POINT INTERVAL ESTIMATION, FROM ONE-ORDER STATISTIC, OF THE LOCATION PARAMETER OF AN EXTREME-VALUE DISTRIBUTION WITH KNOWN SCALE PARAMETER AND OF THE SCALE PARAMETER OF A WEIBULL DISTRIBUTION WITH KNOWN SHAPE PARAMETER


ALBERT H. MOORE
AIR FORCE INSTITUTE OF TECHNOLOGY

H. LEON HARTER
APPLIED MATHEMATICS RESEARCH LABORATORY

Project No. 7071

Distribution of this document is unlimited

OFFICE OF AEROSPACE RESEARCH
United States Air Force

Best Available Copy
NOTICES

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatever, and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

Qualified requesters may obtain copies of this report from the Defense Documentation Center, (DDC), Cameron Station, Alexandria, Virginia.

(Reproduction in whole or in part is permitted for any purpose of the U.S. Gov't)

Distribution of this document is unlimited

Copies of ARL Technical Documentation Reports should not be returned to Aerospace Research Laboratories unless return is required by security considerations, contractual obligations or notices on a specified document.
Point and Interval Estimation, From One-Order Statistic, of the Location Parameter of an Extreme-Value Distribution with Known Scale Parameter and of the Scale Parameter of a Weibull Distribution with Known Shape Parameter

ALBERT H. MOORE AND H. LEON HARTER

Abstract—This paper derives a one-order statistic estimator $\gamma_n$ for the location parameter of the (first) extreme-value distribution of smallest values with cumulative distribution function $F(x; u, b) = 1 - e^{-[e^{(x-u)/b}]}$ using the minimum-variance unbiased one-order statistic estimator for the scale parameter of an exponential distribution, as was done in an earlier paper for the scale parameter of a Weibull distribution. It is shown that exact confidence bounds, based on one-order statistic, can be easily derived for the location parameter of the extreme-value distribution and for the scale parameter of the Weibull distribution, using exact confidence bounds for the scale parameter of the exponential distribution. The estimator for $u$ is shown to be $h \ln \gamma_m + \gamma_m$, where $\gamma_m$ is the $m$th order statistic from an ordered sample of size $n$ from the extreme-value distribution with scale parameter $b$ and $c_m$ is the coefficient for a one-order statistic estimator of the scale parameter of an exponential distribution. Values of the factor $c_m$, which have previously been tabulated for $n = 1(1)20$, are given for $n = 21(1)40$. The ratios of the mean-square-errors of the maximum-likelihood estimators based on $m$ order statistics to those of the one-order statistic estimators for the location parameter of the extreme-value distribution and the scale parameter of the Weibull distribution are investigated by Monte Carlo methods. The use of the table and related tables is discussed and illustrated by numerical examples.

I. INTRODUCTION

In a previous paper, Harter and Moore [1] have derived a maximum-likelihood and an unbiased estimator $\hat{\lambda}_b$ and $\tilde{\lambda}_b$ of the location parameter of the extreme-value distribution with known scale parameter, based on the first $m$ out of $n$ ordered observations. However, in many practical applications an inefficient estimator may be chosen for its inherent simplicity. Harter [2] found the minimum-variance unbiased one-order statistic estimator for the scale parameter $\sigma$ of the exponential distribution, from a sample of size $n$. Moore and Harter [3] tabulated the coefficient $c_m$ of the minimum-variance unbiased one-order statistic estimator for the scale parameter of the exponential distribution from a censored sample of size $m$ from a life test of $n$ items [n = 1(1)20] and showed how it could be used to obtain a one-order statistic estimator for the scale parameter of Weibull populations with known shape parameter. By use of the coefficient of the $m$th order statistic, computed for estimation of the scale parameter of the exponential distribution, it is shown in Section II that a consistent one-order statistic estimator for the scale parameter of the extreme-value distribution with known scale parameter can be obtained. The values of $c_m$ are given in Table I, along with the relative efficiencies of the one-order statistic estimators of the scale parameter $\sigma$ of an exponential distribution as compared with the unbiased $m$-order statistic estimators, for $n = 21(1)40$, $m = 1(1)n$, and $k = \min(m, \sigma)$, where the $r$th order statistic is optimal for the complete sample. In Sections III and IV it is shown that exact confidence bounds, based on one-order statistic, can be derived for the location parameter of the extreme-value distribution and the scale parameter of the Weibull distribution using the coefficients of the exact confidence bounds, found by Harter [4], for the scale parameter of an exponential distribution. In Section V a Monte Carlo comparison of the relative merits of the one-order statistic estimators and the maximum-likelihood estimators is given. In Section VI, the use of Table I and related tables is discussed and illustrated by numerical examples.

II. MATHEMATICAL FORMULATION

If $Y$ has an exponential distribution with scale parameter $\sigma$ and location parameter zero, then $X = b \ln Y$ has the (first) extreme-value distribution with $u = b \ln \sigma$ as location parameter and $b$ as scale parameter. A one-order statistic estimator for the scale parameter of the exponential distribution is given by

$$\hat{\sigma} = c_m y_m$$

where

$$c_m = 1 \left[ \sum_{i=1}^{m} 1/(n - i + 1) \right]$$

and $y_m$ is the $m$th order statistic from an ordered sample of size $n$ from the exponential distribution. Therefore, an
TABLE I

<table>
<thead>
<tr>
<th>N</th>
<th>K</th>
<th>M</th>
<th>(M/4, N)</th>
<th>EFF.</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>3</td>
<td>5</td>
<td>1.0572</td>
<td>97.87</td>
</tr>
<tr>
<td>22</td>
<td>2</td>
<td>4</td>
<td>1.2780</td>
<td>99.17</td>
</tr>
<tr>
<td>23</td>
<td>3</td>
<td>6</td>
<td>1.4080</td>
<td>99.87</td>
</tr>
<tr>
<td>24</td>
<td>4</td>
<td>8</td>
<td>1.5380</td>
<td>99.87</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
<td>10</td>
<td>1.6680</td>
<td>99.87</td>
</tr>
<tr>
<td>26</td>
<td>6</td>
<td>12</td>
<td>1.8080</td>
<td>99.87</td>
</tr>
<tr>
<td>27</td>
<td>7</td>
<td>14</td>
<td>1.9480</td>
<td>99.87</td>
</tr>
<tr>
<td>28</td>
<td>8</td>
<td>16</td>
<td>2.0880</td>
<td>99.87</td>
</tr>
<tr>
<td>29</td>
<td>9</td>
<td>18</td>
<td>2.2280</td>
<td>99.87</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>20</td>
<td>2.3680</td>
<td>99.87</td>
</tr>
<tr>
<td>31</td>
<td>11</td>
<td>22</td>
<td>2.5080</td>
<td>99.87</td>
</tr>
<tr>
<td>32</td>
<td>12</td>
<td>24</td>
<td>2.6480</td>
<td>99.87</td>
</tr>
<tr>
<td>33</td>
<td>13</td>
<td>26</td>
<td>2.7880</td>
<td>99.87</td>
</tr>
<tr>
<td>34</td>
<td>14</td>
<td>28</td>
<td>2.9280</td>
<td>99.87</td>
</tr>
<tr>
<td>35</td>
<td>15</td>
<td>30</td>
<td>3.0680</td>
<td>99.87</td>
</tr>
<tr>
<td>36</td>
<td>16</td>
<td>32</td>
<td>3.2080</td>
<td>99.87</td>
</tr>
<tr>
<td>37</td>
<td>17</td>
<td>34</td>
<td>3.3480</td>
<td>99.87</td>
</tr>
<tr>
<td>38</td>
<td>18</td>
<td>36</td>
<td>3.4880</td>
<td>99.87</td>
</tr>
<tr>
<td>39</td>
<td>19</td>
<td>38</td>
<td>3.6280</td>
<td>99.87</td>
</tr>
<tr>
<td>40</td>
<td>20</td>
<td>40</td>
<td>3.7680</td>
<td>99.87</td>
</tr>
</tbody>
</table>

Note: EFF. stands for efficiency relative to the MT-order statistic.
<table>
<thead>
<tr>
<th>N</th>
<th>A</th>
<th>M</th>
<th>C(k, N)</th>
<th>K</th>
<th>M</th>
<th>C(k, N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>10</td>
<td>4</td>
<td>2.5</td>
<td>10</td>
<td>3</td>
<td>9.65000</td>
</tr>
<tr>
<td>12</td>
<td>11</td>
<td>5</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>12</td>
<td>6</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>13</td>
<td>7</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>14</td>
<td>8</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>15</td>
<td>9</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>16</td>
<td>10</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>17</td>
<td>11</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>18</td>
<td>12</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>19</td>
<td>13</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>20</td>
<td>14</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>21</td>
<td>15</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>22</td>
<td>16</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>23</td>
<td>17</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>24</td>
<td>18</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>25</td>
<td>19</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>26</td>
<td>20</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>27</td>
<td>21</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>28</td>
<td>22</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>29</td>
<td>23</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>30</td>
<td>24</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>31</td>
<td>25</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>32</td>
<td>26</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>33</td>
<td>27</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>34</td>
<td>28</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>35</td>
<td>29</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>36</td>
<td>30</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>37</td>
<td>31</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>38</td>
<td>32</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>39</td>
<td>33</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>40</td>
<td>34</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>41</td>
<td>35</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>42</td>
<td>36</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>43</td>
<td>37</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>44</td>
<td>38</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>46</td>
<td>45</td>
<td>39</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>46</td>
<td>40</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>47</td>
<td>41</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>49</td>
<td>48</td>
<td>42</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>49</td>
<td>43</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>50</td>
<td>44</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>52</td>
<td>51</td>
<td>45</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>53</td>
<td>52</td>
<td>46</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>54</td>
<td>53</td>
<td>47</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>55</td>
<td>54</td>
<td>48</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>56</td>
<td>55</td>
<td>49</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>57</td>
<td>56</td>
<td>50</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>58</td>
<td>57</td>
<td>51</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>59</td>
<td>58</td>
<td>52</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>59</td>
<td>53</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>60</td>
<td>54</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>62</td>
<td>61</td>
<td>55</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>63</td>
<td>62</td>
<td>56</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>63</td>
<td>57</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>65</td>
<td>64</td>
<td>58</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>66</td>
<td>65</td>
<td>59</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>67</td>
<td>66</td>
<td>60</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>68</td>
<td>67</td>
<td>61</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>69</td>
<td>68</td>
<td>62</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>69</td>
<td>63</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>71</td>
<td>70</td>
<td>64</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>72</td>
<td>71</td>
<td>65</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>73</td>
<td>72</td>
<td>66</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>74</td>
<td>73</td>
<td>67</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>75</td>
<td>74</td>
<td>68</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>76</td>
<td>75</td>
<td>69</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>77</td>
<td>76</td>
<td>70</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>78</td>
<td>77</td>
<td>71</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>79</td>
<td>78</td>
<td>72</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>79</td>
<td>73</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>81</td>
<td>80</td>
<td>74</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>82</td>
<td>81</td>
<td>75</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>83</td>
<td>82</td>
<td>76</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>84</td>
<td>83</td>
<td>77</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>85</td>
<td>84</td>
<td>78</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>86</td>
<td>85</td>
<td>79</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>87</td>
<td>86</td>
<td>80</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>88</td>
<td>87</td>
<td>81</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>89</td>
<td>88</td>
<td>82</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>89</td>
<td>83</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>91</td>
<td>90</td>
<td>84</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>92</td>
<td>91</td>
<td>85</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>93</td>
<td>92</td>
<td>86</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>94</td>
<td>93</td>
<td>87</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95</td>
<td>94</td>
<td>88</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>96</td>
<td>95</td>
<td>89</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>97</td>
<td>96</td>
<td>90</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>98</td>
<td>97</td>
<td>91</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>98</td>
<td>92</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>99</td>
<td>93</td>
<td>9.65000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>K</td>
<td>M</td>
<td>C.K.N.</td>
<td>EFF.</td>
<td>V</td>
<td>K</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>-------</td>
<td>------</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>32</td>
<td>19</td>
<td>10</td>
<td>1.14448</td>
<td>0.84</td>
<td>34</td>
<td>28</td>
</tr>
<tr>
<td>32</td>
<td>20</td>
<td>20</td>
<td>1.00481</td>
<td>92.36</td>
<td>34</td>
<td>29</td>
</tr>
<tr>
<td>32</td>
<td>21</td>
<td>11</td>
<td>0.90629</td>
<td>91.51</td>
<td>34</td>
<td>29</td>
</tr>
<tr>
<td>32</td>
<td>22</td>
<td>22</td>
<td>0.80734</td>
<td>89.65</td>
<td>35</td>
<td>1</td>
</tr>
<tr>
<td>32</td>
<td>23</td>
<td>23</td>
<td>0.70832</td>
<td>88.44</td>
<td>35</td>
<td>2</td>
</tr>
<tr>
<td>32</td>
<td>24</td>
<td>24</td>
<td>0.60920</td>
<td>87.05</td>
<td>35</td>
<td>3</td>
</tr>
<tr>
<td>32</td>
<td>25</td>
<td>25</td>
<td>0.50998</td>
<td>85.53</td>
<td>35</td>
<td>4</td>
</tr>
<tr>
<td>32</td>
<td>26</td>
<td>26</td>
<td>0.41074</td>
<td>83.24</td>
<td>35</td>
<td>5</td>
</tr>
<tr>
<td>32</td>
<td>27</td>
<td>27</td>
<td>0.31148</td>
<td>80.36</td>
<td>35</td>
<td>6</td>
</tr>
<tr>
<td>32</td>
<td>28</td>
<td>28</td>
<td>0.21210</td>
<td>77.60</td>
<td>36</td>
<td>7</td>
</tr>
<tr>
<td>32</td>
<td>29</td>
<td>29</td>
<td>0.11270</td>
<td>74.66</td>
<td>36</td>
<td>8</td>
</tr>
<tr>
<td>32</td>
<td>30</td>
<td>30</td>
<td>0.01270</td>
<td>71.34</td>
<td>36</td>
<td>9</td>
</tr>
<tr>
<td>32</td>
<td>31</td>
<td>31</td>
<td>0.01270</td>
<td>67.98</td>
<td>36</td>
<td>10</td>
</tr>
<tr>
<td>32</td>
<td>32</td>
<td>32</td>
<td>0.01270</td>
<td>63.53</td>
<td>36</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3.88089</td>
<td>100.00</td>
<td>36</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2.50801</td>
<td>98.17</td>
<td>36</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1.20801</td>
<td>95.82</td>
<td>36</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
<td>0.70801</td>
<td>92.53</td>
<td>36</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>5</td>
<td>0.51801</td>
<td>90.03</td>
<td>36</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>6</td>
<td>0.41801</td>
<td>86.33</td>
<td>36</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>7</td>
<td>0.31801</td>
<td>82.36</td>
<td>36</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>8</td>
<td>0.21801</td>
<td>77.98</td>
<td>36</td>
<td>19</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>9</td>
<td>0.11801</td>
<td>73.21</td>
<td>36</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>10</td>
<td>0.01801</td>
<td>67.94</td>
<td>36</td>
<td>21</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>11</td>
<td>0.01801</td>
<td>62.65</td>
<td>36</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>12</td>
<td>0.01801</td>
<td>57.17</td>
<td>36</td>
<td>23</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>13</td>
<td>0.01801</td>
<td>51.58</td>
<td>36</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>14</td>
<td>0.01801</td>
<td>45.90</td>
<td>36</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>15</td>
<td>0.01801</td>
<td>40.16</td>
<td>36</td>
<td>26</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>16</td>
<td>0.01801</td>
<td>34.32</td>
<td>36</td>
<td>27</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>17</td>
<td>0.01801</td>
<td>28.43</td>
<td>36</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>18</td>
<td>0.01801</td>
<td>22.44</td>
<td>36</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
<td>19</td>
<td>0.01801</td>
<td>16.35</td>
<td>36</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>20</td>
<td>0.01801</td>
<td>10.26</td>
<td>36</td>
<td>31</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td>21</td>
<td>0.01801</td>
<td>4.17</td>
<td>36</td>
<td>32</td>
</tr>
<tr>
<td>3</td>
<td>22</td>
<td>22</td>
<td>0.01801</td>
<td>0.1</td>
<td>36</td>
<td>33</td>
</tr>
<tr>
<td>3</td>
<td>23</td>
<td>23</td>
<td>0.01801</td>
<td>0.1</td>
<td>36</td>
<td>34</td>
</tr>
<tr>
<td>3</td>
<td>24</td>
<td>24</td>
<td>0.01801</td>
<td>0.1</td>
<td>36</td>
<td>35</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>25</td>
<td>0.01801</td>
<td>0.1</td>
<td>36</td>
<td>36</td>
</tr>
</tbody>
</table>

Table 1 (Cont.)
### Table 1 (Continued)

<table>
<thead>
<tr>
<th>X</th>
<th>N</th>
<th>K</th>
<th>M</th>
<th>C(K, N)</th>
<th>E(K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1763681</td>
<td>99.95</td>
</tr>
<tr>
<td>37</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1606419</td>
<td>99.90</td>
</tr>
<tr>
<td>37</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>1609956</td>
<td>99.84</td>
</tr>
<tr>
<td>37</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>1613502</td>
<td>99.78</td>
</tr>
<tr>
<td>37</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>1616148</td>
<td>99.72</td>
</tr>
<tr>
<td>37</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>1618824</td>
<td>99.66</td>
</tr>
<tr>
<td>37</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>1621547</td>
<td>99.60</td>
</tr>
<tr>
<td>37</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>1624266</td>
<td>99.53</td>
</tr>
<tr>
<td>37</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>1627071</td>
<td>99.47</td>
</tr>
<tr>
<td>37</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>1630317</td>
<td>99.41</td>
</tr>
<tr>
<td>37</td>
<td>13</td>
<td>13</td>
<td>13</td>
<td>1633567</td>
<td>99.35</td>
</tr>
<tr>
<td>37</td>
<td>14</td>
<td>14</td>
<td>14</td>
<td>1636847</td>
<td>99.29</td>
</tr>
<tr>
<td>37</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>1640147</td>
<td>99.23</td>
</tr>
<tr>
<td>37</td>
<td>16</td>
<td>16</td>
<td>16</td>
<td>1643467</td>
<td>99.17</td>
</tr>
<tr>
<td>37</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td>1646807</td>
<td>99.11</td>
</tr>
<tr>
<td>37</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>1650177</td>
<td>99.05</td>
</tr>
</tbody>
</table>

**Note:** The table continues with similar entries for various values of X, N, K, M, C(K, N), and E(K).
estimator for \( u \) is given by

\[
\hat{u} = b \ln \theta
\]

where \( \theta = \exp(Y) \) is the the estimated parameter of the location parameter \( u \) of the extreme-value distribution, since \( u = b \ln \theta \) and \( \theta \) is a consistent estimator for the scale parameter \( \sigma \) of the exponential distribution.

Similarly, as was shown in an earlier paper [3], if \( Y \) represents an exponential distribution with the parameter \( \alpha \) then \( T = Y^2 \) has a Weibull distribution with shape parameter \( K \), scale parameter \( \theta = \alpha^{1/K} \), and

\[
\hat{\theta} = c_m^{1/S_m} = (\ln y_m)^{1/K}
\]

is a consistent estimator for \( \theta \) and \( c_m \) is the \( m \)-th order statistic from a Weibull distribution with shape parameter \( K \).

Note the coefficient \( c_m \) has been tabulated for \( m = 1(1)20 \) by Moore and Harter [3]. The values of the coefficient are given in Table I, in which it is called \( c_m \) where \( K = \text{min}(m,r) \), the \( k \)-th order statistic being optimal for estimation from the first \( m \) order statistics of a sample of size \( n \) and the \( k \)-th order statistic being optimal for estimation from the complete sample.

III. EXACT CONFIDENCE BOUNDS FOR THE LOCATION PARAMETER OF THE EXTREME-VALUE DISTRIBUTION

Harter [4] has obtained exact upper and lower bounds and central confidence intervals for the scale parameter of the one-parameter exponential distribution for a wide range of confidence levels based on the \( m \)-th order statistic \( y_m \) of a sample of size \( n \). The coefficients \( B_m \) \( y_m \) have been tabulated for \( m = 1(1)20 \) for all \( m \) optimal. Let us introduce the notation

\[
D_m = B_m y_m \quad 	ext{and} \quad D_m = R_m y_m.
\]

Now the exact confidence interval based on one order statistic is given by

\[
D_{m-1} y_m < \theta < D_{m-1} y_m.
\]

But \( u = b \ln \sigma \); therefore, by substitution, we obtain the following

\[
b \ln D_{m-1} + x_m < u < b \ln D_{m} + x_m
\]

which gives an exact central confidence interval with the same level of confidence given by the tabulated values of \( y_m \). Therefore we have a simple method of computing exact central confidence intervals or upper and lower confidence bounds for the location parameter of the extreme-value distribution, with scale parameter \( b \), based on one-order statistic.

IV. EXACT CONFIDENCE INTERVALS FOR THE SCALE PARAMETER OF THE WEIBULL DISTRIBUTION WITH KNOWN SHAPE PARAMETER

If the random variable \( T \) has a Weibull distribution with shape parameter \( K \) then it is easily shown that \( Y = T^2 \) has an exponential distribution with \( \theta = \alpha^{1/K} \). In inequality (5) replace \( y_m \) by \( c_m \), the \( k \)-th power of the \( m \)-th order statistic from a Weibull distribution with scale parameter \( K \), and we obtain

\[
D_{m-1}^{c_m} \theta < \sigma < D_{m-1}^{c_m} \theta
\]

Take the \( K \)-th root of each member of (7) and obtain

\[
D_{m-1}^{c_m} \theta < \sigma < D_{m-1}^{c_m} \theta
\]

gives an exact central confidence interval for the scale parameter of Weibull distributions with known shape parameter \( K \).

V. MONTE CARLO STUDY OF RATIOS OF MEAN-SQUARE-ERRORS

It seemed reasonable to the authors to conjecture that the ratios of the mean-square-errors of the \( m \)-order-statistic estimator and of the one-order statistic estimator for both the scale parameter of a two-parameter Weibull distribution with known shape parameter and the location parameter of an extreme-value distribution with known scale parameter are closely approximated by the relative efficiency of the one-order statistic estimator of the scale parameter of an one-parameter exponential distribution as compared with the \( m \)-order statistic estimator, which has been tabulated by Moore and Harter [3] for \( m = 1(1)20 \) and in Table I of the present paper for \( n = 21(1)40 \). It should be noted that one may speak of relative efficiency in the case of the exponential distribution, since the estimators are unbiased, but only of ratios of mean-square-errors in the cases of the Weibull and extreme-value distributions, for which the estimators are biased.) In order to check the validity of the conjecture, a Monte Carlo study of the ratios of mean-square-errors was performed. One thousand random samples each of size \( n = 1(1)40 \) from a one-parameter exponential distribution with scale
parameter was generated in the IBM 7094 computer. These were transformed into samples from a 2-parameter Weibull distribution with shape parameter 2 and from an extreme-value distribution with scale parameter 1. From each sample, the one-order statistic estimate and the order statistic estimate, based on the first order statistics \( m = 1 \) of the scale parameters of the exponential and Weibull distributions and of the location parameter of the extreme-value distribution were computed. For each distribution and for each combination of \( m \) and \( n \), the ratio of the mean-square error of the order statistic to that of the one-order statistic estimates was calculated. Except for fluctuations due to random sampling, the ratio of mean-square errors in the case of the exponential distribution should agree with the tabulated relative efficiencies, and it was found that the agreement was quite good. Moreover, it was found that the ratios of mean-square errors in the cases of the Weibull and extreme-value distributions agreed with the tabulated relative efficiencies almost as well as did those for the exponential distribution, thus confirming the conjecture.

VI. USE OF TABLE I AND RELATED TABLES, WITH NUMERICAL EXAMPLES

Table I gives the coefficient of the optimum single-order statistic (the \( k \)th) in an unbiased estimator of the scale parameter of a one-parameter exponential distribution from the first \( m \) order statistics of a sample of size \( n \) \( m = 21(1)40 \), and the relative efficiency of the one-order statistic estimator as compared with the \( m \) order statistic estimator. It is a condensed extension of the similar table for \( n = 11102 \) given by Moore and Harter [3], which also includes columns giving the variances of the two estimators. These columns have been omitted from Table I to save space, which can be done without loss of information, since the variance of the \( m \) order statistic estimator is simply \( 1/m \) and that of one-order statistic estimator can be found by dividing \( 1/m \) by the relative efficiency. These two tables can also be used to obtain consistent one-order statistic estimators of the scale parameter of a two-parameter Weibull distribution with known shape parameter and of the location parameter of an extreme-value distribution with known scale parameter, together with their approximate “efficiencies” (ratios of mean-square errors) relative to the \( m \) order statistic estimators. Harter [4] has tabulated coefficients of optimum order statistic in exact upper and lower confidence bounds, based on one-order statistic, for the scale parameter of a one-parameter exponential distribution. These may also be used to obtain exact confidence bounds, based on one-order statistic, for the scale parameter of a two-parameter Weibull distribution with known shape parameter and for the location parameter of an extreme-value distribution with known scale parameter.

As an example of the previously mentioned uses of Table I and related tables, consider the following tabulation of data (observed failure times in hours) resulting from a simulated life test on forty components:

| 5  | 33  | 55  | 65  | 82  | 102  | 114  | 142  |
| 40 | 34  | 58  | 65  | 83  | 104  | 116  | 144  |
| 17 | 36  | 58  | 65  | 80  | 106  | 112  | 134  |
| 42 | 34  | 61  | 67  | 71  | 107  | 119  | 158  |
| 32 | 55  | 64  | 68  | 92  | 114  | 139  | 195  |

Suppose the experimenter knows that these data have come from a two-parameter Weibull distribution with shape parameter \( K = 2.0 \), and that he wishes to find a point estimate and 80 percent lower and upper confidence bounds on the scale parameter \( \theta \). Harter and Moore [5] have previously done this for estimates based on the first \( m \) order statistics \( m = 80(8)40 \). From Table I of the present paper and from Table I of Harter [4], one finds that, for a one-parameter exponential distribution, the optimum-order statistic for obtaining a point estimate and the 80 percent lower and upper confidence bounds on the scale parameter is 32, with coefficients 0.64074, 0.553447, and 0.708717, respectively. Substituting these values in (4) and (9), one finds that the point estimate of the scale parameter of the Weibull distribution from which the above sample came is \( \sqrt{0.64074 (110) = 92.9} \), the 80 percent lower confidence bound is \( \sqrt{0.553447 (110) = 86.3} \) and the 80 percent upper confidence bound is \( \sqrt{0.708717 (110) = 101.7} \), as compared with results 93.7, 87.6, and 101.7 obtained from the first 32 order statistics, 93.3, 87.8, and 100.3 obtained from all 40 observations, and the true population parameter of 100.

Now consider the same data transformed to data from an extreme-value distribution with scale parameter \( b = 0.5 \) by using natural logarithms.

Using the tabular values, one finds by substituting in (3) and (6) that the point estimate of the location parameter of the extreme-value distribution is \( 0.5 \ln 0.64074 + 4.754 = 4.538 \), the 80 percent lower confidence bound is \( 0.5 \ln 0.553447 + 4.754 = 4.438 \), and the 80 percent upper confidence bound is \( 0.5 \ln 0.708717 + 4.754 = 4.623 \), as compared with results [1] 4.541, 4.474, and 4.624 based on the first 32 order statistics, 4.537, 4.476, and 4.610 based on all 40 observations, and the true population parameter of 4.605 (= in 100).

REFERENCES

This paper derives a one-order statistic estimator $U, b$ for the location of the (first) extreme-value distribution of smallest values with cumulative distribution function $F(x; u, b) = 1 - \exp\{-\exp[(x-u)/b]\}$ using the minimum-variance unbiased one-order statistic estimator for the scale parameter of an exponential distribution, as was done in an earlier paper for the scale parameter of a Weibull distribution. It is shown that exact confidence bounds, based on one-order statistic, can be easily derived for the location parameter of the extreme-value distribution and for the scale parameter of the Weibull distribution, using exact confidence bounds for the scale parameter of the exponential distribution. The estimator for $u$ is shown to be $b$ in $c_m + x_m$ is the $n$th order statistic from an ordered sample of size $n$ from the extreme-value distribution with scale parameter $b$ and $c_m$ is the coefficient for a one-order statistic estimator of the scale parameter of an exponential distribution. Values of the factor $c_m$, which have previously been tabulated for $n = 1(1)20$, are given for $n = 1(1)40$. The ratios of the mean-square-errors of the maximum-likelihood estimators based on $m$ order statistics to those of the one-order statistic estimators for the location parameter of the extreme-value distribution and the scale parameter of the Weibull distribution are investigated by Monte Carlo methods. The use of the table and related tables is discussed and illustrated by numerical examples.
<table>
<thead>
<tr>
<th>KEY WORDS</th>
<th>LINK A</th>
<th></th>
<th>LINK B</th>
<th></th>
<th>LINK C</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ROLE</td>
<td>WT</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>ROLE</td>
<td>WT</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>ROLE</td>
<td>WT</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Security Classification