM 7.** Let  $C$  and  $D$  be committees of  $\Gamma(N, W)$  that overlap essentially. Write  $E$  for  $C \cup D$ , and write  $E_1$ ,  $E_2$ ,  $E_3$  for  $C \cap D$ ,  $C - D$ , and  $D - C$  respectively (see the diagram). Then  $E$  is also a committee of  $\Gamma(N, W)$ , and the committee game  $\Gamma(E, W_E)$  is either the sum or the product of the games  $\Gamma(E_i, W_{E_i})$ ,  $i = 1, 2, 3$ , where  $W_{E_i}$  denotes the set of nonempty intersections of  $W$  with  $E_i$ .

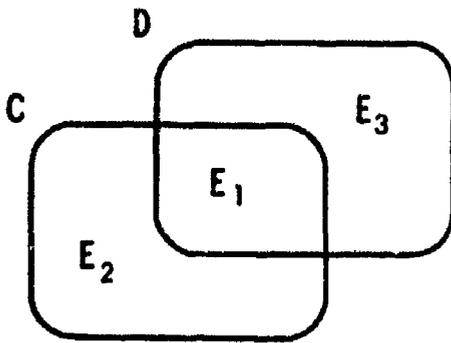


Diagram for Theorem 7

Proof. (a) Assume first that no element of  $W^m$  meets more than one of  $E_1, E_2, E_3$ . To show that  $E$  is a committee we must show that  $(S \cap E) \cup (S' - E) \in W^m$  for all  $S, S' \in W^m(E)$ . Without loss of generality,  $S'$  meets  $C$ .

Subcase i: If  $S$  also meets  $C$ , then we can substitute  $S$  for  $S'$  in  $C$  and obtain

$$(S \cap C) \cup (S' - C) \in W^m.$$

But by our initial assumption  $S$  and  $S'$  cannot meet  $E_3$ ; hence  $S \cap C = S \cap E$  and  $S' - C = S' - E$ , and we have  $(S \cap E) \cup (S' - E) \in W^m$ .

Subcase ii: If  $S$  does not meet  $C$  then  $S$  must meet  $D$ . Find a  $T \in W^m$  that meets  $E_1$  (such a set exists because  $E_1$  contains a nondummy). Then  $T$  meets  $C$ , and we can substitute  $T$  for  $S'$  in  $C$  and obtain

$$(T \cap C) \cup (S' - C) \in W^m.$$

But this set—call it  $R$ —meets  $E_1$  and hence  $D$ , and so we can substitute  $S$  for  $R$  in  $D$  and obtain

$$(S \cap D) \cup (R - D) \in W^m.$$

But  $S \cap D = S \cap E$  and  $R - D = S' - C = S' - E$ , and we again have  $(S \cap E) \cup (S' - E) \in W^m$ . This shows that  $E$  is a committee. Moreover, by our initial assumption, the minimal winning coalitions of  $\Gamma(E, W_E)$  "fit" within the partition  $\{E_1, E_2, E_3\}$ , so that by the

remarks in Section 8 we have

$$\Gamma(E, W_E) = \Gamma(E_1, W_{E_1}) \oplus \Gamma(E_2, W_{E_2}) \oplus \Gamma(E_3, W_{E_3}).$$

(b) Now suppose that the initial assumption of (a) is not satisfied, so that there exists a set  $Q \in W^m$  that meets at least two of  $E_1, E_2, E_3$ . Without loss of generality,  $Q$  meets  $E_3$  and also meets  $C$ . Let  $S_1 \in W^m(E_1)$ ,  $S_2 \in W^m(E_2)$  (such sets exist by the nondummy assumption). Substituting  $S_2$  for  $Q$  in  $C$ , we obtain

$$(S_2 \cap C) \cup (Q - C) \in W^m.$$

Substituting  $S_1$  for this set in  $D$ , we obtain

$$(S_1 \cap E_1) \cup (S_2 \cap E_2) \cup (S_1 \cap E_3) \cup (Q - E) \in W^m.$$

In the committee  $C = E_1 \cup E_2$ , we therefore have

$$(S_1 \cap E_1) \cup (S_2 \cap E_2) \in W_C^m,$$

as well as

$$S_1 \cap C = (S_1 \cap E_1) \cup (S_1 \cap E_2) \in W_C^m$$

and

$$S_2 \cap C = (S_2 \cap E_1) \cup (S_2 \cap E_2) \in W_C^m,$$

all by Theorem 1. Consider these three elements of  $W_C^m$ . Since both  $S_1 \cap E_1$  and  $S_2 \cap E_2$  are nonempty, we must also have  $S_2 \cap E_1$

and  $S_1 \cap E_2$  nonempty, to avoid contradicting minimality. Thus we have shown that every  $S \in W^m$  that meets  $E_1$  also meets  $E_2$ , and vice versa.

$Q$  was chosen unsymmetrically, as between  $C$  and  $D$ , but we now know that there are sets in  $W^m$ , for example  $S_2$ , that meet both  $E_2$  and  $D$ . Using such a set in place of  $Q$ , we can repeat the above argument and establish that every  $S \in W^m$  that meets  $E_1$  also meets  $E_3$ , and vice versa. In other words, every element of  $W^m(E)$  meets all three of  $E_1, E_2, E_3$ .

It is now easily shown by a substitution argument that  $E$  is a committee.  $C$  is therefore a committee of  $\Gamma(E, W_E)$ , by Theorem 4. Applying the test at the end of Section 8 we find that  $C$  is a factoring committee of  $\Gamma(E, W_E)$ . Hence the complementary set  $E - C = E_3$  is also a factoring committee. Similarly  $E_2$ , the complement of  $D$  in  $E$ , is a factoring committee; hence also  $E_2 \cup E_3$  and finally  $E - (E_2 \cup E_3) = E_1$  are factoring committees. Thus we have  $\Gamma(E, W_E) = \Gamma(E_1, W_{E_1}) \otimes \Gamma(E_2, W_{E_2}) \otimes \Gamma(E_3, W_{E_3})$ . This completes the proof.

**COROLLARY.** Let  $G$  be a simple game without dummies. Then either  $G$  is a sum or product, or the maximal proper committees of  $G$  are disjoint.

Proof. Let  $C$  and  $D$  be maximal proper committees,  $C \neq D$ , and assume that  $C \cap D \neq \emptyset$ . We have  $C \not\subseteq D$  and  $D \not\subseteq C$  by maximality. Hence  $C$  and  $D$  overlap essentially, there being no dummies. Hence  $E = C \cup D$  is a committee. But since  $E$  strictly contains the maximal proper committees  $C$  and  $D$ , it can only be an improper committee, i. e., the all-player set of  $G$ . But this means that  $G$  itself is a sum or product.

## 10. THE UNIQUE PRIME DECOMPOSITION

Theorem 7 and its corollary pave the way for a systematic determination of the entire decomposition pattern of a simple game, once the committees are known.\* The process starts with the full game  $G$  (from which we shall assume the dummies have been eliminated), and works down the hierarchy of committees until the individual players are reached.

At the first step there are four mutually exclusive possibilities:  $G$  is a sum;  $G$  is a product;  $G$  is decomposable but not a sum or product;  $G$  is prime.

(a)  $G$  is a sum. Then in the next level we install the components  $\{G_i \mid i = 1, \dots, m\}$  of the finest sum decomposition of  $G$ . Thus, none of the  $G_i$ 's are themselves sums.

It is necessary to prove here that we have not chopped too fine, i. e., that no committee of  $G$  overlaps any of the player-sets  $\{N_i\}$  of the  $\{G_i\}$ . Suppose therefore that  $C$  is a committee that overlaps  $N_1$ , and hence also overlaps the set  $N_2 \cup \dots \cup N_m$ . By Theorem 7, the set  $E = C \cup (N_2 \cup \dots \cup N_m)$  is a committee. Applying the theorem again to  $N_1$  and  $E$ , we see that the whole game  $G$  is the

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\* In principle this is no problem, since the number of potential committee games to be checked out is finite. However, it would be desirable to have an efficient combinatorial algorithm that would discover committees, given a list of the minimal winning coalitions or other description of the game. At present, this can only be done for sums and products.

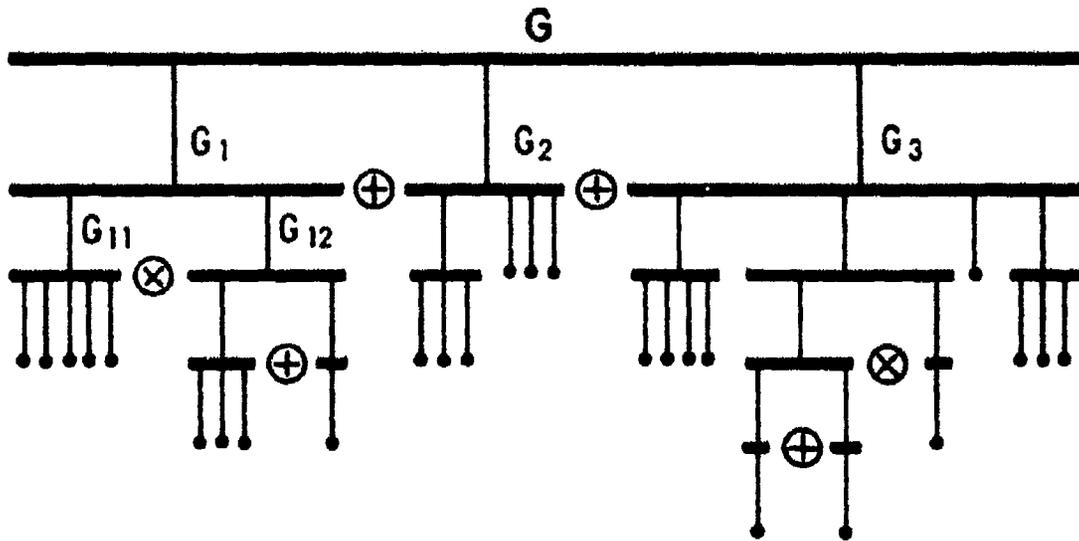
sum or product of games on the three sets  $N_1 \cap C$ ,  $N_1 - C$ , and  $N_2 \cup \dots \cup N_m$ . In fact, it is the sum, since no game is both a sum and a product. But this means that the component  $G_1$  decomposes into a sum, contrary to hypothesis. Hence an overlapping committee like D is impossible, and we can assert that every committee of G that is not a  $G_1$  or a sum of  $G_1$ 's is actually a subcommittee of one of them.

(b) G is a product. Then the components of the finest product decomposition comprise the next level. As above, we can assert that the committees of G that are not products of these components must be committees of the individual components.

(c) G is decomposable, but not a sum or product. Then the corollary to Theorem 7 tells us that the maximal proper committees of G are disjoint, so we can install them at the next level of the hierarchy, along with any "unaffiliated" individual players that belong to no proper committee of G. Again we can assert that all committees of G not yet represented are subcommittees of the components we have just installed.

(d) G is prime. Then all players are "unaffiliated", so we list them in the next level and stop.

The process then continues by analyzing in the same way each game that appears on the second level, and so on until every branch of the tree has terminated with an individual player.



Committee decomposition diagram:

$$G = (A[a, b, c, d, e] \otimes (B[f, g, h] \oplus C[i]))$$

$$\oplus D[E[j, k, l], m, n, o]$$

$$\oplus F[G[p, q, r, s], (H[t] \otimes I[u]) \oplus J[v], w, K[x, y, z]].$$

The accompanying diagram is virtually self-explanatory. The game  $G$  is a sum of three components  $G_1 \oplus G_2 \oplus G_3$ ; the first of these is a product of two components  $G_{11} \oplus G_{12}$ ; the second happens to be a six-person game having a three-person committee; and so on. The only committees not represented directly in the diagram are the partial sums  $G_1 \oplus G_2$ ,  $G_1 \oplus G_3$ , and  $G_2 \oplus G_3$ .

The compound representation of  $G$  written beneath the diagram is more concise, and in its own way just as descriptive. Note that only quotients and individual players are named. "A" is some five-person game, "D", "F", and "G" are four-person games, and so on. (A minor notational change would save us the trouble of writing down the one-person games.)

The important point is that the quotients that appear in the representation are all prime. For when a quotient has a proper committee, then players in the corresponding components of the compound form a proper committee of that game, which necessarily includes the players of at least two components in the representation. In the present case, however, the components are maximal proper committees (or unaffiliated players), so that this cannot happen.

We have thus established our final result:

**THEOREM 8.** Every simple game has a compound representation that uses nothing but prime quotients and the associative operations

$\oplus$  and  $\otimes$  and that is unique except for the  
arbitrariness in the ordering of players  
and components and in the disposition of  
dummy players.

REFERENCES

1. Isbell, J. R., "Homogeneous games III," in Advances in Game Theory: Annals of Mathematics Study No. 52, Princeton University Press, 1964, 255-265.
2. Owen, Guillermo, "Tensor composition of nonnegative games," in Advances in Game Theory: Annals of Mathematics Study No. 52, Princeton University Press, 1964, 307-326.
3. Riker, W. H., and L. S. Shapley, Weighted Voting: A Mathematical Analysis for Instrumental Judgments, The RAND Corporation, P-3318, March 1966. (To appear in Nomos X: Representation (Yearbook of the American Society for Political and Legal Philosophy), Atherton Press, New York.)
4. Shapley, L. S., Simple Games: An Outline of the Descriptive Theory, The RAND Corporation, P-2277, April 1961. (Also published without bibliography in Behavioral Science, 7 (1962), 59-66.)
5. Shapley, L. S., Compound Simple Games I: Solutions of Sums and Products, The RAND Corporation, RM-3192, June 1962.
6. Shapley, L. S., Compound Simple Games II: Some General Composition Theorems, The RAND Corporation, RM-3643, July 1963.
7. Shapley, L. S., "Solutions of compound simple games," in Advances in Game Theory: Annals of Mathematics Study No. 52, Princeton University Press, 1964, 267-305. (Includes most of the contents of [5] and [6].)
8. von Neumann, J., and O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1944, 1947, 1953.

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