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DATA SMOOTHING AND TREND ESTIMATION

Robert D. Turner

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RESEARCH AND ENGINEERING SUPPORT DIVISION
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FOREWORD

This research paper has been written because of the need for explicit characterization of smoothing and prediction techniques for many applications. While the methods of curve-fitting by least-squares have been known for many years, it is frequently difficult to find formulas which describe the errors in estimation based on such methods.

In certain applications, the central problem is to predict the value of a measured quantity which exhibits a systematic trend which can be described in terms of a low-order polynomial. In other applications, the problem is to provide concurrent estimates of the true value of the measured quantity and its rate of change. In both situations, the analytical need exists for measures of the effects of random errors and their interdependency and the consequences of systematic errors stemming from an inadequate model of the underlying trend. This paper represents an initial step toward fulfilling this need. It was motivated by problems of aiming anti-aircraft guns against maneuvering targets and achieving precision weapon delivery by tactical aircraft. It is hoped that the results presented here will find other applications as well.

ABSTRACT

Explicit formulas are presented for estimating position, velocity, and acceleration in low-order polynomial trends, based on least-squares smoothing of sampled data accompanied by statistically uncorrelated measurement errors. Formulas are also given for interpolation and prediction of position and velocity. Expressions for the variances and covariances of consistent position, velocity, and acceleration estimates are given, and the systematic errors accruing from use of a trend estimation basis which is one order lower than the actual trend are presented.

One interesting result is that the normalized correlation between the errors in an estimate of current position and those in an estimate of current velocity approaches $\sqrt{3}/2$ when the number of measurements in the estimates becomes large.

Finally, the problem of implementing real-time least-squares estimation and prediction formulas in practical systems is discussed. It is concluded that arithmetic execution time requirements can be relaxed by generating certain sums recursively, and that data storage requirements can frequently be eased by collapsing the raw data into short-term-average samples.

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I. INTRODUCTION

The purpose of this paper is to document some ostensibly well-known results of elementary statistics, in forms which will hopefully prove useful to persons interested in the smoothing of statistically stationary data sequences. In particular, the results presented describe the properties of estimators based on low-order polynomial least-squares fits.

While such estimators are frequently not optimum for a given setting, they are realizable, so that the results presented here constitute, in one sense, a lower bound on the quality of estimation that can be performed in a given situation. Conversely, if a least-squares fit does happen to be the optimum procedure for a particular problem, the results provide upper bounds on the performance of more "economical" procedures.

In what follows, it is assumed that the data originate as sampled values of a continuous well-defined process, $x(t)$; this process is assumed to be representable with sufficient precision as a Taylor's series in the independent variable t:

$$x(t) = x_o + v_o(t-t_o) + (1/2)a_o(t-t_o)^2 + (1/6)b_o(t-t_o)^3 + \dots \quad (1)$$

The symbol x will be taken to represent a position coordinate and t will be taken to represent time, so that

$$v(t) = dx/dt \quad (2)$$

and

$$a(t) = d^2x/dt^2 \quad (3)$$

represent the velocity and acceleration associated with the coordinate $x(t)$. In some problems, it will be desirable to estimate v and a as well as x.

The approach taken in this paper is to deal with the underlying process in its original form, as given by Eq. 1, rather than to develop the representation of Eq. 1 in the form of a series of orthogonal polynomials. The latter approach is most appropriate when the desired result is simply the best fitting curve for estimation and prediction of $x(t)$, and it is used extensively in statistical literature, e.g., Ref. 1, pp. 186-191. The convenience of the orthogonal-polynomial approach stems from the fact that the problem of solving the simultaneous equations for the coefficients used in the representation is trivial, by virtue of the orthogonality property. In many applications, however, it is important to exhibit the consequences of the terms in Eq. 1 as they stand. That is, position, velocity, and acceleration will frequently be of considerably greater significance in a specific situation than the coefficients of the first, second, and third terms in the orthogonal-polynomial representation. It is for this reason that the more cumbersome direct approach has been adopted. In particular, the direct approach facilitates the computation of variances and covariances of position, velocity, and acceleration estimates and the determination of the consequences of systematic errors in the estimation model. While it is true that such results could have been obtained from the orthogonal-polynomial representation, the current approach is more straightforward; in particular, some of the covariance results are more easily generalized than if the orthogonal-polynomial approach had been adopted.

The <u>true</u> sampled values are unperturbed values of $x(t)$ at $t = t_o + kT$, where k is an integer running from 1 to N (N is therefore the sample size) and T is the interval between adjacent samples. The total period over which the data are observed is

$$T_o = (N-1)T \ . \tag{4}$$

Denoting the true sampled values by x_k, Eq. 1 gives

$$x_k = x_o + v_o kT + (1/2)a_o(kT)^2 + (1/6)b_o(kT)^3 + \cdots \tag{5}$$

For simplicity in writing, x_k will be written in the form

$$x_k = x_o + kr_o + k^2 q_o + k^3 p_o + \cdots \tag{6}$$

with the correspondence

$$\begin{aligned} x_o &= x_o \\ r_o &= v_o T \\ q_o &= (1/2)\ a_o T^2 \\ p_o &= (1/6)\ b_o T^3 \end{aligned} \tag{7}$$

and so forth.

The observed data will be denoted by X_k; it is assumed that the errors in the observed data manifest themselves as

$$X_k = x_k + \xi_k \qquad k = 1, \ldots, N \tag{8}$$

when the ξ_k's constitute a sequence of zero-mean, uncorrelated, identically distributed random variables with variance σ_x^2.

The problem at hand is to develop means for estimating the true values of the position, or velocity, or acceleration (or linear functions of these variables) at some time t, which will be expressed in the form

$$t = t_o + KT$$

(note that K need not be an integer), which may occur prior to, during, or subsequent to, the period of observation. Such estimates will be denoted by the symbols

$$\hat{x}_K, \hat{v}_K, \hat{a}_K, \ldots$$

for the estimated position, velocity, acceleration, etc. at the time corresponding to the choice of K.

The method of obtaining formulas for these estimates will be discussed in greater detail in Sec. II, which begins with an assumption about the nature of x(t). This assumption deals with the point at which the Taylor's series representation of Eq. 1 can be truncated without sacrificing the usefulness of the estimation formula. If it is assumed that the values of x(t) are substantially constant over the time interval for which the estimation formula is to be used, then it is appropriate to employ just the first term of Eq. 1; the estimation formula then takes the form

$$\hat{x}_K = \hat{x}_o \qquad \text{(zero trend)} \ . \tag{9}$$

If it is assumed that x(t) is satisfactorily represented over the estimation interval by the first two terms of Eq. 1, then the estimation formula reads

$$\hat{x}_K = \hat{x}_o + \hat{r}_o K \qquad \text{(linear trend)} \ . \tag{10}$$

If it is assumed that x(t) is satisfactorily represented over the estimation interval by the first three terms of Eq. 1, then the estimation formula reads

$$\hat{x}_K = \hat{x}_o + \hat{r}_o K + \hat{q}_o K^2 \qquad \text{(quadratic trend)} \ , \tag{11}$$

and so forth. For reasons that will become apparent, the three forms of Eqs. 9, 10, and 11 are the only ones that will be considered in this paper.

The least-squares fitting procedure involves choosing the parameters of the selected estimation formula so that the mean-square difference between the observed X_k's and the corresponding $\hat{x}_k$'s is minimized. Once these parameters have been computed, they can be used to provide estimates of velocity, acceleration, and so forth. Specifically, it follows that

$$\hat{v}_K = 0, \ \hat{a}_K = 0 \tag{12}$$

for the zero-trend case;

$$\hat{v}_K = \hat{r}_o/T, \ \hat{a}_K = 0 \tag{13}$$

for the linear-trend case; and

$$\hat{v}_K = (1/T)(\hat{r}_o+2\hat{q}_oK), \ a_K = 2\hat{q}_o/T^2 \tag{14}$$

for the quadratic-trend case.

The remainder of the paper will be concerned with a presentation of formulas for the estimation parameters (Sec. II); an assessment of certain statistical characteristics of the random errors in estimation (Sec. III); a discussion of the systematic errors which result from an inadequate representation in the smoothing model; and finally, a review of some elementary points regarding implementation of the estimation schemes (Sec. V).

Some clarification of the notation used to index the time variable may assist in interpreting the results of the following sections. Three indices are employed:

(1) k, which is an integer, is used to index the times at which the observed data were taken;

(2) K, which is a continuous variable, is used to index the time for which an estimate is to be computed and was chosen so that K = k when the estimation time coincides with the time at which the kth observed datum was obtained;

(3) M, which is also a continuous variable, is used to index the time for which an estimate is to be computed, and was chosen so that M = 0 corresponds to the midpoint of the observation interval. Because the observation interval extends from K = 1 to K = N,

$$M = K - (1/2)(N+1) \ .$$

All three indices are related to real time via the intersample period T; thus,

$$t = t_o + KT$$

$$= t_o + MT + (1/2)(N+1)T \ ,$$

where t_o is the value of t at one intersample period prior to the time of observation of the first datum.

In some applications (prediction), it is convenient to assume that the time origin coincides with the time at which the last (N^{th}) datum was obtained. Denoting this time by t′,

$$t' = t - t_o - NT \ ,$$

whence

$$t' = (K-N)T$$

and

$$t' = MT - T_o/2 \ .$$

II. ESTIMATION FORMULAS

A. METHOD OF DERIVATION

It is assumed that the underlying process x(t) can be represented by a truncated Taylor's series, e.g., a quadratic. The corresponding estimation formula is similarly truncated, as in Eqs. 9, 10, and 11. The mean-squared error is a uniformly weighted average of the squared difference between the observed data and the value yielded by the estimation formula at the corresponding point in time:

$$E = (1/N) \sum_{k=1}^{N} (x_k - \hat{x}_k)^2 . \tag{15}$$

The expression for E that results when the explicit form of the estimator $\hat{x}_K$ is substituted into Eq. 15 is then separately differentiated with respect to each of the (as yet) unknown parameters in $\hat{x}_K$. Because E is a positive quadratic function of the unknown parameters, equating the derivatives to zero yields a set of simultaneous equations which must be fulfilled for E to take on its minimum value. These equations are then solved for the unknown parameters. In solving these equations, the following identities* are helpful:

$$\sum_{k=1}^{N} k = (N/2)(N+1) \tag{16}$$

*See, for example, Ref. 2, pp. 7-8.

$$\sum_{k=1}^{N} k^2 = (N/6)(N+1)(2N+1) \tag{17}$$

$$\sum_{k=1}^{N} k^3 = (N^2/4)(N+1)^2 \tag{18}$$

$$\sum_{k=1}^{N} k^4 = (N/30)(N+1)(2N+1)(3N^2+3N-1) \tag{19}$$

B. RESULTS

1. Zero Trend

The single parameter is $\hat{x}_o$:

$$\hat{x}_o = (1/N) \sum_{k=1}^{N} x_k \ . \tag{20}$$

From Eq. 9, this is also the expression for $\hat{x}_K$. Finally, by assumption,

$$\hat{v}_K = 0 \tag{21}$$

$$\hat{a}_K = 0 \tag{22}$$

2. Linear Trend

The two parameters are $\hat{x}_o$ and $\hat{r}_o$:

$$\hat{x}_o = \left[2/N(N-1)\right] \sum_{k=1}^{N} \left[(2N+1) - 3k\right] x_k \tag{23}$$

$$\hat{r}_o = \left[6/N(N^2-1)\right] \sum_{k=1}^{N} \left[2k - (N+1)\right] X_k \tag{24}$$

Substituting these results into the position estimation formula, Eq. 10, yields

$$\hat{x}_K = \left[2/N(N^2-1)\right] \sum_{k=1}^{N} \left[(N+1)(2N+1-3K) + 3k(2K-N-1)\right] X_k \tag{25}$$

For the special case K = N (estimate of current position), Eq. 25 simplifies to

$$\hat{x}_N = \left[2/N(N+1)\right] \sum_{k=1}^{N} \left[3k - (N+1)\right] X_k \ . \tag{26}$$

The velocity estimate is simply

$$\hat{v}_K = \hat{r}_o/T \ . \tag{27}$$

Using Eq. 4, and the above expression for $\hat{r}_o$,

$$\hat{v}_K = \left[6/N(N+1)T_o\right] \sum_{k=1}^{N} \left[2k - (N+1)\right] X_k \ , \tag{28}$$

where T_o is the total period of observation.

By assumption,

$$\hat{a}_K = 0 \ . \tag{29}$$

3. Quadratic Trend

The three parameters are

$$\hat{x}_o = \frac{3}{N(N-1)(N-2)} \sum_{k=1}^{N} \left[10k^2 - 6(2N+1)k + (3N^2+3N+2)\right] X_k \qquad (30)$$

$$\hat{r}_o = \frac{6}{N(N^2-1)(N^2-4)} \sum_{k=1}^{N} \left[-30(N+1)k^2 + 2(2N+1)(8N+11)k - 3(N+1)(N+2)(2N+1)\right] X_k \qquad (31)$$

and

$$\hat{q}_o = \frac{30}{N(N^2-1)(N^2-4)} \sum_{k=1}^{N} \left[6k^2 - 6(N+1)k + (N+1)(N+2)\right] X_k \qquad (32)$$

Substituting these results into the position estimation formula, Eq. 11, yields

$$\hat{x}_K = \frac{3}{N(N^2-1)(N^2-4)} \sum_{k=1}^{N} \Big\{ 10\left[(N+1)(N+2) - 6K(N+1) + 6K^2\right] k^2 - 2\left[3(N+1)(N+2)(2N+1) - 2K(2N+1)(8N+11) + 30K^2(N+1)\right] k + (N+1)(N+2)\left[(3N^2+3N+2) - 6K(2N+1) + 10K^2\right] \Big\} X_k . \qquad (33)$$

For the special case K = N (estimate of current position), Eq. 33 simplifies to

$$\hat{x}_N = \frac{3}{N(N+1)(N+2)} \sum_{k=1}^{N} \left[10k^2 - 2(4N+3)k + (N+1)(N+2)\right] X_k . \qquad (34)$$

Substituting Eqs. 31 and 32 into the velocity estimation formula of Eq. 14 and invoking Eq. 4, there is obtained

$$\hat{v}_K = \frac{6}{N(N+1)(N^2-4)T_o} \sum_{k=1}^{N} \Big\{ 30\Big[2K - (N+1)\Big] k^2$$

$$-2\Big[30K(N+1) - (2N+1)(8N+11)\Big] k$$

$$+ (N+1)(N+2) \Big[10K - 3(2N+1)\Big] \Big\} X_k . \qquad (35)$$

For the special case K = N (estimate of current velocity), Eq. 35 simplifies to

$$\hat{v}_N = \frac{6}{N(N+1)(N^2-4)T_o} \sum_{k=1}^{N} \Big[30(N-1)k^2 - 2(14N^2-11)\, k$$

$$+ (N+1)(N+2)(4N-3)\Big] X_k . \qquad (36)$$

The acceleration estimate is simply

$$\hat{a}_K = 2q_o/T^2 .$$

Using Eq. 4 in Eq. 32,

$$\hat{a}_K = \frac{60(N-1)}{N(N+1)(N^2-4)T_o^{\,2}} \sum_{k=1}^{N} \Big[6k^2 - 6(N+1)k + (N+1)(N+2)\Big] X_k . \qquad (37)$$

4. Interpretation of Results

Each of the estimation formulas involves a weighted linear average of the X_k's. For the special case K = N and N>>1, the relative weights assigned to the observed data for current position estimation are approximately:

1	zero trend
$2\left[3(k/N) - 1\right]$	linear trend
$3\left[10(k/N)^2 - 8(k/N) + 1\right]$	quadratic trend

These functions are depicted graphically in Fig. 1. It can be seen that as the order of the estimation formula increases, more weight is placed on the most recent data (i.e., those points for k approaching N).

For current velocity estimation (K = N), the relative weights for N>>1 are

$6\left[2(k/N) - 1\right]$	linear trend
$12\left[15(k/N)^2 - 14(k/N) + 2\right]$	quadratic trend

These functions are depicted graphically in Fig. 2. It is seen that in the quadratic case, the more recent data are weighted somewhat more heavily than the oldest data, and that the extremes are weighted more heavily than the intermediate points.

Finally, the asymptotic weighting for the acceleration estimate for the quadratic case is shown in Fig. 3. The data are weighted symmetrically, and the values are substantially greater than for the current position and current velocity estimates.

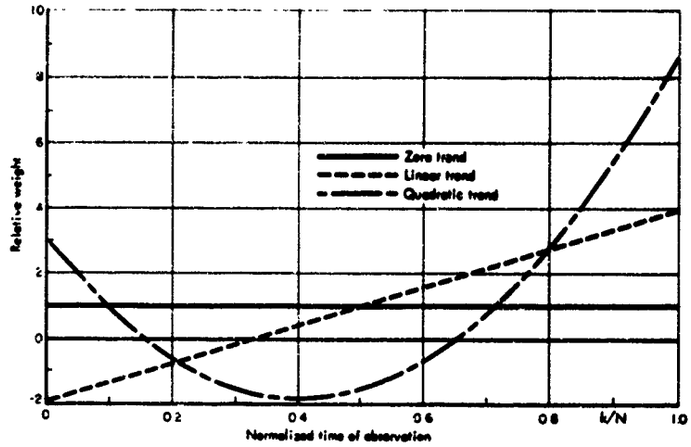

FIGURE 1 Asymptotic Weighting for Current Position Estimation

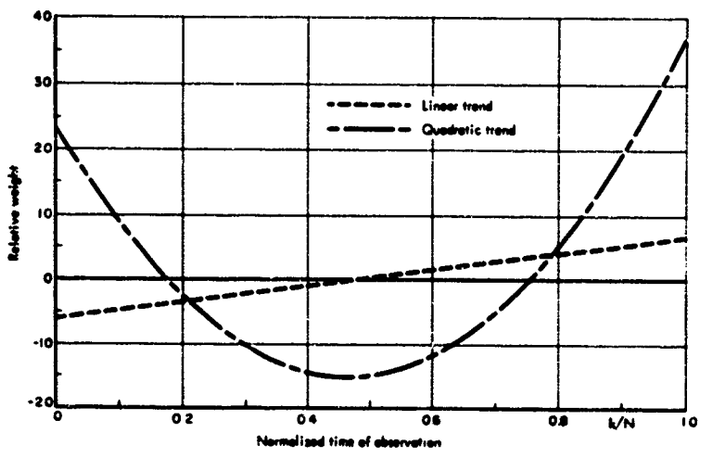

FIGURE 2 Asymptotic Weighting for Current Velocity

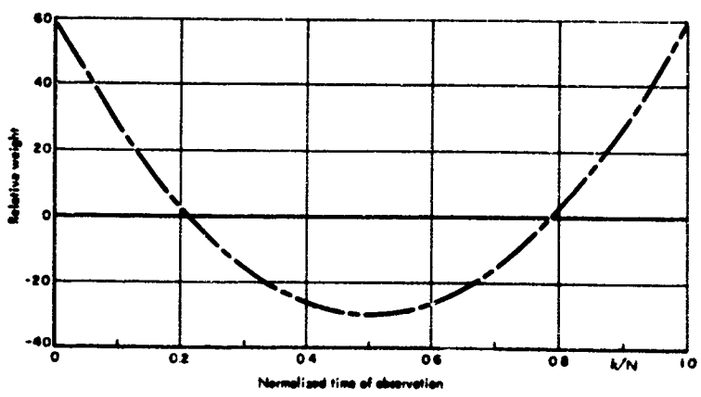

FIGURE 3 Asymptotic Weighting for Quadratic Trend Estimation of Acceleration

III. RANDOM ERRORS

A. GENERAL REMARKS

The estimation formulas presented in the previous section are all of the form

$$\hat{y} = \sum_{k=1}^{N} W_k X_k \ . \tag{38}$$

It can be verified that these formulas have the following property: if the model (zero trend, linear, or quadratic) for the underlying process is correct, and if there are no errors in the observed data, then the estimate itself is correct. In this section, the effect of relaxing the second condition in the preceding sentence will be assessed. That is, it will be presumed, as was mentioned in the Introduction, that the X_k's are in error. These errors are assumed to be additive, statistically uncorrelated, identically distributed, and to have zero means and (finite) variance σ_x^2. Using the notation of Eq. 8, the estimate $\hat{y}$ is given by

$$\hat{y} = \sum_{k=1}^{N} W_k x_k + \sum_{k=1}^{N} W_k \varepsilon_k \ .$$

The ensemble average of $\hat{y}$ is

$$E\left[\hat{y}\right] = \sum_{k=1}^{N} W_k x_k + \sum_{k=1}^{N} W_k E\left[\varepsilon_k\right] \ .$$

Because the ξ_k's are zero-mean random variables, the second sum vanishes. In view of the preceding remarks, the first sum is just the true value of y, so that

$$E\left[\hat{y}\right] = y \ ,$$

which implies in turn that the errors in $\hat{y}$ have zero mean.

The next step is to compute the variance of the errors in $\hat{y}$. It follows from the foregoing results that the variance

$$\sigma^2\left(\hat{y}\right) = E\left[(\hat{y}-y)^2\right] = E\left[\left(\sum_{k=1}^{N} W_k \xi_k\right)^2\right]$$

$$= E\left[\sum_{k=1}^{N}\sum_{n=1}^{N} W_k W_n \xi_k \xi_n\right]$$

$$= \sum_{k=1}^{N}\sum_{n=1}^{N} W_k W_n E\left[\xi_k \xi_n\right] \qquad (39)$$

Because the errors are assumed to be zero mean and uncorrelated,

$$E\left[\xi_k \xi_n\right] = \begin{cases} 0 & k \neq n \\ \sigma_x^{\ 2} & k = n \end{cases}$$

so that Eq. 39 becomes

$$\sigma^2\left(\hat{y}\right) = \sigma_x^{\ 2} \sum_{k=1}^{N} W_k^{\ 2} \ . \qquad (40)$$

The task of computing the variance of the estimation formulas is thus reduced to one of identifying the weights W_k in the formula, and computing the sum indicated in Eq. 40.

It is also of interest to compute the covariance of different errors. Thus, if two estimates $\hat{y}$, given by Eq. 38, and $\hat{z}$, given by

$$\hat{z} = \sum_{k=1}^{N} W_k' X_k$$

are considered, the covariance is given by

$$\begin{aligned} \mathrm{Cov}\left[\hat{y}, \hat{z}\right] &= E\left[\left(\hat{y}-y\right)\left(\hat{z}-z\right)\right] \\ &= E\left[\left(\sum_{k=1}^{N} W_k \varepsilon_n\right)\left(\sum_{k=1}^{N} W_k' \varepsilon_k\right)\right] . \end{aligned}$$

Following the same procedure as for the variance calculation,

$$\mathrm{Cov}\left[\hat{y}, \hat{z}\right] = \sigma_x^2 \sum_{k=1}^{N} W_k W_k' .$$

The normalized correlation coefficient of the errors in $\hat{y}$ and those in $\hat{z}$ is given by

$$\rho\left(\hat{y}, \hat{z}\right) = \frac{\mathrm{Cov}\left[\hat{y}, \hat{z}\right]}{\sigma(\hat{y})\ \sigma(\hat{z})} . \tag{41}$$

B. RESULTS

1. Zero Trend

The variance of the position estimate is

$$\sigma^2\left(\hat{x}_k\right) = \sigma^2\left(\hat{x}_o\right) = \sigma_x^2/N . \tag{42}$$

By assumption, the variances of the velocity and acceleration estimates are zero.

2. Linear Trend

The variances of the estimation parameters $\hat{x}_o$ and $\hat{r}_o$ are

$$\sigma^2\left(\hat{x}_o\right) = \frac{2\sigma_x^{\ 2}}{N}\left(\frac{2N+1}{N-1}\right) \tag{43}$$

$$\sigma^2\left(\hat{r}_o\right) = \frac{12\sigma_x^{\ 2}}{N(N^2-1)} \tag{44}$$

The covariance of $\hat{x}_o$ and $\hat{r}_o$ is

$$\mathrm{Cov}\left[\hat{x}_o,\ \hat{r}_o\right] = \frac{-6\sigma_x^{\ 2}}{N(N-1)} \ .$$

The variance of the position estimate $\hat{x}_K$ can be computed directly, or by noting that by virtue of Eq. 10, it follows that

$$\sigma^2\left(x_K\right) = \sigma^2\left(\hat{x}_o\right) + 2K\ \mathrm{Cov}\left[\hat{x}_o,\ \hat{r}_o\right] + K^2\sigma^2\left(\hat{r}_o\right) \ .$$

In any event,

$$\sigma^2\left(\hat{x}_K\right) = \frac{2\sigma_x^{\ 2}}{N(N^2-1)}\left[(N+1)(2N+1) - 6K(N+1) + 6K^2\right] \ . \tag{45}$$

For the special case K = N, the variance of the estimate of current position reduces to

$$\sigma^2\left(\hat{x}_N\right) = \frac{2\sigma_x^{\ 2}}{N}\left(\frac{2N-1}{N+1}\right) . \tag{46}$$

The variance of the velocity estimate, v_K, is simply

$$\sigma^2\left(\hat{v}_K\right) = \sigma^2\left(\hat{r}_o\right)\Big/T^2 \ .$$

Using Eq. 4,

$$\sigma^2\left(\hat{v}_K\right) = \frac{12\sigma_x^{\ 2}}{NT_o^{\ 2}}\left(\frac{N-1}{N+1}\right) . \tag{47}$$

The normalized correlation coefficient for the errors in the current position estimate and the velocity estimate is

$$\rho\left(\hat{x}_N, \hat{v}_N\right) = \sqrt{\frac{3(N-1)}{2(2N-1)}} . \tag{48}$$

3. Quadratic Trend

The variances of the estimation parameters $\hat{x}_o$, $\hat{r}_o$, and $\hat{q}_o$ are

$$\sigma^2\left(\hat{x}_o\right) = \frac{3\sigma_x^{\ 2}}{N}\left[\frac{3N^2+3N+2}{(N-1)(N-2)}\right] \tag{49}$$

$$\sigma^2\left(\hat{r}_o\right) = \frac{12\sigma_x^{\ 2}}{N}\left[\frac{(2N+1)(8N+11)}{(N^2-1)(N^2-4)}\right] \tag{50}$$

$$\sigma^2\left(\hat{q}_o\right) = \frac{180\sigma_x^{\ 2}}{N(N^2-1)(N^2-4)} \tag{51}$$

The covariances are

$$\mathrm{Cov}\left[\hat{x}_o, \hat{r}_o\right] = -\frac{180\sigma_x^{\ 2}(2N+1)}{N(N-1)(N-2)} \tag{52}$$

$$\mathrm{Cov}\left[\hat{x}_o, \hat{q}_o\right] = \frac{30\sigma_x^{\ 2}}{N(N-1)(N-2)} \tag{53}$$

$$\mathrm{Cov}\left[\hat{r}_o, \hat{q}_o\right] = \frac{-180\sigma_x^{\ 2}}{N(N-1)(N^2-4)} \tag{54}$$

The variance of the position estimate $\hat{x}_K$ is given by

$$\sigma^2\left(\hat{x}_K\right) = \frac{3\sigma_x^{\ 2}}{N(N^2-1)(N^2-4)} \Big[(N+1)(N+2)(3N^2+3N+2)$$

$$- 12(N+1)(N+2)(2N+1)K + 12(7N^2+15N+7)K^2$$

$$- 120(N+1)K^3 + 60K^4\Big] . \tag{55}$$

Setting K = N in Eq. 55, the variance of the estimate of current position reduces to

$$\sigma^2\left(\hat{x}_N\right) = \frac{3\sigma_x^{\ 2}}{N(N+1)(N+2)} (3N^2-3N+2) . \tag{56}$$

The variance of the velocity estimate $\hat{v}_K$ is

$$\sigma^2\left(\hat{v}_K\right) = \frac{12\sigma_x^{\ 2}}{NT_o^{\ 2}} \left(\frac{N-1}{N+1}\right)\left[\frac{(2N+1)(8N+11) - 60K(N+1) + 60K^2}{N^2-4}\right] . \tag{57}$$

where T_o is the total period of observation. The variance of the estimate of current velocity (K = N) is

$$\sigma^2\left(\hat{v}_N\right) = \frac{12\sigma_x^{\ 2}}{NT_o^{\ 2}} \left(\frac{N-1}{N+1}\right) \left[\frac{(2N-1)(8N-11)}{N^2-4}\right] . \tag{58}$$

The normalized correlation coefficient for the errors in the current position estimate and the current velocity estimate is

$$\rho\left(\hat{x}_N, \hat{v}_N\right) = \sqrt{\frac{9(N-1)(N-2)(2N-1)}{(8N-11)(3N^2+3N-2)}} . \tag{59}$$

The variance of the acceleration estimate $\hat{a}_K$ is

$$\sigma^2\left(\hat{a}_K\right) = \frac{720\sigma_x^{\ 2}}{NT_o^{\ 4}} \left[\frac{(N-1)^3}{(N+1)(N^2-4)}\right] . \tag{60}$$

The normalized correlation coefficient for the errors in the current position estimate and the acceleration estimate is

$$\rho\left(\hat{x}_N, \hat{a}_N\right) = \sqrt{\frac{5(N-1)(N-2)}{3(3N^2-3N+2)}} . \tag{61}$$

The normalized correlation coefficient for the errors in the current velocity estimate and the acceleration estimate is

$$\rho\left(\hat{v}_N, \hat{a}_N\right) = \sqrt{\frac{15(N-1)^2}{(2N-1)(8N-11)}} \tag{62}$$

C. INTERPRETATION OF RESULTS: LARGE-SAMPLE LIMITS

For the linear-trend case, the variance of the position estimate $\hat{x}_K$ takes on a minimum value for

$$K_o = \frac{N+1}{2} . \tag{63}$$

For this value of K, the variance is

$$\sigma^2\left(\hat{x}_{K_o}\right) = \sigma_x^{\ 2}/N ,$$

which is the same as the variance of the zero-trend estimate of position. This result implies the following:

(1) If the objective of the estimation procedure is to estimate the true value of the position at the midpoint of the

observation interval, then the variances yielded by the zero-trend formula and the linear-trend formulas will be the same.*

(2) If the zero-trend hypothesis is true, then use of the linear-trend formula for position estimation will yield a variance that is everywhere greater than the variance that would be obtained with the zero-trend formula with the exception of the single point given by Eq. 63. This point is in consonance with the general statistical principle that simply increasing the complexity of the hypothesized trend beyond the necessary level will only degrade the quality of the resulting estimate.

With regard to the quadratic-trend position estimate, the variance takes on three extreme values at the points

$$K_o = \frac{N+1}{2} \tag{64}$$

and

$$K_{\pm 1} = \frac{N+1}{2} \pm \sqrt{\frac{N^2+1}{20}} \quad . \tag{65}$$

The variance at the point given by Eq. 64 is a local maximum, and has the value

$$\sigma^2\left(x_{K_o}\right) = \frac{3\sigma_x^{\ 2}}{4N}\left(\frac{3N^2-7}{N^2-4}\right) . \tag{66}$$

The variances at the points given by Eq. 65 are minima and have the value

$$\sigma^2\left(x_{K_{\pm 1}}\right) = \frac{3\sigma_x^{\ 2}}{5N}\left(\frac{3N^2-2}{N^2-1}\right) . \tag{67}$$

*Furthermore, it will be seen in the next section that the systematic error in the zero-trend formula vanishes for this case when a strictly linear trend is present.

The variance of the quadratic-trend position estimate is equal to the variance of the linear-trend position estimate at the points

$$K = \frac{N+1}{2} \pm \sqrt{\frac{N^2-1}{12}} \tag{68}$$

at which the variances are simply

$$\sigma^2\left(\hat{x}_K\right) = 2\sigma_x^2/N \ . \tag{69}$$

For the quadratic-trend case, the variance of the velocity-estimation error takes on its minimum value for

$$K_o = \frac{N+1}{2}$$

at which point the variance is

$$\sigma^2\left(v_{K_o}\right) = \frac{12\sigma_x^2}{NT_o^2}\left(\frac{N-1}{N+1}\right) , \tag{70}$$

which is just the variance of the velocity-estimate error for the linear-trend case.

For large values of N (i.e., $N \geq 100$), the expressions for the variances and correlation coefficients can be replaced by considerably simpler approximations. Of particular interest are the normalized correlation coefficients. In the linear-trend case, $\rho\left(\hat{x}_N, \hat{v}_N\right)$ approaches $\sqrt{3/2} = 0.8660$. In the quadratic-trend case, the following limiting approximations hold

$$\rho\left(\hat{x}_N, \hat{v}_N\right) \doteq \sqrt{3/2} = 0.8660$$

$$\rho\left(\hat{x}_N, \hat{a}_N\right) \doteq \sqrt{5/3} = 0.7454$$

$$\rho\left(\hat{v}_N, \hat{a}_N\right) \doteq \sqrt{15/4} = 0.9682$$

The fact that the values for these coefficients are rather large indicates that cross terms in error calculations involving (for example) concurrent position and velocity errors cannot be ignored.

Figure 4 presents graphs of the large-sample limit of the standard deviation of the position-estimation error normalized so that the standard deviation for the zero-trend case is unity.

Figure 5 presents graphs of the large-sample limit of the standard deviation of the velocity-estimation error, normalized so that the standard deviation for the linear-trend case is unity.

These graphs indicate that the errors for the quadratic-trend case grow substantially faster than for the linear-trend case when the objective is to predict position or velocity beyond the period of observation ($K/N > 1$). In practical situations, however, it is frequently possible to make the observation period substantially longer in the quadratic-trend case, because systematic errors in the prediction equation will be less for the quadratic case than they would be for the linear case.

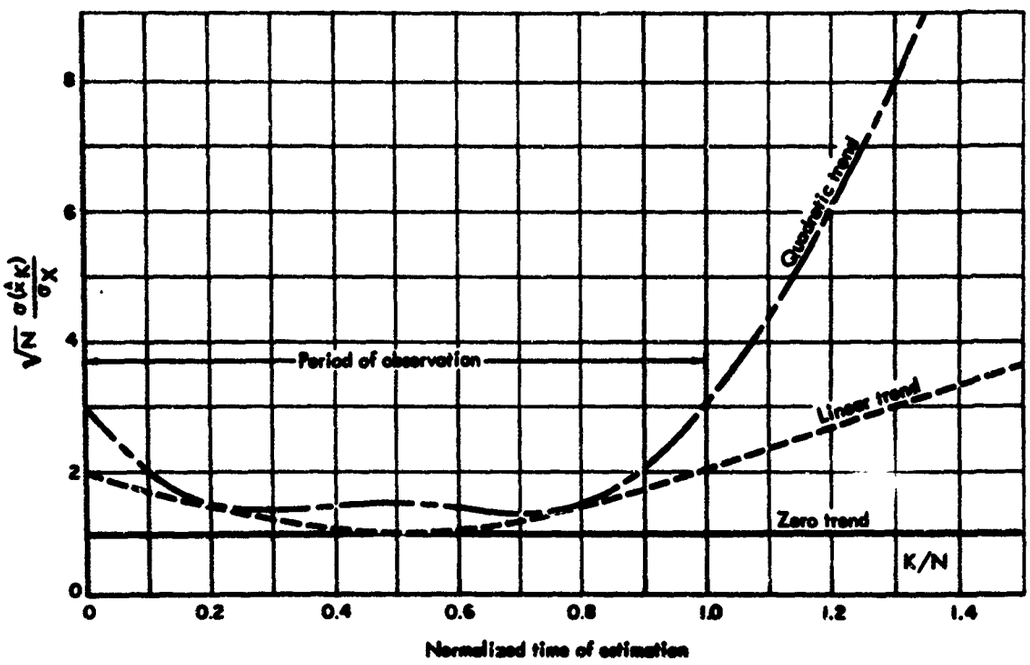

FIGURE 4 Large - Sample Limit of Standard Deviation of Position Estimation Error

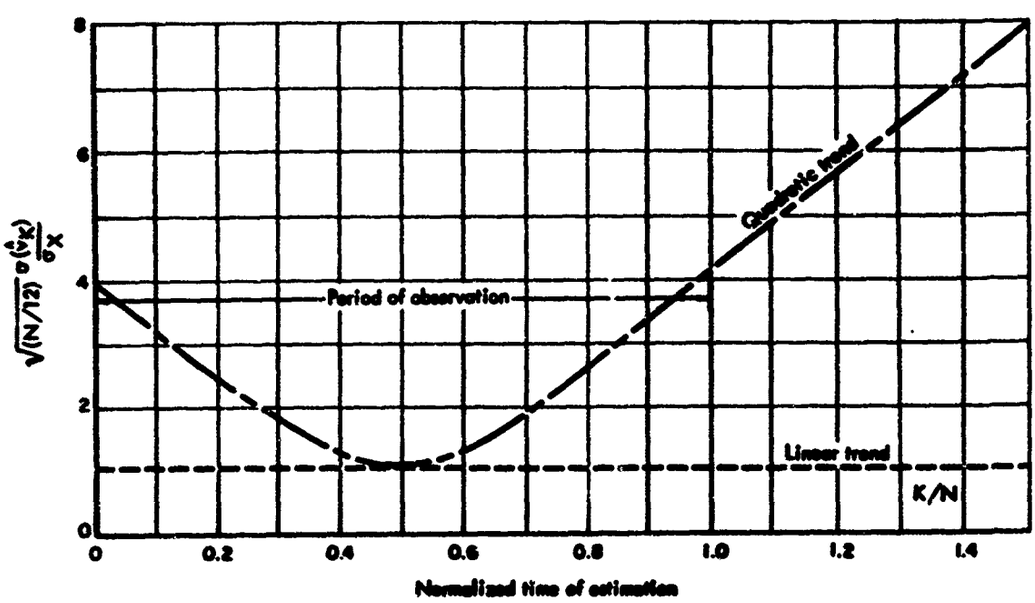

FIGURE 5 Large - Sample Limit of Standard Deviation of Velocity Estimation Error

IV. SYSTEMATIC ERRORS

A. GENERAL REMARKS

In many practical situations, the form assumed for the estimation formula is only an approximation to the actual variations of the underlying process. The significance of higher-order terms in the Taylor's series representation of the underlying process is determined by the time interval over which the estimation rule is to be applied, as well as the magnitude of the coefficients of these terms. For the purpose of interpolation, i.e., estimation for times within the period of observation, the significant time period is the period of observation itself. For purposes of extrapolation or prediction, the prediction interval, i.e., the elapsed time between the period of observation and the time at which the predicted estimate is to be determined, is usually more important.

In any event, the presence of higher-order terms in the representation of the process leads to errors in estimation which cannot be removed simply by improving the quality of the observed data or by taking a larger sample of data. To distinguish these errors from the errors which accrue from imperfections in the observed data (i.e., the random errors discussed in the previous section), they are referred to as systematic errors.

The results presented here are intended to facilitate a preliminary assessment of systematic errors. The approach taken has been to assume that the trend of the true data contains a single term (of the next-higher order) beyond what is accounted for in the particular estimation formula. For example, a cubic trend is assumed for estimation rules based on the quadratic model. As such, the results provide a basis for establishing necessary (but not sufficient) conditions

for determining the validity or accuracy of a particular estimation procedure, as long as the underlying data are characterized by a single, well-behaved trend which is representable with reasonable accuracy via Taylor's series. It should be noted that there are entirely credible situations in which the underlying process is not characterizable in this manner; for example, a linear trend can have an abrupt change in slope, or can change to a quadratic, during the estimation interval. The results presented here are not applicable to such situations, which necessitate either a more detailed analytical model, or computer simulation, for assessment of estimation performance.

B. RESULTS

1. Zero Trend

The position estimate $\hat{x}_K$ is a constant given by Eq. 20. It is assumed that the underlying process exhibits a linear trend, so that the true sample values are of the form

$$x_k = x_o + kr_o \ .$$

The systematic error in the position estimate will be defined as

$$\Delta\hat{x}_K = E\left[\hat{x}_K\right] - x_K$$

and (for this case) is given by

$$\Delta\hat{x}_K = (v_o T/2)(N+1-2K) \ , \qquad (71)$$

where v_o is the true velocity.

In what follows, it is of interest to reference the estimation time index K to the midpoint of the period of observation. This is accomplished by setting

$$M = K - \frac{N+1}{2} \ . \qquad (72)$$

The quantity KT represents the time for which the estimate is to be made, relative to the time t_o mentioned in the Introduction. The quantity MT, in turn, represents the time for which the estimate is to be made, relative to the midpoint of the period of observation. With this shift, Eq. 71 becomes

$$\Delta\hat{x}_K = -Mv_oT \ . \tag{73}$$

2. Linear Trend

The position estimate is given by Eq. 25; it is assumed that the true values of the sampled data exhibit a quadratic trend:

$$x_k = x_o + kr_o + k^2q_o \ .$$

The systematic error of the position estimate is given by

$$\Delta\hat{x}_K = -(a_oT^2/12)\left[6K^2 - 6K(N+1) + (N+1)(N+2)\right] \ , \tag{74}$$

where a_o is the true acceleration. In terms of the shifted index M,

$$\Delta\hat{x}_K = -(a_oT^2/2)\left[M^2 - \frac{(N^2-1)}{12}\right] \tag{75}$$

The velocity estimate is given by Eq. 35. The systematic error is

$$\Delta\hat{v}_K = (a_oT/2)(N+1 - 2K) \tag{76}$$

or

$$\Delta\hat{v}_K = Ma_oT \ . \tag{77}$$

3. Quadratic Trend

The position estimate is given by Eq. 33; it is assumed that the true values of the sampled data exhibit a cubic trend:

$$x_k = x_o + kr_o + k^2q_o + k^3p_o \ .$$

The systematic error of the position estimate is given by

$$\Delta\hat{x}_K = (b_o T^3/120)\Big[(N+1)(N+2)(N+3) - 2K(6N^2+15N+11) + 30K^2(N+1) - 20K^3\Big] , \qquad (78)$$

where b_o is the true value of the third derivative of the underlying process. Expressed in terms of the shifted index M,

$$\Delta\hat{x}_K = M(b_o T^3/120)(3N^2-7-20M^2) \qquad (79)$$

The velocity estimate is given by Eq. 35; the systematic error is

$$\Delta\hat{v}_K = (b_o T^2/60)\Big[30K^2 - 30(N+1)K + (6N^2+15N+11)\Big] \qquad (80)$$

or, in terms of the shifted index M,

$$\Delta\hat{v}_K = (b_o T^2/120)\Big[60M^2 - (3N^2-7)\Big] \quad . \qquad (81)$$

V. IMPLEMENTATION CONSIDERATIONS

In some situations (particularly those involving real-time estimation), direct implementation of the formulas presented in Sec. II can lead to substantial problems from the standpoint of data processing. Specifically, the requirements for data storage and computational speed (or parallel arithmetic hardware) can become burdensome, unless certain simplifications and approximations are adopted.

Consider, for example, the task of obtaining smoothed estimates of the current range, azimuth, and elevation of a radar target from raw data provided by a monopulse tracking radar. Suppose that the radar operates at a pulse repetition frequency of 10,000 pulses per second, and that smoothed estimates of the current values of each of the three coordinates are to be provided at a rate of 100 per second, with a smoothing interval of one second and a computational lag of 10 milliseconds. Under these conditions, it will be necessary to perform three computations of the form

$$\hat{x}_j = \sum_{k=1}^{10,000} W_k X_{k+j}$$

one hundred times per second. In this expression, $\hat{x}_j$ is the jth estimate of the value of the coordinate x (i.e., range, azimuth, or elevation) 10 milliseconds ago; X_{k+j} is the kth measurement of that coordinate in the sample to be used in computing $\hat{x}_j$; and W_k is the weight assigned to X_{k+j}, in accordance with the appropriate estimation formula

from Sec. II. For example, if the quadratic-trend estimation is to be employed, Eq. 34 gives

$$W_k = \frac{30k^2 - 6(4N+3)k + 3(N+1)(N+2)}{N(N+1)(N+2)}$$

in which N = 10,000.

If the computations were to be carried out in the straightforward manner, it would be necessary to perform 30,000 multiplications and additions in 10 milliseconds, which would require the completion of a single multiply-and-add cycle in one microsecond, assuming parallel processing of the three sets of data. Moreover, such computations would entail storage and shifting of 30,000 words of raw data. While both the computational speed and data storage requirements are within the state of the art, they would be considered burdensome for many applications.

The first simplification to be noted stems from the fact that the various estimation formulas can be reduced to simple linear combinations of the following sums:

$$S_0(j) = \sum_{k=1}^{N} x_{k+j}$$

$$S_1(j) = \sum_{k=1}^{N} kx_{k+j}$$

$$S_2(j) = \sum_{k=1}^{N} k^2 x_{k+j}$$

Updating these forms can be accomplished without large-scale computations. For example, if estimates are to be provided at the original data rate, the following simplifications can be used:

$$S_0(j+1) = S_0(j) + X_{N+j+1} - X_{j+1}$$

$$S_1(j+1) = S_1(j) - S_0(j) + NX_{N+j+1}$$

$$S_2(j+1) = S_2(j) - 2S_1(j) + S_0(j) + N^2X_{N+j+1}$$

If these algorithms were to be employed in the most straightforward manner for the computations of the aforementioned example, it would be necessary to perform approximately 3000 operations in the 10-millisecond computation interval. Assuming parallel operation for the three sets of data, 100 microseconds would be available for performing the ten updating operations (seven of which are additions or subtractions). From the standpoint of computational speed requirements, this rate is much more reasonable than the previous requirement, but the data storage requirement is, however, slightly greater.

To reduce both the computational burden and the data storage requirement, it may be permissible to replace the original data with short-term averages. Thus, suppose that the X's are averaged over a short time interval, such as 10 milliseconds, and that the resulting average is regarded as a new raw datum which was observed at the midpoint of the averaging interval, then

$$X_j^* = (1/N_A) \sum_{k=1}^{N_A} X_{k+j} ,$$

where N_A is the number of observations used in computing the short-term average (100 in the example being disucssed). Equation 73 shows that the systematic error in X_j^*, due to a linear trend in the raw data, vanishes, because of the interpretation of X_j^* as having been taken at the midpoint of the short-term averaging interval (i.e., M = 0). The most significant systematic error will therefore arise from

quadratic and higher (even) order trends; from Eq. 75, the order of magnitude of the systematic error will be

$$\Delta X_j^* = a_o T_A^2 / 24 \quad .$$

Thus, if the systematic error is to be less than one foot, the acceleration a_o should not exceed 2.4×10^5 ft/sec^2, or approximately 7500 g. For most applications, a_o will be considerably smaller than this figure, so that the systematic errors in X_j^* will be negligible.

The impact of this on the data processing requirements is quite significant, because it is now only necessary to accumulate three sums of the form of X_j^*; the three additions per 100 microseconds can be done serially instead of in parallel (assuming appropriate buffer storage).

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13. ABSTRACT Explicit formulas are presented for estimating position, velocity, and acceleration in low-order polynomial trends, based on least-squares smoothing of sampled data accompanied by statistically uncorrelated measurement errors. Formulas are also given for interpolation and prediction of position and velocity. Expressions for the variances and covariances of consistent position, velocity, and acceleration estimates are given, and the systematic errors accruing from use of a trend estimation basis which is one order lower than the actual trend are presented.

One interesting result is that the normalized correlation between the errors in an estimate of current position and those in an estimate of current velocity approaches $\sqrt{3}/2$ when the number of measurements in the estimates becomes large.

Finally, the problem of implementing real-time least-squares estimation and prediction formulas in practical systems is discussed. It is concluded that arithmetic execution time requirements can be relaxed by generating certain sums recursively, and that data storage requirements can frequently be eased by collapsing the raw data into short-term-average samples.

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