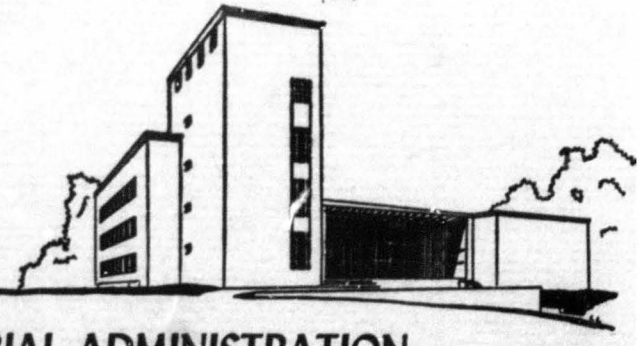
AD 648085

Carnegie Institute of Technology

Pittsburgh 13, Pennsylvania

DDC
RECEIVED
MAR 13 1967
C



GRADUATE SCHOOL of INDUSTRIAL ADMINISTRATION

William Larimer Mellon, Founder

ARCHIVE COPY

Management Sciences Research Report No. 89

STATISTICAL ESTIMATION IN A PROBLEM OF SYSTEM RELIABILITY

by

D. P. Gaver

and

M. Mazumdar*

February, 1967

*Westinghouse Electric Corporation, Pittsburgh, Pa.

Management Sciences Research Group
Graduate School of Industrial Administration
Carnegie Institute of Technology
Pittsburgh, Pennsylvania 15213

This report was prepared as part of the activities of the Management Sciences Research Group, Carnegie Institute of Technology, under Contract NONR 760(24) NR 047-048 with the U. S. Office of Naval Research and under a Ford Foundation Grant and also under partial support of Westinghouse Electric Corporation, Research Labs. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

STATISTICAL ESTIMATION IN A PROBLEM OF SYSTEM RELIABILITY

D. P. Gaver
Carnegie Institute of Technology
and
Westinghouse Electric Corporation
M. Mazumdar
Westinghouse Electric Corporation

I. Introduction to the Problem

In certain applied areas, notably that of systems reliability, but also those of work sampling and communications, it is common to consider a system that randomly occupies one of two states. Thus in the reliability application (upon which we shall concentrate hereafter) a radar system may be operative ("up"), or inoperative ("down"), while in work sampling a worker may be "working" or "resting", and in communications a telephone line or other facility may be "idle" or "busy". Random state occupancy means that the durations of the times spent in the two states exhibits random variability. For example, the tendency of system times-to-failure (up-times) to be approximately exponentially distributed is well-known (see Davis [5]). Similarly, the idle time of a telephone line will be exponential if calls occur in a stationary Poisson fashion, as may frequently be a reasonable assumption. Of course distributions other than the exponential can, and frequently do, arise.

Our purpose in this paper is to discuss the problem of estimating the parameters of the probability distributions specifying the two-state process described. Of interest also is the estimation of functions of these parameters, e.g. "system readiness" and "system reliability" in the reliability context; these notions will be defined later. The estimation problem clearly depends upon the manner in which observations are taken,

i.e. upon the sampling plan. We assume here that at least certain of the observations are available only at rather infrequent times. When observations are made we assume that they either, a), reveal only the state of the system that prevails at the instant of observation, in which case we call them snapshots, or b), consist of continuous recording of the system state throughout intervals of fixed or random duration, in which case we call them patches. In this paper we shall consider mixtures of these plans, in which snapshots and patches are mutually interspersed.

The motivation for considering such sampling plans is essentially that of determining system parameters economically and without the bias that is possible, at least in the reliability area, from the use of operator log-book data. The latter frequently is extremely spotty and unreliable, and a supplement is desirable. Snapshot, or even patch, observations approximate those made when occasional readings on system state are possible. For example, after the time of a military action it is likely to be accurately recalled whether or not a radar was up when needed, and possibly even the approximate length of time it remained so. See Cox ([1], p. 87 ff.) for a discussion of a related machine stoppage question. In a recent paper, [3], Cox has also considered an estimation problem very similar to ours. In activity sampling or time study snapshot observations are frequently taken; it has been suggested by Moder [10] that the efficiency of such studies can be increased by also observing the remaining time in state of the system, particularly if the state observed occurs relatively infrequently. In telephony analogous sampling procedures may also be considered.

An interesting effect that sometimes accompanies system observations is that the system behavior actually changes as a consequence. For one thing, inspections may increase the vigilance and motivation of support personnel, thus inducing the system to spend more time in the up, or desirable, state. On the other hand, too-frequent observations are distracting and time-consuming, and may breed resentment. We shall not attempt here to provide statistical procedures accounting for the possible behavioral impact of inspections or audits, although work in this direction is in progress.

II. The Exponential Model and Maximum Likelihood

Before proceeding to estimate parameters in the general model described it is necessary to specify the latter mathematically. In reliability terminology we assume that the system is alternately up and down, that the i-th up time is U_i, the i-th down time is D_i, and that $\{U_i, i=1,2, \ldots\}$ and $\{D_i, i=1,2, \ldots\}$ are mutually independent sequences of independent and identically distributed random variables. Thus the system is described by a two-state renewal process (Cox [1]) . In this case if the system is observed at widely spaced instants the chance that it is observed up is, by general renewal theory,

$$P\{\text{up in long run}\} = \frac{E[U]}{E[U]+E[D]} \tag{2.1}$$

with the complementary probability applying to the complementary event. It is thus clear that by rare snapshot sampling we can only estimate the ratio of the two means. However, if actual time durations are observed, as in patch sampling, it appears that estimates of both $E[U]$ and $E[D]$ are available. We treat this possibility in terms of the special model, in which up- and down-times are exponentially distributed.

A. Exponential Model

Let the up- and down-times have densities

$$f_U(x) = e^{-\mu x}\mu, \qquad \mu > 0,\ x \geq o,$$

and

$$f_D(y) = e^{-\lambda y}\lambda, \qquad \lambda > 0,\ y \geq 0. \tag{2.2}$$

where $E[U] = \mu^{-1}$ and $E[D] = \lambda^{-1}$.

This simple two-state Markov model occurs in reliability applications, and perhaps in other areas as well. We shall base our estimation procedures upon it. However, we shall also attempt to test the robustness of the estimates thereby derived by treating processes having up and down times actually described by other distributions (e.g. members of the log-normal family) as if they were from the Markov process generated by distributions (2.2). Results of this test, conducted by experimental sampling, are described in Section IV .

B. Maximum Likelihood

Given the two-state renewal model, with distributions (2.2) we can now write down the likelihood function under the assumption of rare observation. Let us, for illustration, suppose that the system is initially observed in the up state, and that starting at that moment a patch of complete up periods (U realizations) and complete down periods (D realizations) is recorded. Following this, a long interval is allowed to elapse and the system is again observed; the state observed recorded at the moment of first observation, and a subsequent patch of up and down realizations again recorded; the process is repeated periodically, always with long intervals elapsing between consecutive patches.

Owing to the assumption of long delays between patches we shall assume that the probability that the system is in an up condition when a new patch observation begins is

$$\frac{E[U]}{E[U]+E[D]} = \frac{\lambda}{\lambda+\mu} \quad ; \tag{2.3}$$

the system is down at patch beginning with the complementary probability. Moreover, the memoryless or Markovian property of the exponential distribution (see Feller [6]), assures that the duration of the first time in state at patch beginning (termed "remaining life", or "forward recurrence time", see Cox [1]) is exponentially distributed, with the parameter appropriate to the state observed at patch beginning. It is this last property that renders patch sampling of the exponential process especially tractable, for in general the forward recurrence time is <u>not</u> distributed in the same manner as is the corresponding time in state.

The above considerations then lead to the likelihood function

$$L(\lambda,\mu) = e^{-\mu x_+}\mu^a \; e^{-\lambda y_+}\lambda^b \; \left(\frac{\lambda}{\lambda+\mu}\right)^{\alpha}\left(\frac{\mu}{\lambda+\mu}\right)^{\beta}, \tag{2.4}$$

where

x_+ = total uptime observed; $x_+ = \sum_{i=1}^{a} x_i$,

x_i being an individual up interval (or forward recurrence time),

a = total number of up intervals (including forward recurrence times),

y_+ = total down time observed; $y_+ = \sum_{i=1}^{b} y_i$,

b = total number of down intervals (including forward recurrence times),

α = total number of patches beginning with system up,

β = total number of patches beginning with system down.

Expression (2.4) can be specialized to account for a number of alternative

sampling plans. Two examples follow.

<u>Case 1</u>: A system's up and down history is continuously recorded through k initial up and down periods. Thereafter, m rare snapshot observations are made, on r of which the system is in an up condition. In this case

$$a = b = k$$

$$\alpha = r, \ \beta = m-r \ .$$

<u>Case 2</u>: A system is observed m times at rare intervals, and each time the system state and the remaining time in that state are recorded. Then

$$a = \alpha = r$$

$$b = \beta = m-r.$$

Returning to the general case, with likelihood (2.4), differentiation of the log-likelihood yields the two equations

$$\frac{\partial \log L}{\partial \lambda} = \frac{\alpha+b}{\lambda} - \frac{\alpha+\beta}{\lambda+\mu} - y_+ = 0 \tag{2.5}$$

and

$$\frac{\partial \log L}{\partial \mu} = \frac{a+\beta}{\mu} - \frac{\alpha+\beta}{\lambda+\mu} - x_+ = 0 \ . \tag{2.6}$$

Eliminating the term involving $(\lambda+\mu)^{-1}$ we obtain

$$\lambda^{-1} = \frac{y_+ - x_+}{\alpha+\beta} + \frac{a+\beta}{\alpha+b}\,\mu^{-1};$$

then substitution into (2.6) produces the quadratic equation in μ:

$$\mu^2[x_+(y_+ - x_+)] + \mu[y_+(\alpha-a) + x_+(2a+b+\beta)] - (a+\beta)(a+b) = 0 \ . \tag{2.7}$$

The appropriate solution is the maximum likelihood estimate

$$\hat{\mu} = \frac{[y_+(a-\alpha)-x_+(2a+b+\beta)]+\sqrt{[y_+(a-\alpha)-x_+(2a+b+\beta)]^2+4(y_+-x_+)(a+\beta)(a+b)x_+}}{2(y_+-x_+)x_+} \quad (2.8)$$

By symmetry, the maximum likelihood estimate of λ is

$$\hat{\lambda} = \frac{[x_+(b-\beta)-y_+(2b+a+\alpha)]+\sqrt{[x_+(b-\beta)-y_+(2b+a+\alpha)]^2+4(x_+-y_+)(b+\alpha)(a+b)y_+}}{2(x_+-y_+)y_+} \quad (2.9)$$

A little algebra shows that the quantities under the radicals are non-negative. For the Cases 1 and 2 above (2.8) and (2.9) become

$$\hat{\mu} = \frac{[y_+(k-r)-x_+(3k+m-r)]+\sqrt{[y_+(k-r)-x_+(3k+m-r)]^2+8(y_+-x_+)(k+m-r)kx_+}}{2(y_+-x_+)x_+} \quad (2.10)$$

$$\hat{\lambda} = \frac{[x_+(k+r-m)-y_+(3k+r)]+\sqrt{[x_+(k+r-m)-y_+(3k+r)]^2+8(x_+-y_+)(k+r)ky_+}}{2(x_+-y_+)y_+} \quad (2.11)$$

Case 2:

$$\hat{\mu} = \frac{m}{x_+ + \sqrt{x_+y_+}}, \quad (2.12)$$

$$\hat{\lambda} = \frac{m}{y_+ + \sqrt{x_+y_+}}. \quad (2.13)$$

The asymptotic variance-covariance matrix can be computed directly. We find first

$$-E\left[\frac{\partial^2 \log L}{\partial\mu^2}\right] = \frac{k}{\mu^2} + \frac{m\lambda}{\mu(\lambda+\mu)^2} \quad (2.14)$$

$$-E\left[\frac{\partial^2 \log L}{\partial\lambda^2}\right] = \frac{k}{\lambda^2} + \frac{m\mu}{\lambda(\lambda+\mu)^2} \tag{2.15}$$

$$-E\left[\frac{\partial^2 \log L}{\partial\lambda\partial\mu}\right] = -\frac{m}{(\lambda+\mu)^2} \tag{2.16}$$

so (2.14) and (2.15) are the diagonal, and (2.16) the off-diagonal, elements in the information matrix. Inversion then gives for

<u>Case 1</u>:

$$\mathrm{Var}\,[\hat{\mu}] = \frac{\mu^2}{k}\left[\frac{m\lambda\mu + k(\lambda+\mu)^2}{2m\lambda\mu+k(\lambda+\mu)^2}\right] \tag{2.17}$$

$$\mathrm{Var}\,[\hat{\lambda}] = \frac{\lambda^2}{k}\left[\frac{m\lambda\mu+k(\lambda+\mu)^2}{2m\lambda\mu+k(\lambda+\mu)^2}\right] \tag{2.18}$$

$$\mathrm{Cov}(\hat{\lambda},\hat{\mu}) = \frac{m\lambda^2\mu^2}{2mk\lambda\mu + k^2(\lambda+\mu)^2} \quad . \tag{2.19}$$

By similar manipulation there results for

<u>Case 2</u>:

$$\mathrm{Var}[\hat{\mu}] = \frac{\mu^2(\mu+2\lambda)}{2m\lambda}$$

$$\mathrm{Var}[\hat{\lambda}] = \frac{\lambda^2(\lambda+2\mu)}{2m\mu}$$

$$\mathrm{Cov}[\hat{\lambda},\hat{\mu}] = \frac{\lambda\mu}{2m} \quad .$$

Under certain conditions, e.g. when k becomes large, $\hat{\lambda}$ and $\hat{\mu}$ can be expected to be approximately normally distributed; also, the covariance tends to zero. A proof of asymptotic normality following the pattern of Cramér ([4], p. 366) could be given, but is omitted.

Although the large-sample properties of maximum likelihood estimators are familiar, it is interesting to investigate samples of

realistic size. We shall carry out such an investigation here by experimental sampling. By this means we are able to get an idea of the adequacy of the maximum likelihood estimators and to evaluate simple empirical adjustments to the latter to improve their performance. Section III contains such results. Sampling experiments also can be expected to reveal possible inadequacies of assumptions such as those

made in writing down the basic likelihood (2.4).

C. <u>First Sampling Experiment</u>

This experiment was conducted for the sampling plan of Case 1 above, with estimates (2.10) and (2.11). The actual system sampled involved parameter values

$\mu = 0.2$ (expected up-time of 5 units)

$\lambda = 1$ (expected down-time of 1 unit)

$k = 5$ (number of initial periods observed)

$m = 10$ (number of later snapshots)

A synthetic system realization was observed continuously through five consecutive up- and down-times, after which merely the state -- up, or down -- was noted; snapshots were taken at intervals of approximately 15 time units. Five hundred realizations were examined. From the data for each realization estimates μ and λ were computed, using (2.10) and (2.11). In addition maximum likelihood estimates of the above parameters were obtained, using merely the outcomes of the $k = 5$ initial period observations, omitting the snapshots; we denote the latter by $\tilde{\mu}$ and $\tilde{\lambda}$. Then, using both sets of estimates, i.e. $(\hat{\mu}, \hat{\lambda})$ and $(\tilde{\mu}, \tilde{\lambda})$ we computed estimates of system performance,

a) Operational Readiness $= R = \dfrac{E[U]}{E[U]+E[D]} = \dfrac{\lambda}{\lambda+\mu}$ (2.20)

b) Operational Reliability $= r(T) = \dfrac{\lambda}{\lambda+\mu} e^{-\mu T}$ (2.21)

by substituting the estimates in for the unknown true parameters. The

results estimate a) the long-run probability that the system will be up when needed, and b) the long-run probability that the system will be up and remain so for T time units thereafter, (e.g. throughout a mission time). For our sampling experiment T = 2 time units.

A summary of the results obtained is given in the table below. These summaries have been computed as follows. Consider the estimate of μ, say, obtained on the i-th run; call it $\hat{\mu}_i$. Then

$$A(\hat{\mu}) = \frac{1}{500} \sum_{i=1}^{500} \hat{\mu}_i \tag{2.22}$$

and

$$V(\hat{\mu}) = \frac{1}{500} \sum_{i=1}^{500} [\hat{\mu}_i - A(\hat{\mu})]^2, \tag{2.23}$$

while

$$M(\hat{\mu}) = \frac{1}{500} \sum_{i=1}^{500} [\hat{\mu}_i - \mu]^2; \tag{2.24}$$

exactly the same procedure applies to $\hat{\lambda}$.

The numbers in parentheses next to the estimated variances for Patch-Snapshot sampling were computed using the asymptotic formulas (2.17) and (2.18). Those in square brackets next to the corresponding means, variances, and mean-square error for Patch sampling were computed using the exact formulas

$$E[\tilde{\mu}] = \frac{k}{k-1} \mu \tag{2.25}$$

and

$$Var[\tilde{\mu}] = \left(\frac{k}{k-1}\right)^2 \frac{\mu^2}{k-2} \tag{2.26}$$

$$M[\hat{\mu}] = \frac{k+2}{(k-1)(k-2)} \mu^2 . \tag{2.27}$$

By and large, agreement between the estimated variances and the exact variances for Patch Sampling is very good. The agreement between the asymptotic variances and sampling variances for Patch-Snapshot Sampling is less good. The difference is probably attributable to the failure of the asymptotic formula for our small sample size ($k = 5$, $m = 10$). An alternative approach would be to use the linearization or "delta method", expanding μ and λ in powers of $\bar{x}$, $\bar{y}$, and $\frac{r}{m}$ and taking expectations. By this means the approximate bias as well as the variance can be determined.

Table 1
Estimates of Failure and Repair Rates
$\lambda=1, \mu=0.2$, $k=5$, $m=10$
(Statistics based on 500 independent realizations)

	Patch-Snapshot	Patch
Averages:	$A(\hat{\mu}) = 0.231$	$A(\tilde{\mu}) = 0.244[.250]$
	$A(\hat{\lambda}) = 1.20$	$A(\tilde{\lambda}) = 1.24\ [1.25]$
Variances:	$V(\hat{\mu}) = 0.011(0.0066)$	$V(\tilde{\mu}) = 0.020[.021]$
	$V(\hat{\lambda}) = 0.41\ (0.16)$	$V(\tilde{\lambda}) = 0.61\ [0.52]$
Mean Square:	$M(\hat{\mu}) = 0.012$	$M(\tilde{\mu}) = 0.022[0.023]$
Error	$M(\hat{\lambda})$; 0.45	$M(\tilde{\lambda}) = 0.67\ [0.58]$

Indications are present in this table that the estimates are biassed upwards, but less so when Patch-Snapshot information is available than with Patch information alone; the bias of the latter can be computed, see (2.25). The difference between $A(\hat{\mu})$ and μ is significant at the two-sided 5% level, as is that between $A(\hat{\lambda})$ and λ. In the following section we discuss ways of

reducing the bias. In spite of the presence of bias, Patch-Snapshot Sampling does considerably reduce both the variances and mean-squared errors of the estimates of λ and μ below those of the estimates obtained from Patch Sampling alone.

It is of interest to study the results of utilizing our estimates of λ and μ to estimate the measures R (see (2.20)) and r(T) (see (2.21)) of operational performance. The following table contrasts the results of using Patch-Snapshot information, and Patch information alone.

Table 2
Estimates of Operational Performance
$\lambda=1$, $\mu=0.2$, $k=5$, $m=10$

	Patch-Snapshot	Patch
Averages:	$A(\hat{R}) = 0.822$ (0.833)	$A(\tilde{R}) = 0.815$ (0.833)
	$A(\hat{r}) = 0.533$ (0.558)	$A(\tilde{r}) = 0.526$ (0.558)
Variances:	$V(\hat{R}) = 0.0057$ (0.005)	$V(\tilde{R}) = 0.0094$ (.00774)
	$V(\hat{r}) = 0.016$ (0.016)	$V(\tilde{r}) = 0.022$ (.0218)
Mean-Square Error	$M(\hat{R}) = 0.0058$	$M(\tilde{R}) = 0.0097$
	$M(\hat{r}) = 0.017$	$M(\tilde{r}) = 0.0226$

The asymptotic variances of $\hat{R}$ and $\hat{r}$, listed in parentheses, were obtained by delta method: for example, from the approximation

$$\hat{R} \doteq R + (\hat{\lambda}-\lambda)\frac{\partial R}{\partial \lambda} + (\hat{\mu}-\mu)\frac{\partial R}{\mu},$$

there results, if $\lambda = E[\hat{\lambda}]$ and $\mu = E[\hat{\mu}]$,

$$\mathrm{Var}\,[\hat{R}] \doteq \left(\frac{\partial R}{\partial \lambda}\right)^2 \mathrm{Var}[\hat{\lambda}] + \left(\frac{\partial R}{\partial \mu}\right)^2 \mathrm{Var}[\hat{\mu}] + 2\,\frac{\partial R}{\partial \lambda}\,\frac{\partial R}{\partial \mu}\,\mathrm{Cov}[\hat{\lambda},\hat{\mu}]\,. \qquad (2.28)$$

In fact we do not have $E[\hat{\lambda}] = \lambda$ and $E[\hat{\mu}] = \mu$ for our sampling experiment. The best estimates of $E[\hat{\lambda}]$ and $E[\hat{\mu}]$ available are $A(\hat{\lambda})$ and $A(\hat{\mu})$ respectively, and expanding around these values can be expected to improve the approximations (although the variances for $\hat{R}$ and $\hat{r}$ in Table 2 are in excellent agreement). The above statements hold true also for the estimates based upon $\tilde{\lambda}$ and $\tilde{\mu}$. When the derivatives in (2.28) are evaluated and the asymptotic variances and covariance (see (2.17), (2.18), and (2.19)) are substituted into (2.18) the results are

<u>Case 1</u>:

$$\mathrm{Var}[\hat{R}] \doteq \frac{2\lambda^2\mu^2}{(\lambda+\mu)^2[2m\lambda\mu+k(\lambda+\mu)^2]} \qquad (2.29)$$

and

$$\mathrm{Var}[\hat{r}] \doteq e^{-2\mu T}\,\frac{\lambda^2\mu^2}{(\lambda+\mu)^2}\left\{\frac{2k+2kT(\lambda+\mu)+T^2[m\lambda\mu+k(\lambda+\mu)^2]}{2km\lambda\mu + k^2(\lambda+\mu)^2}\right\}. \qquad (2.30)$$

By setting $m = 0$ in the latter expressions, variances for Patch sampling result. On the basis of our knowledge of the exact moments of $\tilde{\lambda}$ and $\tilde{\mu}$ (see (2.25) and (2.26)) a somewhat improved approximation to $\mathrm{Var}[\tilde{R}]$ is available. It is now possible to expand around the finite sample expectation, and to use exact variances. The result is $\frac{k}{k-2}$ times (2.29), the latter with $m = 0$. When $k=0$ the sampling plan reduces to consideration of m snapshot observations only, and the maximum likelihood estimator of R is simply $\frac{r}{m}$, with mean R and variance $\frac{R(1-R)}{m}$. It is interesting that when $k = 0$ in our approximate formula (2.29) for

$\mathrm{Var}[\hat{R}]$ the latter delivers the <u>exact</u> variance just quoted.

Although Case 2 will not be explored to any great extent, the form of the estimate of R implied by (2.12) and (2.13) is easy to obtain and appears rather striking:

<u>Case 2</u>

$$R = \frac{\sqrt{x_+}}{\sqrt{x_+} + \sqrt{y_+}} \tag{2.31}$$

and

$$\mathrm{Var}[\hat{R}] = \frac{R(1-R)}{2m} \tag{2.32}$$

Addition of the excess life information thus reduces the variance of our estimate of R to exactly one-half of that of the corresponding snapshot estimate. If k Patch observations are alone available, then (2.29) provides that

$$\mathrm{Var}[\hat{R}] = 2\,\frac{R^2(1-R)^2}{k} \quad . \tag{2.33}$$

The variances associated with the above estimates should be of aid in selecting a sampling procedure, if such an option is available.

III. Modifications of Maximum Likelihood Estimates

Maximum likelihood estimates possess many desirable large-sample properties. If, however, one is interested in estimates that minimize such a plausible measure of loss or estimate ineffectiveness as the mean-squared error, then it is known that modifications in the "raw" maximum likelihood estimates are sometimes effective. An example of a modification (to remove bias) is furnished by the familiar practice of dividing by (sample size-1) rather than (sample size) when estimating the Gaussian variance.

In this section we use a sampling experiment to evaluate the effect of some heuristic or empirical modifications in the Patch-Snapshot estimates. The modifications are suggested by first considering the simple components of the problem -- Patch information or Snapshot information alone -- and adapting the results to the problem at hand.

A. Patch estimate unbiassing

The maximum likelihood estimate of μ based on a Patch sample of k,

$$\tilde{\mu} = \frac{k}{x_+}$$

is biassed. Since the density of u_+ is gamma we find directly that

$$E[\tilde{\mu}] = kE[\frac{1}{x_+}] = k \int_0^\infty \frac{1}{x} e^{-\mu x} \frac{(\mu x)^{k-1}}{(k)} \mu dx = \frac{k}{k-1} \mu . \qquad (3.1)$$

Thus we can easily remove the bias of $\tilde{\mu}$, and likewise of $\tilde{\lambda}$, by simply multiplying $\tilde{\mu}$ and $\tilde{\lambda}$ by $\frac{k-1}{k}$. This suggests the following heuristics for

improving Patch-Snapshot estimates (2.10) and (2.11):

A-1: Replace k by k-1 in the formulas (2.10) and (2.11);

A-2: Replace x_+ by $\frac{k}{k-1} x_+$, and y_+ by $\frac{k}{k-1} y_+$ in (2.10) and (2.11).

B. Patch estimates with smallest mean square error

Consider estimates of the form $c\mu$, where μ is the Patch Sample estimate. Then, again using the gamma distribution of x_+, it may be shown that

$$\frac{k-2}{k} \tilde{\mu} = \frac{k-2}{x_+}$$

is the estimate of minimum mean-squared error. This suggests considering the procedure

B-1: Replace k by k-2 in (2.10) and (2.11).

B-2: Replace x_+ by $\frac{k}{k-2} x_+$, and y_+ by $\frac{k}{k-2} y_+$ in (2.10) and (2.11).

The above adjustments are but a few of the many possibilities, and are simply proposed for experimental investigation. Their apparent effects upon bias and mean-squared error will shortly be examined. Before doing this, we consider the effect of modifying the part of the likelihood function effected by Snapshots alone.

C. Snapshot estimate that minimizes the maximum mean-squared error

Suppose Patch information is ignored. Then it is known, see Lehmann [9], that the estimate of a binomial parameter $p(= \frac{\lambda}{\lambda+\mu})$ that minimizes the maximum mean-squared error is, in present notation, given by

$$p_{m.m.} = \frac{r}{m} \frac{\sqrt{m}}{1+\sqrt{m}} + \frac{1}{2(1+\sqrt{m})} \quad . \tag{3.2}$$

This suggests that we replace r by

$$\frac{r\sqrt{m}}{1+\sqrt{m}} + \frac{1}{2}\,\frac{m}{(1+\sqrt{m})}\ , \tag{3.3}$$

in the estimating formulas (2.10) and (2.11). However, the value of this modification can perhaps be questioned in advance if there is evidence that p is much above one-half; see the comparison of risks of the minimax and the minimum variance unbiassed estimate, $\frac{r}{m}$, (Lehmann [9], p. 4-24). The sampling experiment tends to bear out our suspicions for the present example.

Still further modifications are suggested if prior information is formally incorporated via Bayes' Theorem.

D. Snapshot generalized maximum likelihood estimate, beta prior

Given Snapshot information, and, in addition, a beta prior distribution, then the modified likelihood function appears as follows

$$L(r;\gamma,\ \delta) = p^{r+\gamma}(1-p)^{m-r+\delta}\ ; \tag{3.4}$$

where $p = \frac{\lambda}{\lambda+\mu}$, and the generalized maximum likelihood estimate p_g is the value of p that maximizes $L(r;\gamma,\ \delta)$:

$$p_g = \frac{r+\gamma}{m+\ +\delta}\ . \tag{3.5}$$

Let us suppose that rather diffuse information about the value of $p = \frac{\lambda}{\lambda+\mu}$ is available from previous experience. Consider the priors with

$$\begin{aligned} &1)\quad \gamma = 4,\ \delta = 1 \\ &2)\quad \gamma = 8,\ \delta = 2; \end{aligned} \tag{3.6}$$

both have modal values at 0.8. While 1) is the more diffuse, spreading

p-value probability more uniformly over [0, 1] than does 2), neither appears to strongly pre-judge the issue.

The heuristic modifications suggested are then

D-1: Replace r by $m(\frac{r+4}{m+5})$ in (2.10) and (2.11)

D-2: Replace r by $m(\frac{r+8}{m+10})$.

The estimate (2.32) can be interpreted in a strictly Bayesian manner as the mean of a posterior distribution for p. $L(r; \gamma, \delta)$ is proportional to that posterior when the prior is of the beta form; we get (2.32) precisely when the beta prior is proportional to $p^{\gamma-1}(1-p)^{\delta-1}$.

E. Generalized maximum likelihood estimate, beta prior

Examination of the likelihood function (2.4) shows that a beta prior for $\frac{\lambda}{\lambda+\mu}$ may be applied directly to it to create a posterior density for λ and μ. If the prior is proportional to $p^{\gamma}(1-p)^{\delta}$, then in the general likelihood (2.4) we only need to change α to $\alpha+\gamma$, and β to $\beta+\delta$ to obtain the posterior. In order to obtain the values of λ and μ that are the modal values of the posterior (generalized maximum likelihood estimates) the above changes may be incorporated into (2.8) and (2.9). In particular, for Case 1 the parameters of (2.4) become

$$a = b = k$$

$$\alpha = r+ \ , \ \beta = m-r+\delta = m+\gamma+\delta-(r+\gamma)$$

and the generalized estimates are obtained by replacing r by $r+\delta$ and m by $m+\delta+\theta$ in (2.10) and (2.11). We shall examine the modifications

E-1: Replace r by r+4 and m by m+5 in (2.10) and (2.11);

E-2: Replace r by r+8 and m by m+10 in (2.10) and (2.11).

These correspond to the beta priors of (2.33).

The above modifications are but a few of those that suggest themselves. For example, we have not studied the effect of introducing prior probabilities for λ and μ in the Patch part of the likelihood function (2.4). Natural priors for this purpose are independent gamma densities:

$$\varphi(\mu) = e^{-\mu\xi} \frac{(\mu\xi)^{u-1}}{(u)!} \xi \tag{3.7}$$

and

$$\psi(\lambda) = e^{-\lambda\eta} \frac{(\eta\lambda)^{v-1}}{(v)!} \eta , \tag{3.8}$$

where the prior expectations and variances of μ and λ are given by

$$E[\mu] = \frac{u}{\xi} , \quad \mathrm{Var}\ [\mu] = \frac{E^2[\mu]}{u} , \tag{3.9}$$

$$E[\lambda] = \frac{v}{\eta} , \quad \mathrm{Var}[\lambda] = \frac{E^2[\lambda]}{v} . \tag{3.10}$$

Densities (2.34) and (2.35) incorporate conveniently with the part of the likelihood involving $e^{-\mu x+} \mu^a e^{-\lambda y+} \lambda^b$. Specification of the priors can be accomplished by selecting values for the above expectations and variances and solving for the parameters ξ, u, η, and v. Such specification may, as if often suggested in Bayesian analysis, reflect subjective attitudes as well as past performance of similar equipments. Given the parameters, then it is

easily seen that the generalized maximum likelihood estimates for Patch-Snapshot sampling result by replacing x_+ by $x_+ + \xi$, y_+ by $y_+ + \eta$, a by a+u-1, and b by b+v-1 in the expressions (2.10) and (2.11). It appears that a natural bivariate prior for Patch-Snapshot sampling is proportional to

$$e^{-[\mu\xi+\lambda\eta]} \frac{(\mu\xi)^{u-1}(\lambda\eta)^{v-1}}{(\mu+\lambda)^{w}} \tag{3.11}$$

but in the latter μ and λ are not independent. Specification by choice of parameters ξ, η, u, v, and w will not be attempted.

The five-hundred realization sampling experiment, described in Section II, C, was utilized to evaluate the effects of the various modifications. The following table summarizes the results.

Table 3
Sampling Study of Maximum Likelihood Estimate Modifications
Exponential Distributions

	Patch-Snapshot						Patch					
	$A(\hat{\mu})$	$A(\hat{\lambda})$	$V(\hat{\mu})$	$V(\hat{\lambda})$	$M(\hat{\mu})$	$M(\hat{\lambda})$	$A(\tilde{\mu})$	$A(\tilde{\lambda})$	$V(\tilde{\mu})$	$V(\tilde{\lambda})$	$M(\tilde{\mu})$	$M(\tilde{\lambda})$
No Modif.	.231	1.198	.011	.41	.012	.45	.244	1.24	.020	.61	.022	.67
A-1	.184	.953	.0067	.25	.0069	.25	.195	.993	.013	.39	.013	.39
A-2	.185	.958	.0069	.26	.0071	.26	.195	.993	.013	.39	.013	.39
B-1	.136	.710	.0037	.14	.0077	.22	.146	.745	.0071	.22	.010	.28
B-2	.139	.719	.0039	.15	.0076	.23	.146	.745	.0071	.22	.010	.28
C	.255	1.07	.012	.28	.015	.29	.244	1.24	.020	.61	.022	.67
D-1	.234	1.178	.010	.37	.012	.40	.244	1.24	.020	.61	.022	.67
D-2	.236	1.169	.010	.35	.012	.38	.244	1.24	.020	.61	.022	.67
E-1	.233	1.157	.0095	.32	.011	.34	.244	1.24	.020	.61	.022	.67
E-2	.234	1.127	.0087	.26	.0099	.27	.244	1.24	.020	.61	.022	.67

(R = 0.833 r = 0.558)

	$A(\hat{R})$	$A(\hat{r})$	$V(\hat{R})$	$V(\hat{r})$	$M(\hat{R})$	$M(\hat{r})$	$A(\tilde{R})$	$A(\tilde{r})$	$V(\tilde{R})$	$V(\tilde{r})$	$M(\tilde{R})$	$M(\tilde{r})$
No Modif.	.822	.533	.0057	.016	.0058	.017	.815	.526	.0094	.022	.0097	.023
A-1	.822	.581	.0057	.014	.0058	.015	.815	.572	.0094	.020	.0097	.020
A-2	.822	.580	.0057	.014	.0058	.015	.815	.572	.0094	.020	.0097	.020
B-1	.823	.634	.0060	.013	.0061	.018	.815	.623	.0094	.018	.0097	.022
B-2	.822	.632	.0057	.012	.0058	.018	.815	.623	.0094	.018	.0097	.022
C	.794	.493	.0055	.015	.0070	.019	.815	.526	.0094	.022	.0097	.023
D-1	.821	.528	.0045	.014	.0047	.015	.815	.526	.0094	.022	.0097	.023
D-2	.820	.525	.0041	.013	.0043	.015	.815	.526	.0094	.022	.0097	.023
E-1	.821	.528	.0041	.013	.0042	.014	.815	.526	.0094	.022	.0097	.023
E-2	.819	.524	.0031	.011	.0033	.012	.815	.526	.0094	.022	.0097	.023

DISCUSSION OF TABLE 3.

A. Modification of the rate estimates

1. Consider first the <u>bias</u>. Use of $\sqrt{\frac{V(\mu)}{500}}$ as an estimate of the standard deviation of $A(\hat{\mu})$ together with the normal approximation provides strong evidence that

a) The unmodified Patch-Snapshot estimate $\hat{\mu}$ (and also $\hat{\lambda}$) is biassed upwards. Likewise, so are the modified estimates D, D-1, D-2, E-1, and E-2. The magnitude of this upward bias is very much the same for modifications other than C; the bias of C is relatively large for the estimate of $\hat{\mu}$, and small for that of $\hat{\lambda}$.

b) The modified estimates A-1, A-2, and in particular B-1, and B-2, are biassed downwards. Modifications A-1 and A-2 are closer, on the average, to the true mean than the unmodified estimate, and also closer than modifications B-1 and B-2.

c) The unmodified Patch estimate $\tilde{\mu}$ (and also $\tilde{\lambda}$) is biassed upwards to a greater degree than the Patch-Snapshot estimates. As anticipated, modifications A-1 and A-2 effectively remove this bias. Applied to Patch-Snapshot they tend to over-remove it (see b)), which is not surprising.

2. Consider the <u>mean-squared error</u>.

a) The mean-squared errors, $M(\hat{\mu})$ and $M(\hat{\lambda})$, of the Patch-Snapshot estimates $\hat{\lambda}$ and $\hat{\mu}$ seem to be reduced by a factor of approximately 1.65 by application of modifications A-1 and A-2. Modifications B-1 and B-2 have about the same effect upon mean-squared error,

but they have much greater effect upon variance: the B-modifications reduce the variance of the estimates by approximately a factor of 3.

b) Modification C has a more beneficial effect upon mean and mean-squared error of $\hat{\lambda}$ than upon $\hat{\mu}$. Modifications D-1 and D-2 have only very slightly beneficial effects, again more importantly upon $\hat{\lambda}$ than upon $\hat{\mu}$. The same is true of E-1 and E-2.

c) Of the modifications considered, the B-modifications produce the smallest mean-squared errors for the Patch estimates. The A-modifications are, however, effective in reducing mean-squared error and are exactly unbiassed.

B. The effect of the modifications on estimates of probabilities R and r

1. Consider the <u>bias</u>.

a) Use of the sample variance leads to the conclusion that $\hat{R}$ and $\tilde{R}$, the Patch Snapshot and Patch estimates of R, are biassed downwards. None of the suggested modifications significantly effect this bias, save for C. which somewhat increases its magnitude. Very probably the fact that the priors involved in the D and E modifications have a mode below 0.833 (at 0.8) has a tendency to further increase the bias. The bias of $\hat{R}$ is somewhat smaller than that of $\tilde{R}$.

b) The Patch-Snapshot estimate $\hat{r}$ appears to be biassed downwards when no modification is made, but use of the sample variance does not enable us to state this with high confidence. Again $\tilde{r}$ has a greater bias than $\hat{r}$. The A modifications seem to remove this effect,

perhaps replacing it by a smaller, upward, bias. The B modifications apparently go too far.

2. Consider the mean-squared error.

a) Only the D and E modifications seem to have a beneficial effect upon the mean-squared error of $\hat{R}$; these effectively cut the mean squared error in half. A smaller, but still desirable, impact upon $\hat{r}$ is noticeable.

b) If sampling were Snapshot alone, then the estimate of R is the number of successes $\frac{r}{m} = \bar{R}$, and

$$E[R] = \frac{\lambda}{\lambda+\mu} \quad , \quad Var[r] = \frac{\lambda\mu}{m(\lambda+\mu)^2} .$$

Thus $\bar{R}$ is unbiassed, and for our example

$$V(\bar{R}) = M(\bar{R}) = 0.0139 . \tag{3.12}$$

Comparison with $V(\hat{R})$ and $V(\tilde{R})$ indicates that Patch Snapshot reduces the above mean square error by about a factor of two.

IV. Robustness

The exponential assumptions (2.2) made in deriving our maximum likelihood estimates are often plausible, especially in reliability problems. However, one seldom can test such assumptions thoroughly, and it seems worthwhile to investigate the behavior of our estimates when, in fact, other -- perhaps equally plausible -- distributions govern the observations. To this end we have utilized a sampling experiment, first generating five hundred realizations for each of three sets of alternative distributions, then sampling from these realizations as was done before, and finally computing estimates of μ, λ, R, and r. Modifications were also considered.

Specifically, suppose that the up times of the two-state process have the log-normal distribution:

$$P\ U \leq x = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{(\ln x - m_u)/\sigma_u} e^{-z^2/2} dz \tag{4.1}$$

and that the down times also have the log-normal distribution with parameters m_d and σ_d^2. We can express the moments of U (and D) in terms of $m_u(m_d)$ and $\sigma_u^2(\sigma_d^2)$; and vice-versa, thus

$$m_u = \log \left\{ \frac{E^2[U]}{\sqrt{E[U^2]}} \right\}, \quad \sigma_u^2 = \log \left\{ \frac{E[U^2]}{E^2[U]} \right\} . \tag{4.2}$$

Given $E[U]$ and $E[U^2]$, and the corresponding moments for down time one can utilize (4.1) to draw appropriately centered and scaled random normal deviates. Let $\tilde{U}$ be such a number; then a realization of U having desired mean and variance is achieved by computing

$$U = e^{\tilde{U}} .$$

From such log-normally distributed realizations, a realization of the up-down process history may be obtained.

As was remarked earlier, one use for such histories is to investigate the behavior of maximum likelihood estimators computed under the exponential specification when, in fact, the log-normal specification prevails. The results of such investigation are tabled and discussed below. We take, as in the exponential situation,

$$\begin{aligned} E[U] &= 5 \\ E[D] &= 1. \end{aligned} \tag{4.3}$$

Now for the exponential distribution the following relationship between first and second moments exists:

$$E[U^2] = 2\, E^2[U] . \tag{4.4}$$

We can easily determine log-normal distributions with this property. From (4.2) the underlying normal must in this case have parameters

$$m_u = \log E[U] - \frac{1}{2} \log 2, \quad \sigma_u^2 = \log 2. \tag{4.5}$$

A log-normal distribution with parameters (4.5) will agree with an exponential with mean $E[U]$ up to second moments. In what follows we shall refer to this situation as Case E, (E = exponential).

Contrasted with the above will be examined one in which the coefficient of variation is one-half:

$$E[U^2] = \frac{3}{2} E^2[U], \tag{4.6}$$

so

$$m_u = \log E[U] - \frac{1}{2} \log \left(\frac{3}{2}\right) , \qquad \sigma_u^2 = \log \frac{3}{2} . \tag{4.7}$$

This is called Case H (H = half-exponential).

The last case to be considered is that in which the coefficient of variation is twice that of the exponential:

$$E[U^2] = 3\, E^2[U] , \tag{4.8}$$

so

$$m_u = \log E[U] - \frac{1}{2} \log 3, \qquad \sigma_u^2 = \log 3. \tag{4.9}$$

This is called Case T (T = twice-exponential).

Comparisons of the behavior of our estimates when the distributions of Cases, E, H, and T prevail are given in the next tables.

Table 4
Sampling Study of Maximum Likelihood Estimates of Rates λ and μ
Modifications and Robustness
Exponential and Log-Normal Distribution

	Patch-Snapshot						Patch					
	$A(\hat{\mu})$	$A(\hat{\lambda})$	$V(\hat{\mu})$	$V(\hat{\lambda})$	$M(\hat{\mu})$	$M(\hat{\lambda})$	$A(\tilde{\mu})$	$A(\tilde{\lambda})$	$V(\tilde{\mu})$	$V(\tilde{\lambda})$	$M(\tilde{\mu})$	$M(\tilde{\lambda})$
No Modif; Exp.	.231	1.20	.011	.41	.012	.45	.244	1.24	.020	.61	.022	.67
L.-N.,E	.222	1.16	.0066	.20	.0071	.23	.230	1.19	.0093	.23	.010	.27
L.-N.,T	.243	1.30	.013	.39	.015	.48	.257	1.35	.021	.51	.024	.63
L.-N.,H	.213	1.07	.0039	.10	.0041	.11	.215	1.10	.0043	.11	.0045	.12
A-1; Exp.	.184	.953	.0067	.25	.0069	.25	.195	.993	.013	.39	.013	.39
L.-N.,E	.177	.926	.0043	.13	.0048	.14	.184	.955	.0059	.15	.0062	.15
L.-N.,T	.193	1.03	.0084	.25	.0085	.25	.206	1.080	.013	.33	.013	.33
L.-N.,H	.170	.857	.0026	.070	.0035	.090	.172	.882	.0027	.069	.0035	.083
B-1; Exp.	.136	.710	.0037	.14	.0077	.22	.146	.745	.0071	.22	.010	.28
L.-N.,E	.132	.692	.0025	.078	.0072	.17	.138	.716	.0033	.083	.0072	.16
L.-N.,T	.143	.770	.0047	.14	.0079	.19	.154	.810	.0074	.18	.0095	.22
L.-N.,H	.127	.642	.0016	.043	.0069	.17	.129	.662	.0015	.039	.0066	.015
C; Exp.	.255	1.07	.012	.28	.015	.29	.244	1.24	.020	.61	.022	.67
L.-N.,E	.245	1.04	.0074	.14	.0094	.14	.230	1.19	.0093	.23	.010	.27
L.-N.,T	.269	1.15	.015	.27	.020	.30	.26	1.35	.021	.51	.024	.63
L.-N.,H	.234	.964	.0040	.070	.0051	.072	.215	1.10	.0043	.11	.0045	.12
D-1; Exp.	.234	1.18	.010	.37	.012	.40	.244	1.24	.020	.61	.022	.67
L.-N.,E	.226	1.14	.0062	.17	.0069	.19	.230	1.19	.0093	.23	.010	.27
L.-N.,T	.247	1.28	.013	.34	.015	.42	.257	1.35	.021	.51	.024	.63
L.-N.,H	.216	1.06	.0034	.083	.0036	.087	.215	1.10	.0043	.11	.0045	.12
E-1; Exp.	.233	1.16	.0095	.32	.011	.34	.244	1.24	.020	.61	.022	.67
L.-N.,E	.225	1.13	.0060	.16	.0066	.18	.230	1.19	.0093	.23	.010	.27
L.-N.,T	.245	1.25	.012	.32	.014	.38	.257	1.35	.021	.51	.024	.63
L.-N.,H	.216	1.05	.0034	.084	.0037	.086	.215	1.10	.0043	.11	.0045	.12

DISCUSSION OF TABLE 4.

a) Use of $\sqrt{\frac{V(\hat{\mu})}{500}}$ and $\sqrt{\frac{V(\hat{\lambda})}{500}}$ and the normal approximation provide strong evidence that for Patch-Snapshot sampling the L.-N. cases E, T, and H are all biassed upwards, just as is the original exponential. The upward bias seems actually to be somewhat smaller for the cases L.-N., E and L.-N., H than for the exponential case itself. If the parameters were estimated knowing that the L.-N. specification were correct still better properties would be expected for the estimates, but there are indications that only when L.-N., T prevails does any degradation occur in mean-squared error. The latter effect is possibly attributable to the positive skewness of L.-N., T, which is greater than that for the exponential. The degree of bias exhibited by the Patch-Snapshot estimates seems also to be smaller than that of the Patch estimates alone.

The mean-squared errors of our estimates (2.10) and (2.11) in the L.-N. cases seem satisfactory; for only L.-N., T do $M(\hat{\mu})$ and $M(\hat{\lambda})$ exceed the values for the exponential case. Again an improvement over the mean-squared errors for Patch estimates alone seems to exist, although this improvement is not dramatic.

b) For the L.-N. specification the A-modification appears to over-reduce the upward bias mentioned in a) above, at least for L.-N., E and H. It is nearly correct for L.-N., T. The net effect is to reduce or leave essentially unchanged the bias, except for L.-N., H; in the latter case the bias is increased. But in <u>all</u> cases the mean-squared error is brought down by the A-modification, sometimes considerably.

c) The B-modification tends to over-correct the bias, without producing a compensatory effect upon mean-squared error. In case L. N., H the mean-squared error seems to have been increased.

Lastly, we examine the maximum likelihood estimates of R and r when the log-normal is the actual underlying distribution. A summary appears in Table 5.

Table 5
Sampling Study of Maximum Likelihood Estimates of Probabilities R and r; Modifications and Robustness
Exponential and Log-Normal Distributions

	Patch-Snapshot						Patch					
	$A(\hat{R})$	$A(\hat{r})$	$V(\hat{R})$	$V(\hat{r})$	$M(\hat{R})$	$M(\hat{r})$	$A(\tilde{R})$	$A(\tilde{r})$	$V(\tilde{R})$	$V(\tilde{r})$	$M(\tilde{R})$	$M(\tilde{r})$
No Modif.; Exp.	.822	.533	.00565	.0159	.00577	.0166	.815	.526	.00941	.0216	.00975	.0226
L.-N., E	.828	.542	.00458	.0128	.00460	.0131	.824	.536	.00724	.0175	.00732	.0180
L.-N., T	.827	.528	.00693	.0212	.00697	.0222	.816	.518	.0129	.0289	.0132	.0306
L.-N., H	.827	.547	.00351	.00914	.00356	.00927	.829	.548	.00375	.00966	.00377	.00978
A-1; Exp.	.822	.581	.00571	.0144	.00583	.0149	.815	.572	.00941	.0203	.00975	.0204
L.-N., E	.828	.590	.00476	.0117	.00479	.0127	.824	.583	.00724	.0161	.00732	.0167
L.-N., T	.828	.577	.00678	.0188	.00682	.0191	.816	.564	.0129	.0269	.0132	.0269
L.-N., H.	.826	.594	.00393	.00891	.00398	.0102	.829	.595	.00375	.00875	.00377	.0100
B-1; Exp.	.823	.634	.00600	.0128	.00612	.0185	.815	.623	.00941	.0183	.00975	.0225
L.-N., E	.828	.643	.00522	.0107	.00525	.0178	.824	.635	.00724	.0142	.00732	.0200
L.-N., T	.828	.577	.00678	.0188	.00682	.0191	.816	.564	.0129	.0269	.0132	.0269
L.-N., T	.827	.594	.00351	.00824	.00356	.00946	.829	.595	.00375	.00875	.00377	.0100
C; Exp.	.794	.493	.00549	.0146	.00701	.0190	.815	.526	.00941	.0216	.00975	.0226
L.-N., E	.800	.501	.00436	.0117	.00544	.0151	.824	.536	.00724	.0175	.00732	.0180
L.-N., T	.800	.487	.00697	.0201	.00812	.0252	.816	.518	.0129	.0289	.0132	.0306
L.-N., H	.799	.507	.00306	.00765	.00424	.0103	.829	.548	.00375	.00966	.00377	.00978
D-1; Exp.	.821	.528	.00454	.0141	.00469	.0150	.815	.526	.00941	.0216	.00975	.0226
L.-N., E	.827	.536	.00356	.0111	.00361	.0116	.824	.536	.00724	.0175	.00732	.0180
L.-N., T	.825	.521	.00584	.0192	.00590	.0206	.816	.518	.0129	.0289	.0132	.0306
L.-N., H	.826	.542	.00240	.00710	.00246	.00737	.829	.548	.00375	.00966	.00377	.00978
E-1; Exp.	.821	.528	.00408	.0131	.00424	.0140	.815	.526	.00941	.0216	.00975	.0226
L.-N., E	.826	.536	.00332	.0105	.00338	.0110	.824	.536	.00724	.0175	.00732	.0180
L.-N., T	.826	.522	.00501	.0177	.00507	.0191	.816	.518	.0129	.0289	.0132	.0306
L.-N., H	.824	.541	.00254	.00725	.00263	.00757	.829	.548	.00375	.00966	.00377	.00978

DISCUSSION OF TABLE 5.

a) There is an apparent tendency for the estimates R and r to be biassed downwards; the bias of the latter being greater than that of the former. Notice that the biases are actually not as serious for L.-N. cases as for the exponential. The various modifications of the estimates of λ and μ seem to have very little effect upon the magnitude of the bias. Modifications A, B, and C tend to somewhat increase the mean-squared error of R. The D and E modifications somewhat reduce the mean-squared error of R. Certainly the latter mean-squared error is brought considerably below that of the "distribution-free" estimate (3.12).

b) The bias of the unmodified estimates r, and likewise of $\tilde{r}$, is in a downwards direction. In part this may be the effect of the downward bias of the R estimates, and in part that of the upward bias of the estimates of μ. Apparently the A and B modifications over-compensate for this effect. For the present numbers the simple arithmetic average of the unmodified and the A-modified estimates has smaller bias than either component alone; this may be an accident, but further heuristics are suggested as a consequence. Modifications D and E force the bias, and also the mean-squared error, still further downwards. It would be of interest to combine modifications, applying A and D together for example, but this experiment has not yet been conducted.

Tables 4 and 5 suggest that, at least for the particular situation considered in our sampling experiment, the maximum likelihood estimates based on an exponential specification behave well also when the data is log-normal.

Certainly no hard and fast conclusions are possible as a result of the sampling experiments. However, possibly useful indications concerning the behavior of our maximum likelihood estimates are obtained where none seem otherwise available.

REFERENCES

[1] Cox, D.R., Renewal Theory, Methuen Monograph, John Wiley and Sons, Inc., New York; 1962.

[2] Cox, D.R., "Some problems of statistical analysis connected with congestion", Proceedings of the Symposium on CONGESTION THEORY, University of No. Carolina Press, Chapel Hill, No. Carolina; 1965.

[3] Cox, D.R., "A note on the analysis of a type of reliability trial," J. SIAM, Applied Mathematics; Vol. 14, No. 5, Sept. 1966, pp. 1133-1142.

[4] Cramér, H., Mathematical Methods of Statistics, Princeton Univ. Press, Princeton, N.J.; 1946.

[5] Davis, D.J., "An analysis of some failure date," J. American Statistical Assn., Vol. 47, No. 258, June 1952, p. 113.

[6] Feller, W., An Introduction to Probability Theory and its Applications, Vol. I, John Wiley and Sons, New York, 1957.

[7] Good, I.J., The Estimation of Probabilities, Research Monograph No. 30, The M.I.T. Press, Cambridge, Mass.; 1965.

[8] Hodges, J.L., and Lehmann, E.L., "Some problems in minimax point estimation", Annals of Math. Stat., Vol. 21, 1950, pp. 182-197.

[9] Lehmann, E.L., Notes on the Theory of Estimation (Chaps. 1-5), Univ. of California, Berkeley 4, Calif. (reprinted 1962).

[10] Moder, J., and Gaver, D.P., "Estimation of elemental standard times using attribute-variable data", to be submitted to the Journal of Industrial Engineering.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Graduate School of Industrial Administration Carnegie Institute of Technology	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP Not applicable

3. REPORT TITLE

STATISTICAL ESTIMATION IN A PROBLEM OF SYSTEM RELIABILITY

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(Last name, first name, initial)*

Gaver, D. P. and Mazumdar, M.*

6. REPORT DATE February, 1967	7a. TOTAL NO. OF PAGES 36	7b. NO. OF REFS 10
8a. CONTRACT OR GRANT NO. NONR 760(24) b. PROJECT NO. NR 047-048 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Management Sciences Research Report No. 89 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* None	

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES Also under partial support of Westinghouse Electric Corporation, Research Laboratory	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch Office of Naval Research Washington, D. C. 20360

13. ABSTRACT

The statistical estimation of the reliability of a simple system, subject to exponential failures and subsequent repairs, is considered. Observations are assumed taken at isolated instants (snapshots), and for also continuous periods of time (patches); the information is combined by maximum likelihood. Procedures for improving small-sample properties of estimates are studied by Monte Carlo sampling experiments. The robustness of the estimates is similarly considered.

*Westinghouse Electric Corp., Pgh. Pa.

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

Unclassified

Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
readiness						
reliability						
failures						
repairs						
probability						
statistical estimation						
robustness						
work sampling						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

Unclassified

Security Classification