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AN ANALYTICAL DESCRIPTION OF THE BEHAVIOR OF GRANULAR MEDIA*

by

Jay B. Weidler** and Paul R. Paslay***

Normality of the plastic strain increment vector to the loading surface is not always observed for granular media. Beginning from a thermodynamic basis and considering only isothermal processes, a series of physically appealing assumptions is employed to increasingly delimit the possible directions of the plastic strain increment vector. The results of the analysis indicate that on certain portions of the loading surface normality should hold while on the remainder, envelopes or fans are possible. Finally, several alternative restrictions are offered which provide a unique stress-strain relation at every point on the loading surface.

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Introduction

A granular media is qualitatively described as an aggregate of particles whose resistance to inelastic flow in enhanced by an increase in the hydrostatic component of the external forces. The voids of such a material are considered to be interconnected and these interstices may be dry, partially saturated or completely filled with liquid. It will be presumed for the system in question that drainage is rapid enough so that the pressure in the liquid remains constant under a quasi-static load change. The particles comprising the aggregate are considered as rigid bodies of unspecified size or shape. The over-all size of the particle is assumed, however, to be small enough with respect to the size of the system to permit treatment by the methods of continuum mechanics. The theory to be presented here allows for cohesion between particles, and it presumes time-independent and isothermal behavior.

Historically a mathematical formulation of the division between attainable and unattainable stress states in granular media was proposed first by Coulomb [1]*. This was generalized later by Mohr [2]. The form of the line demarking the regions is shown in Figure 1 for a two-dimensional (τ - p) stress space. Until recently the major aim of those working in the mechanics of granular media has been in finding states of stress in equilibrium with the applied loads for generalized inelastic flow conditions. The extent of this endeavor is described in a recent book by Sokolovskii [3].

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* Numbers in brackets designate references listed in the bibliography.

In contrast to the above mentioned effort the literature concerned with the generalization of the stress-strain relationship had to be deferred until the theory of plasticity had been well-developed. Generalizations of a linearized, two-dimensional, Mohr-Coulomb criterion (shown in Figure 2) to include stress-plastic strain increment relations were proposed by Drucker and Prager [4], Shield [5], Drucker, Gibson, and Henkel [6], Jenike and Shield [7], and Drucker [8], [9]. All of the above employed the concept of normality [10] to determine the direction of the plastic strain increment vector.

Experimentally it is observed that granular media are inelastically deformed almost immediately upon application of stress and subsequently work harden until the stress state reaches the Mohr-Coulomb envelope. At this point if the material was initially in a loose state, it begins to flow, while if the initial state was dense, work softening begins. Although much effort has been devoted to experimentally determining the shape of the limiting envelope, little work has been reported describing the shape of the intervening loading surfaces. Consequently correlation of theory to experiment has rested largely on observation of the direction of the strain-increment vector at stress-states on the limiting surface. In Figure 3 if normality holds, then the strain increment vector must indicate a volume increase. Numerous experiments give evidence that the rate of volume increase lies closer to the direction of zero volume change than to the direction required by normality.

Although this phenomenon can be explained through the use of corners on the loading surface, an attempt will be made in the subsequent discussion to present an alternative explanation. In granular media the resistance to deformation is furnished by two types of friction. The first of these arises on the contact surfaces of adjoining particles termed surface friction. The

second, denoted as interlocking friction refers the resistance offered by the interference of the particles themselves to changes of their relative positions. It has been previously pointed out that normality is not a law of nature and that in frictional systems, normality does not hold in general [11], [12].

Perhaps the most significant alteration incorporated in the theory presented here is the dependence of the loading surface on the specific volume of the material. This phenomenon has been discussed by Jenike and Shield [7]. If, for a given stress state, the flow makes the material more dense, then work-hardening will generally occur while if the flow is such that the volume increases, the material will generally work soften. Whenever a theory includes this feature, the direction of the plastic strain increment vector assumes the additional significance that besides determining the flow state it determines the subsequent loading surface.

The purpose of this paper then is to develop a stress -- plastic strain increment relation which accommodates the observed phenomena.

General Considerations

The theory to be presented is deduced by observing the restrictions resulting from a series of physically motivated assumptions. These assumptions are presented in a sequence which begins from a thermodynamic basis and continues to the point where a definite theory can be established. Each of these assumptions implies its predecessors but at the same time the plausibility of each assumption decreases. In the ensuing development tensile stresses are taken as positive.

The theory, as was previously mentioned, is based on isothermal behavior with changes of state accomplished quasistatically. The state of the material is assumed to be known when the stress and plastic strain tensors are known. Elastic strains are neglected and the initial state is one where the stress and plastic strain tensors both vanish. A loading surface is assumed to exist that encloses all stress states which do not induce plastic flow.

The configuration of the loading surface is determined by the state and is defined by a relationship of the form:

$$F(\sigma_{ij}, \varepsilon_{ij}^{p}) = 0 \tag{1}$$

Here the ε_{ij}^{p} are also the components of the total as well as the plastic strain tensors, since the elastic components are assumed equal to zero. The direction and sense of the vector in nine dimensional strain space with components, $d\varepsilon_{ij}^{p}$, are uniquely determined by the stress state σ_{kl}. It is postulated that for each stress state on the loading surface it is possible to induce plastic flow and for each $d\varepsilon_{ij}^{p}$ vector there is at least one point on the loading surface.

Finally all functions, including the loading surface, are assumed to possess sufficient continuity for the ensuing development. Specifically such notions as corners on the loading surface are excluded.

Conjugate Points and the Clausius Inequality

The behavior of any material must be such that the laws of thermodynamics are not violated. The first such law is a balance equation between work, heat flow, and internal energy. Since the measurement of the heat flow in a

granular medium is exceedingly difficult, the first law does not play a central role in the development of an analytical theory of behavior. The first law can thus be regarded as an auxiliary equation which determines the heat flow in any process. As the state of the system is defined by σ_{ij} and ε_{ij} and the internal energy is a state variable, it is clear that the internal energy per unit current volume, U , may be expressed by

$$U = U(\sigma_{ij}, \varepsilon_{ij}) \tag{2}$$

In general one would anticipate that when the elastic strain is neglected U would be dependent only on the plastic strain. This restriction is introduced later. Thus the first law, or balance of energy is given by

$$dQ = dU - \sigma_{ij}\, d\varepsilon_{ij} \tag{3}$$

where dQ is the differential of the heat flow per unit current volume into the system and $\sigma_{ij}\, d\varepsilon_{ij}$ is the work per unit current volume done on the system. Since $d\varepsilon_{ij}$ equals $d\varepsilon_{ij}{}^{p}$ (elastic strains being neglected), Equation (3) may be rewritten as

$$dQ = dU - \sigma_{ij}\, d\varepsilon_{ij}{}^{p} \tag{4}$$

This result gives no restrictions on the behavior of granular media. It only gives a means for finding dQ for a differential change of state in an element of the material.

The second law of thermodynamics may be conveniently stated in terms of a cycle. Il'yushin [13] has used this concept to develop a thermodynamic postulate for elastic-plastic materials. A cycle is defined as a continuous change of state where the initial and final states coincide. Before examining

the consequences of the second law, the concept of a cycle in a granular system must be clarified. The requirement of returning to the initial state in a cycle implies a zero net change in the plastic strain. It further implies by Equation (1) that the initial and final loading surfaces are the same. Consequently a cycle in a granular system requires that the original packing or an equivalent packing can be restored or established by a properly chosen sequence of stress states. Isotropically work hardening materials provide an example where this requirement on cycling cannot be satisfied by isothermal processes. Here the size of the loading surface, an additional state variable, always increases even though the net plastic strain can be made equal to zero. In reality, permanent changes to the granular system do occur during plastic deformation, but it is assumed that they are of secondary importance to the over-all behavior. Such changes include modification of surface friction factors, breakage or abrasion of grains, and permanent deformation or rupture of cohesive bonds.

Having discussed the concept of a cycle in general terms, the following thought experiment illustrating the concept will be useful in the subsequent development of the theory.

First applying a loading

$$\sigma_{ij} = \kappa \Sigma_{ij} \tag{5}$$

where Σ_{ij} is a constant stress vector and κ is a parameter which has a zero value in the reference state and increases monotonically to $\kappa^{(1)}$ as the stress state reaches the initial loading surface. Hence

$$\sigma_{ij}^{(1)} = \kappa^{(1)} \Sigma_{ij} \tag{6}$$

$$F(\sigma_{ij}^{(1)}, 0) = 0 \tag{7}$$

By a previous postulate the direction and sense of the strain vector are determined by $\sigma_{ij}^{①}$. For this thought experiment a differential amount of strain, $d\varepsilon_{ij}^{p}$, is induced to which there corresponds the nine component vector, $d\vec{\varepsilon}^{p}$, in strain space. The assumption that corresponding to this there is a differential change in stress $d\sigma_{ij}^{①}$ is substantiated experimentally for most soils and is accepted in what follows. Consequently

$$F(\sigma_{ij}^{①} + d\sigma_{ij}^{①}, d\varepsilon_{ij}^{p}) = 0 \tag{8}$$

defines the new yield surface. In order to complete this cycle the element must be returned to zero stress and zero strain as these are the state variables. Consequently the negative of the existing plastic strain must be induced. A previous assumption, insuring the availability of all strain vectors, indicates that for at least one stress state, say $\sigma_{ij}^{②}$, on the new loading surface, the initial change of plastic strain will be opposite to $d\vec{\varepsilon}^{p}$. Such a stress state is said to be the conjugate stress state to $\sigma_{ij}^{①}$. The next increment of the loading program is to change the stress state from $\sigma_{ij}^{①} + d\sigma_{ij}^{①}$ to $\sigma_{ij}^{②}$ in such a way that no additional plastic flow occurs. It is assumed that such a loading path can be found. At this point a differential change of strain is induced which eliminates the existing plastic strain. This is possible since the initial plastic strain increment vector at $\sigma_{ij}^{②}$ is directed opposite to $d\vec{\varepsilon}^{p}$. The final step in the loading program is to return to zero stress without inducing further plastic deformation. This gives the same initial and final states to the cycle; i.e., $\sigma_{ij} = \varepsilon_{ij} = 0$. This cycle is presented schematically in Figure 4. The above cycle of loading may be extended to include finite changes of strain as well as cycles beginning from an arbitrary state.

It is now of interest to investigate the consequences of applying the second law of thermodynamics to such cycles. For an isothermal process the second law of thermodynamics requires that

$$\oint dQ \leq 0 \tag{9}$$

where $\oint$ implies integration around a closed cycle and dQ is, as mentioned before, the heat flow per unit current volume into the system. Actually what is meant by an isothermal process is not defined since temperature has not been defined. In this sense it may be more appropriate to regard Equation (9) as a postulate. Now since the internal energy change in any closed cycle is zero, Equation (9) and Equation (4) yield

$$\oint \sigma_{ij}\, d\varepsilon_{ij}^{p} \geq 0 \tag{10}$$

Thus for the cycle depicted schematically in Figure 4, Equation (10) may be written as

$$\sigma_{ij}^{①}\, d\varepsilon_{ij}^{p} - \sigma_{ij}^{②}\, d\varepsilon_{ij}^{p} \geq 0 \tag{11}$$

if products of the second order in the differentials of stress and strain are neglected. On factoring out $d\varepsilon_{ij}^{p}$, the result

$$(\sigma_{ij}^{①} - \sigma_{ij}^{②})\, d\varepsilon_{ij}^{p} \geq 0 \tag{12}$$

may be interpreted as requiring the dot product of the vector $(\sigma_{ij}^{①} - \sigma_{ij}^{②})$ and the vector $d\varepsilon_{ij}^{p}$ to be greater than or equal to zero. In other words the projection of the vector $d\vec{\varepsilon}^{p}$ on $(\vec{\sigma}^{①} - \vec{\sigma}^{②})$ in the nine dimensional stress-strain space must be positive or zero. A schematic representation of this is shown in Figure 5 where the admissible directions of $d\varepsilon^{p}$ are depicted.

If for σ_{ij}①, two conjugate points, σ_{ij}② and σ_{ij}③, exist the limitations on the direction of $d\vec{\varepsilon}^p$ are indicated schematically in Figure 6. Since for each stress state on the loading surface, it is possible to induce plastic flow, the conjugate stress states must be located so that a plane can be passed through σ_{ij}① such that all conjugate points are in the plane or on one side of it.

A Less Restrictive Concept of State

It is clear from the preceding section that more definite assumptions must be made before the direction of $d\vec{\varepsilon}^p$ can be uniquely determined for a stress point on the loading surface. Now any sufficiently restrictive postulate, as long as it satisfies Clausius' Inequality, Equation (10), can establish a definite stress-strain increment relationship; for example, see Drucker [10]. The approach to be followed in this paper is to continue to make further assumptions, usually in thermodynamic terminology, finally reaching a point where, for each point on the loading surface, $d\vec{\varepsilon}^p$ is uniquely determined. The first assumption in this process is presented in this section.

The preceding analysis depended on a closed cycle which had the same initial and final state. In the case of granular media it appears to be of interest to explore the consequences of specifying the state in a less restrictive manner than was done in the preceding section. Let the state be now determined if σ_{ij} and v, the specific volume, are known.

Since the differential change in v is given by

$$dv = v\, d\varepsilon_{ii}^{p} \tag{13}$$

and as the configuration of the loading surface must be the same for equivalent states, i.e.

$$F(\sigma_{ij}, v) = 0 \tag{14}$$

then the process of a closed cycle, similar to that of Figure 4, is shown as Figure 7. Other than the modified definition of the state and Equation (14) replacing Equation (1), the preceding theory is unaltered.

In this case it is anticipated that more than one conjugate point exists for every point on the loading surface. The reader should note that at a conjugate point in this cycle $d\varepsilon_{ii}^{p②} = - d\varepsilon_{ii}^{p①}$ but that, in general, $d\varepsilon_{ij}^{p②} \neq - d\varepsilon_{ij}^{p①}$ In fact it is anticipated that any point on certain regions of the loading surface would be a conjugate point for $\sigma_{ij}^{①}$.

The restriction placed on the closed cycle shown in Figure 7 by Equation (10) is

$$\sigma_{ij}^{①} d\varepsilon_{ij}^{p①} + \sigma_{ij}^{②} d\varepsilon_{ij}^{p②} \geqslant 0 \tag{15}$$

for every conjugate stress state $\sigma_{ij}^{②}$.

It is clear that this less restrictive concept of state will not, by itself, lead to a definite specification of $d\vec{\varepsilon}^{p}$. Subsequent assumptions, however, will incorporate this modification. Owing to the fact that the results presented in this section are a special case of the preceding section, all results are, thus far, consistent.

Discussion of the Strain Energy Function

In the previous section the state variables were reduced to σ_{ij} and v. Implied in this assumption was a concept that the only permanent deformation

altering the response of the material to subsequent changes of stress was a change in the specific volume.

The assumption made in this section is that the internal energy, U , is a function of the specific volume only, i.e.,

$$U = U(v) \tag{16}$$

and

$$dU = \frac{dU}{dv}\, dv = \frac{dU}{dv}\, v d\varepsilon_{ii}{}^{p} \tag{17}$$

Physically, the basis for this assumption lies in the mechanisms available in granular media for storing internal energy. If an aggregate of particles decreases in volume, it is possible for some of the particles to be wedged between their neighbors in such a manner that on a release of stress, residual forces remain in the system. Should subsequent stressing alter the existing volume then it is plausible that these residual forces could be augmented or released.

To place some bounds on the possible variations of U with v , the pressure-volume curve typical to granular media is shown as Figure 8. This curve provides a limit on $\frac{dU}{dv}$, i.e.

$$0 \leq -\frac{dU}{dv}\, v \leq p^{*} \tag{18}$$

so that Figure 9 results. p^{*} represents that value of p where the stress path $-p\delta_{ij}$ intersects the current loading surface. Now since p^{*}, the hydrostatic pressure is positive, $\frac{dU}{dv}$ must be less than or equal to zero and hence the internal energy -- specific volume curve has a shape as depicted in Figure 10.

With this assumption on the internal energy, it is of interest to apply a load to the system causing a differential change of strain $d\varepsilon_{ij}^{p}$ and consequently a change in specific volume v. As $\frac{dU}{dv}$ is anticipated to be negative, there will be, by Equation (17), energy released from storage if dv is positive and energy stored if dv is negative. During this process the consequences of requiring that the sum of the work done by the stress field and the energy released from storage be positive semi-definite may be expressed as

$$\sigma_{ij}\, d\varepsilon_{ij}^{p} - dU \geq 0 \tag{19}$$

or using Equation (18) as

$$\left(\sigma_{ij} - \frac{dU}{dv}\, v\delta_{ij}\right) d\varepsilon_{ij}^{p} \geq 0 \tag{20}$$

A schematic representation of Inequality (20) is shown as Figure 11. In this figure the reader should note that $p = p^*$ serves to govern the magnitude of the vector $- v \frac{dU}{dv} \delta_{ij}$ as $0 \leq v \frac{dU}{dv} \leq p^*$. This requirement is a result of a postulate concerning the way in which energy is stored and released in a soil and has no deeper meaning than this. Comparison of Equation (4) and Inequality (20) shows the equivalence of this requirement to that obtained by postulating that during isothermal flow the heat flow from the soil is positive -- semi-definite.

If Inequality (20) is written for $\sigma_{ij}^{①}$, where $\sigma_{ij}^{①}$ is a stress state on the loading surface, then for a differential change in plastic strain $d\varepsilon_{ij}^{p①}$ (recall $v > 0$)

$$\sigma_{ij}^{①}\, d\varepsilon_{ij}^{p①} - \frac{dU}{dv}\, v\, d\varepsilon_{ii}^{p①} \geq 0 \tag{21}$$

Now for $\sigma_{ij}^{(2)}$ a stress state which is a conjugate to $\sigma_{ij}^{(1)}$, Inequality (20) can be written for a change in plastic strain $d\varepsilon_{ij}^{p(2)}$ such that

$$d\varepsilon_{ii}^{p(2)} = - d\varepsilon_{ii}^{p(1)} \tag{22}$$

as

$$\sigma_{ij}^{(2)} d\varepsilon_{ij}^{p(2)} + \frac{dU}{dv} v \, d\varepsilon_{ii}^{p(1)} \geq 0 \tag{23}$$

Inequalities (21) and (23) imply

$$\sigma_{ij}^{(1)} d\varepsilon_{ij}^{p(1)} + \sigma_{ij}^{(2)} d\varepsilon_{ij}^{p(2)} \geq 0 \tag{24}$$

which is equivalent to Inequality (15). Consequently the restriction of Inequality (20) implies all previously derived restrictions.

As the direction for $d\vec{\varepsilon}^{p}$ is still not uniquely determined the following section introduces further limitations to be imposed on the soil behavior.

Stability Considerations

A direct postulate for stability of soil would be unreasonable as experiments indicate, that soil is, in the usual sense, unstable for certain kinds of loading. However, it seems hopeful to try to make some limitation on the material behavior corresponding to a stability postulate. As previous reasoning has been based on a storage of energy (change of internal energy) determined by the change of specific volume, it is appealing to introduce this concept into a "stability" postulate.

One approach which leads to a definite restriction is to postulate something about the energy balance when a set of loads is superimposed on an

initial set of loads. Presume that the initial loading induces the stress state σ_{ij}^*. Obviously this initial stress must lie in or on the loading surface corresponding to the initial state. Now the superimposed stress state is applied so that the total stress σ_{ij} is at some point on the yield surface. If a differential change of plastic strain, $d\varepsilon_{ij}^p$, is induced then it is postulated that the sum of the work done by the superimposed stresses, $\sigma_{ij} - \sigma_{ij}^*$, during the change and a fraction, α, of the energy released be greater than zero. When energy is stored due to a volume decrease, or when the volume is unaltered, it is only required that the work done by the arbitrary superimposed stresses be positive-semi-definite. This leads to

$$(\sigma_{ij} - \sigma_{ij}^*)d\varepsilon_{ij}^p + \left\langle -\alpha \frac{dU}{dv} v \, d\varepsilon_{ii}^p \right\rangle \geq 0 \tag{25}$$

where the notation

$$\begin{aligned} \langle x \rangle &= x \quad \text{if} \quad x > 0 \\ \langle x \rangle &= 0 \quad \text{if} \quad x \leq 0 \end{aligned} \tag{26}$$

is used and

$$0 \leq \alpha \leq 1 \tag{27}$$

As the point σ_{ij}^* may be any point on or inside the yield surface, $(\sigma_{ij} - \sigma_{ij}^*)$ may be interpreted as a vector whose tail is in or on the yield surface and whose tip is on the yield surface. Anticipating that $\frac{dU}{dv}$ is negative the stability requirement may be rewritten in the following form

$$(\sigma_{ij} - \sigma_{ij}^* - \delta_{ij} \, \alpha \frac{dU}{dv} v \, \frac{\langle d\varepsilon_{kk}^p \rangle}{d\varepsilon_{kk}^p})d\varepsilon_{ij}^p \geq 0 \tag{28}$$

which yields the graphical interpretation shown in Figure 12.

When $d\varepsilon_{ii}^{p} \geq 0$ the stability requirement Inequality (28) simply requires the projection of de_{ij}^{p} on $(\sigma_{ij_A} - \sigma_{ij}^{*} - \alpha \frac{dU}{dv} v\ \delta_{ij})$ to be positive.

When $d\varepsilon_{ii}^{p} < 0$ the above requirement reduces to $(\sigma_{ij_A} - \sigma_{ij}^{*})\ d\varepsilon_{ij}^{p} \geq 0$. In other words, referring to Figure 12, the boundary of the shaded region coincides with the loading surface. Consequently when $d\varepsilon_{ii}^{p} < 0$ the stability requirement leads to normality of $d\vec{\varepsilon}^{p}$ with the loading surface. The sense of $d\vec{\varepsilon}^{p}$ is outward.

Further when $d\varepsilon_{ii}^{p} < 0$ at point P on the limit surface then $(\sigma_{ij_P} - \sigma_{ij}^{*}) \cdot d\varepsilon_{ij}^{p} \geq 0$ implies, in geometrical terminology, that all points enclosed by the loading surface are on one side of the tangent plane to the loading surface P. This latter property will be referred to as convexity.

Now consider point A in Figure 12. Either $d\varepsilon_{ii} < 0$ or not. Assume that $d\varepsilon_{ii}^{p} < 0$, then outward normality must hold. According to Figure 12 if $d\vec{\varepsilon}^{p}$ were normal to the loading surface, it would have a projection on the $\sigma_{ij} = -p\delta_{ij}$ line in the direction of increasing tensile stress so that $d\varepsilon_{ii}^{p} \geq 0$ contrary to the assumption. Consequently at point A $d\varepsilon_{ii}^{p} \geq 0$. The admissible directions for $d\vec{\varepsilon}^{p}$ deduced from Inequality (28) are shown in Figure 13.

Point B in Figure 12 is inside the shaded area if $d\varepsilon_{ii}^{p} \geq 0$. Clearly no non-trivial $d\vec{\varepsilon}^{p}$ is possible such that Inequality (28) is satisfied if $d\varepsilon_{ii}^{p} \geq 0$ so that $d\varepsilon_{ii}^{p} \leq 0$. It had previously been assumed that plastic flow could be induced for every point on the loading surface. Consequently outward normality of $d\vec{\varepsilon}^{p}$ to such points as B must exist.

Finally consider point C. If $d\varepsilon_{ii}^{p} < 0$ is assumed the outward normality condition of $d\vec{\varepsilon}^{p}$ to the loading surface results whereas if $d\varepsilon_{ii}^{p} \geq 0$ is assumed the conditions on the direction of $d\vec{\varepsilon}^{p}$ deduced from Inequality (28) and shown in Figure 14 result. Consequently either $d\varepsilon_{ii}^{p} \geq 0$ or $d\varepsilon_{ii}^{p} < 0$ may occur at point C.

It is now of interest to determine under what conditions Inequality (28) implies satisfaction of Inequality (20). In the case that the volume increases the two inequalities are equivalent if

$$\sigma_{ij}^* = v \frac{\partial U}{\partial v} (1-\alpha)\delta_{ij} \tag{29}$$

If the volume decreases the two inequalities are equivalent if

$$\sigma_{ij}^* = v \frac{\partial U}{\partial v} \sigma_{ij} \tag{30}$$

In other words, satisfaction of Inequality (28) implies satisfaction of Inequality (20) if the two values for σ_{ij}^* given by Equations (29) and (30) are admissible (i.e. inside the loading surface). Since $0 \leq -v \frac{\partial U}{\partial v} \leq p^*$ the restrictions do not seem unreasonable and consequently will be assumed satisfied in the following. This means then that satisfaction of Inequality (28) automatically implies satisfaction of all previous requirements which have been presented, i.e., Inequalities (10), (15) and (20)

The above reasoning, thermodynamic and otherwise, imposes restrictions on the kind of incremental stress-strain relationship that can be used with a specified loading surface. In the case where $\alpha \frac{dU}{dv}$ is identically zero the conditions, similar to plasticity, show that normality between $d\vec{\varepsilon}^p$ and the corresponding tangent plane to the loading surface must be specified everywhere. This result is significant because stipulation of the loading surface then automatically gives the incremental stress-strain relation. In the present case where $\alpha \frac{dU}{dv}$ is less than zero the incremental stress-strain relation is determined for some points on the loading surface (point B types) and delimited for the remainder.

The result of the above restrictions then, unlike plasticity, is that further restraints must be introduced before a definite direction for $d\vec{\varepsilon}^p$ corresponding to each stress point on the loading surface can be established. In order to be of practical use in solving problems a definite choice of direction for $d\vec{\varepsilon}^p$ at each point on the loading surface must be established.

In attempting to introduce further postulates concerning soil behavior those postulates compatible with the previous set of assumptions should be considered first. That is, it would be appealing to insure that the previous set of postulates could be satisfied. The next section presents several possible restrictions which lead to definite choices for $d\vec{\varepsilon}^p$ which are compatible with the theory thus far developed.

Possible Restrictions Leading to a Definitive Incremental Stress-Strain Relationship

The three restrictions to be presented in this section which lead to definitive incremental stress-strain relationships are so chosen that the preceding theory is automatically satisfied. This is accomplished by only considering directions of $d\vec{\varepsilon}^p$ admissible from the preceding theory (hereafter called admissible directions) and then making a statement regarding them which leads to a unique choice of $d\vec{\varepsilon}^p$ at each point on the loading surface. It is convenient to define the length of a $d\vec{\varepsilon}^p$ vector to be

$$\text{length of } d\vec{\varepsilon}^p = |d\vec{\varepsilon}^p| = \sqrt{d\varepsilon_{ij}^p \, d\varepsilon_{ij}^p} \tag{31}$$

The first choice presented can be conveniently stated in the following proposition:

"Of all the admissible directions of $d\vec{\varepsilon}^p$ at a point on the loading surface the actual $d\vec{\varepsilon}^p$ maximizes the plastic work $\sigma_{ij}d\varepsilon_{ij}^p$, done by the external loading when compared to all other admissible $d\vec{\varepsilon}^p$ of equal length." Figure 15 shows schematically, the consequences of such a proposition ($d\vec{\varepsilon}^p$ vectors marked by 1). The points, B,C,A,A',A" correspond to points B,C,A,A,A of Figures 12, 13, and 14. This choice of $d\vec{\varepsilon}^p$ was found by determining which projection of $d\vec{\varepsilon}^p$ on the vector $\vec{\sigma}$ was a maximum and yet lay within the fans provided by the previous section.

The second alternative is expressed by the postulate

"Of all the admissible $d\vec{\varepsilon}^p$ at a point on the loading surface, the actual $d\vec{\varepsilon}^p$ maximizes the expression $\{\frac{\sigma_{ij}d\varepsilon_{ij}^p}{d\varepsilon_{kk}^p}\}$ when compared to all other admissible $d\vec{\varepsilon}^p$." The quantity maximized here is seen to be the ratio of the plastic work and the volume change. The $d\vec{\varepsilon}^p$ determined by this postulate are shown again in Figure 15 by the vectors marked 2.

The final postulate presented is expressed by the proposition:

"Of all the admissible vectors $d\vec{\varepsilon}^p$ at a point of the loading surface the actual $d\vec{\varepsilon}^p$ maximizes the dissipation $(\sigma_{ij} - \frac{dU}{dv} v\, \delta_{ij})d\varepsilon_{ij}^p$ when compared to all other admissible $d\vec{\varepsilon}^p$ of equal length."*

This result is shown again on Figure 15 by the vectors marked 3.

Which of these three postulates and possibly some others not presented is the correct one can only be answered by further analytical and experimental work. The validity of all the previous assumptions as to their applicability to problems of granular media can again only be answered by further effort.

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* This formulation was pointed out to the authors by Professor D. C. Drucker in a private communication.

Discussion:

In order to improve upon the currently used plastic deformation theories of granular media, a theory is presented here which attempts to overcome some of the known deficiencies in previous theories. Clearly the development of a more sophisticated theory almost always brings mathematical difficulites so that a balance must be maintained between sophistication and tractability. The authors feel that adoption of an appropriate specific loading surface function in the framework of the above theory will lead to the ability to solve several technically important problems [14], [15].

Several assumptions made in the theory should be verified experimentally by comparison of predicted and measured results for technically important problems. Direct verification of individual assumptions, such as the stability postulate used here, is not usually possible. Often confidence in such postulates can be strengthened by finding relatively simple mechanical models for the material which satisfy the postulate. Although no such model has yet been conceived, except for $\alpha = 0$, the authors feel that the postulate may nevertheless have meaning when $0 < \alpha \leq 1$.

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FIGURE CAPTIONS

Figure 1 - Division Between Attainable and Unattainable Stress States in a Granular Media

Figure 2 - Two-dimensional Linearized Form of the Mohr-Coulomb Yield Criterion

Figure 3 - Observed Discrepancy in Direction of $d\vec{\epsilon}^p$ on Limit Surface

Figure 4 - Schematic Diagram Illustrating Concept of a Loading Cycle in Stress-Strain Space

Figure 5 - Schematic Diagram Showing How $\oint \sigma_{ij} d\epsilon_{ij}^{\ p} \geq 0$ Restricts the Directions of $d\vec{\epsilon}^p$ in Terms of the Stress $\sigma_{ij}^{①}$ and One of its Conjugate $\sigma_{ij}^{②}$

Figure 6 - Case of Two Conjugate Stress States

Figure 7 - Schematic Diagram Showing Loading Cycle in Stress-Strain Space for Material Whose State is Given by σ_{ij}, v

Figure 8 - Typical Specific Volume Versus Hydrostatic Pressure Curve for Granular Media

Figure 9 - Bounds on $\frac{dU}{dv}$ v Provided by Hydrostatic Pressure -- Specific Volume Curve

Figure 10 - Dependence of Internal Energy on Specific Volume if $\frac{dU}{dv} \leq 0$

Figure 11 - Admissible $d\vec{\epsilon}^p$ Which Satisfy Inequality (21)

Figure 12 - A Schematic Way of Determining Admissible $(\sigma_{ij_A} - \sigma_{ij}^* - \delta_{ij}\ \alpha\ \frac{dU}{dv}v$ Vectors. Any Vector Originating in the Shaded Region is Admissible. (For $\frac{dU}{dv} < 0$)

Figure 13 - Determination of Admissible Directions for $d\vec{\varepsilon}^p$ at Point A (for $\frac{dU}{dv} < 0$)

Figure 14 - Determination of Admissible Directions for $d\vec{\varepsilon}^p$ at Point C. In this case $d\varepsilon_{ii}^p \leq 0$ Leads to Outward Normality. Either $d\varepsilon_{ij}^p \leq 0$ or $d\varepsilon_{ij}^p > 0$ May Satisfy the Condition Imposed by Inequality (29) (for $\frac{dU}{dv} < 0$)

Figure 15 - Three Possible Restrictions Leading to Definite Incremental Stress-Strain Relationships. (1) Maximum Work, (2) Maximum of Work Divided by Volume Change, (3) Maximum Dissipation (For $\frac{dU}{dv} < 0$)

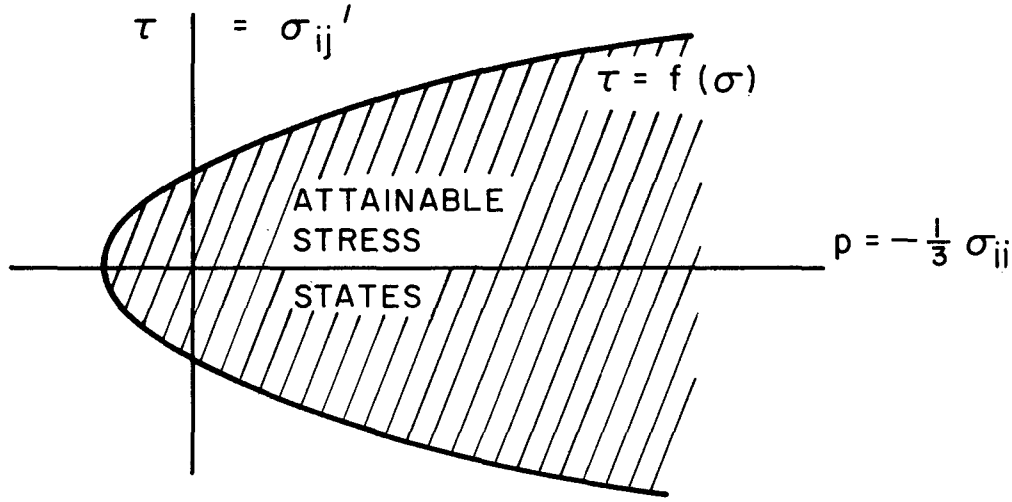

FIG. 1 DIVISION BETWEEN ATTAINABLE AND UNATTAINABLE STRESS STATES IN A GRANULAR MEDIA

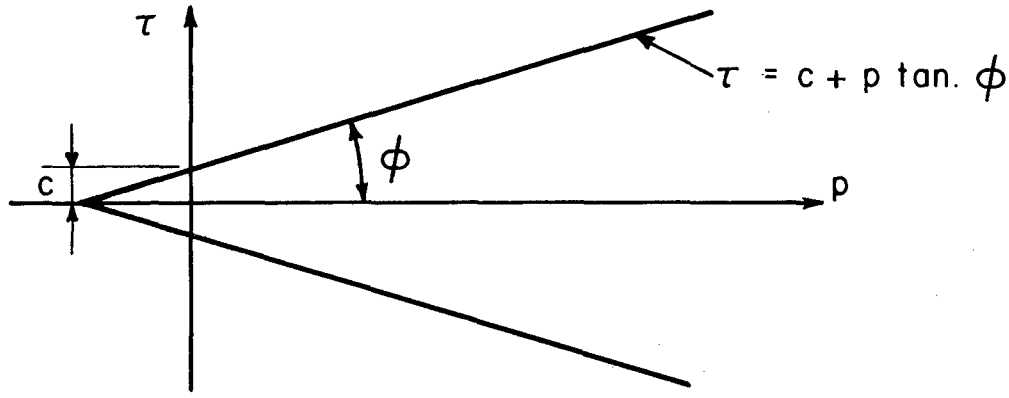

FIG. 2 TWO-DIMENSIONAL LINEARIZED FORM OF THE MOHR-COULOMB YIELD CRITERION

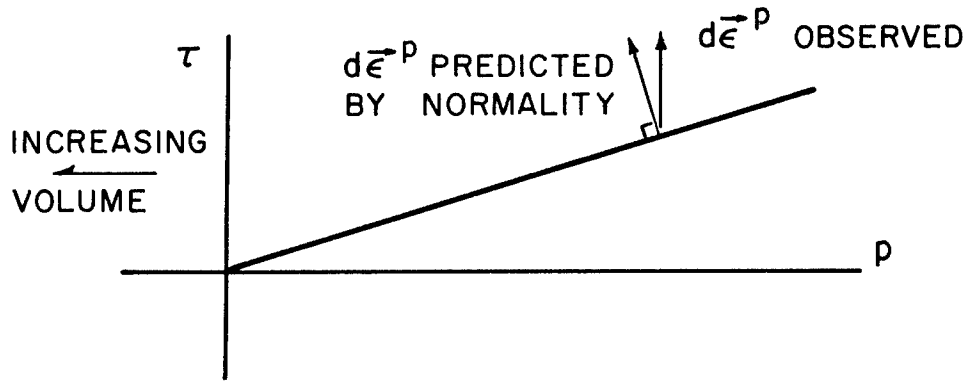

FIG. 3 OBSERVED DISCREPANCY IN DIRECTION OF $d\vec{\epsilon}^{\,p}$ ON LIMIT SURFACE

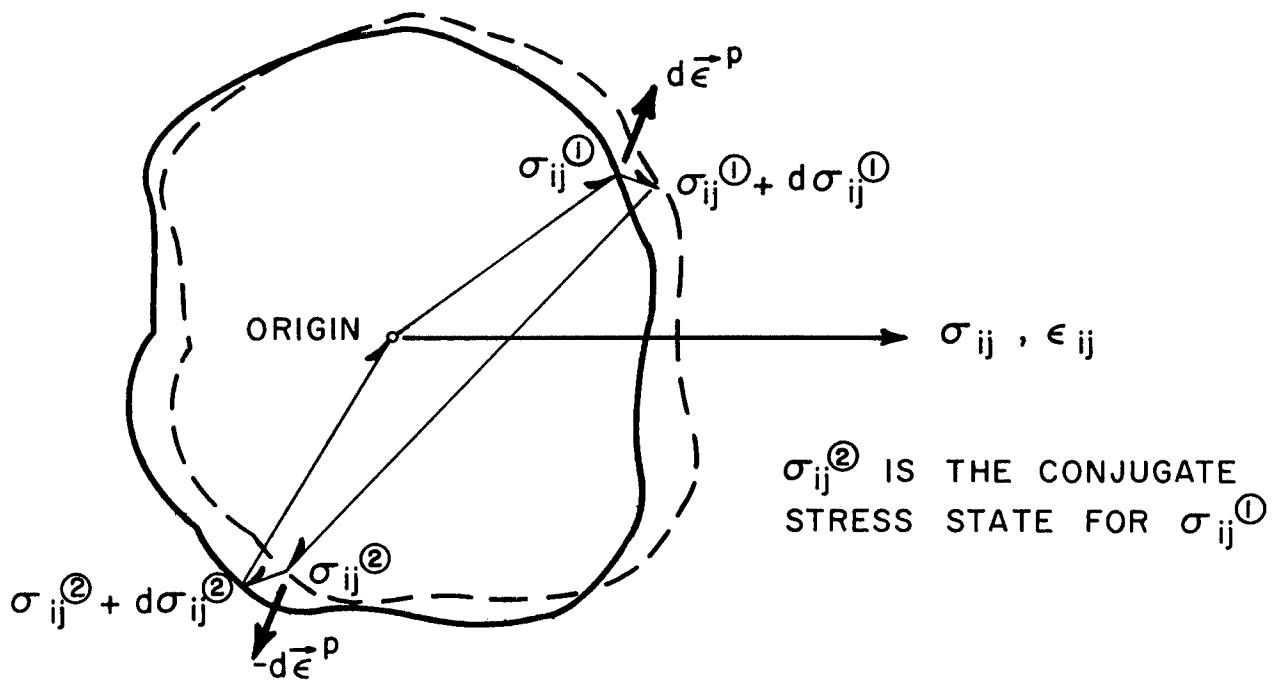

FIG. 4 SCHEMATIC DIAGRAM ILLUSTRATING CONCEPT OF A LOADING CYCLE IN STRESS-STRAIN SPACE

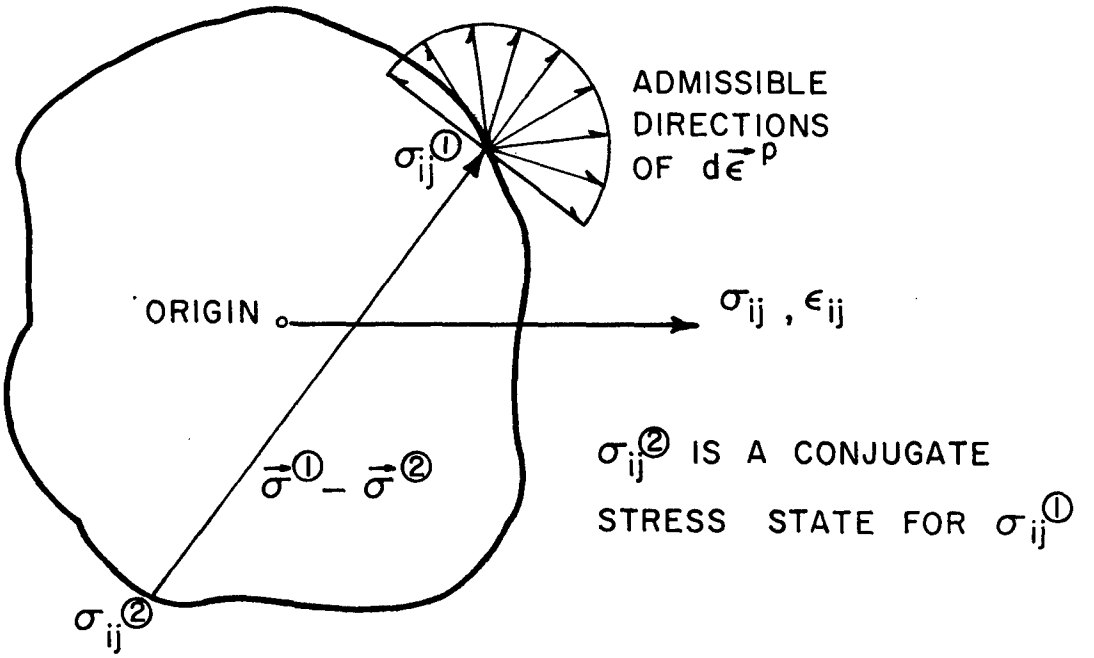

FIG. 5 SCHEMATIC DIAGRAM SHOWING HOW $\oint \sigma_{ij}\, d\epsilon_{ij}^{\,p} \geq 0$ RESTRICTS THE DIRECTIONS OF $d\vec{\epsilon}^{\,p}$ IN TERMS OF THE STRESS $\sigma_{ij}^{①}$ AND ONE OF ITS CONJUGATES $\sigma_{ij}^{②}$

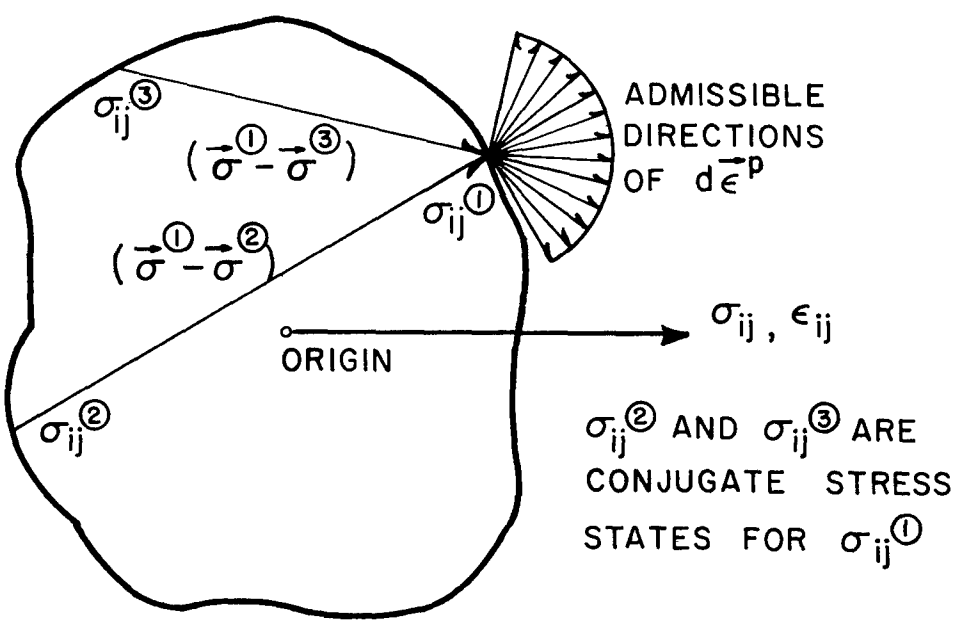

FIG. 6 CASE OF TWO CONJUGATE STRESS STATES

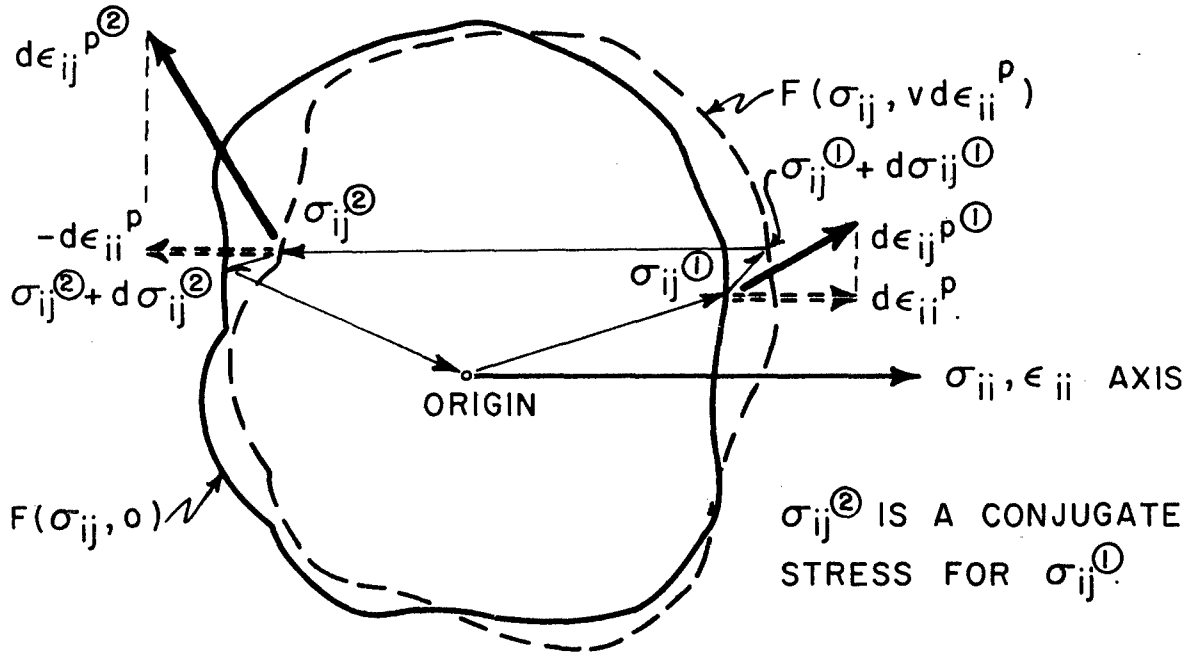

FIG. 7 SCHEMATIC DIAGRAM SHOWING LOADING CYCLE IN STRESS–STRAIN SPACE FOR MATERIAL WHOSE STATE IS GIVEN BY σ_{ij}, v

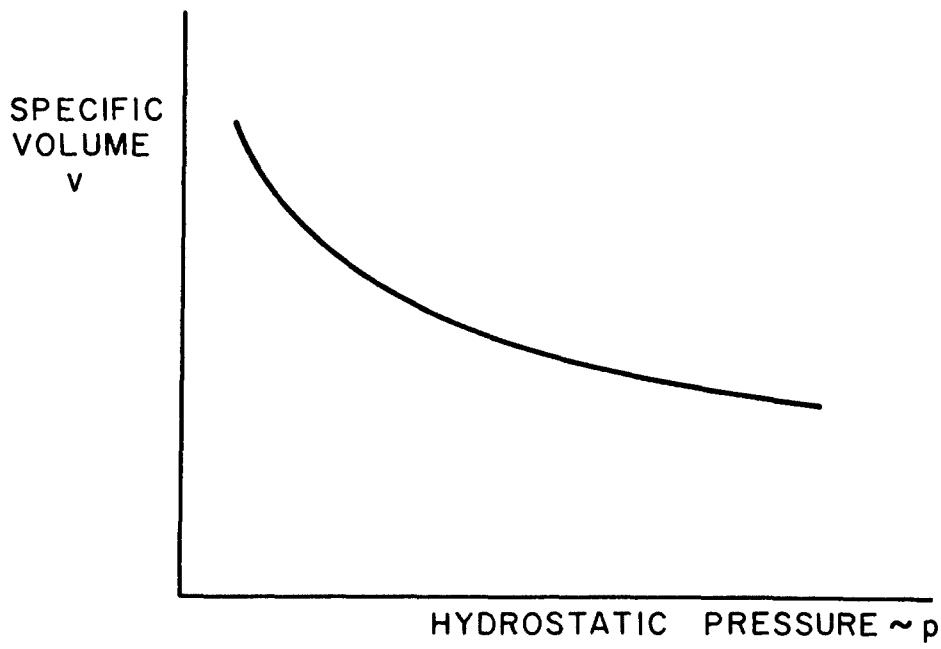

FIG. 8 TYPICAL SPECIFIC VOLUME VERSUS HYDROSTATIC PRESSURE CURVE FOR GRANULAR MEDIA

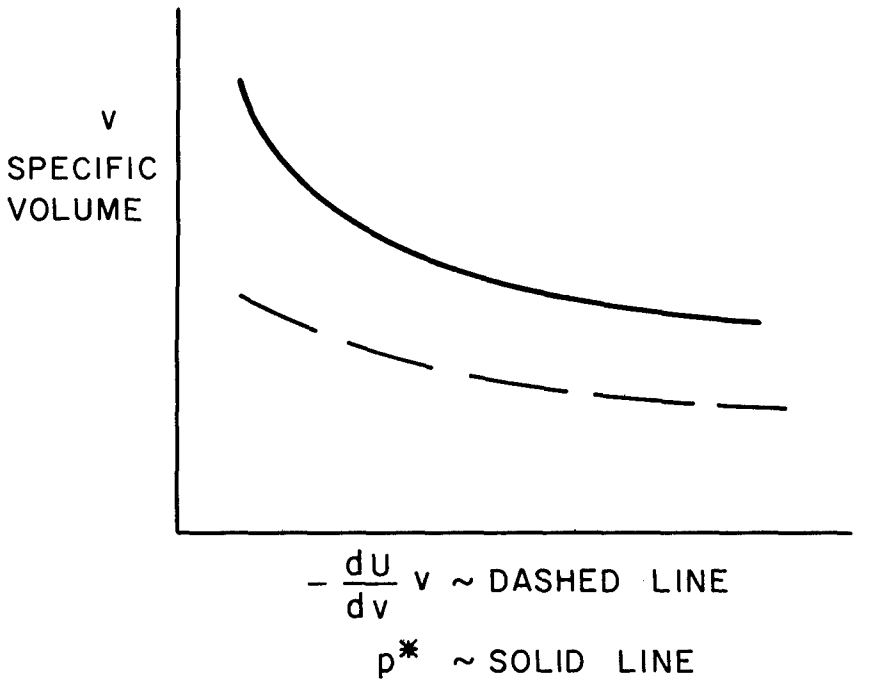

FIG. 9 BOUNDS ON $\frac{dU}{dv}v$ PROVIDED BY HYDROSTATIC PRESSURE -- SPECIFIC VOLUME CURVE

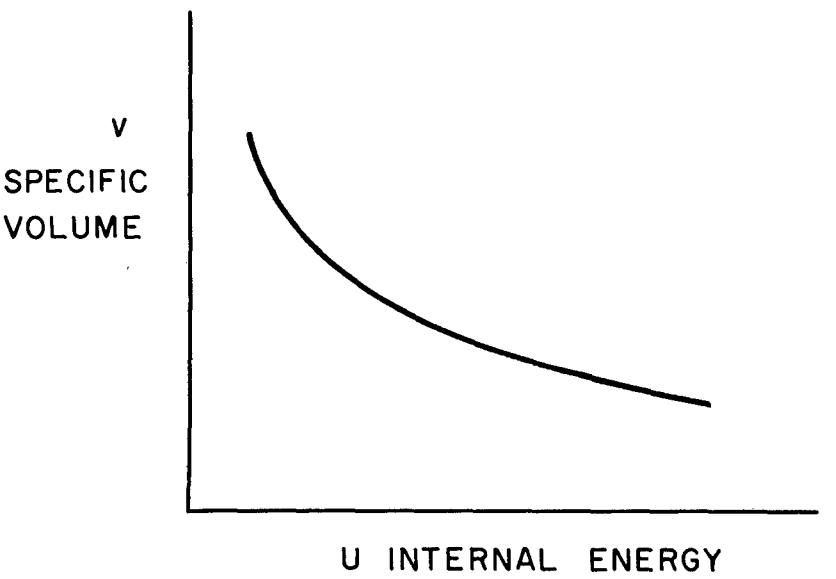

FIG. 10 DEPENDENCE OF INTERNAL ENERGY ON SPECIFIC VOLUME IF $\frac{dU}{dv} \leq 0$

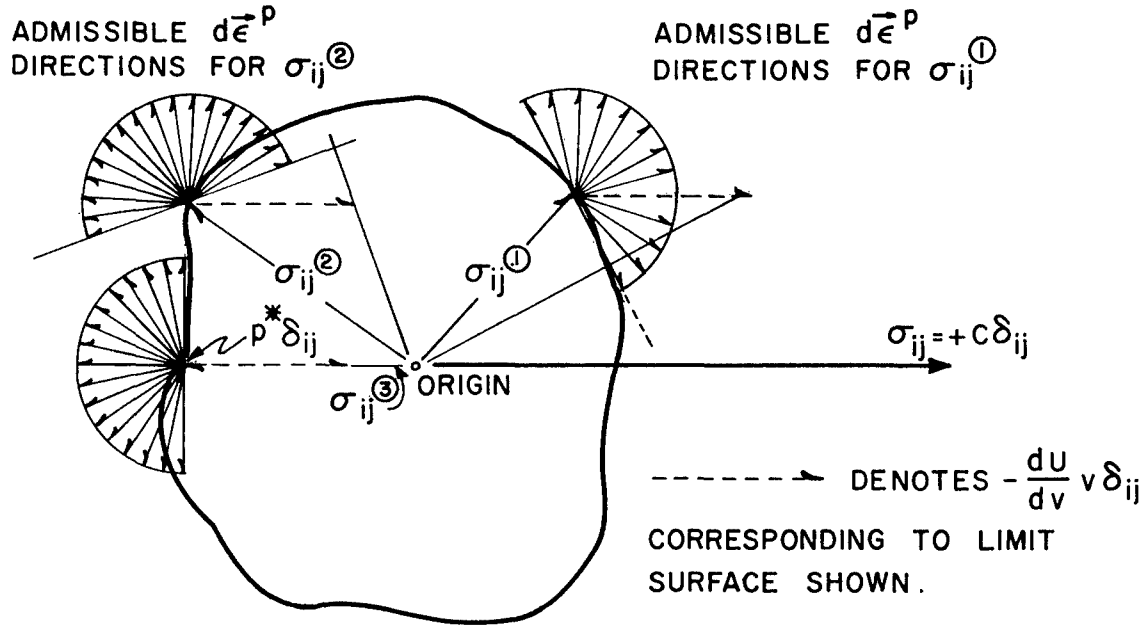

FIG. 11 ADMISSIBLE $d\vec{\epsilon}^p$ WHICH SATISFY INEQUALITY (21)

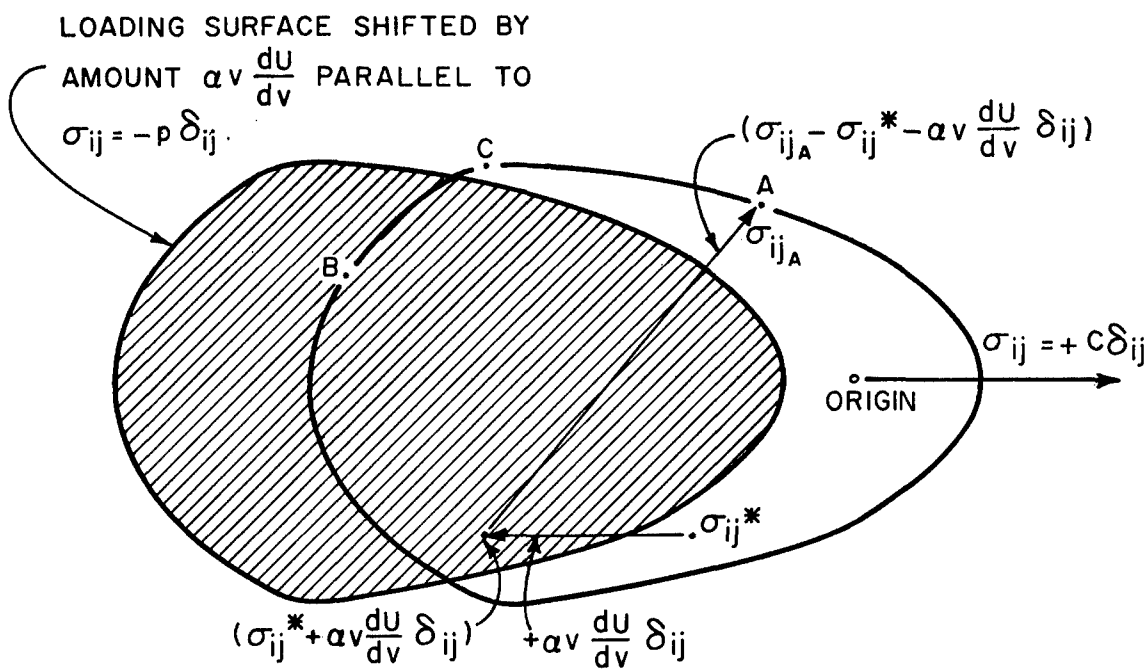

FIG. 12 A SCHEMATIC WAY OF DETERMINING ADMISSIBLE $(\sigma_{ij_A} - \sigma_{ij}^* - \alpha v \frac{dU}{dv} \delta_{ij})$ VECTORS. ANY VECTOR ORIGINATING IN THE SHADED REGION IS ADMISSIBLE. (FOR $\frac{dU}{dv} < 0$)

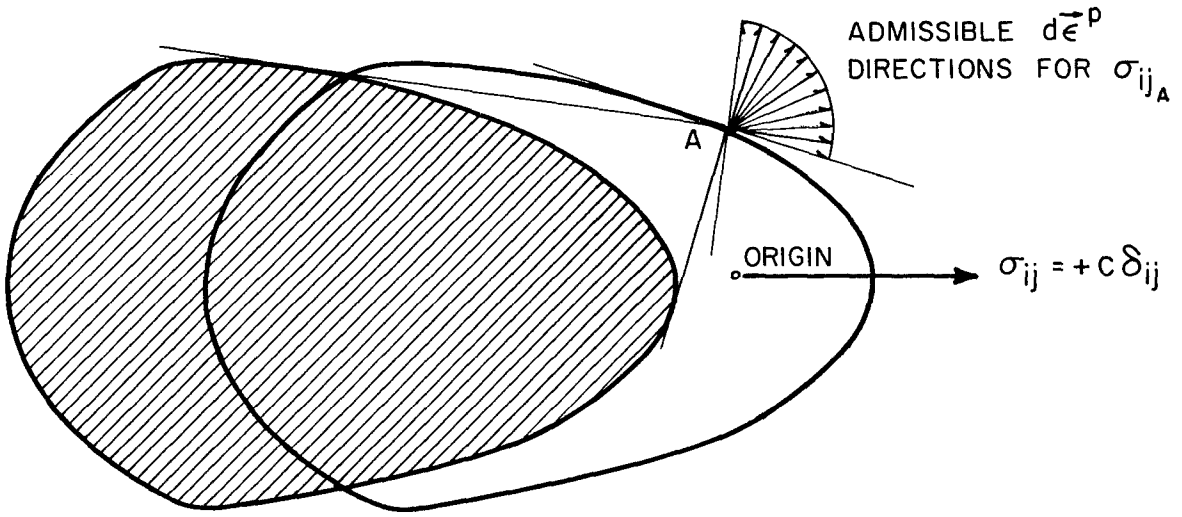

FIG. 13 DETERMINATION OF ADMISSIBLE DIRECTIONS FOR $d\vec{\epsilon}^{\,p}$ AT POINT A. (FOR $\frac{dU}{dv} < 0$)

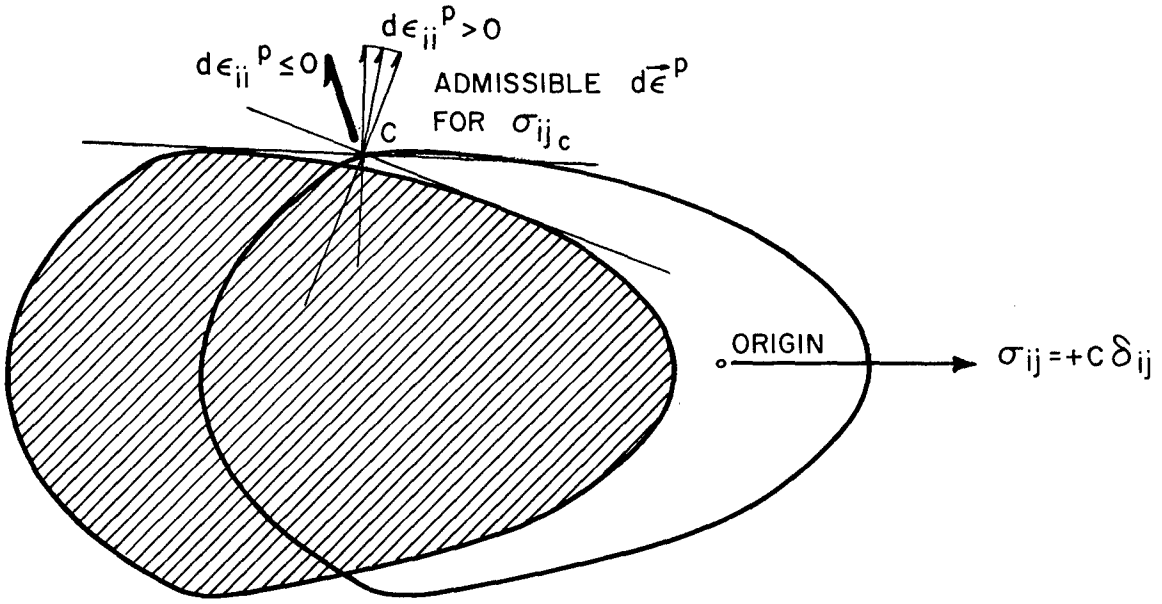

FIG. 14 DETERMINATION OF ADMISSIBLE DIRECTIONS FOR $d\vec{\epsilon}^{\,p}$ AT POINT C. IN THIS CASE $d\epsilon_{ii}^{\,p} \leq 0$ LEADS TO OUTWARD NORMALITY. EITHER $d\epsilon_{ii}^{\,p} \leq 0$ OR $d\epsilon_{ii}^{\,p} > 0$ MAY SATISFY THE CONDITION IMPOSED BY INEQUALITY (28) (FOR $\frac{dU}{dv} < 0$)

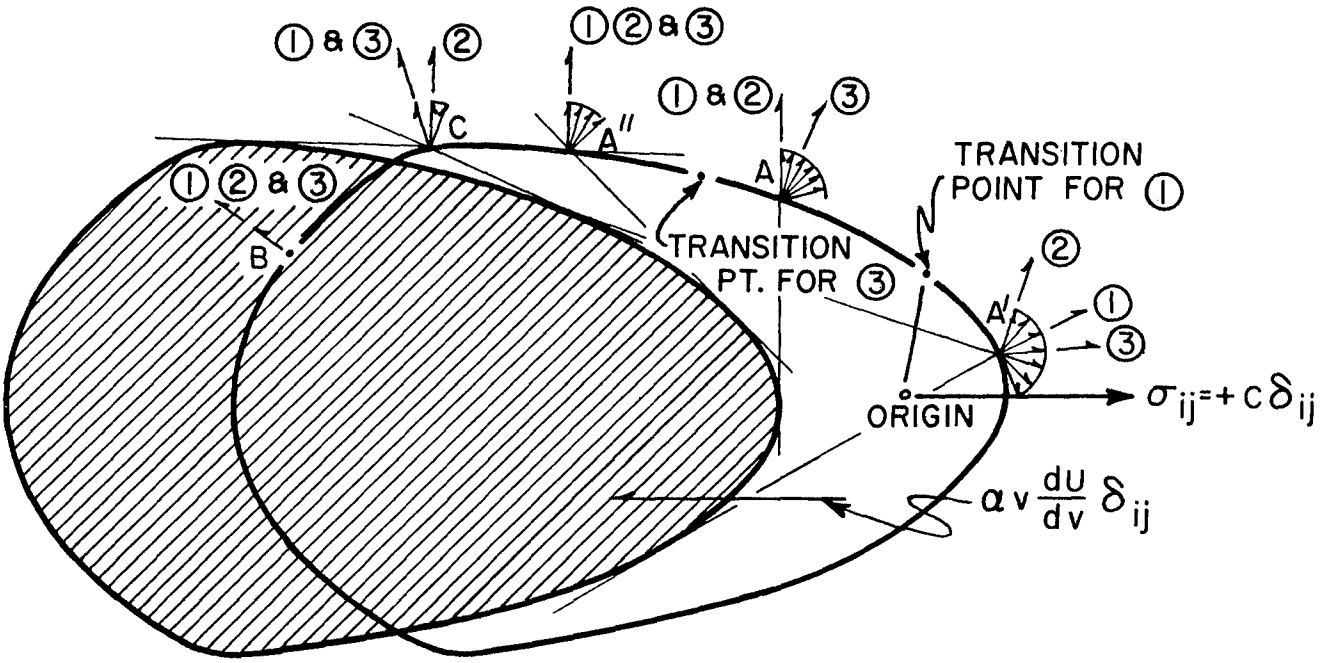

FIG. 15 THREE POSSIBLE RESTRICTIONS LEADING TO DEFINITE INCREMENTAL STRESS-STRAIN RELATIONSHIPS. (1) MAXIMUM WORK, (2) MAXIMUM OF WORK DIVIDED BY VOLUME CHANGE, (3) MAXIMUM DISSIPATION (FOR $\frac{dU}{dv} < 0$)