Navigation with High-Altitude Satellites: A Study of the Effects of Satellite-User Geometry on Position Accuracy

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NAVIGATION WITH HIGH-ALTITUDE SATELLITES:
A STUDY OF THE EFFECTS OF SATELLITE-USER GEOMETRY
ON POSITION ACCURACY

CAROLE D. SULLIVAN

Group 66

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ABSTRACT

An error analysis of a hyperbolic navigation system using high-altitude satellites revealed that certain satellite arrangements result in long, narrow corridors on the earth (singular regions), in which navigation errors are very large. The study showed that in these singular regions the satellite-user geometry results in navigation equations which are sensitive to measurement errors and thus cannot be solved accurately for all three user coordinates.

Accepted for the Air Force
Franklin C. Hudson
Chief, Lincoln Laboratory Office
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I. INTRODUCTION

It has become apparent that high-altitude satellites have possibilities for navigation as well as communication. Although use of such satellites can allow the user to obtain an accurate position fix, Schweppe* pointed out that, for certain arrangements of the satellites and the user, the error in the user's determination of his position in three coordinates is extremely large. It is the purpose of this report to determine the extent of such singular regions, i.e., regions in which a user cannot make a unique three-coordinate fix, and to explore the reasons for their existence.

II. SYSTEM CONCEPT

The navigation system considered is an hyperbolic system in which a user receives timing signals transmitted from each of three synchronous-altitude satellites. The satellite clocks, from which the timing signals are derived, are assumed to be perfectly synchronized to a master clock. The user measures his height as well as the time of arrival of each of the satellite signals. In the noiseless case (that is, the user has made all his measurements perfectly and knows the satellite positions exactly), by using the difference in the time of arrival of the timing signals emitted by two of these satellites, the user can position himself on the locus of points satisfying this condition, that is, on an hyperbola of revolution. By taking one of these satellites and a third, he can determine another hyperboloid. The intersection of these two figures is not sufficient to determine the user's position in three coordinates. The user knows, however, he is on the surface of a sphere with radius equal to the radius of the earth plus his measured height. The intersection of the sphere and the hyperboloids will determine his position in three coordinates.

In practice, the user is not able to make perfect measurements and, therefore, the effects of measurement errors must be considered.

III. ERROR ANALYSIS

The quantities observed by the user are (a) the time of arrival of timing signals from each of the three satellites (as indicated on the user's clock) and (b) the user's height or distance from the center of the earth. Denote the user's position in three-dimensional space (origin at the center of the earth) by \( \mathbf{p}_u \), a three-dimensional column vector (Fig. 1). Similarly, denote the satellite positions by the vectors \( \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \). If the user's clock were synchronized to the master

---

* F. C. Schweppe (private communication).
clock governing the satellite transmissions and there were no errors in the user’s measurements, the time of arrival of a signal from a satellite would provide the user with the distance between himself and the satellite. That is, the user would observe indirectly

\[ s_j = |p_u - p_j| , \quad j = 1, 2, 3 . \]

If the user clock were offset from the master clock by an unknown constant \( \tau_o \), the user would really observe

\[ |p_u - p_j| + \tau_o = s_j + \tau_o , \quad j = 1, 2, 3 \]

where \( \tau_o \) has the dimension of length, since it has the effect of changing the apparent distance between the user and the satellite. The user height is

\[ h_u = |p_u| \]

and in the absence of any noise, the observables are

\[
\begin{pmatrix}
  s_1 + \tau_o \\
  s_2 + \tau_o \\
  s_3 + \tau_o \\
  h_u
\end{pmatrix}
\]

where \( r \) has the dimension of length, since it has the effect of changing the apparent distance between the user and the satellite. The user height is

\[ h_u = |p_u| \]

and in the absence of any noise, the observables are

\[
\begin{pmatrix}
  s_1 + \tau_o \\
  s_2 + \tau_o \\
  s_3 + \tau_o \\
  h_u
\end{pmatrix}
\]

The quantities that \( y \) depends on then are \( p_u, p_1, p_2, p_3, \) and \( \tau_o \). We express these quantities as a column vector, called the state vector \( x \).

\[
x = \begin{pmatrix}
p_u \\
p_1 \\
p_2 \\
p_3 \\
\tau_o
\end{pmatrix}
\]

(2)

so that, as long as there is no noise,

\[
y = h(x) = \begin{pmatrix}
h_1(x) \\
h_2(x) \\
h_3(x) \\
h_4(x)
\end{pmatrix} = \begin{pmatrix}
|p_u - p_1| + \tau_o \\
|p_u - p_2| + \tau_o \\
|p_u - p_3| + \tau_o \\
|p_u|
\end{pmatrix}
\]

(3)

The observables are, however, obscured by additive noise which can be expressed in vector form as

\[
\mathbf{r} = y + n = h(x) + n
\]

(4)
We assume that the a priori knowledge of the components of the state vector is in the form of a probability density. In particular, we take $\mathbf{x}$ to have statistically independent Gaussian components with mean $E[\mathbf{x}] = \mathbf{x}_m$ and covariance $W = E[(\mathbf{x} - \mathbf{x}_m)(\mathbf{x} - \mathbf{x}_m)']$

where $(\cdot)'$ denotes matrix transpose and $E$ denotes the statistical expectation. The noise vector $\mathbf{n}$ is also taken to be Gaussian with zero mean. Under these assumptions, the problem of calculating the user position from noisy observables can be viewed as forming a statistical estimate of the state vector $\mathbf{x}$. In particular, the maximum a posteriori probability estimate of the state vector is formed, i.e., the estimate of $\mathbf{x}$ that maximizes $p(\mathbf{x}|\mathbf{r})$. Let $\hat{\mathbf{x}}$ denote this estimate. The error analysis of such a navigation scheme consists of calculating the covariance matrix of the error vector $\mathbf{e} = (\mathbf{x} - \hat{\mathbf{x}})$.

In order to do this, the vector function $\mathbf{h}(\mathbf{x})$ is first linearized about the user's true position to make this a linear estimation problem. Let the matrix $\mathbf{H}$ denote the linear transformation required to obtain the linearized version of the general nonlinear function $\mathbf{h}(\cdot)$, i.e.,

$$\mathbf{h}(\mathbf{x}) \approx \mathbf{h}(\mathbf{x}_o) + \mathbf{H}(\mathbf{x} - \mathbf{x}_o) \quad (5)$$

where

$$\mathbf{H} = \begin{bmatrix} \frac{\partial h_i}{\partial x_j} \end{bmatrix}_{ij} = \left( \frac{\partial h_i}{\partial x_j} \right)_{ij} \quad (\mathbf{x} = \mathbf{x}_o)$$

This linear approximation of the nonlinear function $\mathbf{h}(\cdot)$ is accurate when the difference between the estimate $\hat{\mathbf{x}}$ and $\mathbf{x}_o$ is small, as it should be for an accurate navigation system.

The resulting error covariance matrix $V = (v_{ij}) = E[\mathbf{e}\mathbf{e}']$ has been calculated as

$$V = W - WH[N + WH]'^{-1}HW = [H'N^{-1}H + W^{-1}]^{-1} \quad (6)$$

where $W$ is as previously defined and $N$ is the covariance matrix of the noise vector, i.e.,

$$N = (n_{ij}) = E[n_i n_j]$$

where $n_k$ is an element of the noise vector $\mathbf{n}$.

Since we are interested in the estimate of $\mathbf{p}_u$, only a submatrix of $V$ is pertinent to the actual navigation errors. The upper left $3 \times 3$ submatrix of $V$ is the error covariance matrix of the components of the position estimate $\hat{\mathbf{p}}_u$. As our measure of the accuracy of navigation, we adopt the root-mean-squared (RMS) error between the estimate $\hat{\mathbf{p}}_u$ and the true position $\mathbf{p}_u$.

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* The statistical expectation of a matrix is taken to be the matrix of expected values of each of the elements.

A computer program was written to calculate the $V$ matrix and from it the RMS position error for any positions of the satellites and the user. The input to the program consists of the a priori standard deviation of the components of the state vector $x$ and the observation noise $n$. These are taken to be

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (RMS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ranging error</td>
<td>10 meters</td>
</tr>
<tr>
<td>Height error</td>
<td>10 meters</td>
</tr>
<tr>
<td>User position error</td>
<td>$10^6$ meters</td>
</tr>
<tr>
<td>Satellite position error</td>
<td>$10^2$ meters</td>
</tr>
<tr>
<td>Clock error</td>
<td>$10^6$ meters</td>
</tr>
</tbody>
</table>

where the square of the ranging error and the square of the height error are the diagonal elements of the covariance matrix $N$, the other elements of this matrix being zero, since the observation errors are taken to be statistically independent. The squares of the remaining three parameters are the diagonal elements of the $W$ matrix, which is also diagonal for the same reason. In order to eliminate any a priori knowledge of user position and clock error, the variances of these quantities should approach infinity. This limit was well approximated by using $10^6$ meters as the a priori variance of these quantities, which proved to be of inestimable value in view of the finite word length of the digital computer used for the calculations. The other parameters lead to reasonable navigation accuracies and are realizable (hopefully) with practical equipment.* The program calculates the RMS error between the maximum a posteriori estimate of $p$ and the actual user's position for any user-satellite geometries.

Consider the case where the satellite positions are as follows:

<table>
<thead>
<tr>
<th>Satellite</th>
<th>Latitude</th>
<th>Longitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>30.0</td>
</tr>
<tr>
<td>2</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>0.0</td>
<td>-30.0</td>
</tr>
</tbody>
</table>

that is, when all three satellites are along the equator. The RMS errors for the user at various positions between 60° latitude and the equator, and 40° longitude and 0° longitude are given in Fig. 2. At 60° latitude, the RMS error is of the order of $10^2$ meters for all longitudes and increases steadily as the user approaches the equator; at the equator, it has a value of $10^6$ meters, but this is the a priori assumption of the position error (Fig. 3). The observations and calculations have not given the user any information about his position. This is a singular region of the type discovered by Schweppe.

We have seen that the RMS error varies markedly over the different user positions. The question then arises as to why certain arrangements of satellites and user cause large errors in position determination.

IV. ANGLE BETWEEN HYPERBOLAS

Consider again how a user would determine his position if this were a deterministic case (i.e., no noise and no errors in satellite position knowledge). He would calculate two hyperbolas of revolution and a sphere, all three of which he knows he must be on (Fig. 4). The intersection of these figures, therefore, provides his position fix. Suppose, however, instead of being a

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deterministic case, there is some error in the user's measurements and thus an error $\rho$ in his
calculations of the position of one hyperbola. This error would produce an error in position
determination

$$\epsilon = \frac{\rho}{\sin \theta}$$  \hspace{1cm} \hspace{1cm} (7)$$

where $\theta$ denotes the angle formed by the hyperboloids intersected with the plane of the earth at
the point of the user. If $\theta$ is close to $90^\circ$, $\sin \theta$ is close to one and the error $\epsilon$ in the user's
determination of his position is nearly equal to the error $\rho$ in his calculations. However, as
$\theta \to 0$, $\sin \theta \to 0$ and $\epsilon \to \infty$, which indicates that the error $\rho$ would be greatly magnified for small
angles. The hyperboloids and the sphere intersect in such a way that the user, unable to measure
perfectly, cannot determine his position accurately in three coordinates. The error in his
position determination would be entirely dependent on his a priori knowledge.

To calculate $\theta$, let $\mathbf{p}_u$ be a three-dimensional vector (as in Sec. III), origin at the center
of the earth, denoting the user's position with coordinates $(x_u, y_u, z_u)$, and let $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$
represent the three satellites with coordinates $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and $(x_3, y_3, z_3)$, respectively. Then the distance between the user and each of the satellites is

$$|\mathbf{p}_u - \mathbf{p}_i| = \sqrt{(x_u - x_i)^2 + (y_u - y_i)^2 + (z_u - z_i)^2} = s_i \hspace{1cm} i = 1, 2, 3$$  \hspace{1cm} (8)$$

Since the time required for the user to receive the timing signals is a function of the distance
between the user and the satellite, we can describe the hyperboloid on which the user is
located as the locus of points such that the difference in the distance between that point and two
other points is constant. One hyperboloid is determined by satellite 1, satellite 2, and $|s_1 - s_2|$;
and another by satellite 2, satellite 3, and $|s_2 - s_3|$. From Eq. (8), the two hyperboloids are

$$f_1(x, y, z) = |s_1 - s_2| = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$$

$$- \sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2}$$  \hspace{1cm} (9)$$

$$f_2(x, y, z) = |s_2 - s_3| = \sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2}$$

$$- \sqrt{(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2}.$$  \hspace{1cm} (10)$$

If $f$ is a function of three independent variables $(x, y, z)$, the gradient of the function is
defined as

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$
where \( \mathbf{i}, \mathbf{j}, \text{ and } \mathbf{k} \) are unit vectors along the positive x, y, and z axes, respectively. Geometrically, \( \nabla f \), evaluated at the point \((x_0', y_0', z_0')\), is a vector whose direction is normal to the level surface \([i.e., \text{ the set of all points such that } f(x, y, z) = c] \text{ at the point } (x_0', y_0', z_0')\). In particular, \( f_1(x, y, z) \) in Eq. (9) defines a family of hyperboloids, the value of \( |s_1 - s_2| \) defining the particular hyperboloid. The gradient of \( f_1 \) evaluated at a point \((x_0', y_0', z_0')\) will be a vector whose direction is normal to the particular hyperboloid passing through \((x_0', y_0', z_0')\). Likewise, we could find a vector whose direction is normal to the hyperboloid defined by \( f_2(x, y, z) \) in Eq. (10) passing through the point \((x_0', y_0', z_0')\).

If \( \nabla f_1 \) and \( \nabla f_2 \) are the gradients of the functions \( f_1 \) and \( f_2 \), then the angle \( \theta \), formed by the intersections of the hyperboloids and the tangent plane of the sphere, can be found by computing the angle projected in the tangent plane between \( \nabla f_1 \) and \( \nabla f_2 \) evaluated at the user's position.

Let \( \mathbf{E} \) be a unit vector normal to the plane tangent to the sphere at the point of the user. Then

\[
\mathbf{u}_1 = \frac{\nabla f_1 \times \mathbf{E}}{|\nabla f_1 \times \mathbf{E}|} \quad (11)
\]

and

\[
\mathbf{u}_2 = \frac{\nabla f_2 \times \mathbf{E}}{|\nabla f_2 \times \mathbf{E}|} \quad (12)
\]

are unit vectors in the plane tangent to the sphere, and

\[
\cos \theta = \mathbf{u}_1 \cdot \mathbf{u}_2
\]

or

\[
\theta = \cos^{-1} (\mathbf{u}_1 \cdot \mathbf{u}_2) \quad (13)
\]

When Eq. (13) is evaluated for satellite-user geometries where navigation accuracy is known to be poor, the values of \( \theta \) should be near or exactly zero. In particular, if this expression is evaluated for the satellite geometry used in Sec.III at the user positions that had the maximum RMS position error, that is, when the user is located at any point along the equator, \( \cos \theta \) does equal one or the angle between the hyperbolas in the plane tangent to the earth equals zero (see Fig. 5).

Since it is more difficult to solve this equation for non-equatorial cases, a computer program was written. Input to the program consists of the longitude and latitude of the three
satellites and the user; \( 1/\sin \theta \) is the output quantity rather than \( \theta \), since the error magnification is seen in Eq. (7) to depend on \( 1/\sin \theta \).

There is a direct correspondence between the user locations resulting in large values of \( 1/\sin \theta \) and user locations with large RMS position errors for the same satellite geometry. The expression \( 1/\sin \theta \) had a minimum value at 60° latitude for all longitudes and increased steadily until, along the equator, \( \theta \) became zero and \( 1/\sin \theta \) could not be computed. It was also at 60° latitude that the minimum RMS position error occurred, and along the equator that the maximum occurred.

V. RESULTS

It is interesting to explore satellite-user geometries with less symmetry than the equatorial case of Fig. 5. The two computer programs were run, therefore, for other satellite and user positions.

The following satellite positions provide a case with only east-west symmetry.

<table>
<thead>
<tr>
<th>Satellite</th>
<th>Latitude</th>
<th>Longitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satellite 1</td>
<td>0.0</td>
<td>30.0</td>
</tr>
<tr>
<td>Satellite 2</td>
<td>10.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Satellite 3</td>
<td>0.0</td>
<td>-30.0</td>
</tr>
</tbody>
</table>

Navigation accuracy for user positions from +60° to -60° latitude and from -40° to +40° longitude was studied. A singular region again appeared. The maximum RMS error of the user at -40° longitude occurred at latitude +41°; at 0° longitude, it appeared at 49° latitude; and at 40° longitude, the maximum was at 41° latitude. The effect of moving the satellite north was to move the singular region in that direction while the southern hemisphere became free from any singular points. The actual values of the errors are meaningful only in a relative sense because the region was not sampled finely enough to conclude that the worst point was found. The results of the program to calculate the angle between the hyperboloids for this same geometry again showed the maximum value of \( 1/\sin \theta \) occurring at exactly the same user positions as the maximum RMS position errors [see Fig. 6(a-b)].

If satellite 2 is now moved to 15° latitude, 15° longitude, all symmetry is destroyed, and user positions from -40° to +40° longitude and +75° to -75° latitude must be studied. (The increase in the range of latitude is needed because of the more northerly position of the satellite.) Again, there is a narrow locus of singular points with the worst point this time occurring (user longitude 0°) at user latitude 68°. This, however, is nearly out of the region of mutual visibility* of all three satellites [see Fig. 7(a-b)]. The region of singular points could probably be moved entirely out of the region where the user could see all three satellites by further adjustment of the northern satellite.

We can conclude that the region of singular points exists whether the satellites are arranged symmetrically or unsymmetrically. The arrangement of the satellites does, however, have an effect on the location of the region of large errors.

* In this report, visibility curves are computed using an elevation angle of 7°.
Taking the positions of the satellites to be

<table>
<thead>
<tr>
<th>Satellite</th>
<th>Latitude</th>
<th>Longitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satellite 1</td>
<td>0.0</td>
<td>45.0</td>
</tr>
<tr>
<td>Satellite 2</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Satellite 3</td>
<td>0.0</td>
<td>-45.0</td>
</tr>
</tbody>
</table>

we find that the singular points again occur on the equator as expected, and that the maximum values of the RMS position errors are the same along the equator as for the previous equatorial case because of the a priori assumptions. If, however, we compare the RMS values of the user positions off the equator for the 30° and 45° cases, we find that the corresponding values for the 30° case are 2.1 times larger than those for the 45° case, and that 1/sin θ is 0.5 times larger for the 30° case [see Fig. 8(a-b)]. The effect of having the equatorial satellites 90° instead of 60° apart was a decrease in the width of the region of large errors. However, the region of mutual visibility when the two satellites are 90° apart is so limited that three satellites would not be sufficient to cover the North Atlantic, for example; whereas, this could be done if the satellites were only 60° apart.

Thus, if high-altitude satellites are to be used for navigation, consideration must be given to arranging the satellites so that the user will not be in a singular region, that is, so that the user will be able to determine his position accurately. But consideration must also be given to making the area of mutual visibility of the user and the satellites large enough for the system to be practical.

VI. SUMMARY

From the preceding, the following conclusions can be drawn:

(1) It appears that all satellite geometries result in long, narrow corridors in which navigation accuracy is poor.

(2) The cause of this singular region is geometric in nature; that is, the two hyperboloids and the sphere which locate the user intersect in such a way as to greatly magnify any error in the user’s measurements.

(3) A system with all satellites on the equator always yields a singular region along the equator. Thus a practical system would utilize at least one non-equatorial satellite.

(4) If two of the satellites are on the equator and the third is located far enough north, the singular region can be shifted outside the range of mutual visibility.

(5) If two of the satellites are on the equator, the singular region is located in the same hemisphere as the third satellite; i.e., the opposite hemisphere will be free from singular points.

ACKNOWLEDGMENT

The author wishes to thank Thomas J. Goblick, Jr., for his valuable assistance in the preparation of this report and Barney Reiffen for his very helpful comments.
Fig. 1. Hyperbolic navigation system.
<table>
<thead>
<tr>
<th>USER POSITION</th>
<th>LAT</th>
<th>LONG</th>
</tr>
</thead>
<tbody>
<tr>
<td>60.0</td>
<td>9.3120 02</td>
<td>9.3120 02</td>
</tr>
<tr>
<td>57.0</td>
<td>9.4120 02</td>
<td>9.4120 02</td>
</tr>
<tr>
<td>54.0</td>
<td>9.5120 02</td>
<td>9.5120 02</td>
</tr>
<tr>
<td>51.0</td>
<td>9.6120 02</td>
<td>9.6120 02</td>
</tr>
<tr>
<td>48.0</td>
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</tr>
<tr>
<td>0.0</td>
<td>11.3120 02</td>
<td>11.3120 02</td>
</tr>
</tbody>
</table>

Fig. 2. RMS position errors for all equatorial satellites. (All values in meters.)
Fig. 3. Cross section of RMS position error vs latitude (longitude = 0°).
Fig. 4. Intersection of hyperboloid with tangent plane of earth.
Fig. 5. 1/sinθ for all equatorial satellites.
VALUES OF $\theta$ IN THE RANGE

- $+ < 60$
- $\times 60 - 69$
- $\bigcirc 70 - 79$
- $\square 80 - 100$
- $\blacklozenge > 100$ AND $\theta = 0^\circ$

WHERE $Q = \frac{100}{3} \log \left( \frac{1}{\sin \theta} \right) + 33$

Fig. 6(a). $1/\sin \theta$ for an east-west symmetric satellite geometry.
Fig. 6(b). RMS position errors for an east-west symmetric satellite geometry.
VALUES OF Q IN THE RANGE

+  < 60
×  60 - 69
◎  70 - 79
□  80 - 100
★ > 100 AND Θ = 0°

WHERE $Q = \frac{100}{3} \log \left( \frac{1}{\sin \Theta} \right) + 33$

Fig. 7(a). \(1/\sin \Theta\) for an unsymmetric satellite geometry.
Fig. 7(b). RMS position errors for an unsymmetric satellite geometry.
Fig. 8(a). RMS position errors for satellites at 45-0, 0-0, -45-0.
Fig. 8(b). RMS position errors for satellites at 30-0, 0-0, -30-0.
An error analysis of a hyperbolic navigation system using high-altitude satellites revealed that certain satellite arrangements result in long, narrow corridors on the earth (singular regions), in which navigation errors are very large. The study showed that in these singular regions the satellite-user geometry results in navigation equations which are sensitive to measurement errors and thus cannot be solved accurately for all three user coordinates.