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ERROR PERFORMANCE OF DIFFERENTIAL PHASE SHIFT TRANSMISSION OVER A TELEPHONE LINE

AUGUST 1967

J. J. Bussgang

M. Leiter

Prepared for

DEPUTY FOR SURVEILLANCE AND CONTROL SYSTEMS
AEROSPACE INSTRUMENTATION PROGRAM OFFICE
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

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Project 705B
Prepared by
THE MITRE CORPORATION
Bedford, Massachusetts
Contract AF19(628)-5165

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FOREWORD

The report was prepared by the Range Communications Planning and Technology Subdepartment of The MITRE Corporation, Bedford, Massachusetts, under Contract AF 19(628)-5165. The work was directed by the Development Engineering Division under the Aerospace Instrumentation Program Office, Air Force Systems Command, Electronics Systems Division, Laurence G. Hanscom Field, Bedford, Massachusetts. Captain J. J. Centofanti served as the Air Force Project Monitor for this program, identifiable as ESD (ESSI) Project 5932, Range Digital Data Transmission Improvement.

REVIEW AND APPROVAL

Publication of this technical report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

C. V. Horrigan
for LOTIS R. HILL, Colonel, USAF
Director of Aerospace Instrumentation
Program Office

ABSTRACT

This paper considers differential phase-shift transmission over a band-limited channel with specified phase and amplitude characteristics. The problem treated is the prediction of feasible data rates and corresponding error rates over such channels.

An expression is given for the probability of error. This expression takes into account intersymbol and interchannel interference when differential phase-shift keying modulation is used. A distortion index which is a measure of phase error is defined, and the concept of *phase opening* which parallels the concept of *eye opening* commonly employed in the amplitude-shift keying modulation is introduced.

This analysis examines effects of various parameters on the performance of a high data-rate transmission system over a telephone line. Results indicate that for a carefully equalized line a single tone system should be used, and that higher order phase alphabets are to be preferred as the bit rate increases.

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SECTION I

INTRODUCTION

In this paper, the error performance of m-ary differential phase-shift keying (PSK) transmission of digital data over toll-grade voice channels is considered, with special emphasis on high data rates. The study takes into account the frequency characteristics (amplitude and delay distortion versus frequency) of the voice channel. It is assumed that the channels are of sufficiently high grade that they can support data rates in excess of 2400 bits/sec. Error performance of the system is analyzed as a function of the number of tones, the number of possible phase-shifts in each tone, the rate of transmission, the envelope shaping of the pulses, and the frequency separation. Several examples are presented.

In a band-limited channel, high data rates cause intersymbol interference. The effects of interference are different for each data sequence. Thus, one must compute the probability of error for each data sequence and find the average probability of error. The number of sequences that must be considered can be so large as to render the time required for the computation of the average probability of error excessive, even for a high-speed digital computer. Therefore, in addition to the direct evaluation of probability of error, an alternate method of evaluating the effects of the various parameters is examined; i.e., the notion of a distortion index introduced by Lucky[1] for amplitude-shift keying is extended to phase-shift keying. This index requires much less computation than the probability of error does, and appears to be monotonic with error performance.

SECTION II

THE PSK TRANSMITTER

One of the m distinct characters is transmitted every T seconds. The time interval, T, and the rate of character transmission, $1/T$, are known as the *keying interval* and the *keying rate* respectively. In a phase-shift keying system, the data waveform is a segmented sinewave (or tone) at a frequency $\omega_s/2\pi$ which is an integer multiple of the keying rate. The fundamental keying interval of duration T is known also as a baud or a slot. The m different characters correspond to the specified differences in the initial phases of the sinewave segments in successive keying intervals. Assuming that all the m characters are equally probable, and that the successive characters are selected independently, the m phase-differences are chosen to be uniformly spaced, i.e., the ℓth character corresponds to a phase-difference $2\pi\ell/m$, $\ell = 0, 1, \ldots, m-1$.

In order to specify the transmitted character, a pair of successive sinewave segments must be considered. To illustrate a pair of waveforms, the two adjacent keying intervals $-T/2 < t < T/2$ and $T/2 < t < 3T/2$ are arbitrarily chosen as reference and datum. Let ℓ_0 and ℓ_1 be the numbers corresponding to the transmitted characters in each of these two intervals respectively; i.e., the successive transmitted phase-differences are $2\pi\ell_0/m$ and $2\pi\ell_1/m$. An additional phase-shift of π/m radians is purposely introduced with each keying interval in order to minimize interference and improve synchronization. If this is done, then the successive transmitted phases are $2\pi\ell_0/m$ and $2\pi(\ell_0 + \ell_1)/m + \pi/m$. In addition, the transmitted waveforms are assumed modified by an amplitude shaping function, $k(t)$, prior to transmission. Furthermore, successive waveforms are allowed to overlap by δT seconds at each end, i.e., the sinewave

segments are $(1 + 2\delta)T$ long. Thus the transmitted waveforms in the reference and datum intervals are represented by:

$$s_R(t) = k(t) \cos\left(\omega_s t + 2\pi \ell_0/m\right)$$

$$-\left(\frac{1}{2} + \delta\right) T \leq t < \left(\frac{1}{2} + \delta\right) T \qquad \text{(1a)}$$

$$s_D(t) = k(t) \cos\left[\omega_s t + 2\pi\left(\ell_0 + \ell_1\right)/m + \pi/m\right]$$

$$\left(\frac{1}{2} - \delta\right) T \leq t < \left(\frac{3}{2} + \delta\right) T \qquad \text{(1b)}$$

More generally, in the notation used here, the rth keying interval is $(r - 1/2)\,T \leq t < (r + 1/2)\,T$, and the rth transmitted waveform is represented by:

$$s_r(t) = k(t) \cos\left[\omega_s t + 2\pi\left(\ell_r + L_{r-1}\right)/m + \pi/m\right]$$

$$\left(r - \frac{1}{2} - \delta\right) T \leq t < \left(r + \frac{1}{2} + \delta\right) T \qquad \text{(2)}$$

where $2\pi L_{r-1}/m$ represents the phase transmitted in the interval $r - 1$ and ℓ_r represents the transmitted character in interval r.

In later sections of this paper the Fourier transform of $s_r(t)$ is required and is denoted by $S_r(\omega)$. The computation of $S_r(\omega)$ can be carried out by inspection by treating each sinewave segment as a product of a rectangular window of duration $(1 + 2\delta)T$ and a sinewave of infinite duration.

Thus

$$S_r(\omega) = F\left(\omega - \omega_s\right) \exp\left[j\pi\left(2\ell_r + 2L_{r-1} + 1\right)/m\right]$$

$$+ F\left(\omega + \omega_s\right) \exp\left[-j\pi\left(2\ell_r + 2L_{r-1} + 1\right)/m\right] \qquad (3a)$$

where

$$F(\omega) = \frac{K(\omega)}{\omega} \sin\left[\left(\frac{1}{2} + \delta\right)\omega T\right] \qquad (3b)$$

and $K(\omega)$ is the Fourier transform of $k(t)$.

SECTION III

THE TRANSMISSION CIRCUIT

The transmission circuit treated in this report is grossly labeled as the "telephone line." The subscriber using a "telephone line" usually has no clear description of the physical circuit. Most often the actual circuit includes open wire lines, frequency multiplex equipment, microwave or coaxial lines, and frequently intermediate repeaters. The properties of such a circuit are usually described by the telephone company in terms of bounds on the amplitude and delay distortion characteristics in the central portion of the frequency region of interest. In addition, upper bounds on the frequency translation error and on noise hits are occasionally specified. Sometimes a lower bound on the expected signal-to-noise ratio (SNR) may also be given. These specifications are, in fact, implied when the toll service is described as, e.g., Schedule 4, Type A, B, or C.[2]

The analyst concerned with the performance of modulation techniques over "telephone lines" is limited by the imprecise nature of this description. Accurate predictions of performance are particularly difficult since echoes due to even small local irregularities in the frequency characteristics are known to severely affect performance.[3] Since actual, precise characteristics are seldom explicitly given, the analyst can generally draw only certain broad conclusions from a few basic distortions of the idealized characteristics.

It is generally true that, at conventional data rates, the contribution of the thermal noise to the error performance of a voice-grade telephone line is small, regardless of the modulation technique. At these data rates the performance of telephone circuits is limited by "impulse noise" (due

primarily to switching transients) and not by thermal noise.[4] "Impulse noise" is a term used to characterize noise which has a bursty occurrence of high amplitude, short-duration peaks. Furthermore, the noise due to crosstalk from other "telephone lines" is usually insignificant and causes so few errors that its effects can be generally ignored. However, as the data rate is increased, errors are caused by intersymbol and intertone interference, and thermal noise determines the nature of the transition from good performance to poor performance.

The properties of the "telephone line" are represented by the band-limited transfer function $H(\omega)$,

$$H(\omega) = \begin{cases} A(\omega) \exp \left\{ - j\phi(\omega) \right\} & \omega_a < \omega < \omega_b \\ 0 & \text{elsewhere} \end{cases}$$

where $A(\omega)$ and $\phi(\omega)$ are the amplitude and phase characteristics respectively, and ω_a and ω_b are the bandwidth limits of angular frequency.

SECTION IV

THE RECEIVED SIGNAL

The "telephone line" causes time spreading of the transmitted pulse. The output of the telephone line in the nth keying interval due to the transmitted sinewave segment $s_r(t)$ in the rth keying interval is denoted by $z_{n,r}(t)$. The Fourier transform $Z_{n,r}(\omega)$ is readily found and is simply

$$Z_{n,r}(\omega) = \left[S_r(\omega)H(\omega)\right] * \left(\left\{ \exp\left[-j\left(n - \frac{1}{2}\right)\omega T\right] - \exp\left[-j\left(n + \frac{1}{2}\right)\omega T\right] \right\} / j\omega \right) \tag{5}$$

where $*$ is the symbol for convolution.

The Fourier transform of the total received signal in the nth time slot due to an infinite string of transmitted sinewave segments is given by

$$Z(\omega) = \sum_{r=-\infty}^{\infty} Z_{n,r}(\omega) \tag{6}$$

This summation for $r \neq n$ represents the intersymbol interference.

In a PSK system employing more than one data tone on the line, interchannel interference must be included, in addition to the intersymbol interference, i.e., the effect of signals in all the time intervals of other tone frequencies must be considered. Let successive tones be numbered by $p \epsilon (1, 2, \ldots M)$ and the transmitted tone designated by s. Let the Fourier transform of the received signal in the slot n of the tone p due to a waveform in the slot r of the tone s be designated by $Z^{(p)}_{n,r}(\omega \mid s)$.

When $p = s$ this response represents self-interference, i.e., intersymbol interference, and when $p \neq s$ there is intertone interference.

If $\ell_r + L_{r-1}$ denotes the transmitted phase in the rth slot of tone s, then the term $Z^{(p)}_{n,r}(\omega|s)$ can be written as a sum of real and imaginary parts:

$$Z^{(p)}_{n,r}(\omega|s) = \mathrm{Re}\left\{Z^{(p)}_{n,r}(\omega|s)\right\} + j\,\mathrm{Im}\left\{Z^{(p)}_{n,r}(\omega|s)\right\} \tag{7a}$$

where

$$\mathrm{Re}\left\{Z^{(p)}_{n,r}(\omega|s)\right\} = -I_1(\omega)\cos 2\pi\left(\ell_r + L_{r-1}\right)/m$$

$$- I_2(\omega)\sin 2\pi\left(\ell_r + L_{r-1}\right)/m \tag{7b}$$

$$\mathrm{Im}\left\{Z^{(p)}_{n,r}(\omega|s)\right\} = - I_3(\omega)\sin 2\pi\left(\ell_r + L_{r-1}\right)/m$$

$$+ I_4(\omega)\cos 2\pi\left(\ell_r + L_{r-1}\right)/m \tag{7c}$$

and where

$$I_{\frac{1}{2}}(\omega) = \int_{\omega_a}^{\omega_b} d\omega' \frac{A(\omega')}{\omega - \omega'}\left[K(\omega' + \omega) \pm K(\omega' - \omega)\right]$$

$$\sin\left[(\omega - \omega')\frac{T}{2}\right]\cos\left[n(\omega - \omega')T + \phi(\omega')\right] \tag{7d}$$

$$I_{\frac{3}{4}}(\omega) = \int_{\omega_a}^{\omega_b} d\omega' \frac{A(\omega')}{\omega - \omega'} \left[K(\omega' + \omega) \pm K(\omega' - \omega) \right]$$

$$\sin\left[(\omega - \omega')\frac{T}{2}\right] \sin\left[n(\omega - \omega')T + \phi(\omega')\right] \tag{7e}$$

The expressions for the real and imaginary parts of $Z^{(p)}_{n,r}(\omega|s)$ in this form exhibit the two kinds of terms which must be computed to evaluate the performance of the systems: (1) the four integrals which involve the frequency characteristics of the channel and the envelope shaping of the transmitted signal, all of which are fixed for a given transmission system; and (2) cosine and sine coefficients which are a function of the transmitted data sequence alone.

The Fourier transforms of the total received data and reference waveforms in the nth and $(n + 1)^{th}$ interval of the desired tone p of an M tone system is given by

$$Z^{(p)}_{D}(\omega) = \sum_{s=1}^{M} \sum_{r=-\infty}^{\infty} Z^{(p)}_{n,r}(\omega|s) \tag{8a}$$

$$Z^{(p)}_{R}(\omega) = \sum_{s=1}^{M} \sum_{r=-\infty}^{\infty} Z^{(p)}_{n-1,r}(\omega|s) \tag{8b}$$

The summation for r is carried out over both the past and the future intervals in order to allow for the case when the specified idealized transfer function is not physically realizable.

SECTION V

THE PSK RECEIVER

The reason for assuming the particular model of the receiver that is treated in this paper is that this model describes the principles embodied in many current equipments. The receiver is designed to make, on the average, the fewest errors when the transmitted signal is perturbed only by additive white Gaussian noise. For each $i \in \{1, 2, \ldots, m\}$ the receiver computes $\Pr\{r_D(t), r_R(t) | i\}$, i.e., the joint likelihood of $r_D(t)$ and $r_R(t)$ conditional on character i having been transmitted. It is then decided that the transmitted character was that i for which $\Pr\{r_D(t), r_R(t) | i\}$ is maximum. Note that the joint likelihood is a monotonic function λ_i given by Reference [5]:

$$\lambda_i = \int_{-T/2}^{T/2} dt_1 \int_{T/2}^{3T/2} dt_2 \, r_D(t_1) r_R(t_2) \cos\left[\hat{\omega}_p(t_1 - t_2) + 2\pi i/m\right] \tag{9}$$

where $\hat{\omega}_p$ is an estimate at the receiver of the transmitted carrier tone frequency ω_p.

The physical embodiment of the operations indicated by the test statistic, λ_i, can be obtained by three equivalent techniques: (1) matched filtering and envelope detection, (2) matched filtering and product detection, and (3) multiplication and low-pass filtering.[5]

SECTION VI

ERROR PERFORMANCE

INTRODUCTION TO DECISION PROBABILITIES

In a previous paper, [6] the concept of the "decision probability" was introduced. The decision probability is the probability of choosing one out of just two (and not m) alternatives. For a PSK system this probability is the conditional probability of choosing phase-difference $2\pi k/m$ over phase-difference $2\pi i/m$, given that an arbitrary phase-difference $2\pi \ell/m$ was transmitted on tone p. For an m-phase system, this probability will be denoted by $\Pr\{k > i|\ell; m\}$, where the inequality sign indicates that phase-difference k is chosen over i. Decision probabilities are relatively easy to compute because they are probabilities of the received phase-difference falling in half-phases of a phase diagram. They are of interest because they can be used to determine the probability of a character error which involves the choice of one out of all m alternatives. Moreover, they give the probability of a binary digit-error. Thus a two-hypothesis problem rather than an m-hypothesis problem is solved, and yet the desired m-hypothesis results are obtained. In particular, for m-even, the probability of a character error $P_c(m)$ in an m-phase system is tightly bound by [5]

$$2\Pr\{1 > 0 \mid 0;m\} - \frac{1}{m} P_c(2) < P_c(m) < 2\Pr\{1 > 0|0;m\} \qquad m > 2 \tag{10a}$$

where

$$P_c(2) = \Pr\{1 > 0|0;2\} \tag{10b}$$

is usually known exactly. For $m \geq 4$, the upper bound is a good approximation to $P_c(m)$. Denote the probability of a binary digit-error for an

m-phase system by $P_b(m)$; for the usual Gray code this probability can be expressed exactly in terms of the decision probabilities,[5] i.e.,

$$P_b(m) = \Pr\{1 > 0 \mid 0;m\} \qquad m = 2, 4 \tag{11a}$$

For $m = 8$ the probability $P_b(m)$ is tightly bounded by decision probabilities,[5]

$$\frac{2}{3}\left[\Pr\{1 > 0 \mid 0;8\} + P_r\{2 > 1 \mid 0;8\} - P_c(2)\right] < P_b(8) < \frac{2}{3}\left[\Pr\{1 > 0 \mid 0;8\} + \Pr\{2 > 1 \mid 0;8\}\right] \tag{11b}$$

These results are applicable to any maximum likelihood choice between m-even equally-probable phase-differences. Similar expressions can be derived for m-odd and/or when the phase-differences are not equally-probable. Thus, computation of appropriate decision probabilities permits character error and bit error probabilities to be evaluated.

COMPUTATION OF DECISION PROBABILITIES

The SNR's occurring on telephone lines are sufficiently large so that at low data rates the effect of thermal noise is to cause error rates typically below 10^{-7}. Thus at low data rates, where the distortion due to the frequency characteristic is small, the impulse noise which causes error rates around 10^{-5} dominates over thermal noise. However, at high data rates thermal noise does have an effect on performance. It tends to cause a less abrupt transition in performance from the state of no intersymbol or interchannel interference to the state in which intersymbol interference is the dominant cause of errors. The probability of error in a telephone

line environment is computed by taking advantage of the previous result for the probability of error for a channel perturbed by white-Gaussian noise. [5]

At very high SNR (i.e., 40 db or more, which is typical for a telephone line) a thresholding effect exists, i.e., either relatively error-free operation or bottoming due to intersymbol and intertone interference. The error rate does not improve further as SNR is increased.

In a previous paper [5] $\Pr\{k > i|\ell;m\}$ was obtained as a representation of the decision probability for a channel perturbed by additive-white-Gaussian noise ($N_0/2$ watt/Hz for all frequencies),

$$\Pr\{k > i|\ell;m\} = -\frac{1}{\pi}\int_0^\infty dy \exp\left[-\frac{a}{N_0} f(y)\right] h(y)$$

$$\left\{-\frac{3}{2}\cos\left[bg(y)/N_0\right] + y \sin\left[bg(y)\right]/N_0\right\} \tag{12a}$$

where

$$f(y) = \left(y^2 + \frac{3}{4}\right)/\left(y^2 + \frac{9}{4}\right)$$

$$g(y) = y/\left(y^2 + \frac{9}{4}\right) \tag{12b}$$

$$h(y) = 1/\left[\left(y^2 + \frac{1}{4}\right)\left(y^2 + \frac{9}{4}\right)\right]$$

and where

$$a = \frac{2}{T}\left(\left|Z_D^{(p)}(\hat{\omega}_p)\right|^2 + \left|Z_R^{(p)}(\hat{\omega}_p)\right|^2\right)$$

$$b = \frac{4}{T} \operatorname{Re} \left\{ Z_D^{(p)}\left(\hat{\omega}_p\right) \overline{Z}_R^{(p)}\left(\hat{\omega}_p\right) \right.$$

$$\left. \frac{[\exp(j2\pi i/m) - \exp(j2\pi i/m)] \sin \pi(i + k - 2\ell)/m}{\cos 2\pi(i - \ell)/m - \cos 2\pi(k - \ell)/m} \right\} \tag{12c}$$

The angular frequency $\hat{\omega}_p$, is the estimate at the receiver of the carrier frequency ω_p of the incoming keyed data tone. It is important to note that since a and b are a function of the particular data sequence, the value of the integral in Equation (12a) can be different for each data sequence. To compute decision probabilities exactly, $\Pr\{k > i \mid \ell;m\}$ must be computed for all data sequences. An alternate approach will be considered in the next section.

In his analysis of 2-phase clairvoyant PSK, Lewandowski [7] transformed an integral of the same form as the rhs of Equation (12a) into a difference of two Marcum Q-functions. Taking advantage of this result $\Pr\{k > i \mid \ell;m\}$ can be expressed in terms of Q-functions as follows:

$$\Pr\{k > i \mid \ell;m\} = \frac{1}{2} \left\{ 1 - Q\left[\sqrt{\frac{a(1 + b/a)}{4 N_0}}, \sqrt{\frac{a(1 - b/a)}{4 N_0}}\right] + Q\left[\sqrt{\frac{a(1 - b/a)}{4 N_0}}, \sqrt{\frac{a(1 + b/a)}{4 N_0}}\right] \right\} \tag{13a}$$

where

$$Q(\alpha, \beta) = 1 - \int_0^{\beta} x\, I_0(\alpha x) \exp\left[-\left(x^2 + \alpha^2\right)/2\right] dx \tag{13b}$$

While Equation (13) appears neater than Equation (12), there appears to be no clear advantage to representing error performances in terms of the Q-function.

Unfortunately, current tables of Q-functions are not generally available for large arguments of Q and/or with the accuracy required for the range of interest here. For example, a routine for computing Q-function with relative accuracy $\Delta Q/Q$ of 10^{-5} is not sufficient in many problems. This can be seen by noticing that

$$Q\left[\sqrt{\frac{a(1 + b/a)}{4 N_0}}, \sqrt{\frac{a(1 - b/a)}{4 N_0}}\right] \tag{14}$$

in Equations (13) is approximately unity when $\Pr\{k > i|\ell;m\}$ is small, so that a routine with relative accuracy $\Delta Q/Q$ of 10^{-5} can be used only for $\Pr\{k > i|\ell;m\}$ greater than 10^{-5}.

A convenient approximation for the decision probabilities is available in terms of the well-tabulated error function.[5]

$$\Pr\{k > i|\ell;m\} \cong \frac{1}{2}\left\{1 - \operatorname{erf}\left[\frac{b/N_0}{2\left(1 + a/N_0\right)^{1/2}}\right]\right\} \tag{15a}$$

where erf(x) is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-x^2\right) dx \tag{15b}$$

The approximation improves as $b/a \to 0$. In addition when b/a has the maximum value of one, Equation (12) reduces[5] simply to, $\frac{1}{2}\exp\left(-a/N_0\right)$. The approximation, Equation (15), is found to be adequate for computations considered in this study when $m \geq 4$, but when $m = 2$ the exact expression, Equation (12), is required.

DISTORTION INDEX

In a band-limited channel, interference from symbols adjacent in time and frequency limits error performance. The shape of the interference, and hence its effect, depends on the particular character sequence. Hence in order to compute the average probability of a binary digit error one must separately compute the probability of a digit error for every possible sequence of waveforms. Since interference decreases with time and frequency separation, in practice only up to some K intervals that cause interference above the noise level must be considered. Although each computation requires only the numerical solution of four integrals, the number of necessary computations for an m-phase system is m^K. When the data rate is high enough compared to the bandwidth, K can become so large that m^K becomes enormous. However, in some cases, it may only be desired to compare the performance as a function of the number of phases/tone, the number of tones, and the data rate. Therefore, it is useful to define a quality index, which would be monotonic with probability of error and easier to compute than the probability of error itself.

The largest error in the differential phase angle between the reference and data waveform that any data sequence can produce was chosen as such an index. Since good (in terms of performance) sequences for some channels may be poor sequences for other channels, the problem of choosing a different "worst" sequence for each channel remains.

For an amplitude-shift keying (ASK) system, the worst sequence is the one having the maximum level in all the other keying intervals. The ASK distortion index is the *eye opening* of the baseband signal at the center of the keying interval for this sequence. It has been shown that this index is

monotonic with probability of error.[1] The value of the index is meaningful only for channels and data rates for which the worst sequence still has an *open eye*.

As an analogy for a single tone PSK system, the "worst" sequence is defined as the one whose characters selected in order give the maximum increase in phase error. The following procedure is used: The phase-difference of a pair of received reference and data waveforms is computed for each of the m possible transmitted phase differences. The two symbols which produce the maximum phase error due to interference are retained. Then m more computations are made by considering the additional interference due to each of the m possible phases in the adjacent slot. Then that signaling phase which gives the maximum phase error for the three-symbol sequences is retained. The procedure is continued slot by slot until the added interference due to extending sequences is negligible. Specifically,

$$D_\phi = \left|\hat{\alpha}_D - \hat{\alpha}_R - 2\pi\ell/m\right| \tag{16a}$$

is sequentially computed where

$$\hat{\alpha}_D = \tan^{-1} \frac{\mathrm{Re}\, Z_D^{(p)}\left(\hat{\omega}_p\right)}{\mathrm{Im}\, Z_D^{(p)}\left(\hat{\omega}_p\right)} \tag{16b}$$

$$\hat{\alpha}_R = \tan^{-1} \frac{\mathrm{Re}\, Z_R^{(p)}\left(\hat{\omega}_p\right)}{\mathrm{Im}\, Z_R^{(p)}\left(\hat{\omega}_p\right)} \tag{16c}$$

and where $2\pi\ell/m$ is the transmitted phase difference. After each computation another slot is included in $Z_D^{(p)}\left(\hat{\omega}_p\right)$ and $Z_R^{(p)}\left(\hat{\omega}_p\right)$ while that

character sequence which maximized the previous D_ϕ is retained. The same procedure is generalized to multi-tone systems.

Thus, as symbols are added, the already selected sequence is not reconsidered. Therefore, there is no guarantee that the selected sequence is "worst" in the absolute sense. However, the results generated by this procedure have been found to be fully indicative of probabilities of error. Thus the computational simplification appears well worthwhile.

In a N-level ASK system, the distortion index D_A is related to the *eye opening* I_A by

$$I_A = 1 - (N - 1)D_A \tag{17}$$

In a m-phase PSK system, a parallel notion is that of a *phase opening* I_ϕ which is defined in terms of the phase distortion index D_ϕ by

$$I_\phi = \frac{\pi}{m} - D_\phi \ . \tag{18}$$

The angle I_ϕ represents a margin for noise after intersymbol interference is taken into account. The angle I_ϕ can, of course, be conveniently expressed in degrees rather than radians.

SECTION VII

EXAMPLE

In order to illustrate the methods outlined in the preceding section, consider the transmission characteristics of a nominally 3 kHz circuit shown in Figure 1. These characteristics are representative of a carefully equalized telephone line. Let the amplitude shape, k(t), be the commonly used raised-cosine-shape overlapping into adjacent slots on each side by one quarter of a keying interval (see Figure 2). This particular shape was found by Rappaport[8] to be optimum, i.e., to generate least intersymbol interference for a quaternary (m = 4) differential phase system at 2400 bits/sec.

The probability of error in the binary data stream was first computed for 2-phase, 4-phase, and 8-phase transmissions on a single tone located

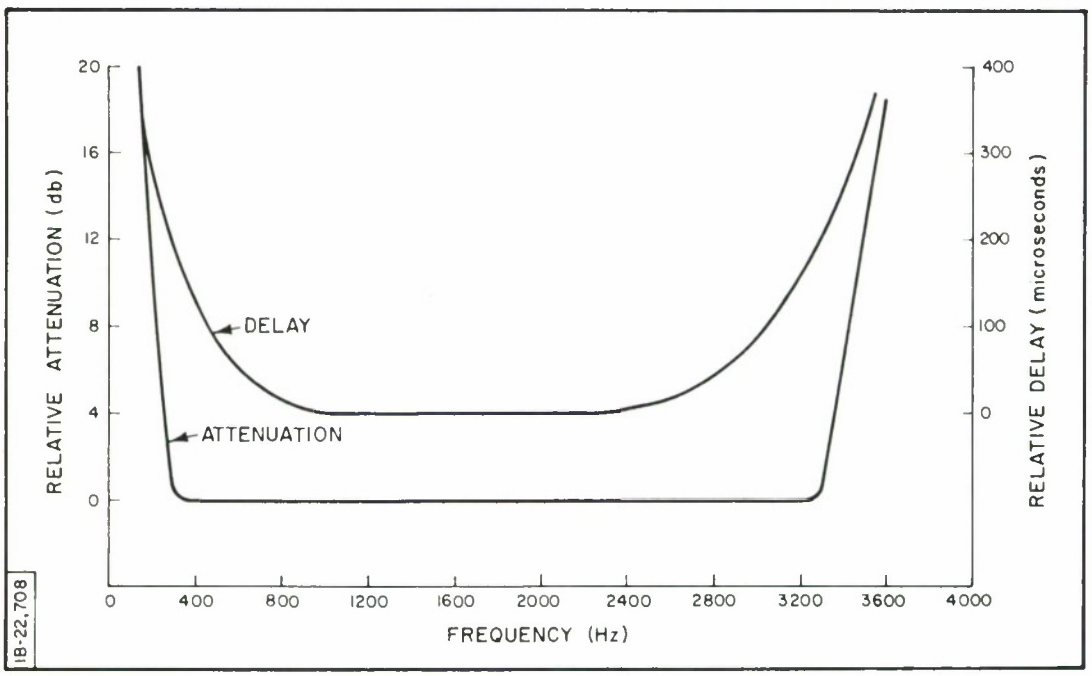

Figure 1. Characteristics of a Carefully Equalized Telephone Line

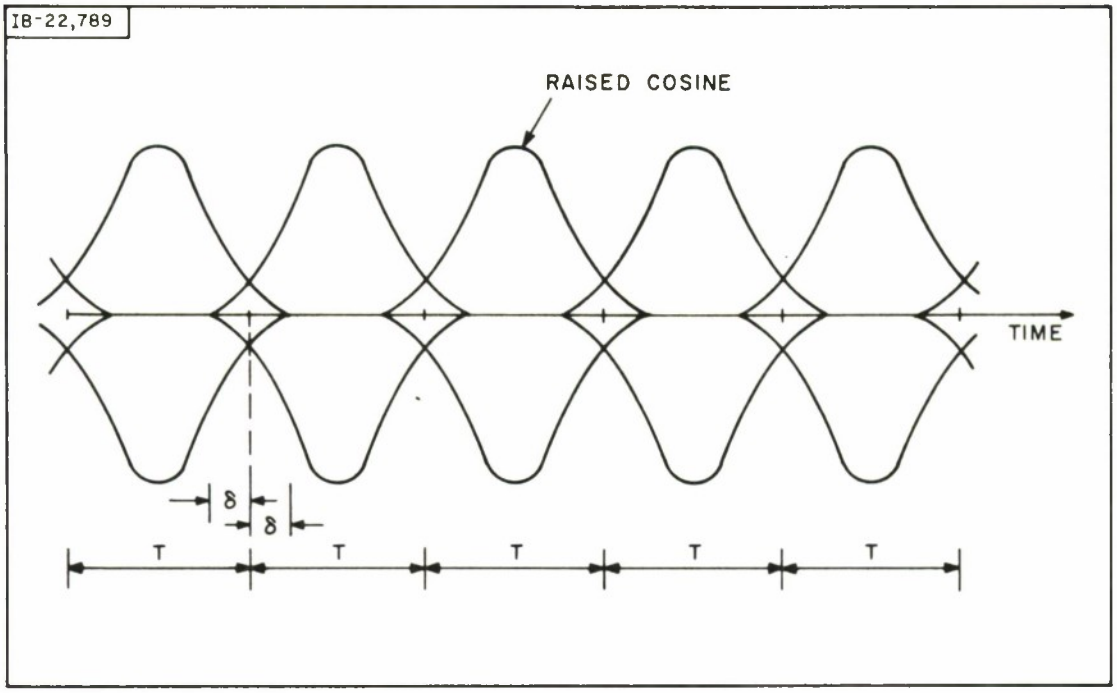

Figure 2. Example of an Envelope Shaping Function k(t)

at the center of the band. These results are shown in Figures 3, 4 and 5. It can be seen that the circuit will support 2-phase transmission at rates up to approximately 3600 bit/sec, 4-phase up to 6000 bit/sec and 8-phase up to 7200 bit/sec. Note that 40 db is a typical SNR and that for 2- and 4-phase transmission, an error rate better than 10^{-5} is obtainable. At 7200 bit/sec and 40 db SNR, the 8-phase system has an error rate of approximately 10^{-4}. Transmission of two tones was next considered and the effect of frequency spacing between tones on the error rate was examined. Results for 4-phase DPSK at 5600 bit/sec are plotted in Figure 6. It can be seen that at this high data rate the error rate is very sensitive to proper frequency spacing. The optimum spacing is approximately 1450 Hz which is slightly above the character rate of 1400/sec. This optimum point arises from having the band-limited nature of the channel counteract the inter-tone

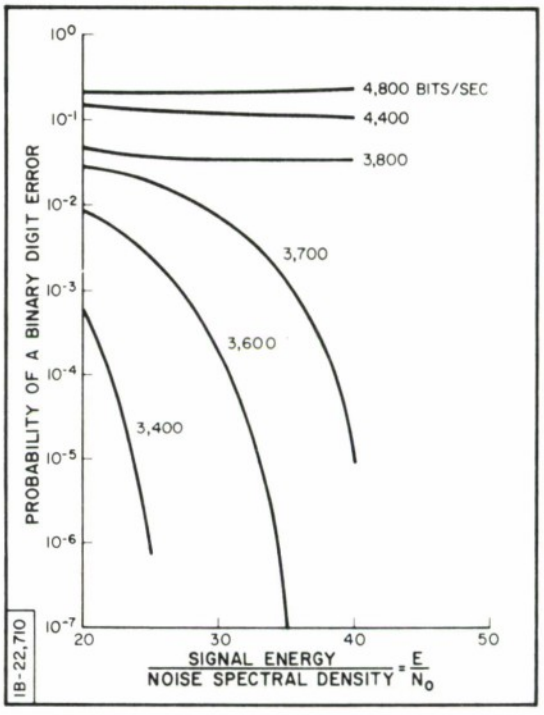

Figure 3. Probability of Error versus SNR for a 1-Tone, 2-Phase System with a Raised Cosine Envelope

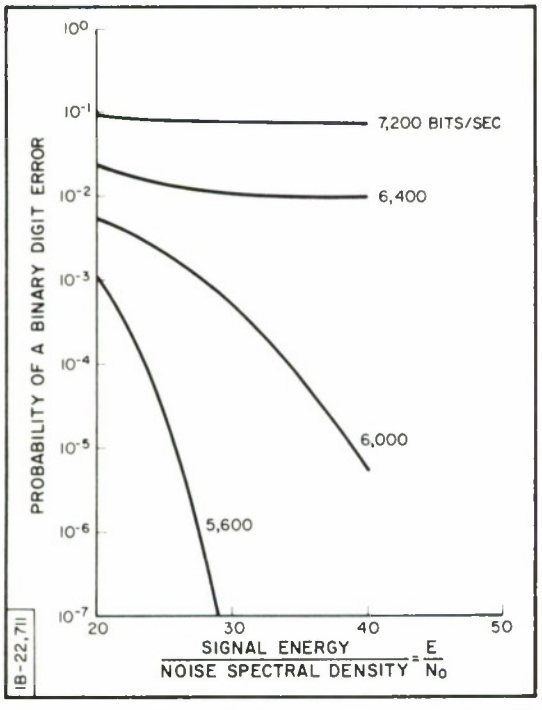

Figure 4. Probability of Error versus SNR for a 1-Tone, 4-Phase System with a Raised Cosine Envelope

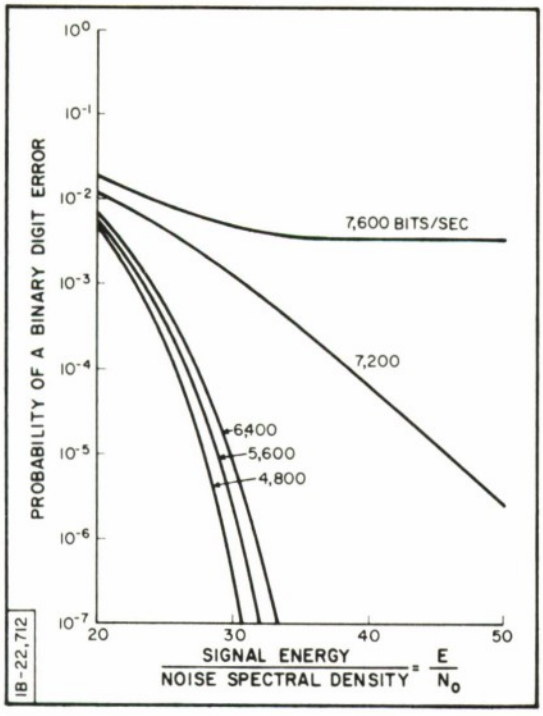

Figure 5. Probability of Error versus SNR for a 1-Tone, 8-Phase System with a Raised Cosine Envelope

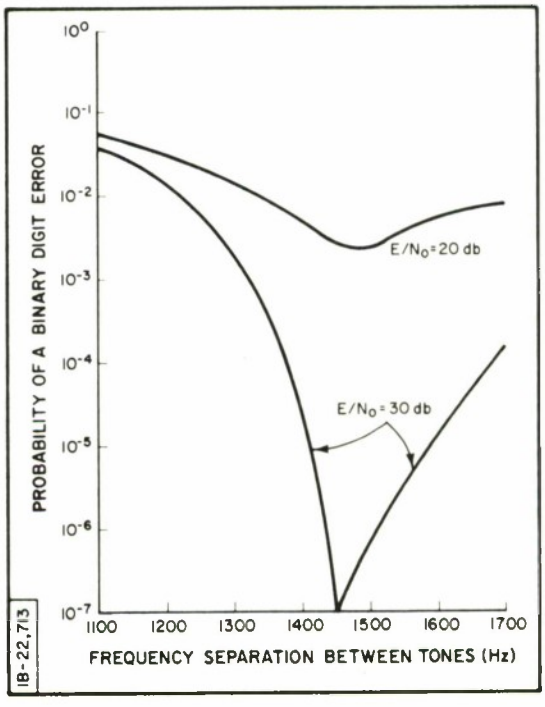

Figure 6. Probability of Error versus Tone Separation for a 2-Tone, 4-Phase System at a Data Rate Equal to 5600 bits/sec

interference. When tones are closer than 1320 Hz the intertone interference is large enough to cause an irremovable phase error, hence "bottoming."

Even with the best spacing, a two tone system was inferior to a single tone system. The same conclusion can be reached taking the conventional "orthogonal" spacing (character rate) and examining the error rate as the number of tones is increased. Since the overall data rate and SNR is kept constant, as tones are added, the keying rate and SNR of each tone decrease. It has been concluded from Figure 7 that, at a high data rate, DPSK over this telephone line is best operated with a single tone.

Continuing with the same example, the *phase opening*, I_ϕ, is plotted in Figure 7 against the data rate for 2-phase, 4-phase and 8-phase single tone systems. The computed curves indicate that, regardless how high the SNR is, there is very little phase margin for noise above 3600 bit/sec for a 2-phase system, 6000 bit/sec for a 4-phase system, and 7500 bit/sec for an 8-phase system. This confirms the probability of error calculations in Figures 3 and 4 which are a great deal more difficult to obtain than phase opening calculation. An interesting aspect of the curves in Figure 8 is that they suggest at what bit rates one should switch to a higher level phase-alphabet. For example, at rates below 3400 bit/sec, a 2-phase alphabet would be recommended. Between 3400 and 5800 bit/sec a 4-phase system would offer better performance. Above 5800 bit/sec an 8-phase system appears superior. Of course, at these high data rates, the *phase opening* is very small and hence extremely large SNR's would be required.

In order to verify that this cross-over in performance is also exhibited by the probability of error curves, 2- and 4-phase systems were compared at data rates around 3400 bit/sec and SNR's of 20 db. At 3200 bit/sec the error rates were 2.0×10^{-6} and 1.0×10^{-6} for 4- and 2-phase transmission,

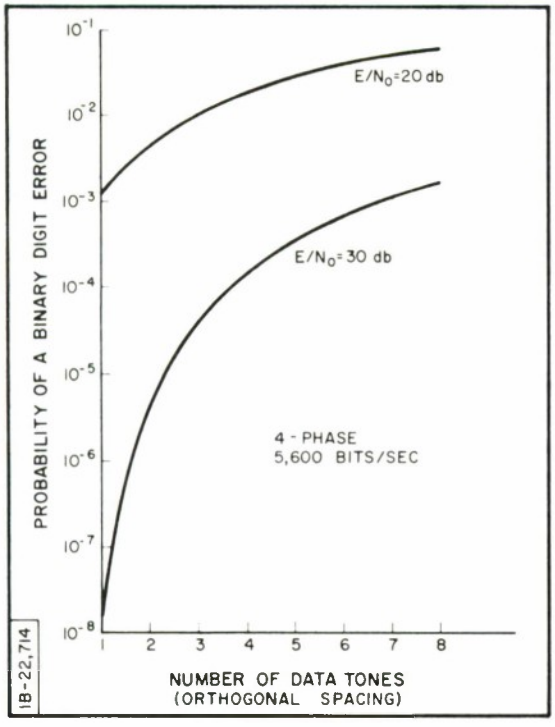

Figure 7. Probability of Error versus Number of Data Tones

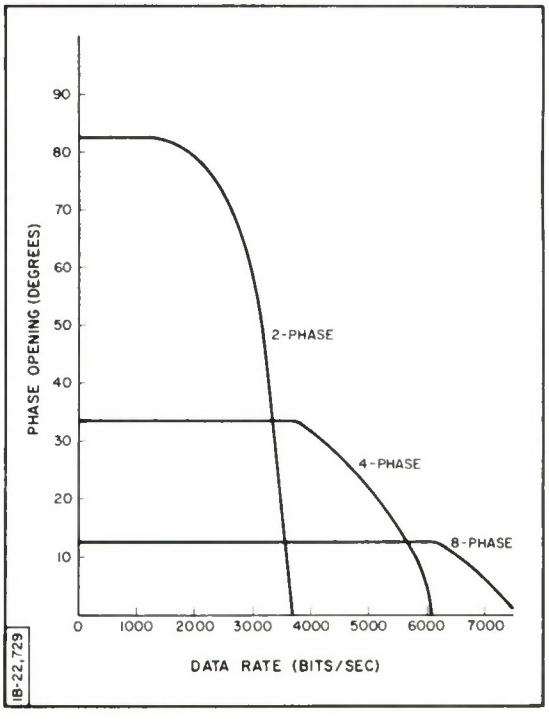

Figure 8. Phase Opening versus Data Rate for a Raised Cosine Pulse

respectively. As 3400 bit/sec, the 4-phase error rate remained approximately the same but the 2-phase error rate rapidly deteriorated. This confirms a cross-over slightly above 3200 bit/sec for SNR = 20 db. A higher SNR would produce a cross-over at a higher data rate. The cross-over in error rate performance around 5800 bit/sec suggested by Figure 8 and 4- and 8-phase transmission can be confirmed by examining Figures 4 and 5.

In order to investigate what is the effect of an unequalized transmission circuit on digital phase-shift transmission, a time delay and amplitude characteristic representative of a Schedule 4 Type B circuit was assumed (see Figure 9). The phase opening was computed for 2-phase, 4-phase, and 8-phase transmissions over this circuit at different data rates. The results shown in Figure 10 indicate that 2-phase transmission could be supported

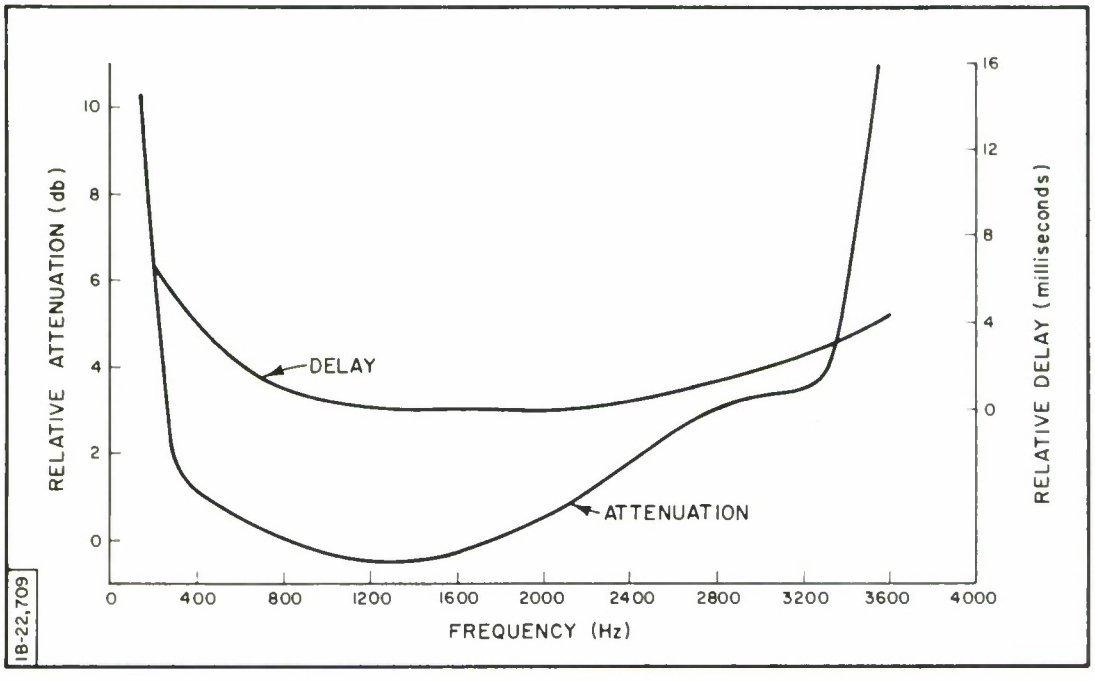

Figure 9. Characteristics Representative of a Schedule 4B Telephone Line

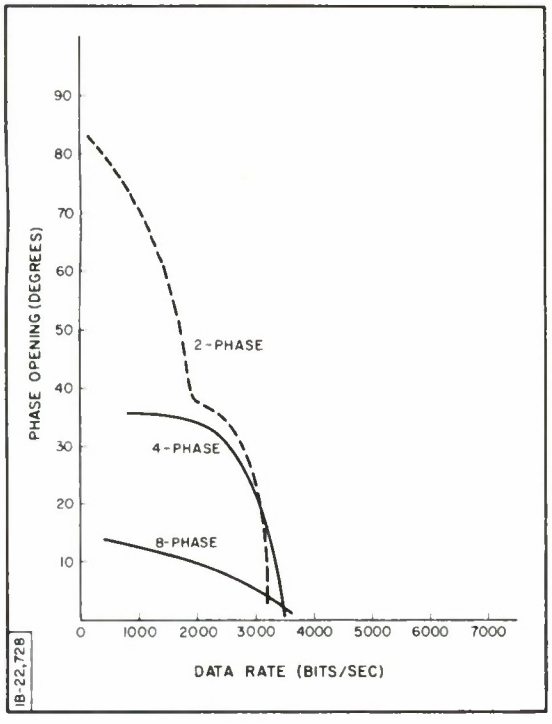

Figure 10. Phase Opening versus Data Rate for a Raised Cosine Pulse

up to rates slightly above 3000 bit/sec. The interesting effect, however, is that the higher order phase alphabets cannot be expected to increase the data rates over such a line.

SECTION VIII

CONCLUSIONS

The results of this study indicate that the analysis of digital phase-shift transmission over band-limited circuits can be greatly simplified by computing the phase opening for the worst data sequence rather than the average probability of error.

Furthermore, higher order alphabets can increase the available data rate, but only over a well-equalized line. Of course, at the higher data rates, larger SNR's are required to maintain the same error rate.

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13. ABSTRACT

This paper considers differential phase-shift transmission over a band-limited channel with specified phase and amplitude characteristics. The problem treated is the prediction of feasible data rates and corresponding error rates over such channels.

An expression is given for the probability of error. This expression takes into account intersymbol and interchannel interference when differential phase-shift keying modulation is used. A distortion index which is a measure of phase error is defined, and the concept of phase opening which parallels the concept of eye opening commonly employed in the amplitude-shift keying modulation is introduced.

This analysis examines effects of various parameters on the performance of a high data-rate transmission system over a telephone line. Results indicate that for a carefully equalized line a single tone system should be used, and that higher order phase alphabets are to be preferred as the bit increases.

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