Best Available

Copy

for all Pictures

AD646656

AD

USAAVLABS TECHNICAL REPORT 66-69

A CONTRIBUTION TO THE THEORY OF JET-WAKES AND VORTICES IN FREE AND CONFINED SURROUNDINGS

By

George Basil Fragoyannis

November 1966

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

CONTRACT DA 44-177-AMC-264(T)

MISSISSIPPI STATE UNIVERSITY
STATE COLLEGE, MISSISSIPPI

Distribution of this document is unlimited

FEB 14 1967

ARCHIVE COPY

Disclaimers

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission, to manufacture, use, or sell any patented invention that may in any way be related thereto.

Disposition Instructions

Destroy this report when no longer needed. Do not return it to originator.

ACCESSION for
CFSTI
DDC
UNANNOUNCED
JUSTIFICATION
BY
DISTRIBUTION/AVAILABILITY CODES
DIST. | AVAIL. and/or SPECIAL
1

DEPARTMENT OF THE ARMY
U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA 23604

This report has been reviewed by this command and is considered to be technically sound. It is published for the exchange of information and the stimulation of ideas.

Task 1P125901A14203
Contract DA 44-177-AMC-264(T)
USAAVLABS Technical Report 66-69
November 1966

A CONTRIBUTION TO THE THEORY OF JET-WAKES AND VORTICES IN FREE AND CONFINED SURROUNDINGS

Aerophysics Research Report No. 67

by

George Basil Fragoyannis

Prepared by

The Aerophysics Department
Mississippi State University
State College, Mississippi

for

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

Distribution of this
document is unlimited

ABSTRACT

The primary purpose of this investigation is the study of the flow field of steady, viscous, incompressible jet-wakes and vortices submerged in free and confined surroundings.

Starting with an extensive account of the present ideas and development on the subject flows, the general principles of the motion of a free laminar jet ~~have been~~ was analyzed and reviewed. From these principles an understanding of the very complex behavior of a turbulent jet ~~has been~~ obtained.

Using a linearized form of the Navier-Stokes equations of motion, a set of solutions capable of describing the laminar axial, rotational, and radial velocity profiles of a jet was deduced.

Next, it was indicated that by introducing a "viscosity function" to the simplified equations of laminar motion, solutions of the same general analytic form, but with a different nondimensional variable, can be obtained to describe the turbulent motion.

For the confined jets and vortices, two different approaches have been used to simplify and solve the equations of motion. The set of solutions obtained is of the same nondimensional form. Using the elementary solutions of this set, analytic expressions for the different characteristics involved in the design of a jet ejector were calculated. Also, a process for the estimation of the different parameters for the optimum design of a jet pump or a thrust augmenter was indicated.

The comparison of the calculated general analytic expressions of this study to existing experimental results indicates that the assumptions and processes used to predict the different velocity profiles of free motions are reasonable.

On the basis of the presented theoretical analysis and available experimental results, it was concluded that the laws describing jet-wakes and vortices for laminar and turbulent motion are of the same general nature.

CONTENTS

	Page
ABSTRACT	iii
LIST OF ILLUSTRATIONS	vi
LIST OF SYMBOLS	viii
CHAPTER ONE. INTRODUCTION	1
CHAPTER TWO. LAMINAR, VISCOUS, AXIALLY SYMMETRIC JET-WAKES AND VORTICES	4
General Remarks	4
Axial Velocity	8
Rotational Velocity	14
The Form of the Solutions	22
CHAPTER THREE. TURBULENT, VISCOUS, AXIALLY SYMMETRIC JET-WAKES AND VORTICES	25
General Remarks	25
Axial Velocity	25
Rotational Velocity	34
CHAPTER FOUR. CONFINED JETS AND VORTICES IN AXIALLY SYMMETRIC SURROUNDINGS	39
General Remarks	39
Jet Ejectors	39
Confined Vortices	50
CHAPTER FIVE. COMPARISON OF PRESENT THEORY WITH EXISTING THEORIES AND EXPERIMENTAL RESULTS	55
CHAPTER SIX. SUMMARY AND CONCLUDING REMARKS	61
REFERENCES	93
DISTRIBUTION	98
APPENDIXES	
I. Two-Dimensional Wakes and Jets	100
II. Axially Symmetric Jet Under Pressure Gradient	106

ILLUSTRATIONS

Figure		Page
1	Jet and Wake Axial Velocities	63
2	Jet Ejector, Nomenclature and Flow Regimes	64
3	Tip Vortex in Ground Proximity	65
4	Wake of a Cylinder	66
5	Wake of a Half-Sphere	66
6	Cornish's Modified Coles' Wake Law and Present Theory	67
7	Faris' and Coles' Wake Law and Modified Present Theory	68
8	Faris' Experimental Value and Present Theory	69
9	Comparison of Görtler, Tollmien, and Present Theories	70
10	Comparison of Different Theoretical Velocity Profiles to Reichardt's Experiments	71
11	Reichardt's Experimental Results for Axisymmetric Jet and Present Theory	72
12	Reichardt's Experimental Results for Two-Dimensional Jets	73
13	Comparison of Excess Velocity Experimental Profiles and Present Theory	74
14	Experimental Two-Dimensional Excess Velocity Profile and Present Theory	75
15	Velocity Distribution in a Two-Dimensional Turbulent Jet Due to Forthmann	76
16	Present Theory Excess Axisymmetric Velocity Profile and Experimental Results of References 58 and 66	77
17	Velocity Distribution in a Two-Dimensional Wake Behind Circular Cylinders, Present Theory-Measurement Due to Schlichting	78

Figure		Page
18	Comparison of Axisymmetric Wake Experimental Results of Hall-Hislop to Present Theory ----------	79
19	Theoretical and Experimental Velocity Profiles in the Inlet Length of an Ejector ----------	80
20	Theoretical and Experimental Velocity Profiles in the Inlet Length of an Ejector ----------	81
21	Comparison of Free-Jet Theoretical Profiles to Experimental Confined Profile ----------	82
22	Comparison of Theoretical and Experimental Center Line Velocity Distribution in an Ejector Mixing Chamber ----------	83
23	Comparison of Theoretical and Experimental Center Line Velocity Distribution for a Free Jet ---------	84
24	Comparison of Theoretical and Experimental Center Line Velocity Distribution ----------	85
25	Comparison of Theoretical and Experimental Rotational Velocity in a Wing-Tip Vortex ----------	86
26	Confined Vortex Experimental and Theoretical Rotational Velocity Profiles ----------	87
27	Comparison of Theoretical and Experimental Rotational Velocity Distribution for Confined Vortex ----------	88
28	Axial Velocity Characteristic Form ----------	89
29	Rotational Velocity Characteristic Form ----------	90
30	Modified Characteristic Axial Velocity Form -------	91
31	Modified Characteristic Rotational Velocities -----	92

SYMBOLS

A	constant used to describe axial velocity profile
B	constant used to describe rotational velocity profile
C	constant of integration
F	nondimensional velocity function component
I	modified Bessel function of first kind
J	Bessel function of first kind
M	axial momentum
P	local static pressure
R	rotational velocity momentum
$R(z)$	ejector radius function
R	constant ejector radius
S	nondimensional variable in Laplacian coordinates system
U	axial velocity
U_o	free-stream velocity
U_{MAX}	maximum axial velocity
U_s	secondary entrained flow velocity
U_p	primary ejector velocity
u	axial defect velocity
u_1, u_2	modified form of axial velocity
u_m	mean axial velocity
$\bar{u}$	turbulent axial velocity component
U	radial velocity
$\bar{U}$	turbulent radial velocity component
ω	rotational or vortical velocity

W_1, W_2 modified forms of rotational velocity

x nondimensional variable

a, c constants

b characteristic thickness

f nondimensional component of axial velocity function

g nondimensional component of Reynolds stress

k, n constant of integer or noninteger order

ℓ characteristic thickness

m mass flow

r distance measured in radial direction

z distance measured along the axis of symmetry

$\alpha, \beta, \gamma, \varepsilon, \epsilon$ constants

θ nondimensional variable

λ separation constant

ν kinematic viscosity

$\nu(z, r)$ viscosity function

ξ nondimensional variable

ρ fluid density

σ dimensional constant

τ shearing stress

$\Lambda(z)$ momentum transfer length

Φ nondimensional axial velocity component

$\bar{\Phi}$ velocity component in Laplacian coordinates system

φ thrust augmentation factor

ψ velocity function in the ejector chamber

x nondimensional parameters describing the motion in an ejector chamber

x_K nondimensional parameters describing the motion in an ejector chamber

<u>Subscripts</u>

i denotes initial conditions

a denotes axial conditions

f denotes final conditions

$0, 1, .. n$ as indicated

CHAPTER ONE. INTRODUCTION

The mass or flow entrainment process is one of the predominate phenomena in the flow motion of jets, wakes, vortices and boundary layers. It is also the basis of many important practical applications, especially when it takes place in confined surroundings, where in many cases the entrained flow is noticeably increased.

In the past, a large number of theoretical analyses have been developed on some parts of the general problem concerning the motion of laminar or turbulent jet-wakes and vortices submerged in either free or confined surroundings. The aim of almost every one of these analyses was to use some form of the Navier-Stokes equations of motion to deduce analytic expressions capable of predicting the particular motion under consideration. In this respect - as indicated in the following chapters - the agreement obtained between the developed theories and the experimental results is generally satisfactory, although in some cases (and for certain regions of the velocity field) the agreement is poor.

But since the theoretical analyses of the past have dealt only with some particular type of motion, the presently existing knowledge of jet-wakes and vortices does not readily allow a comparison or evaluation of a general nature to be drawn relative to the mechanism of the entrainment process. Obviously, this situation exists primarily because of the wide range of the possible motions and the many, and often different, assumptions made in the development of the above-mentioned theoretical analyses. This is especially true for the case of turbulent flow, which lacks a universally applicable "turbulence theory" that could be valid for all these motions. As a result, the turbulent case presents one of the more complicated problems in modern incompressible, viscous fluid dynamics.

A considerable effort in the direction of a more precise understanding of the mixing process involved in all the previously mentioned types of flow was developed in the past by the Aerophysics Department of Mississippi State University. In a series of published papers by Cornish (reference 53) and Faris (reference 51), it was indicated that the semiempirical two-dimensional Coles' wake function - with some modifications - could describe the velocity profile of a free and confined vortex. However, the universal application of this function has not been indicated, and it has not been demonstrated with sufficient theoretical reasoning that this function may have a more universal applicability.

For the above reasons and in view of the present need for more comprehensive and applicable techniques in predicting the flow field components, the present report is devoted to the study and development of generalized analytic expressions capable of describing the types of motion under consideration.

To be more specific, the primary purpose of this investigation is the study of the flow field of steady, viscous incompressible jets submerged in free and confined surroundings (jet-ejectors). Also, the similarity and universal nature of the laws describing jet-wakes and vortices have been investigated; and on the basis of the results obtained, some basic relations between the different jet-ejector parameters have been established.

In the first phase of the present research, the general principles of the motion of free laminar jet-wakes and vortices have been analyzed and reviewed. Furthermore, for a simplified form of the Navier-Stokes equations capable of uniquely describing these types of motions, a set of solutions has been obtained which can be used to predict the axial, rotational and radial velocity profiles.

As indicated in the past, in many cases it is possible to extend laminar solutions to describe turbulent cases. In order to verify this possibility, the Navier-Stokes equations of motion have been reconsidered using the existing "statistical" and "free turbulent" theories.

Studies have indicated that by introducing some specific function, namely, the "viscosity function", to the simplified equations of laminar motion, solutions of the same general analytic form - but with a different nondimensional variable - can be obtained. Comparison with existing experimental results indicates that, under this assumption, one may predict the velocity profiles as accurately as when any of the other more elaborate, previously developed theories are used.

In the second phase of the present research, after a critical review of existing jet-ejector theories and experimental results, and on the basis of the experience obtained from the study of free jets and vortices, the general equations of motion and the boundary and physical conditions of axially symmetric confined jets and vortices have been established.

Two different approaches have been indicated which may be used to simplify and solve the differential equations of motion, one being applicable very close to the initial plane, and the other being valid at some distance from the initial plane. The solutions obtained are of the same general nondimensional form.

Using an assumed nondimensional velocity profile, analytic expressions for the different characteristics involved in a jet-ejector design have been calculated.

Finally, in the first of two appendixes included at the end of this study, it has been verified that the assumptions and processes used for axially symmetric motion may be used successfully to describe the corresponding two-dimensional problems. In the second appendix

some techniques have been indicated that can possibly be used to predict the components of a confined jet in an arbitrary pressure gradient.

CHAPTER TWO
LAMINAR, VISCOUS, AXIALLY SYMMETRIC JET-WAKES AND VORTICES

GENERAL REMARKS

Previous Developments

It is well known that the general equations of motion of an axially symmetric jet-wake or vortex are not amenable to a complete mathematical solution. In the past, approximations have been used to simplify these equations to an integrable form. In this respect, first reference is made to the work of Schlichting (references 1 and 2), who presented a solution for the case of a laminar, viscous, isobaric jet submerged in a stationary surrounding far from the origin. Schlichting, assuming a general form for the stream function, used some of the physical conditions of the problem to simplify the equations to an integrable form. Due to the restrictions imposed and the processes followed, the analytic expression obtained for the axial velocity profile is of single-valued form and can be used to describe with sufficient accuracy the core region of the jet.

Görtler (reference 3) extended Schlichting's analysis to include rotational motion. In this study of the decay of swirl in an axially symmetric jet far from the orifice, Görtler utilized the method of separation of variables in the rotational momentum equation. The derived infinite number of solutions were restricted by using some of the physical conditions of the problem, but the obtained solutions cannot describe vortical flow that behaves like a logarithmic vortex for large values of γ .

Newman (reference 7), using a small perturbation analysis, has simplified the general equations of motion to describe the flow in a viscous trailing vortex. The single-valued form of the profile used to describe rotational velocity does not satisfy some of the physical conditions of the problem, since it becomes an unbounded expression for the rotational mass flow. In the following review of the problem, the general equations of motion will be simplified to yield an integrable form, and then, by assuming a form for the velocity profile, a set of solutions capable of predicting all of the types of motion of current interest will be deduced.

Equations of Motion

For the case of steady, axially symmetric flow, the general Navier-Stokes equations of motion (1) may be simplified by use of the usual boundary layer approximations of Prandtl which yield the following equations in cylindrical coordinates:

$$u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right). \tag{1}$$

$$u\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial r} - \frac{w^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right). \tag{2}$$

$$u\frac{\partial w}{\partial z} + v\frac{\partial w}{\partial r} + \frac{wv}{r} = \nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rw)\right). \tag{3}$$

The continuity equation becomes

$$\frac{\partial(ru)}{\partial z} + \frac{\partial(rv)}{\partial r} = 0, \tag{4}$$

where z denotes distance measured along the axis of symmetry and r the distance from that axis.

The above simplified form of the general equations of motion may represent the velocity and pressure field of a jet-wake or vortex emerging into either stationary or moving external surroundings under the proper boundary conditions associated with the particular motion.

Equation (1) is the simplified form of Schlichting (reference 2), which represents the case of a laminar, circular jet mixing with the surrounding isobaric fluid at rest (P = constant). Equations (2) and (3) are taken from Görtler's study of the decay of swirl in an axially symmetric jet (reference 3).

A brief discussion concerning these solutions may be found in References 1 and 4. The solutions obtained are valid for the case of the mixing of an isobaric jet with the surrounding fluid at rest at some distance downstream from the initial plane.

For the case of a nonstationary surrounding fluid, the axial velocity u may be replaced by the algebraic sum of the free-stream velocity u_0 and the defect velocity u (see Figure 1).

$$u = u_0 + \alpha u, \tag{5}$$

where α = 1 for the case of a jet and α = -1 for the case of a vortex or wake.

Substituting equation (5) into equations (1), (2), (3), and (4) gives

$$\alpha u_0 \frac{\partial u}{\partial z} + \alpha^2 u \frac{\partial u}{\partial z} + \alpha v \frac{\partial u}{\partial r} = \frac{-1}{\rho}\frac{\partial P}{\partial z} + \frac{\alpha \nu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right), \tag{1-a}$$

$$(u_0 + \alpha u)\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial r} - \frac{\omega^2}{r} = \frac{-1}{\rho}\frac{\partial P}{\partial r} + \nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v)\right), \tag{2-a}$$

$$(u_0 + \alpha u)\frac{\partial \omega}{\partial z} + v\frac{\partial \omega}{\partial r} + \frac{\omega v}{r} = \nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r \omega)\right), \tag{3-a}$$

and

$$\alpha \frac{\partial (r u)}{\partial z} + \partial \frac{(r v)}{\partial r} = 0. \tag{4-a}$$

An exact solution of the above system of partial differential equations can only be obtained some distance downstream from the initial plane where the equations of motion may be linearized with satisfactory accuracy. The linearization process can be performed either in the initial plane or after some suitable transformation, since the accuracy of the solution of the equations of motion - equations (1), (2), and (3) - depends upon the general assumptions and the processes used to obtain the final result and not upon when the initial assumption of linearity is performed.

If a small perturbation analysis, similar to the one used by Schlichting (reference 5) or Pai (reference 6) for the equation of axial velocity and the analysis of Newman (reference 7) for the equation of rotational velocity, is used, it may be assumed that the defect velocity is smaller than that of the free-stream, and that the radial velocity and variation of pressure along the z axis is very small. Thus,

$$u_0 \frac{\partial u}{\partial z} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right), \tag{1-b}$$

$$\frac{\omega^2}{r} = \frac{-1}{\rho}\frac{\partial P}{\partial r}, \tag{2-b}$$

and

$$u_0 \frac{\partial \omega}{\partial z} = \nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r \omega)\right). \tag{3-b}$$

As indicated in the following chapters, the partial differential equations (1-b) and (3-b), under a suitable transformation, may be transformed to ordinary differential equations of the same general mathematical form.

The Form of the Possible Solutions

The mathematical form of the preceding partial differential equations is very similar to those describing different physical phenomena, e.g., equation (1-b) is identical to the one of heat transfer describing the radial heat flow of a cylinder at variable temperature and is similar to the well-known equation of Stokes' first problem of a suddenly accelerated plane wall.

The solution of these equations under the proper boundary and physical conditions forms one of the many boundary value problems. The most common method used for the solution of the above system of equations is to assume some general form of solution and then to transform these partial differential equations to ordinary ones which can be integrated using standard mathematical techniques.

In the past, for some of the heat conduction and wave motion problems, the method of separation of variables has produced good results. This method, when applied to equation (1-b), gives a solution of the form

$$u = \sum_{m=1}^{\infty} e^{-\frac{\nu}{u_0}\lambda_m^2 z} \cdot J_0(\lambda_m r), \tag{6}$$

where J_0 is the Bessel function of zero order and of the first kind, since the solution of the second kind is always unbounded at $r = 0$. The above form of solution, as indicated in the next chapter, presents some difficulties when the boundary condition and the constancy of the momentum are to be considered.

But as is known, there are still other methods and assumptions for reducing or solving the above equations. When one uses different methods, completely different forms of solutions may be obtained in which each form may have a different feature at the time that the boundary and physical conditions of the specific problem are used.

In the past, for many physical problems of the above nature (references 1, 2, 4, 5, 7, and 8), a form

$$u = \frac{1}{z} f\left(\frac{r}{z}\right) \text{ or } \omega = \frac{1}{r} f\left(\frac{r}{z}\right)$$

was used to deduce a single-value solution.

In searching for more general expressions of the different velocity components, it may be assumed that

$$r^a z^b f(r^\gamma z^\epsilon)$$

satisfies both equations (1-b) and (3-b), and then, using dimensional considerations, a solution of a more general form may be found.

AXIAL VELOCITY

Solution of Axial Momentum Equation

Following the above brief discussion, a solution of the form

$$u = r^a z^b f(\theta) \tag{7}$$

may be sought where

$$\theta = r^\gamma z^\epsilon \tag{8}$$

and a, b, γ, and ϵ are constants as yet not defined. Substitution of the velocity function, equation (7), into the linearized equation (1-b) gives, after some simplifications,

$$\gamma^2 (r^\gamma z^\epsilon)^2 f''(\theta) + \left[\gamma(2a+\gamma) - \epsilon\left(\frac{r^2 u_0}{z\nu}\right)\right](r^\gamma z^\epsilon) f'(\theta) + \left[a^2 - b\left(\frac{r^2 u_0}{z\nu}\right)\right] f = 0,$$

or

$$\gamma^2 \theta^2 f''(\theta) + \left[\gamma(2a+\gamma) - \epsilon\left(\frac{r^2}{z}\frac{u_0}{\nu}\right)\right]\theta f'(\theta) + \left[a^2 - b\left(\frac{r^2}{z}\frac{u_0}{\nu}\right)\right] f = 0.$$

If the velocity field is to have a universal or self-preserving profile, one may take

$$\theta = r^{\gamma} z^{\epsilon} \equiv \frac{r^2}{z}, \tag{9}$$

where $\epsilon = -1$ and $\gamma = 2$.

Thus,

$$\theta^2 f''(\theta) + \left[a + 1 + \frac{\theta u_o}{4\nu}\right]\theta f'(\theta) + \left[\frac{a^2}{4} - \frac{\beta \theta u_o}{4\nu}\right] f(\theta) = 0. \tag{1-d}$$

This equation, under a transformation of the independent variable similar to the one used by Blasius (reference 8) for the case of the boundary layer along a plate, i.e.,

$$\theta = \frac{4\nu x}{u_o}, \tag{10}$$

takes the nondimensional form

$$x^2 f''(x) + x f'(x)\left[1 + a + x\right] + f(x)\left[\frac{a^2}{4} - \beta x\right] = 0.$$

Finally, by transforming

$$f(x) = x^{-\frac{a}{2}} \phi(x),$$

the following is obtained:

$$x\phi'' + (1 + x)\phi' - \phi\left(\frac{a}{2} + \beta\right) = 0. \tag{1-f}$$

The form of the solution of this equation that is of Laguerre's type will depend on the value of the constants a and β, and can be rewritten in Laplacian coordinates

$$\bar{\phi}' + \bar{\phi}\left[\frac{1}{s} + \left(\frac{a}{2} + \beta\right)\left(\frac{1}{s} - \frac{1}{s+1}\right)\right] = 0.$$

This equation has the solution

$$\bar{\phi} = \left(\frac{s+1}{s}\right)^{\frac{a}{2}+b} \cdot \frac{1}{s} .$$

This will give the following results in the initial coordinate system:

<u>Case 1:</u>

For $\frac{a}{2} + b = -m \qquad m > 0$

where m is a positive integer,

$$\phi_m(x) = e^{-x} \sum_{K=1}^{m} \frac{(-1)^{K-1}(m-1)!\,x^{K-1}}{(K-1)!\,(m-K)!\,(K-1)!} . \tag{11}$$

<u>Case 2:</u>

For $\frac{a}{2} + b = m > 0,$

$$\phi_m(x) = \frac{e^{-x}}{m!} \frac{d^m}{dx^m}\left(x^m e^x\right). \tag{12}$$

<u>Case 3:</u>

For the particular case that

$$\frac{a}{2} + b = -\frac{1}{2} ,$$

Bessel's solution of the form is obtained,

$$\phi(x) = Ae^{-\frac{x}{2}}\, I_o\left(-\frac{x}{2}\right), \tag{13}$$

where I_o is a Bessel function of second kind of zero order.

It must be noticed that, if m is not an integer, the solutions will be similar to those given for Cases 1 and 2 and can be obtained by binomial expansion of $\left(\frac{s+1}{s}\right)^m$.

The corresponding velocity function of the (z , τ) coordinates system is

$$u = A z^{k+\frac{a}{2}} \phi(x) \tag{14}$$

for Cases 1, 2, and 3, respectively.

Any linear combination of velocity profiles given by the general expression

$$u = \sum_{m=1}^{\infty} A_m z^{b_m + \frac{a_m}{2}} \phi_m(x) \tag{15}$$

is a solution of the equation of motion, equation (1-b), where the final form will depend on the boundary conditions and the momentum considerations.

Momentum Considerations

There is a constant associated with the axial motion that can be obtained by multiplying both sides of equation (1) by r, integrating with respect to r between the limits 0 and ∞, and assuming that

$$\lim_{r \to \infty} \left[r \frac{\partial u}{\partial r} \right] = 0.$$

Momentum is given by

$$M = 2\pi \int_0^{\infty} r \left(P + \rho u^2 \right) dr, \tag{16}$$

which can be written

$$M = 2\pi \int_0^{\infty} \left[P + \rho \left(u_0 + \alpha u \right)^2 \right] r\, dr, \tag{16-a}$$

or, neglecting the higher powers of the defect velocity, this constant for the present case can be expressed as

$$M_a = 4\pi \rho u_0 \int_0^{\infty} \alpha u r\, dr. \tag{16-b}$$

Therefore, it must follow that

$$M_a = 4\pi\rho u_0 \int_0^\infty \left[\alpha \sum_{m=1}^{\infty} A_m z^{b_m + \frac{a_m}{2}} \phi_m(x)\right] r\,dr \tag{17}$$

or

$$M_a = 8\nu\pi\rho\alpha \sum_{m=1}^{\infty} A_m \int_0^\infty z^{b_m + \frac{a_m}{2} + 1} \phi_m(x)\,dx \tag{17-a}$$

is a constant value.

To satisfy this condition, solutions must be sought that produce momentum integrals independent of axial distance z. It can be seen that the only solution of equation (1-b) which produces a constant momentum is the one for which $b_m + \frac{a_m}{2} = -1$.

This solution has the form

$$u = A_1 z^{-1} e^{-x}, \tag{18}$$

or

$$u = u_0 + A_1 z^{-1} e^{-x}. \tag{18-a}$$

Any other expressions of the form

$$\int_0^\infty \phi_m(x)\,dx = 0, \tag{18-b}$$

which obviously will not alter the magnitude of the momentum integral, may be added to the solution (18-a).

The above conditions are satisfied by the velocity functions that are given by equations (15) and (11), in which constant momentum is produced for $m = 1$ and zero momentum for $m = 2, 3, 4, \ldots$ Thus, a solution of general form that satisfies the constancy of the momentum is

$$u = A_1 z^{-1} e^{-x} + e^{-x} \sum_{n=z}^{\infty} A_m z^{-m} \sum_{K=1}^{m} \frac{(-1)^{K-1}(n-1)!\,x^{K-1}}{(K-1)!(m-K)!(K-1)!} \tag{19}$$

where K = 2. 3, ... It may be noticed here that, for $n > 1$, and not an integer, the integral in equation (18) does not vanish and the momentum becomes a function of z.

Boundary Conditions

So far, the only restriction imposed in the solutions has been the independence of the momentum with respect to the axial distance z. The final form of the solution is directly related to the following boundary conditions that are the same for the case of the axial velocity of a jet-wake or vortex at some distance downstream from the initial plane.

$$u\Big|_{r \to \infty} = 0, \tag{20}$$

$$\frac{\partial u}{\partial r}\Big|_{r=0} = 0, \tag{21}$$

and

$$u\Big|_{\substack{r=0 \\ z=C_1}} = C_2, \tag{22}$$

where C_1 and C_2 are constants.

It may now be verified that any solution of equation (19) satisfies the above boundary conditions and that solutions of equations (15) and (12) or (13) are unbounded for r approaching infinity. Therefore, the only solution that satisfies the boundary conditions and the constancy of the axial momentum is the one given by equation (19).

Whenever the velocity function is given by more than one term, unavoidable difficulties arise in the calculation of the arbitrary constants A_m, since only A_1 can be calculated from the momentum integral.

$$A_1 = \frac{M_a}{8\pi\nu\rho\alpha}. \tag{23}$$

Some additional information for the calculation of these constants may be obtained experimentally; i.e., the value of the velocity along the z axis where

$$u\Big|_{r=0} = \sum_{m=1}^{\infty} A_m z^{-m}. \tag{24}$$

In tensor notation, the solution of the system is

$$A_m z_i = u_i,$$

where u_i is the axial velocity along the z axis at a distance z_i from the origin. This will give the value of the arbitrary constants A_m.

ROTATIONAL VELOCITY

Solution of the Rotational Momentum Equation

The rotational velocity ω may be calculated from the simplified equation (3-b) by using an approach similar to the one used for axial velocity. Again, assuming a solution of the form

$$\omega = r^a z^b f(\theta), \tag{25}$$

where $\theta = r^\gamma z^\epsilon$, substituting into equation (3-b), and simplifying, it is found that

$$\gamma^2\theta^2 f'' + \left[\gamma(2a+\gamma) - \epsilon\left(\frac{u_0 r^2}{\nu z}\right)\right]\theta f' + \left[a^2 - 1 - b\left(\frac{u_0 r^2}{\nu z}\right)\right] f = 0. \tag{2-d}$$

If the velocity field has a self-preserving or universal profile,

$$\theta = \frac{r^2}{z} \quad \text{and} \quad \gamma = 2, \quad \epsilon = -1,$$

and using as before

$$\theta = \frac{4\nu}{u_0} x$$

and transforming

$$f(x) = x^{-\frac{a+1}{2}} \phi(x),$$

an equation mathematically similar to equation (1-f) is obtained.

$$x\phi'' + x\phi' - \phi\left[\frac{a}{2} + \beta + \frac{1}{2}\right] = 0. \qquad \text{(2-f)}$$

In view of the boundary conditions of the problem, it is assumed that

$$\phi(x) = 0 \quad \text{at} \quad x = 0.$$

Transforming into Laplacian coordinates,

$$\bar{\phi}'(s+1)s + \bar{\phi}\left[2s + \frac{3}{2} + \frac{a}{2} + \beta\right] = 0, \qquad \text{(2-g)}$$

or

$$\bar{\phi} = \frac{1}{s^2}\left(\frac{s+1}{s}\right)^{\frac{a}{2}+\beta-\frac{1}{2}}.$$

This will give the following results in the initial coordinates:

<u>Case 1</u>:

For $\frac{a}{2} + \beta - \frac{1}{2} = m$ $\qquad m$ a positive integer,

$$\phi_m = \sum_{K=0}^{m} \frac{m!\, x^{K+1}}{K!\,(m-K)!\,(K+1)!}. \qquad (26)$$

<u>Case 2</u>:

For $\frac{a}{2} + \beta - \frac{1}{2} = -m - 2 \qquad m \geq 0,$

$$\phi_m = \sum_{K=0}^{m} \frac{(-1)^m (-1)^K m!\, e^{-x} x^{m-K+1}}{K!\,(m-K)!\,(K+1)!}. \qquad (27)$$

Also, for the case

$$\frac{a}{2} + \beta - \frac{1}{2} = -1,$$

or

$$\frac{a}{2} + \beta - \frac{1}{2} = -\frac{3}{2},$$

the following relations are obtained:

$$\phi = 1 - e^{-x} \tag{28}$$

and

$$\phi = x e^{-\frac{x}{2}} \left[I_0\left(-\frac{x}{2}\right) + I_1\left(-\frac{x}{2}\right) \right]. \tag{29}$$

It should be noted that when m is not an integer, the solution is very similar to the one given by equations (26) and (27). In this case the final form will be restricted on the basis of the given boundary conditions and additional physical conditions which the velocity profiles are required to satisfy.

Physical Conditions

Starting first by selecting the velocity profiles which have physical meaning, the conditions concerning the bounded characteristics of the physical quantities related to the motion will be investigated.

Following Görtler (reference 3) in this connection, it may be recalled that the pressure is regarded as constant along the z axis, but that there is a pressure variation along the radial distance from the z axis that can be calculated by integrating equation (2-b). Thus,

$$P \Big|_0^\infty = \rho \int_0^\infty \frac{\omega^2}{r} \, dr \tag{30}$$

or

$$\Delta P = \frac{1}{2} \rho \int_0^\infty \frac{\omega^2}{x} \, dx$$

which must be of a finite form in the domain of definition. On the other hand, there is another constant associated with this type of vortical motion that will depend on the degree of swirling in the described motion. This constant may be derived by multiplying equation (3) by r^2, integrating with respect to r between the limits 0 and ∞, letting u, ω $ruV \to 0$ as $r \to \infty$, and setting $V = \omega = 0$ when $r = 0$. The integral expressing angular momentum for a unit axial width is then obtained.

$$R = 2\pi \rho \int_0^\infty r^2 u \omega \, dr. \tag{31}$$

The constancy of the angular momentum is to be expected due to elementary mechanical principles, and for the present case

$$R = 2\pi\rho \int_0^\infty (u_0 + \alpha u)\, \omega\, r^2\, dr. \tag{31-a}$$

If

$$u_0 \omega >> \alpha u \omega,$$

$$R = 2\pi\rho u_0 \int_0^\infty \omega r^2 dr. \tag{31-b}$$

Finally, the circumferential mass flow for unit axial width,

$$m = \rho \int_0^\infty \omega\, dr,$$

may be assumed to possess a finite value.

Now, the question arises as to the possibility of satisfying all of the above conditions and of deriving a condition which is sufficiently restrictive to enable solutions given for different values of m (of integer and noninteger order) to be obtained for which the physical profile is described with sufficient accuracy.

To examine this question it may be assumed, using a process similar to the one given by Görtler (3), that the integral

$$\int_0^\infty x^K \omega\, dx,$$

or more generally

$$\int_0^\infty r^K \omega^\lambda\, dr,$$

might exist in bounded form where K and λ are arbitrary constants. This last condition yields the relation

$$a + 2\beta = -\frac{K+1}{\lambda}$$

between the arbitrary constants.

The general form of the velocity profile calculated using the above process is given by the function

$$\omega = B z^{\beta + \frac{a}{2}}\, x^{-\frac{1}{2}}\, \phi.$$

Thus, the corresponding pressure difference given by equation (30) is

$$\Delta P = \frac{1}{2} B^2 \rho Z^{2\beta + a} \int_0^\infty \frac{\phi^2}{x^2}\, dx. \tag{30-a}$$

Also, the circumferential momentum flux relation will be

$$R \simeq \frac{8\pi \rho \nu^{\frac{3}{2}}}{u_0^{1/2}} B Z^{\beta + \frac{a}{2} + \frac{3}{2}} \int_0^\infty \phi\, dx, \tag{31-c}$$

and the circumferential mass flow is

$$m = B\rho \left(\frac{\nu}{u_0}\right)^{\frac{1}{2}} Z^{\beta + \frac{a}{2} + \frac{1}{2}} \int_0^\infty \frac{\phi}{x}\, dx. \tag{32-a}$$

It may then be observed that any function given by equation (26) is unbounded as $x \rightarrow \infty$, and it cannot satisfy the integral restriction given above. Also, the particular solutions given by equations (28) and (29) are unbounded in integral form. Equation (28) is identical to the one given by Newman in Reference 7.

$$\omega = \frac{B}{2\pi r}\left(1 - e^{-x}\right). \tag{33}$$

This type of flow at large values of r describes a swirl profile that corresponds to a logarithmic vortex.

The logarithmic vortex is also a solution of the equation of motion and may be obtained from equation (2-f) for the particular case that

$$\frac{a}{2} + \beta + \frac{1}{2} = 0$$

and

$$\phi = C.$$

Although these two forms of the vortex have been used extensively in the past, it may be easily verified that none of the physical conditions previously described can be satisfied, as these functions are unbounded for extreme values of r.

Starting from the momentum integral in equation (31-c), one may observe that for the particular case when

$$\beta + \frac{a}{2} = -\frac{3}{2},$$

the momentum is indipendent of the axial distance Z and is of constant value.

For this case, the constant of integration is

$$B_o = R u_o^{1/2} / 8\pi \rho \nu^{3/2} .$$

The above is also true for all integrals of ϕ for which

$$\beta + \frac{a}{2} = -\frac{3}{2} - m,$$

where m is an integer and $m \geq 1$ are of zero value.

The corresponding solutions of the equation of motion are, for

$$\beta + \frac{a}{2} = -\frac{3}{2}; \; \omega_o = B_o z^{-3/2} x^{1/2} e^{-x}, \tag{34-a}$$

$$\beta + \frac{a}{2} = -\frac{5}{2}; \; \omega_1 = B_1 z^{-5/2} (1 - \frac{x}{2}) x^{1/2} e^{-x}, \tag{34-b}$$

$$\beta + \frac{a}{2} = -\frac{7}{2}; \; \omega_2 = B_2 z^{-7/2} (1 - x + \frac{x^2}{6}) x^{1/2} e^{-x}, \tag{34-c}$$

$$\beta + \frac{a}{2} = -\frac{9}{2}; \; \omega_3 = B_3 z^{-9/2} (1 - \frac{3}{2} x + \frac{1}{2} x^2 - \frac{x^3}{24}) x^{1/2} e^{-x}, \tag{34-d}$$

and, in general,

$$\omega = e^{-x} \sum_{K=0}^{m} \frac{B \; (-1)^m (-1)^K m! \; x^{(m+\frac{1}{2}-K)} z^{-m-3/2}}{K! \; (m-K)! \; (m-K+1)!} . \tag{34}$$

Thus, any combination or linear superposition of equation (34-a), $m = 0$, with any number of forms of equation (34) for which $m > 0$, satisfies the constancy of the angular momentum.

The corresponding values of the integral involved in equation (32-a) for the circumferential mas flow are for $m = 0, +1, +2....$

$$\int_0^\infty \frac{\phi}{x} dx \Rightarrow 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..,$$

respectively. So the quantity expressed by any of the above combinations decreases with the axial distance for $z > 1$ according to an inverse power law.

Finally, the corresponding values of the integral $\int_0^\infty \frac{\phi^2}{x^2} dx$ involved in equation (30) for the pressure differences yield constant values.

Thus, there exists an infinite number of solutions which satisfy the differential equation of motion and the physical conditions imposed previously. But, it can be seen that these conditions are not sufficient to deduce the solutions which have physical meaning; in addition to the boundary condition, the condition that circumferential velocity is always greater than or equal to zero must be considered. The meaning and possibility of the existence of "negative velocities" will be discussed in Chapter Five.

The Boundary Conditions

The swirling velocity must banish for

$$\lim_{r \to 0} (\omega) = 0 \tag{35}$$

and

$$\lim_{r \to \infty} (\omega) = 0 . \tag{36}$$

Also, the swirling velocity must always be greater than or equal to zero.

$$\omega \geqslant 0 .$$

The first two conditions are satisfied by any solution given in equation (34) for m of integer order. The last condition is satisfied by equation (34-a), and can be satisfied by linear combinations of equation (34) only for certain regions of the axial distance z and for some carefully calculated values of the integration constants B_m. Because the algebraic equation

$$\sum_{K=0}^{m} \frac{(-1)^m (-1)^K m! \; x^{m-K+1}}{K!\,(m-K)!\,(m-K+1)!} = 0 \tag{37}$$

has m distinct positive roots, the velocity profile described by a single function $\phi_m(x)$ will represent a velocity that changes sign m times for a radius variation between zero to infinity, a condition that cannot be expected to exist under laminar motion.

The Static Pressure

Using equation (34), one may calculate the static pressure drop along the r axis to be

$$\Delta P = \frac{\rho}{2}\sum_{m=0}^{\infty} B_m^2 z^{-3-2m} \int_0^{\infty} \frac{\phi_m^2}{x^2}\,dx. \tag{38}$$

Also, the static pressure for a given value of x may be calculated from

$$P = \frac{\rho}{2}\sum_{m=0}^{\infty} B_m^2 z^{-3-2m} \int_0^{x} \frac{\phi_m^2}{x^2}(x)\,dx. \tag{39}$$

The Radial Velocity

Using the continuity equation (4-a), the radial velocity U for some point (r, z) is given by

$$U = -\frac{\alpha}{r}\int_0^{r} r\,\frac{\partial u}{\partial z}\,dr$$

or

$$U = -\alpha\frac{z}{2r}\cdot\frac{4\nu}{u_0}\int_0^{x}\frac{\partial u}{\partial z}\,dx.$$

Substituting U from equation (15) gives

$$U = -\frac{az}{2r}\cdot\frac{4\nu}{u_0}\sum_{m=1}^{\infty} A_m z^{-m-1}\int_0^{x}\left[(-m)\phi_m(x) - x\phi'(x)\right]dx + G(z), \tag{40}$$

where $G(z)$ is a function of z or a constant that, according to the boundary conditions, is equal to zero. For the particular case when the axial velocity U is given for $b_m + \frac{a_m}{2} = -1$,

then

$$U = \alpha\frac{A_r}{2z^2}\,e^{-\frac{u_0 r^2}{4\nu z}}.$$

THE FORM OF THE SOLUTIONS

The Initial Conditions

Up to this point, some of the possible boundary and physical conditions that restrict the motion have been considered without restriction being made relative to the value of the velocity components at the initial plane. It may be verified that all the analytic expressions given for the velocity components cannot be defined if the initial plane is at $z = 0$, as $z = 0$ is a singular point for all those functions. On the other hand, the types of simplifications used in the Navier-Stokes equations of motion are such that they can be used only at some distance downstream from the initial plane. Thus, a question arises as to how velocity profiles described by functions of equations (19) and (34) are produced in some initial plane. First, it may be observed that the mathematical treatment of the present problem holds true for a more general coordinate system $z + z_i$, where z_i is an arbitrary negative or positive constant that is expected to be different from the theoretical origin of the velocity field. This constant can be estimated experimentally by comparing measurements taken at any two points along the z axis.

It may be assumed that at $z = z_i$,

$$u = u_i(r) \quad \text{and} \quad \omega = \omega_i(r).$$

Then, following a process similar to the one used by Taylor (reference 9) for the case of the dissipation of eddies, the initial distribution of the velocity is now

$$u_i(r) = e^{-x_i} \sum_{m=1}^{\infty} A_m z_i^{-m} \sum_{K=1}^{m} \frac{(-1)^{K-1}(m-1)!\, x_i^{K-1}}{(K-1)!\,(m-K)!\,(K-1)!} \tag{41}$$

and

$$\omega_i(r) = e^{-x_i} \sum_{m=0}^{\infty} B_m z_i^{-m-\frac{3}{2}} \sum_{K=0}^{m} \frac{(-1)^m (-1)^K m!\, x^{m+\frac{1}{2}-K}}{K!\,(m-K)!\,(m-K+1)!}, \tag{42}$$

where

$$x_i = \frac{r^2 u_0}{4\nu z_i}.$$

The Nondimensional Velocity Profile

Since the system of motion under consideration has no preferred length or dimension for reference, the obtained velocity profile can be expressed in a nondimensional form by using some conveniently chosen reference quantities as follows:

For the vortical velocity: A characteristic region for the vortical velocity at a given axial distance z is one of maximum value. This maximum can be calculated by letting

$$\frac{\partial \omega}{\partial r} = 0.$$

For the case of a vortex of the simple form given by equation (34-a) for

$$\beta + \frac{a}{2} = -1, \qquad r_1 = \left(\frac{2\nu z}{u_0}\right)^{1/2}, \tag{43}$$

where r_1 is the vortex radius for which the velocity is at its maximum value, the corresponding nondimensional velocity is

$$\frac{\omega}{\omega_{MAX}} = C\,\frac{r}{r_1}\,e^{-\frac{1}{2}\left(\frac{r}{r_1}\right)^2}. \tag{44}$$

These values are of the same nature as the ones given by Newman (reference 7), which probably were obtained by estimating the maximum of the velocity profile

$$\omega = \frac{B_e}{r}\,e^{-x}. \tag{45}$$

This is also a solution of the equation of motion resulting by superposition of an antirotating logarithmic vortex to equation (33). The nondimensional vortex profile is given for this case by

$$\frac{\omega}{\omega_{MAX}} = C_1\left(\frac{1}{\xi}\right)e^{-\frac{\xi^2}{2}} \quad \text{where} \quad \xi = \frac{r}{r_1}. \tag{46}$$

But, in general, if the velocity is expressed by a more complicated function, e.g., equation (34), much more sophisticated criteria must be used to obtain a nondimensional form; and since these criteria are related to the integration constant, they can be evaluated only experimentally.

For the axial velocity: Using the same process for the axial defect velocity and considering that the maximum value occurs at $X = 0$,

it may be seen from equation (19) that u_{mAX} is a function of z only and may be written as

$$u_{mAX} = \sum_{m=1}^{\infty} A_m z^{-m}. \tag{47}$$

For such a general solution, (equation 19), difficulties similar to those described in the above paragraph exist; but for the particular case that $m = 1$, the axial velocity defect takes the nondimensional form

$$\frac{u_0 + \alpha u(x)}{u_0 + \alpha u(0)} = e^{-x} \quad \text{or} \quad \frac{u}{u_{mAX}} = e^{-x}, \tag{48}$$

when solutions (18-a) and (18) are used, respectively.

The Validity of the Obtained Solutions

In the preceding chapter, the mathematical solutions of the linearized equations of motion of the axially symmetric laminar flow of a jet-wake or vortex have been described.

Solutions of integral order of the induced variable m have been indicated for every velocity component. The physical conditions concerning the motion with the appropriate boundary conditions have been applied to restrict some of the solutions for integral values of the induced variable m.

Now, it is of particular interest to examine the question of the range of the validity of the obtained solution. Of course, this range is expected to be directly related to the extent that laminar flow exists for this type of motion.

If, as a criterion, the Reynolds number based on the jet diameter and some of the velocity components is used, it may be observed that the Reynolds number very quickly exceeds its critical value so that the flow is expected to be turbulent.

As indicated in the past by Görtler, Schlichting, Squire, Newman, and others (references 1, 3, 7, and 10, respectively), it is possible to extend the obtained solutions of laminar flow to turbulent motion, since for this case the differential equations of motion are almost of the same form. Therefore, it will be useful to examine the behavior of a turbulent axisymmetric motion.

CHAPTER THREE
TURBULENT, VISCOUS, AXIALLY SYMMETRIC JET-WAKES AND VORTICES

GENERAL REMARKS

It has been noted that in most practical applications the flow motion is turbulent for very low Reynolds numbers. Such a turbulent or eddying motion is primarily characterized by a fluctuating velocity component which is superimposed on the principal velocity, the effect of which is equivalent to a great increase in the coefficient of viscosity. To describe such a motion, Boussinesq (reference 11) suggests that turbulent flow can be treated by assuming an increased viscosity due to eddying motion, which can be considered constant through a given flow field. According to this theory, eddy viscosity is not a property of the fluid but depends on its mean velocity. It has been proved that in some cases (references 1 and 12), Boussinesq's assumption is accurate enough to predict the velocity distribution of a free jet as accurately as any other more elaborate theory.

There are currently two principal approaches to the study of turbulent flow. The first, which goes under a general title of "Free Turbulence Theory", includes Prandtl's mixing length (references 13 and 14), Taylor's vorticity transfer (reference 15), Reichardt's inductive theory (reference 16), and others (references 17 and 18). All of these theories establish a functional relation between Reynolds stresses produced due to the mixing motion and the mean-velocity components of the fluid by means of semiempirical hypotheses. The second approach, which goes under the title of "Statistical Theory of Turbulence", is based on methods of statistical mechanics and describes the flow variables by using statistical mean values (references 19 through 22). It is not the aim of this report to discuss the range of applicability of each of these methods. A rather extensive discussion is included in Goldstein (reference 23), Schlichting (reference 1), and Townsend (reference 22).

But as will be seen in the following text, for some cases using the different above-mentioned theories, or a simple assumption of the "viscosity function", solutions of exactly the same form and of good agreement with the experimental results may be obtained. The significance of this fact will be indicated in Chapter Five of this report.

AXIAL VELOCITY

The Equation of Motion

The Navier-Stokes equations of motion for an incompressible viscous fluid - equations (1), (2), and (3) - may be transformed to describe

the turbulent flow by replacing the laminar velocities by

$$U = U_m + u', \tag{49}$$

where U_m is the mean and u' the fluctuating velocity components, respectively.

By taking mean values of equations (1), (2), and (3), considering axial symmetry at the boundaries and the mean values connected with the flow, the equations of mean motion in cylindrical polar coordinates are

$$u_m \frac{\partial u_m}{\partial z} + U_m \frac{\partial u_m}{\partial r} + \frac{\partial \overline{u'^2}}{\partial z} +$$

$$\frac{1}{r} \frac{\partial (r\overline{u'U'})}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u_m}{\partial r}\right)\right], \tag{50}$$

and

$$u_m \frac{\partial U_m}{\partial z} + U_m \frac{\partial U_m}{\partial r} + \frac{\partial (\overline{u'U'})}{\partial z} +$$

$$\frac{1}{r} \frac{\partial (\overline{U'^2} r)}{\partial r} - \frac{\overline{\omega'^2}}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial U_m}{\partial r}\right)\right], \tag{51}$$

where, again, z is the axis of symmetry and r is the radial coordinate. Continuity is also

$$\frac{\partial (u_m r)}{\partial z} + \frac{\partial (U_m r)}{\partial r} = 0. \tag{52}$$

Discussion of Previous Developments

Before proceeding to develop a simplified approach for the solution of the equation of turbulent motion, some of the work performed in the past will be indicated and discussed very briefly. However, it should be noted here that none of the turbulent theories (references 13

through 15) were originally developed for axisymmetric motion, so that any development based on two-dimensional relations between turbulent shear stress and mean motion may be regarded as being without any "theoretical basis".

Townsend (reference 22), using assumptions similar to the one used for laminar flow with the fact that $\overline{u'^2}$ is very nearly equal to $\overline{w'^2}$ everywhere (except at $r = 0$, where the two terms are equal), and that the gradient of the mean value r is large compared with z, has further simplified the above equations (50) and (51) to obtain

$$U_m \frac{\partial U_m}{\partial z} + V_m \frac{\partial U_m}{\partial r} + \frac{\partial(\overline{u'^2} - \overline{v'^2})}{\partial z} + \frac{1}{r}\frac{\partial \overline{u'v'}r}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_m}{\partial r}\right). \tag{53}$$

If the normal stress component is neglected - a condition that is almost valid near the axis of symmetry of a jet or a wake, but is of some non-negligible value near the edge of the boundary - then this gives

$$U_m \frac{\partial U_m}{\partial z} + V_m \frac{\partial U_m}{\partial r} + \frac{1}{r}\frac{\partial \overline{u'v'}r}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_m}{\partial r}\right). \tag{54}$$

The corresponding equation of motion for the case of an axisymmetric jet or wake in a co-flowing surrounding may be obtained if the mean velocity is replaced by

$$U_m = U_0 + \alpha U, \tag{54-a}$$

where U is the defect velocity component, and an order of magnitude evaluation is applied. Then, following Reference 22, it is assumed that

$$U_m = U_0 + \alpha U_2 f_{12}\left(\frac{r}{e_0}\right), \tag{55}$$

and

$$\overline{u'v'} = U_2^2 g_{12}\left(\frac{r}{e_0}\right). \tag{56}$$

In order to have a universal velocity profile,

$$\ell_0 \sim (z - z_0)^{1/3} \tag{57}$$

and

$$U_2 \sim (z - z_0)^{-\frac{2}{3}}. \tag{58}$$

The solution indicated by Townsend is

$$u_o - U_m = A(z - z_1)^{-1} e^{-\frac{u_o r^2}{4V(z - z_1)}}, \tag{59}$$

which may be expressed in a nondimensional form by dividing equation (59) by the maximum defect velocity occurring at r = 0. Thus,

$$\frac{u_o - u_m}{u_o - u_{mAX}} = e^{-\frac{u_o r^2}{4V(z - z_1)}}, \tag{60}$$

where z_1 is defined as some new virtual origin. This velocity profile is identical to the one given for the laminar jet equation (11) for $m = 1$.

An explanation for this similarity is that at sufficient distance downstream from the initial plane, the turbulent velocity field develops independently of the mean-velocity field and the turbulent motion becomes negligible compared to the mean velocity of the motion. Thus, the mean-flow velocity may be approximated by the corresponding velocity of a continuously developing laminar flow.

Although Reichardt's theory (references 16 and 24) was listed as a free turbulence theory, it differs from most free turbulence theories. Instead of attempting to establish some semiempirical relation between Reynolds stress and turbulent velocity components, it assumes in the equation of motion, equation (50), that the terms with pressure gradient and molecular viscosity may be neglected.

Therefore, the equation of axial momentum may be written in the form

$$u_m \frac{\partial u_m}{\partial z} + v_m \frac{\partial u_m}{\partial r} = 0. \tag{61}$$

Now, by multiplying the equation of continuity, equation (52), by u_m and combining,

$$\frac{\partial u_m^2}{\partial z} + \frac{\partial u_m v_m}{\partial r} + \frac{u_m v_m}{r} = 0. \tag{62}$$

Furthermore, Reichardt has assumed that the lateral transport of the mean-velocity momentum is proportional to the transverse gradient of

the horizontal component of the mean-velocity momentum

$$u_m U_m = \Lambda(z) \frac{\partial u_m^2}{\partial r}, \tag{63}$$

where $\Lambda(z)$ is a parameter analogous to Prandtl's mixing length with the dimensions of length. Substituting into equation (62) gives

$$\frac{\partial u_m^2}{\partial z} = \Lambda(z) \left[\frac{\partial^2 u_m^2}{\partial r^2} + \frac{1}{r} \frac{\partial u_m^2}{\partial r} \right]. \tag{64}$$

At this point, a set of solutions may be obtained by assuming that

$$u_m^2 = r^a z^b f(r^\gamma z^\epsilon),$$

and by following the process described for axial defect velocity. The nondimensional parameter will be

$$x = \frac{r^2 \epsilon}{z \Lambda(z)}.$$

However, Reichardt has transformed equation (1) on the basis that in an axisymmetric jet the dynamic pressure is inversely proportional to the square of some characteristic thickness b, a condition that is identical to the one obtained using the small perturbation analysis equation.

Introducing a nondimensional parameter $\xi = r/b$, and assuming that the momentum transfer length is

$$\Lambda(z) = b \frac{db}{dz}, \tag{65}$$

and that a mean dynamic pressure function is

$$u_m^2 = \frac{c^2}{b^2} f(\xi),$$

Reichardt calculated the elementary solution

$$u_m^2 = \frac{c^2}{b^2} e^{-\xi^2/2}. \tag{66}$$

As was expected, due to the mathematical form of equations (46) and (64), the nondimensional solutions are identical; but a difference between the actual velocity profiles exists.

To smooth out this difference, an assumption similar to that of Townsend (reference 22) may be used in which, from energy equilibrium of large eddies, it may be shown that

$$\frac{b}{u_{MAX}} = 3\,\frac{z - z_0}{u_0 R_T'} \,. \tag{67}$$

R_T' is an equilibrium flow constant given by

$$R_T' = \frac{u_{MAX}\, b}{\nu} \,. \tag{68}$$

Thus,

$$z - z_0 \sim \frac{b^2 u_0}{\nu} \,, \tag{69}$$

and Reichardt's solution and the solution obtained using the process described for laminar jets for $\gamma = 2$ and $\epsilon = -1$ are approximated. Although this report is primarily concerned with the axially symmetric flow motions in Appendix I, it may be verified that the assumptions and processes used for axially symmetric motion may be used successfully to describe the corresponding two-dimensional problem.

Finally, mention may be made of the work of Faris (reference 51), who developed a semiempirical relation capable of predicting very accurately the axial velocity profiles by extending the wake law suggested by Cornish and Coles for the turbulent entrainment process of the outer boundary layer.

Solution of the Axial Momentum Equation

An alternative approach for the solution of the equation of turbulent motion, equation (54), can be developed by assuming an approach analogous to Boussinesq's "mixing coefficient" for Reynolds stress.

$$\tau_{zr} = \rho \nu(z) \frac{\partial u_m}{\partial r} \,, \tag{70}$$

where ρ is the actual density of the fluid and $\nu(z)$ is a "viscosity function" which is a function of distance z measured from some initial point.

By combining equation (70) with the general expression for turbulent shear,

$$\tau_{zr} = -\rho\overline{u'v'}, \tag{71}$$

we have

$$\overline{u'v'} = -\nu(z)\frac{\partial u_m}{\partial r}. \tag{72}$$

Then, substitution into equation (54) gives

$$u_m\frac{\partial u_m}{\partial z} + v_m\frac{\partial u_m}{\partial r} = \frac{2\nu(z)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_m}{\partial r}\right) \tag{73}$$

and an equation similar to the equation of axial laminar velocity, equation (1). Obviously, the expression used for Reynolds stress does not affect the mathematical form of the equation of motion and, in this respect, is of the same nature as when it is assumed that $\overline{u'v'} = 0$ and when some "viscosity function" is introduced to the laminar flow case in order to describe mean-velocity motion.

Now, replacing the mean velocity from equation (54-a) and applying an order-of-magnitude evaluation, the resulting partial differential equation may be transformed to an ordinary one by assuming a solution of the form given by equations (7) and (8).

$$\theta^2 f''(\theta) + \left[\frac{2a+\gamma}{\gamma} - \frac{\epsilon}{\gamma^2}\frac{r^2 u_o}{z\,2\nu(z)}\right]\theta f'(\theta) + \left[\frac{a^2}{\gamma} - \frac{\beta}{\gamma^2}\frac{r^2 u_o}{z\,2\nu(z)}\right] f(\theta) = 0. \tag{74}$$

If the mean-velocity profile is expected to have a universal or self-preserving character,

$$[6]^{\varepsilon-\gamma}\cdot\theta = [6]^{\varepsilon-\gamma}\frac{r^{\gamma}}{z^{\varepsilon}} \sim \frac{\varepsilon}{\gamma^2}\frac{r^2}{z}\frac{u_0}{2V(z)} = x,$$

where $\epsilon = -\varepsilon$, $\varepsilon = |\epsilon|$, and 6 is a constant having the dimensions of length. Thus

$$V(z) = \frac{\varepsilon}{\gamma^2}\cdot\frac{r^2}{z}\cdot\frac{u_0}{2}\,6^{-(\epsilon-\gamma)}\cdot\frac{z^{\varepsilon}}{r^{\gamma}}.$$

Therefore, if $V(z)$ is a function of z, $\gamma = 2$. Now, by introducing a nondimensional parameter

$$x = \frac{\varepsilon\, r^2\, u_0}{\gamma^2 z\, 2V(z)}, \tag{75}$$

and transforming the nondimensional equation by

$$f = x^{-\frac{a}{\gamma}}\,\phi(x),$$

then

$$x\phi'' + [1+x]\,\phi' - \left[\frac{a}{\gamma} + \frac{b}{\varepsilon}\right]\phi = 0. \tag{76}$$

This equation gives solutions similar to the one for laminar flow if, in equations (11), (12), and (13), X is replaced by x and $\frac{a}{2} + b$ by $\frac{a}{\gamma} + \frac{b}{\varepsilon}$. Also, the velocity profile is given by

$$u = A z^{b + \frac{a\varepsilon}{\gamma}}\,\phi(x), \tag{77}$$

or, in general,

$$u = \sum_{n=1}^{\infty} A_m z^{m\varepsilon}\,\phi_m(x) \tag{77-a}$$

where $\phi_m(X)$ is given by equation (11) with $\frac{a}{\gamma} + \frac{b}{\varepsilon} = -m$.

If it is assumed that the constancy of the momentum is given in this case by equation (16-a), then $\frac{b}{\varepsilon} + \frac{a}{\gamma} = -\frac{2}{\gamma}$. (78)

Thus, the solution in Laplacian coordinates is

$$\bar{\phi} = \left(\frac{S+1}{S}\right)^{-\frac{2}{\gamma}} \cdot \frac{1}{S} \, . \tag{79}$$

The condition of constancy of momentum is satisfied for all positive integral values of m as in the laminar case. The form of the velocity function depends on the value of γ, or, more generally, on the "viscosity function". If "viscosity" is a function of z only, then $\gamma = 2$.

When $\varepsilon = 1$, a velocity profile similar to the laminar profile in equation (11) for $m = 1$ is obtained. If $\varepsilon = 2$, a nondimensional solution is given by

$$u_1 = e^{-x}, \tag{80}$$

where

$$x = C_1 \frac{r^2}{z^2} \, .$$

But, if "viscosity" is a function of both variables r and z, a condition that may lead to the equation (74) due to the form of the simplified equation of motion, then a relation between ε and γ must be established in addition to determining the value of γ.

For instance, using the relation resulting from Schlichting's analysis of the turbulent wake (reference 5) as a basis for establishing a relationship between ε and γ,

$$2\varepsilon = \gamma,$$

and under a more general assumption that

$$1 \leq \gamma \leq 2$$

as expected from References 5, 6, 7, and 16, an infinite number of solutions for the limiting cases may be obtained. The solution for $\gamma = 2$ has already been given by equation (18) and for $\gamma = 1$ with $m = 2$,

$$u_2 = \frac{A}{z}(1 - x)e^{-x}, \tag{81}$$

where x now becomes $x = C_2 \frac{r}{z^{1/2}}$. It may be verified that the last solution satisfies all conditions of the problem. Again, for these cases, $U_0 + \alpha u$ may be expressed by the same function.

The values of the constant of integration, the induced constant,

the virtual origin of z, or the characteristic thickness b must be determined experimentally. Obviously, all of these quantities may be included in a conveniently chosen "viscosity function".

Before comparing the calculated velocity profiles to existing measurements for axially symmetric wakes and jets, it will be useful to investigate whether the semiempirical profile of the turbulent boundary layer wake (as modified by Cornish and Faris (reference 51)) to describe jets may be approximated by some of the possible solutions of the linearized equations of motion indicated above.

We may then verify that a velocity profile of the simple form

$$u_1 \sim e^{-x},$$

where x is given by equation (75), successfully approximates the proposed profile to give

$$e^{-x} \sim 1 - \beta_{3D}.$$

Here, β_{3D} is Coles' wake function (54). But superimposing the solutions

$$u_1 + \frac{1}{2} u_2,$$

one may obtain a very good agreement. Thus, Coles' wake law may be regarded as a solution of the linearized equations of motion.

ROTATIONAL VELOCITY

Introductory Remarks

Since in the case of the axial velocity component the laminar velocity profile can be extended to describe turbulent motion with the aid of the "viscosity function" or "momentum transfer length", it may be expected that by using similar techniques the mean-vortical velocity profile may be approximated.

Unfortunately, existing experimental data are not of a nature which allows verification of the necessary assumptions that lead to profiles representing the actual case. Also, in view of the infinite variety of solutions that may be obtained for a form of equations (2) or (54), the following discussion will be limited to the indication of some of the different possible processes that have been used in the past or that may be followed in an alternative way for estimation of a sufficiently accurate mean-vortical velocity profile.

In general, it may be assumed that at a position downstream, the velocity profile behaves as the mean velocity, and that the turbulent velocity components are of negligible order.

It may also be assumed that the swirling velocity does not affect the first approximation of the axial flow velocity. Then, by replacing the kinematic viscosity in equation (3-b) by its value given in one of the existing theories (references 13, 14, and 15), it may be possible to solve the problem.

Görtler (reference 3) indicated this process by using the apparent kinematic viscosity from Prandtl's second hypothesis.

$$\varepsilon = Kb\left(\bar{u}_{MAX} - \bar{u}_{MIN}\right).$$

For the present case,

$$\varepsilon_o = Kb\bar{u}(z,0)$$

and is constant for a given distance from the initial plane.

Squire (reference 10) considered the growth of a viscous vortex and concluded that there is no justification for the introduction of complicated formulas for eddy viscosity or mixing length. Also, since the circulation is a principal parameter characterizing the motion, apparent kinematic viscosity may be determined from the summation $\nu + \varepsilon$, where $\varepsilon = \alpha B$ is constant and B is the circulation of the line vortex. Newman (reference 7), in discussing the agreement of the vortical velocity profile given by equation (33) with some experimental results of tests carried out by Mowforth (reference 25), concludes that it is clear that this type of motion cannot be described in terms of a constant viscosity for every downstream station, and that eddy viscosity is approximately constant at the vortex core. For relatively large radii, however, deviation from the theoretical profile is very large.

Solution of the Rotational Momentum Equation

On the basis of the above brief discussion and since the most uncertain parameter in the description of turbulent flow is the assumption concerning viscosity or relating stress to rate of deformation, it will be preferable to search for solutions of the equation of rotational momentum that must be of some general character and of such form that the solutions may be adapted to describe actual mean-rotational profiles.

If, as in the case of the axial jet, a characteristic width b and

a viscosity function $V(rz)$ are introduced, it may be assumed that a solution exists of the form $\omega = r^a b^\beta f(r^\gamma b^\epsilon)$ and an equation of the form below may be obtained by following the same elementary process as for axial motion.

$$f'' x^2 + f' x\left[\frac{2a+\gamma}{\gamma} + x\right] + f\left[\frac{a^2-1}{\gamma^2} - \frac{\beta x}{\varepsilon}\right] = 0, \tag{82}$$

where for the present case

$$x = \frac{r^\gamma}{b^\varepsilon}[C]^{\varepsilon-\gamma} \sim \frac{\varepsilon}{\gamma^2}\frac{u_o}{V(rz)}\frac{db}{dz}\frac{r^2}{b^2}, \tag{83}$$

$\varepsilon = -\epsilon = |\epsilon|$, and C has the dimensions of length. Then, by letting

$$f = \bar{x}^{\frac{a+1}{\gamma}}\phi(x), \tag{84}$$

and by transforming into the Laplacian coordinates,

$$\bar{\phi} = \frac{(S+1)^{\frac{a}{\gamma}+\frac{\beta}{\varepsilon}-\frac{1}{\gamma}}}{S^{\frac{a}{\gamma}+\frac{\beta}{\varepsilon}+\frac{1}{\gamma}+1}}. \tag{85}$$

Before transforming to the original (r, b) space, the existing relations may be indicated between the constants a, β, γ, and ε in order to satisfy the three integral conditions which are given for the laminar case by equations (30), (31), and (32), assuming presently that they may be extended for turbulent flow.

As the velocity profile for this case will be given by

$$\omega_m = \sum_{m=0}^{\infty} B_m b^{-(m+\frac{1}{\gamma}+1)\varepsilon} x^{-\frac{1}{\gamma}} \phi_m(x), \tag{86}$$

where ϕ_m is given by (27) with $m = \frac{a}{\gamma} + \frac{\beta}{\varepsilon} + \frac{1}{\gamma} + 1$,

the angular momentum integral is then

$$R = C_1 B b^{\beta + \frac{\varepsilon a}{\gamma} + \frac{3\varepsilon}{\gamma}} \int_0^\infty x^{\frac{2-\gamma}{\gamma}} \phi(x)\, dx. \tag{87}$$

Thus, the constancy of the angular momentum implies

$$\frac{\beta}{\varepsilon} + \frac{a}{\gamma} = -\frac{3}{\gamma}. \tag{88}$$

Therefore,

$$\bar{\phi} = \frac{S^{-1+\frac{2}{\gamma}}}{(S+1)^{4/\gamma}},$$

for which the general solution is

$$u'_m = B b^{-\frac{3\varepsilon}{\gamma}} x^{-\frac{1}{\gamma}} \phi(x). \tag{89}$$

If viscosity and characteristic thickness are functions of (z) only, then

$$\gamma = 2$$

and

$$w_m = B b^{-\frac{3\varepsilon}{2}} x^{-\frac{1}{2}} e^{-x}, \tag{90}$$

where $x = C^2 \frac{r^2}{b^3}$. The simplest case, as was indicated for axial velocity, is for $\varepsilon = 1$, where, if the expressions deduced from energy equilibrium for large eddies are used, then

$$w_m = B \frac{r}{b^2} e^{-C_1 \frac{r^2}{b^2}}. \tag{91}$$

The nondimensional velocity profile may be calculated by dividing by the maximum velocity occurring at

$$\frac{\partial w_m}{\partial r} = 0,$$

which is of the form

$$\frac{w_m}{w_{m(MAX)}} = C_3 \frac{r}{r_1} e^{-C_4 \frac{r^2}{r_1^2}} \tag{92}$$

where r_1 is the radius where maximum velocity occurs.

However, for other more complicated "viscosity functions" using the same approach, profiles of different analytic form may be calculated. Also, by combining solutions of the general form, equation (89), a measured experimental profile satisfying all of the existing boundary and physical conditions may be approximated. Since presently available experimental results do not permit a well-founded decision for the existing relation between γ and ε or for the value of either of them, the present analysis is limited at this point. Hopefully, future experiments will give the necessary information for the estimation of the different constants relating the nondimensional parameters in equations (90) and (84). Then, a realistic form of the analytic expression for the velocity profile may be formulated.

CHAPTER FOUR
CONFINED JETS AND VORTICES IN AXIALLY SYMMETRIC SURROUNDINGS

GENERAL REMARKS

It is well known that the jet entrainment process is the basis of many important practical applications, especially when the motion of the jet takes place in confined surroundings. In some cases, the presence of the surrounding walls noticeably increases the entrained mass flow.

In this chapter, some of the characteristics of the axisymmetric jet-mixing phenomena of jet ejectors and vortex chambers will be examined. In this respect, the existing knowledge and experimental data of the flow with jet and vortex profiles in a cylindrical channel will be used. Also, a form of the flow in the inlet length of a circular pipe will be used which will help in developing a better understanding of the confined mixing process. Such a process has not been completely covered in the up-to-date literature where, in general, assumptions valid for two-dimensional motion are used to describe three-dimensional problems.

JET EJECTORS

Summary of Previous Developments

During the past two decades, extensive theoretical and experimental research has been conducted by several investigators under the above general topic. H. B. Helmbold (references 27, 28, and 29), in "Contribution to the Jet Pump Theory", has given much useful theoretical and experimental information concerning the existing integral and analytic relations between the different flow characteristics and also some of the possible processes that may be used in order to calculate some of the goemetric dimensions of a jet pump.

Cruse and Tontini (reference 30) have reviewed the theoretical work performed in References 27 and 31 and have presented analytic methods for the design of a constant-pressure mixing tube and for the calculation of an effective mixing length.

O. V. Yakovlevskiy (reference 41) has studied the mixing of a jet in a channel with variable cross section and gives analytic expressions of the velocity profile for the two cases of constant diameter and conically shaped tubes. Yakovlevskiy's velocity profiles are in agreement with existing experimental results.

Under the topic "Steady Flow Ejector Research", Lockheed Aircraft Company (references 32 and 33) has performed an extensive study of the different geometric and flow parameters affecting the general design of a jet ejector to be used as a thrust-augmentation device.

Finally, general information concerning the confined jet flow, as it may be applied to the design of a jet pump, thrust augmenter, or noise reducer, may be found in References 34 through 40.

The Velocity Distribution at the Ejector Chamber

To calculate an analytic expression representing the velocity profiles at different lateral sections along the length of the mixing chamber and from the point where the jet "primary flow" exists to the co-flowing external "secondary flow" of the mixing tube, the Navier-Stokes equations of motion for an axially symmetric flow (1) must be written.

If it is assumed that the rotational velocity is zero and the well-known Prandtl boundary layer approximations are applied, then

$$u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{1}{\rho r}\frac{\partial (r\tau)}{\partial r}. \tag{93}$$

Again, the most uncertain parameter in equation (93) is the relationship existing between shearing stress and the rest of the flow parameters. It has been noted previously that all the existing free turbulence theories (references 13, 14, and 15) are valid only for two-dimensional flow because the assumptions on which they are based are such that they cannot be extended for three-dimensional problems. Therefore, it will again be assumed that the viscous term is of the form

$$\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{94}$$

where ν is a "viscosity function" that corresponds to a turbulent shearing stress

$$\tau = \frac{\rho}{r}\int \nu \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) dr. \tag{94-a}$$

When ν is a function of z only, equation (94-a) takes the form

$$\tau = \rho \nu \frac{\partial u}{\partial r}. \tag{94-b}$$

By substituting equation (94) into equation (93),

$$u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right). \tag{95}$$

This equation, with the continuity equation (4), forms a system of two equations from which the three unknown functions u, v, and P must be calculated. Thus, in order to proceed to a solution, some "arbitrary relation" between the function must be used. For this reason, the existing boundary and physical conditions that will limit our final decision as to the arbitrary relation may be considered.

The momentum for this case is given by the integral, equation (16), where the limits of integration are $R(z)$ and zero.

The relation $r = R(z)$ is the function describing the boundary of the mixing tube. If this theorem is applied over a closed control surface, it may be stated as follows: the difference of the momentum over two lateral cross sections is equal to the axial component of the pressure force on the tube wall.

The mass flow must also remain constant along the length of the channel so that

$$m = \int_0^{R(z)} 2\pi\rho u r\, dr.$$

The boundary conditions are:

$$\text{at} \quad r = 0, \; v = 0, \tag{96}$$

$$\text{at} \quad r = 0, \; \frac{du}{dr} = 0, \tag{97}$$

$$\text{and at} \quad r = R(z), \; u = v = 0. \tag{98}$$

In addition to all of the above, and for the study of a free jet, the assumption that the velocity profiles at different lateral cross sections have a self-preserving or universal form has to be considered in order to lead to a solution.

In this respect the work of Abramovich (reference 42), who has experimentally verified that there exists an interesting analogy between the velocity fields of a free and a confined jet, may be considered.

It was found that the dimensional velocity field at every cross section of the mixing chamber was of the same form as the central part of a free jet hypothetically bounded by the walls of the chamber. Abramovich concluded that this was not surprising if the universality of

the existing general laws of turbulent mixing are considered.

In the up-to-date literature, two types of jet ejectors are described: the constant-pressure and the constant-diameter types. Here, the variable-pressure type (not necessarily constant-diameter) will be examined, and the constant-pressure type will then be taken as a particular case.

A simplified form of equation (95) may be deduced if, at some point along the radius of the ejector and for the region under consideration, the velocity becomes a value U_m such that

$$U_m \frac{\partial U_m}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z}, \tag{99}$$

where approximately

$$\frac{\partial U_m}{\partial r} \cong 0.$$

Now, by replacing the velocity function U by

$$U = U_m + \alpha u \tag{100}$$

in equation (93), and by using equation (99) and applying an order of magnitude evaluation,

$$U_m >> u,$$

an expression that is almost correct at a few ejector diameters downstream from the initial plane may be written. Thus,

$$u \frac{\partial U_m}{\partial z} + U_m \frac{\partial u}{\partial z} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right), \tag{101}$$

or introducing a function

$$\psi = u \cdot U_m, \tag{102}$$

$$\frac{\partial \psi}{\partial z} = \frac{\nu}{r U_m}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right). \tag{103}$$

An alternate approach which will simplify equation (93) in the region very close to the initial plane is to assume that the secondary flow can be regarded as a free-stream relative to the primary flow of the ejector, and that the pressure at some jet diameters downstream from the initial plane may be regarded as unaffected by the confining walls of the mixing chamber.

For this case, the jet will behave like a free jet and, according to existing experience (reference 44), the laminar flow linearization

techniques may be used to give an equation of the form

$$u_s \frac{\partial u}{\partial z} = \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \tag{104}$$

where u is the defect velocity and u_s is the undisturbed velocity of the secondary flow.

Equations (103) and (104) are mathematically similar to equation (1-b) and to the heat conduction equation describing the radial flow of a hollow cylinder (reference 43). The boundary and initial conditions are similar to the same heat transfer problem.

Thus, the corresponding process for the present problem is to assume that equation (104) describes the velocity field at the first few jet ejector diameters with the initial conditions:

$$u_s = C_s \text{ , for } z = z_i, \; r_e < r < R_c, \tag{105-a}$$

and

$$u_p = C_p \text{ , for } z = z, \; 0 \leq r < r_e, \tag{105-b}$$

where u_s and u_p are the constant velocities of primary and secondary flow, respectively.

From the lateral section at the axial distance z_K up to the exit of the mixing chamber, equation (103) is used with velocity distribution at the initial plane z_K given by the solution of equation (104) evaluated at the surface z_K.

The problem of a free jet close to the initial plane was first considered by Pai (reference 44), who used a method of separation of variables similar to the one used for heat transfer problems (reference 43).

The solution obtained may be of the form

$$\frac{u}{u_p} = r_e \int_0^{R(z)} e^{-\lambda^2 \alpha^2 z} J_0(\lambda r) J_1(\lambda r_0) \, d\lambda, \tag{106}$$

where J_0 and J_1 are Bessel functions of zero and first order, respectively, and which satisfy initial conditions expressed by a Weber-Schafheitlin discontinuous integral.

However, if the simplified equations (103) and (104) can be used

for the regions of confined jet flow as indicated previously, then so can the mathematically similar equations of the general form

$$r^{a} z^{\beta} f(r^{\gamma} z^{\epsilon}). \tag{107}$$

Following the same process as in Chapters Two and Three for the present case, solutions of the general form - equations (26), (27), and (29) - are obtained. Whenever the conditions for momentum and mass constancy are not very restrictive, as when the integral limits are between zero and some defined point $R(z)$, the function describing the velocity profile does not have to be bounded in the limit $(r \to \infty)$.

General Case of Ejector Design

The more general case for the design of an ejector is when we assume that the boundaries are described by a function not yet defined. Then, in order to solve the equation of motion, it may be assumed that the solution is of the general form

$$\psi \quad \text{or} \quad u = r^{a} R^{\beta}(z) f(r^{\gamma} R^{\epsilon}(z)). \tag{108}$$

For the case of a turbulent jet, equation (108) may be substituted into equations (103) or (104) to obtain equation (74), where the nondimensional parameters are

$$[6]^{\varepsilon-\gamma} \cdot \theta = \frac{r^{\gamma}}{[R(z)]^{\varepsilon}} \cdot [6]^{\varepsilon-\gamma} \sim x, \tag{109}$$

and

$$x = \frac{\varepsilon}{\gamma^{2}} \frac{r^{2}}{R(z)} \frac{u_{o}}{V(r,z)} \frac{d[R(z)]}{dz}. \tag{110}$$

Here, u_o is either u_s or u_m when x is to be used for equations (104) or (103), respectively.

The solution for equations (103) or (104) is then

$$u \quad \text{or} \quad \psi = R(z)^{\beta + \frac{a\varepsilon}{\gamma}} \phi_1(X), \tag{111}$$

where for equation (104)

$$\phi_1(x)\Big|_{x=0} = A_1 ,$$

from which the maximum velocity is

$$u_{max} = \left[R(z)\right]^{\beta + \frac{a\varepsilon}{\gamma}} . \tag{112}$$

Thus, the mass flow is

$$m = C\pi\rho \int_0^{x|R(z)} A x^{\frac{2-\gamma}{\gamma}} R(z)^{\beta + \frac{a\varepsilon}{\gamma} + \frac{2\varepsilon}{\gamma}} \phi(x)\,dx + \rho U_s \pi R^2(z), \tag{113}$$

and it is necessary that

$$\beta + \frac{a\varepsilon}{\gamma} = -\frac{2\varepsilon}{\gamma} ,$$

for the mass flow will be independent of the axial distance.

The solution of the equation of motion in Laplacian coordinates is, therefore,

$$\bar{\phi} = \frac{1}{s}\left(\frac{s+1}{s}\right)^{-\frac{2}{\gamma}} . \tag{114}$$

If the integral of the function $\phi(x)$ over a region $R(z) \sim 0$ must be independent of $R(z)$, γ must equal ε so that the upper limit of the integral will be

$$x\Big|_{R(z)} = C .$$

The above expression may be considered equal to 1 without losing the generality of the equation.

The mass flow integral may now be written as

$$m = 2\pi\rho \int_0^1 x^{\frac{2-\gamma}{\gamma}} \phi(x)\,dx . \tag{115}$$

If friction on the wall of the mixing chamber is neglected, the momentum integral evaluated at two different lateral sections $R(z_1)$

and $R(z_2)$ will give the value of the axial component of the pressure force

$$R^{\frac{2\varepsilon}{\gamma}} \int_0^{x|_{R(z)}} x^{\frac{2-\gamma}{\gamma}} \left[P + \rho (u_m + \alpha u)^2 \right] dx \Bigg|_{x|_{R(z_2)}}^{x|_{R(z_1)}} \tag{116}$$

Using the same process, similar nondimensional relations between the different parameters describing the velocity field at some distance from the initial plane may be stated where the functions $R(z)$, the induced constants, the function $\phi(x)$, the constant of integration, and the virtual origin of z may be different for every case.

However, if the solution of equations (104) and (103) must satisfy the previously stated initial conditions of equations (105-a) and (105-b) as well as the variation of the velocity in the z direction so that it will be continuous at the lateral section z_K,

$$\left. \frac{\partial u}{\partial z} \right|_{z = z_K}$$

must have the same value for both velocity functions and the problem becomes extremely complicated. The deduction of general expressions for the ejector characteristics is insuperable.

In the following section, a simplified approach will be presented that may be used for the calculation of some of the parameters involved for an ejector design.

Simplified Approach for Ejector Design

One very important conclusion relating to the form of the possible solution of equations (103) and (104) is that the nondimensional velocity profile expressed by equation (114) is of the same general form for both cases.

On the basis of this observation and by using Abramovich's (reference 42) experimental verification, it may be assumed with sufficient accuracy that the elementary nondimensional form of equation (114) in the plane may be obtained if $\gamma = 2$. This makes it identical to the one that can be deduced for the case of a free jet. Now

$$\frac{u - u_s}{u_m - u_s} = e^{-x}, \tag{117}$$

or for

$$\gamma = 1,$$

$$\frac{u - u_s}{u_m - u_s} = e^{-x}(1-x); \tag{118}$$

where

$$x = \frac{r^2}{R^2(z)} \cdot C,$$

and u_m is the axial velocity at a given lateral section. u is the absolute rate of flow, and u_s is the nominal velocity of the secondary jet flow that can be regarded constant over the region under consideration.

Using the notation indicated in Figure 2, the excess mass flow is

$$m_f = (u_f - u_s) F_f = 2\pi\rho \int_0^{R(z)} (u - u_s)\, r\, dr. \tag{119}$$

From equation (118), we have

$$m_f = \frac{\pi \rho R^2(z)}{C} \int_0^{x|_{R(z)=1}} (u_m - u_s) e^{-x}(1-x)\, dx; \tag{120}$$

and finally, the excess velocity along the z axis is given by

$$u_m - u_s = m_f C_2 \cdot \frac{1}{R^2(z)}, \tag{121}$$

where

$$C_2 = \frac{\pi \rho e^{-1}}{C}.$$

Using the velocity profile given by equation (118), we have

$$u - u_s = \frac{m_f C_2}{R^2(z)} e^{-x}(1-x), \tag{122}$$

where m_f and C_2 may be estimated experimentally. A similar expression may be deduced from equation (117).

From the momentum equation, the pressure at any section of the region under consideration may be determined.

The excess momentum between any of the lateral sections and the exit of the mixing chamber may be written as

$$M_f(u_f-u_s)+P_fF_f = \int(u-u_s)\,dM + P.F. \tag{123}$$

The velocity momentum from the above equation may be simplified by writing

$$I = \int(u-u_s)\,dM = \frac{\pi\rho R^2(z)}{C}\int_0^1(u-u_s)u\,dx,$$

and by adding and subtracting the momentum of the secondary jet.

$$I = \frac{\pi\rho R^2(z)}{C}\left[(u_m-u_s)^2\int_0^1\left(\frac{u-u_s}{u_m-u_s}\right)^2dx+u_s\int_0^1(u-u_s)\,dx\right]. \tag{124}$$

Now, by using equation (118), we have

$$I = R^2(z)(u_m-u_s)\left[(u_m-u_s)C_3+u_sC_4\right], \tag{125}$$

where

$$C_3 = \frac{\pi\rho}{C}\int_0^1 e^{-2x}(1-x)^2\,dx$$

and

$$C_4 = \frac{\pi\rho}{C}\int_0^1 e^{-x}(1-x)\,dx.$$

Also,

$$M_f(u_f-u_s) \simeq \pi\rho R^2(z)u_f(u_f-u_s). \tag{126}$$

By substituting equations (125) and (126) into equation (123), the axial pressure force due to the pressure difference between those two sections is obtained.

The above analysis is based on a velocity profile given by equations (108) or (111) that exists when the boundaries of the mixing chamber are given by

$$r = R(z).$$

For the particular case of a constant-diameter chamber, a velocity function of equation (7) may be used to deduce expressions similar to

those already given.

The pressure gradient for a constant-diameter mixing tube was calculated in Reference 42, where two nondimensional variables were introduced on the basis of existing similarities of free and confined jets.

The variable relating the radius of the chamber to the free jet is

$$x_K = \frac{R}{b},$$

and the other which relates the free jet to the local radius may be written as

$$x = \frac{r}{b}.$$

Then, using simplified relations between the different velocity field components as established in References 4, 27, 38, 39, and 42, an approximate expression was obtained for the pressure difference.

$$P_f - P = \rho f(x_K), \tag{127}$$

where $f(x_K)$ is a given function of x_K and also of expressions for the mixing length and the mass ratio of the ejector.

Finally, if the ejector is to be used as a thrust augmenter, the thrust augmentation factor becomes particularly important.

The thrust augmentation factor φ is defined by the formula

$$\varphi = \frac{2\int_0^{R(z)} u^2 r\, dr}{R_e U_p^2}. \tag{128}$$

If the absolute value of the velocity is given by equation (115), then

$$\varphi = \frac{\int_0^{x|_{R(z)}} x^{\frac{2-\gamma}{\gamma}} \left[R(z)\right]^{\beta+\frac{a\varepsilon}{\gamma}+\frac{2\varepsilon}{\gamma}} \phi^2(x)\, dx}{C R^2_e U_p^2}$$

for $\gamma = 2$; and for a velocity satisfying the constancy of the mass flow, equation (113),

$$\varphi = \frac{C_3}{R^2(z)\,\pi\,\rho\,R_e^2\,u_p^2} \,. \tag{129}$$

An additional equation to be solved for the case of a high-augmentation factor ejector is

$$\frac{\partial}{\partial z}\left[\int_0^{R(z)} u^2\, r\, dr\right] = 0. \tag{130}$$

This may also be used for the estimation of the ejector boundary function $R(z)$.

Remarks

The indicated solutions in Chapter Three are based on linearized partial differential equations of motion which may be transformed to ordinary ones by using suitable operations. A process of solving the complete equation of motion is indicated in Appendix II.

CONFINED VORTICES

Introductory Remarks

Numerous theoretical and experimental investigations of the topic "confined vortex flow" have been developed in the past primarily because some very important applications can be developed on the basis of the behavior of this type of motion as indicated in References 45, 46, and 47.

In general, the flow field in a vortex tube is the result of the interaction of the primary core flow with the secondary boundary layer flow of the surrounding and the end wall of the tube. In this respect, extensive theoretical and experimental investigations have been developed in References 48, 45, 49, and others. In the present section the discussion will be restricted to the vortex produced in a cylindrical chamber similar to the one used in Reference 50, and an attempt will be made to determine whether the nondimensional solution of the linearized equation of rotational motion, as obtained in Chapters One and Two, may be extended to describe the considered confined vortex flow.

Rotational Velocity Distribution in a Vortex Chamber

An analytic expression of the rotational velocity profile at different lateral sections of a vortex tube may be deduced by solving the general Navier-Stokes equation of motion for axially symmetric flow. If, for the sake of simplicity, it is assumed that Prandtl's boundary layer approximations are valid, for this case, equations (1), (2), and (3) will represent this type of motion. A further simplification of the above equations may be obtained by processes similar to those in previous chapters that represent the variation of the rotational velocity independent of the axial velocity, a condition that may be regarded as unreasonable at this stage of development. However, if it is assumed that equation (86) represents the velocity profile in a confined vortex chamber, the velocity function is

$$\omega = b^{\beta + \frac{\varepsilon a}{\gamma}} \cdot x^{-\frac{1}{\gamma}} \phi(x),$$

where the nondimensional parameter is given by equation (84) and the "viscosity function" by equation (94-a). Then, solutions of equation (85) may be obtained.

The physical conditions represented by the integral equations (30), (31), and (32), due to the fact that the upper limit of integration is some defined function or constant, do not restrict the motion so definitively as for the case of a free vortex. However, the constancy of the circumferential mass flow may be expected.

The circumferential mass flow for a given axial width is

$$m = \frac{\rho B b^{\beta + \frac{\varepsilon a}{\gamma} + \frac{\varepsilon}{\gamma}}}{\gamma} \int_0^{x_1 R(z)} \frac{\phi(x)}{x} dx, \tag{131}$$

and the independency of the axial distance may be expected for

$$\frac{\beta}{\varepsilon} + \frac{a}{\gamma} = -\frac{1}{\gamma}. \tag{132}$$

This relation between the constants will give a solution in the Laplacian coordinate [illegible]

$$\bar{\phi}(S) = \frac{(S+1)^{-\frac{2}{\gamma}}}{S} \tag{133}$$

Thus, by using different values of γ, an infinite number of solutions may be obtained. The simplest solution in the (r, z) space is

for $\gamma = 2$.

$$\phi(x) = 1 - e^{-x}. \tag{134}$$

The corresponding velocity function is

$$\omega = \frac{B}{r}\left(1 - e^{-x}\right). \tag{135}$$

The singularity of the velocity function and the mass flow expression at ($r = 0$) or ($x = 0$) is removable. The mass flow will then be

$$m = B_1\left(\frac{y}{1!} - \frac{y^2}{2 \cdot 2!} + \cdots\right), \tag{136}$$

where

$$y = x\Big|_R ,$$

which will be constant if y is independent of z. This condition may be obtained under proper arrangement of the nondimensional variable x by taking $\gamma = \varepsilon$.

But, if turbulent nondimensional free vortex profiles may be extended to describe the confined motion, it may be possible that the experimental verification of Abramovich (reference 42), relative to the existing analogy between the axial velocity of a free and a confined jet, may also exist for the case of the vortical velocity profile.

Now, it will be particularly interesting, before examining the possibility of using the central part of the elementary nondimensional free vortex velocity profile as hypothetically bounded by the wall of the vortex chamber, to describe the confined velocity profile as indicated by the results obtained in Reference 50.

Thompson, using a semiempirical analysis, has indicated that the vortical velocity profile is of the form

$$\frac{\omega}{\omega_{max}} = \frac{B(\beta_{3D})}{r} + r, \tag{137}$$

where β_{3D} is Cornish's modified Coles' wake function (reference 51) which, according to the theory developed in this report, may be

approximated by the exponential function

$$e^{-x} \sim 1 - \beta_{3D}.$$

Thus, equation (137) becomes

$$\frac{w}{w_{max}} = \frac{B}{r}\left(1-e^{-x}\right)+r, \tag{138}$$

which may be recognized as a solution of the linearized equation of motion, equation (3-b). The velocity given by equation (138) does not satisfy some of the imposed physical conditions of the problem, but shows sufficient agreement with experimental results.

Using Thompson's experimental results, the above-stated possibility of extending Abramovich's concept for this type of motion has also been investigated. It was found that the corresponding free vortex profile may be used to describe the core of confined vortex motion in the middle section of the vortex chamber where the end wall interaction is not large.

It was also found that a velocity profile of equation (135) or equation (33) may be used to describe this type of motion with sufficient accuracy.

In general, as for jet ejector design, two nondimensional parameters may be used to describe the confined motion. One is

$$x_K = \frac{R}{b},$$

which relates the radius of the vortex to that of the tube. The other parameter is

$$X = \frac{r}{b},$$

which relates the local to the free radius.

The constancy of the circumferential mass flow may be assumed only if

$$\frac{R}{b} \sim \text{constant}.$$

Similarly, for the nondimensional velocity profile, where

$$x \sim \frac{r}{r_1},$$

the mass flow may be constant only if $\frac{R}{\tau_1} \sim$ constant and $\omega_{MAX} \sim$ constant. These conditions are valid only if τ_1 = constant, which can be expected if no losses occur during the motion.

CHAPTER FIVE
COMPARISON OF PRESENT THEORY WITH EXISTING THEORIES AND EXPERIMENTAL RESULTS

In Chapter Four, a set of solutions of the linearized equations of motion of free and confined jet-wakes and vortices was developed that, as was indicated, may be used to describe turbulent motion using a suitable "viscosity function", "virtual origin" or "characteristic thickness".

In this chapter, these solutions will be compared to other theoretical profiles and to collected and developed experimental data.

It was verified (see Figures 6 and 7) that the two-dimensional wake law of Cornish and Coles (references 53 and 54), which governs the turbulent entrainment process at the outer edge of a turbulent boundary layer, may be approximated very accurately by equation (11), when X is replaced by $\mathcal{X}$ as given by equation (66) or equation (84).

Using the same analytic expression, equation (11), and a different "virtual origin", the semiempirical Coles' wake function as modified by Cornish and Faris (reference 51) may be deduced. This function and the solution presented in this paper may be used to describe the experimental results obtained by Faris (reference 51) with very good accuracy for the case of an axisymmetric jet emerging in a stationary surrounding (see Figure 8).

Comparing the velocity profiles obtained by Tollmien (reference 62), Görtler (reference 12), and the present theory (Figure 9), it can be verified that all of these are in very good agreement at the core region, but that there is a small difference at the outer boundaries of the jet. The same difference may be observed between the above theoretical profiles and the experimental results of Reichardt. An explanation for this overestimation is that at the outer jet region the normal stress which has been neglected in all previous analyses has some significant value if it is compared to the other fluid motion components at the outer region.

However, if two or three forms of the general solution, equation (11), are used, the small defect may be smoothed out and the agreement of this theory with the experimental results thereby becomes much better (Figures 10 and 11).

The analytic expression by Reichardt (reference 16) for the dynamic pressure profile of a free axisymmetric jet, obtained by using his "inductive theory of turbulence", is of the same form as the elementary solution given by equations (11 and 84) when the nondimensional parameter

$$\chi = \left(\frac{r}{z}\right)^2 \cdot \frac{1}{0.02}$$

is used.

Both theories are in good agreement with the experimental results presented in Reference 16 (see Figures 10, 11, and 12). Finally, the present theory agrees with the experimental results obtained by Forthmann and Reichardt for the axial velocity distribution of a two-dimensional turbulent jet (references 16 and 55) as can be verified in Figures 12 and 15.

Therefore, it has been verified that, although present theory is basically developed by the use of a small perturbation analysis for the case of a jet in a co-flowing surrounding some distance downstream of the initial plane, in practice it can be used to predict the motion of a jet in a stationary medium as accurately as any of the much more elaborate processes used in the past.

For the case of a jet in a co-flowing surrounding, there are very few existing experimental results. However, in Figure 16 it is indicated that the experimental data obtained in References 58 and 66 for the case of a jet spreading through the co-flowing air coincides with the present theoretical results. Also, a satisfactory agreement is obtained with the results of References 5, 59, and 61 (see Figures 13, 14, and 17. The small defect at the core region where theory overestimates the actual velocity of the jet may be regarded as within the accuracy of any experimental measurement. The velocity distribution obtained by using the present theory predicts very accurately the experimental results shown in Reference 56 for an axisymmetric wake (Figure 18) as well as the experimental results of Reference 5 for the two-dimensional wake behind a circular cylinder.

For the confined jet, the results obtained by Reference 39 can be approximated by the same general velocity profile given by equation (11) (see Figures 19 and 20), where for this case the nondimensional parameter χ is given by equation (110). However, the existing agreement becomes poor downstream of the initial plane as the boundary layer of the wall of the mixing chamber approaches nonnegligible thickness relative to the diameter of the tube. If, instead of the general solution, the core part of the elementary solution given by equation (48) as stated by Abramovich (reference 42) is used, it is apparent that the velocity profile at the broader part of an ejector chamber is described with sufficient accuracy (see Figure 21).

For all the above cases, the maximum velocity distribution along the z axis may be approximated by a function of the general form of equation (24), where K may be an integer or a noninteger. The elementary distribution for a two-dimensional jet is obtained with $K = 1/2$,

and for free and confined jets with $K = 1$. These two elementary distributions predict existing experimental results (Figure 22).

However, the concept of virtual origin $z = z + z_i$ permits one to use an expression of the form

$$u_{MAX} = \frac{C}{(z + z_i)^K}$$

(indicated in Figures 22 and 23) to describe the maximum velocity variation some distance downstream of the initial plane.

Thus, the obtained solutions for the case of two- and three-dimensional jets and wakes that are of the same nondimensional general form agree very accurately with the existing experimental results. This indicates that the entrainment processes in these types of motion are of a universal nature.

Also, in many cases the velocity profile deduced from laminar flow analysis may identically describe turbulent motion. The reason for this similarity is that at a sufficient distance downstream from the initial plane, the turbulent velocity field develops independently of the mean-velocity field, and the turbulent motion becomes negligible when compared to the mean velocity of the motion. Thus, the velocity field at that region may be approximated by the corresponding velocity of a continuously developed laminar flow. When the laminar velocity profile in the intermediate region fails to describe the motion, a conveniently estimated "viscosity function" may be used to modify the laminar solution to one describing turbulent, not fully developed motion. The accuracy that can be obtained using this technique is as good as that of any of the other more elaborate methods.

For vortical velocity, as indicated in the preceding parts of this report, the existing experimental data are not of a nature which allows verification of the necessary assumption that leads to profiles representing the actual case. However, physical conditions used to restrict the set of existing solutions to those representing the actual motion are not necessarily strictly valid.

The agreement obtained between the elementary solution, equation (34-a), and experimental results of Reference 52 is indicated in Figure 25, and is not sufficient to permit the deduction of any definite conclusion. However, the existing defect between theory and experimental values may be due to existing difficulties involved in measuring flow quantities such as the vortical velocity, and the defect may also be due to some misadjustment of the general solution due to the use of conditions which do not exist for actual motion. In this respect, if the imposed condition

$$W \geqslant 0$$

is neglected,

two or three of the solutions given by equation (34) may be superimposed to approximate the actual velocity profile very accurately.

For confined vortex motion, the results obtained by Reference 50 may be approximated by the solutions given by equations (135) or (138), both of which are solutions of the linearized equation of motion and which, under proper definition of the variable χ, may satisfy all of the existing restrictions of the motion. Also, the extension of Abramovich's verification for the case of the vortex motion may be valid with sufficient accuracy only when a generalized solution of equation (34) is used. The elementary solution of this equation describes very accurately the core of the vortex but fails to accurately describe its outer regions (see Figures 26 and 27).

In the previous discussion, it has been indicated that the existence of some "negative velocity" or "back flow" may permit the prediction, with sufficient accuracy, of the outer part of axial or vortical jet or wake motions. Here, this possibility will be investigated very briefly.

In Figures 28 and 29, the first few terms of the solution given by equations (11) and (34) are plotted separately, assuming that

$$A_m = B_m = 1$$

and

$$z = 1.$$

All of these, except for the elementary solution, present some negative regions at the outer part of the jet or the vortex.

The superposition of these solutions may produce velocity profiles of the form indicated in Figures 30 and 31. Such velocity profiles have been obtained theoretically in the past by Terazawa and Shigemitsu (references 63 and 57) for the decay of two-dimensional vortex motion. In searching for experimental evidence that may indicate the existence of such velocity profiles, the experimental work of Reference 64 on the formation and growth of vortices behind plates may be recalled where, in many of the obtained pictures, the outer part of the main vortex is split into secondary vortices representing regions of "back flow" or counterrotating motion.

The same result was obtained for the case of a trailing vortex in Reference 65 (see Figure 3). Although in this case the vortex is formed in ground proximity, there is a noticeable ground effect in the lower part of the outer region of the vortex. Similarly, a "negative velocity" region can be obtained at the boundary of a jet or at the wake of a cylinder or sphere.

In Figures 4 and 5, the velocity field in the wake of a cylinder is

indicated by using the newly developed flow visualization technique, smoke generation on a hot wire (reference 67), where the region of back flow at the middle part of the wake is very large.

Thus, in general, it is reasonable to use velocity profiles of equation (11) or equation (34) with the nondimensional variable χ given by equation (66) or equation (84) to describe jet-wake and vortex motion.

At this point it may be noted that the obtained solution of the simplified equations of laminar motion, with the aid of the "viscosity function", can predict with very good accuracy all the existing experimental results, and are of reasonable agreement with all the existing theories.

The question now arises as to why all these theories based on so questionable an assumption as "mixing length", "eddy viscosity", and "viscosity function" produce results that are so close to the experimental results. To answer this question, the important role of dimensional considerations and of boundary and physical conditions in the solution of all the problems must first be realized. If, for instance, the process and results in the analysis of jets or wakes of this report and of References 2, 3, 5, 8, and 16 are evaluated, it is quickly observed that both the form of the transformed partial differential equation of motion and the final solution obtained are largely based on these considerations and conditions. Also, the quantities of primary importance related to the described phenomenon are not very highly affected by any of the assumptions made in relating stress to rate of strain. Thus, the same results are obtained following more than one assumption when, in many cases, these assumptions are contradictory ones.

However, it must be noted that, at the present time, none of the semiempirical formulas used by turbulence theories (references 13, 14, 15, and 17) can produce analytic expressions for the velocity profiles of the different flow motions, since each one of them is valid only for a specific problem or a category of problems. A generally valid theory for the representation of turbulent motion does not exist, and it is probably more attractive, at least from an engineering point of view, to treat the equation of motion by assuming an unknown "viscosity function" that can be determined by using the experimental results. This could be done in such a way that the final analytic expression for the velocity profiles would agree with those results. This approach can be used instead of introducing some semiempirical relation which, in many cases, would complicate the differential equation of motion and finally produce results that would be valid in a limited flow field region. Furthermore, all the previously mentioned theories are developed for two-dimensional motion, and they cannot be used for the present axisymmetric motion.

In closing this discussion, the existing similarities of the axial and rotational momentum equations of motion in the conveniently transformed coordinate system that can be used identically to describe jet-wakes and vortices should be emphasized. Also, the obtained solutions, equations (11) and (34), are of the same nondimensional characteristic form, and may be used for free and confined cases by proper adjustment to the corresponding boundary and physical conditions for every case. So, on the basis of the present theoretical analysis and all the existing and above-mentioned experimental results, it may be stated that the laws governing the entrainment process of jet-wakes and vortices for laminar and turbulent motions and for free and confined surroundings are of the same universal nature.

Finally, if critical experiments are performed, as suggested in the body and the appendixes of this report, to supply the additional information necessary for the calculation of the different induced constants, the given analytic relations between the flow or jet ejector component and parameters will be an extremely useful tool for any engineering application.

CHAPTER SIX. SUMMARY AND CONCLUDING REMARKS

On the basis of an extensive account of the present ideas and development on the subject of jet-wakes and vortex flow, the general principles of the motion of a free, laminar jet have been analyzed and reviewed, and from an extension of these principles, an understanding of the very complex behavior of a turbulent jet has been obtained.

Also, under a more general assumption that the term "jet" means the motion of a fluid on a side of a tangential separation surface, the similarities of the entrainment process of the flow field of jet-wakes and vortices were investigated. More specifically, using a linearized form of the Navier-Stokes equations of motion, a set of solutions was obtained which describes the axial, rotational and radial laminar velocity profiles.

In order to verify the possibility of extending laminar solutions to describe turbulent cases, the general equations of motion have been reconsidered using the existing "statistical" and "free turbulent" theories.

It was then proved that, by introducing a "viscosity function" to the simplified equations of laminar motion, solutions of the same general analytic form, but with a different nondimensional variable, can be obtained which are capable of describing the turbulent cases.

For the case of confined jets and vortices, two different approaches have been used to simplify and solve the equations of motion, one valid very close to the initial plane, and one valid some distance downstream. The set of solutions obtained is of the same nondimensional form. Using the elementary solutions of this set, analytic expressions for the different characteristics involved in the design of jet ejectors were calculated. Also, a process for the estimation of the different parameters for the optimum design of a jet pump or thrust augmenter was included.

It was finally indicated that the generalized nondimensional analytic expressions, calculated by this study to describe the different field components of free or confined jet-wakes and vortices, are in very good agreement with all the existing experimental results. On the basis of this agreement, it is concluded that the linearization techniques used herein and the introduction of the "viscosity function" may be used to predict all those velocity fields as accurately as with any of the other elaborate methods of the past.

Also, by comparing the conveniently transformed basic equations of motion and the final form of the solutions obtained to describe laminar and turbulent jet-wakes and vortices in free and confined surroundings, the similarity of the mechanisms of the entrainment

process of these flow fields proposed by Cornish (reference 68) is supported.

Finally, for vortex motion and confined jets, some critical experiments are suggested in the body and appendixes of the present study, which would be useful in supplying some additional information necessary for the calculation of the different induced constants, or for the estimation of the validity of some of the boundary and physical conditions used.

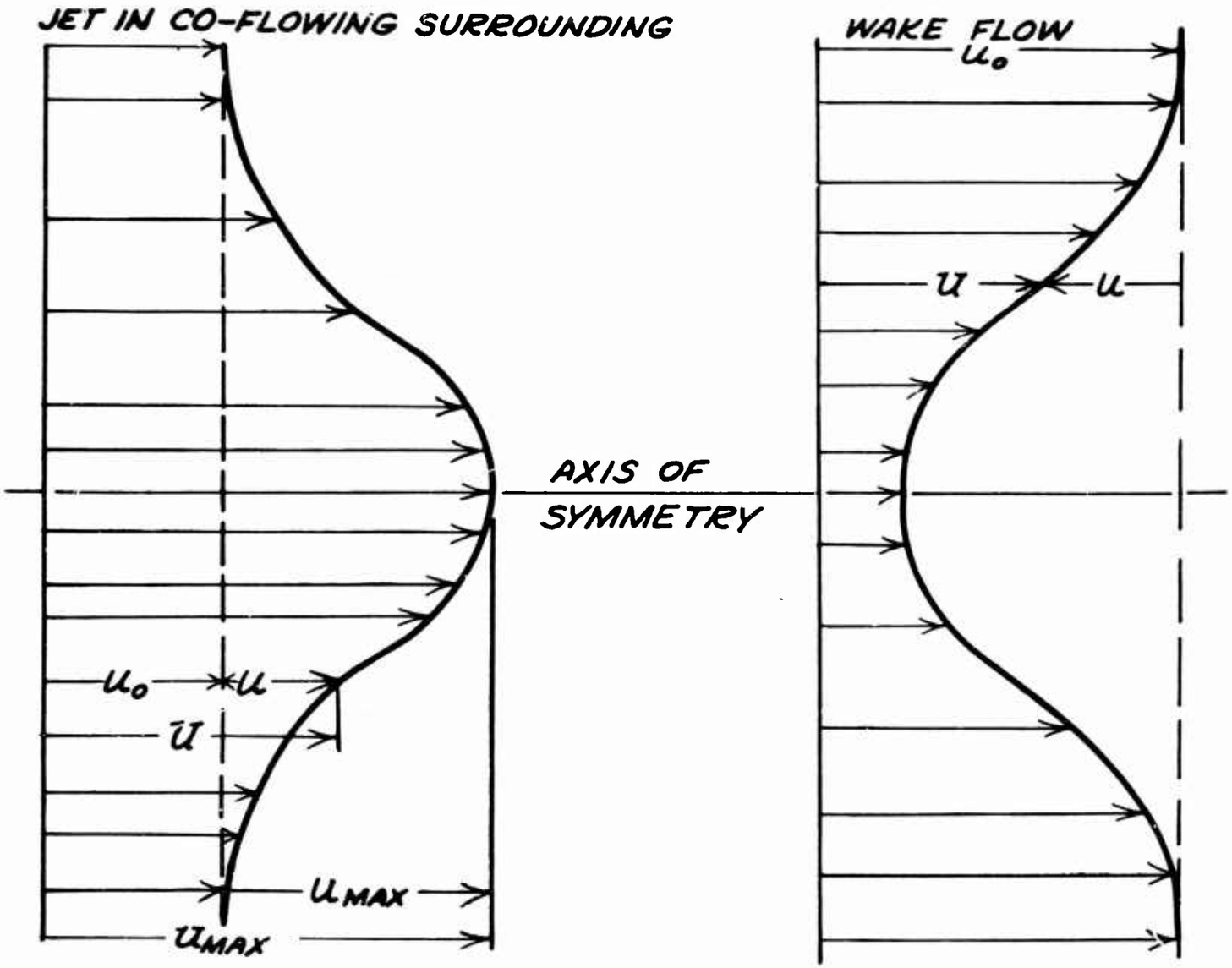

Figure 1. Jet and Wake Axial Velocities.

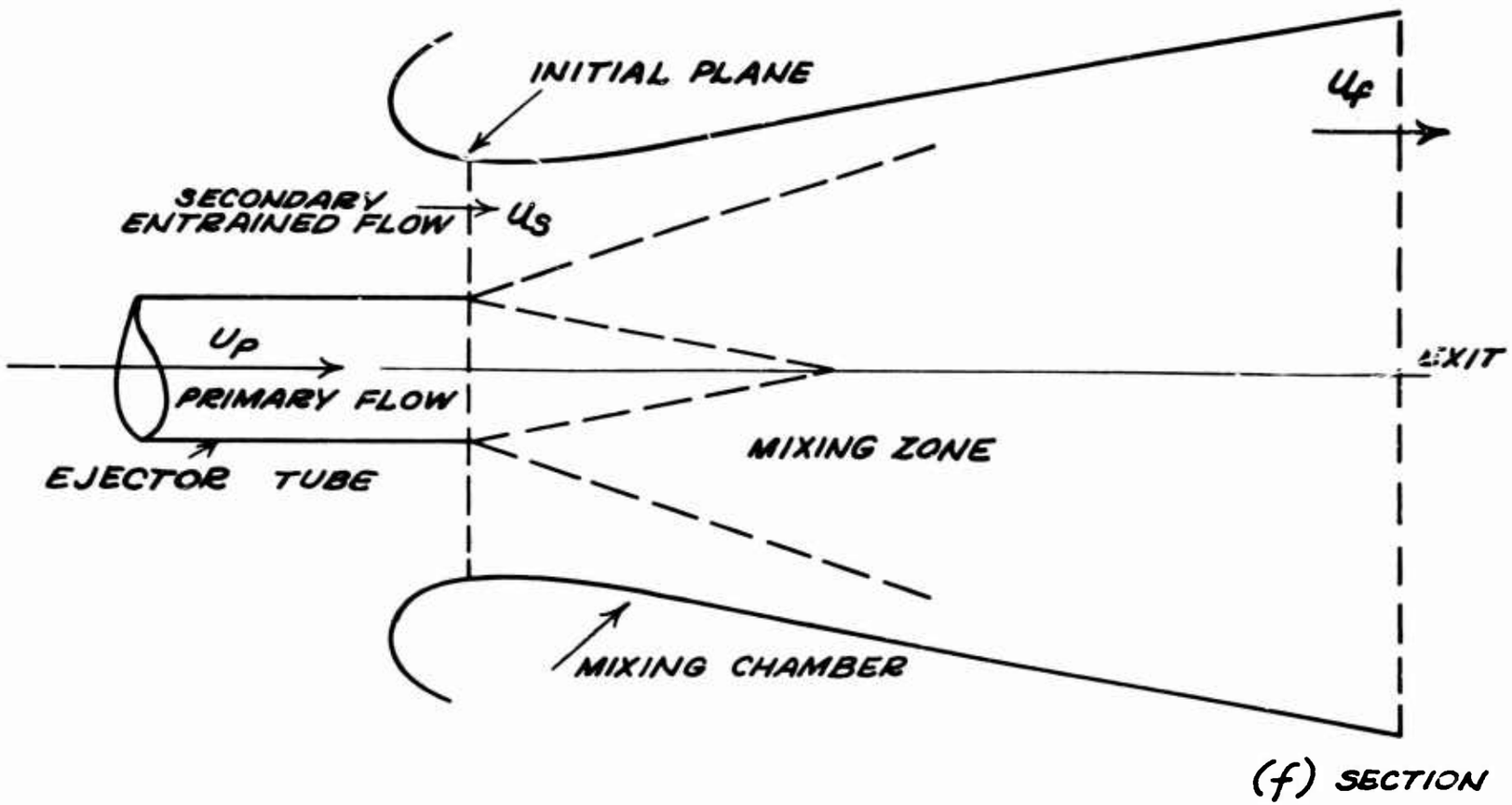

Figure 2. Jet Ejector, Nomenclature and Flow Regimes.

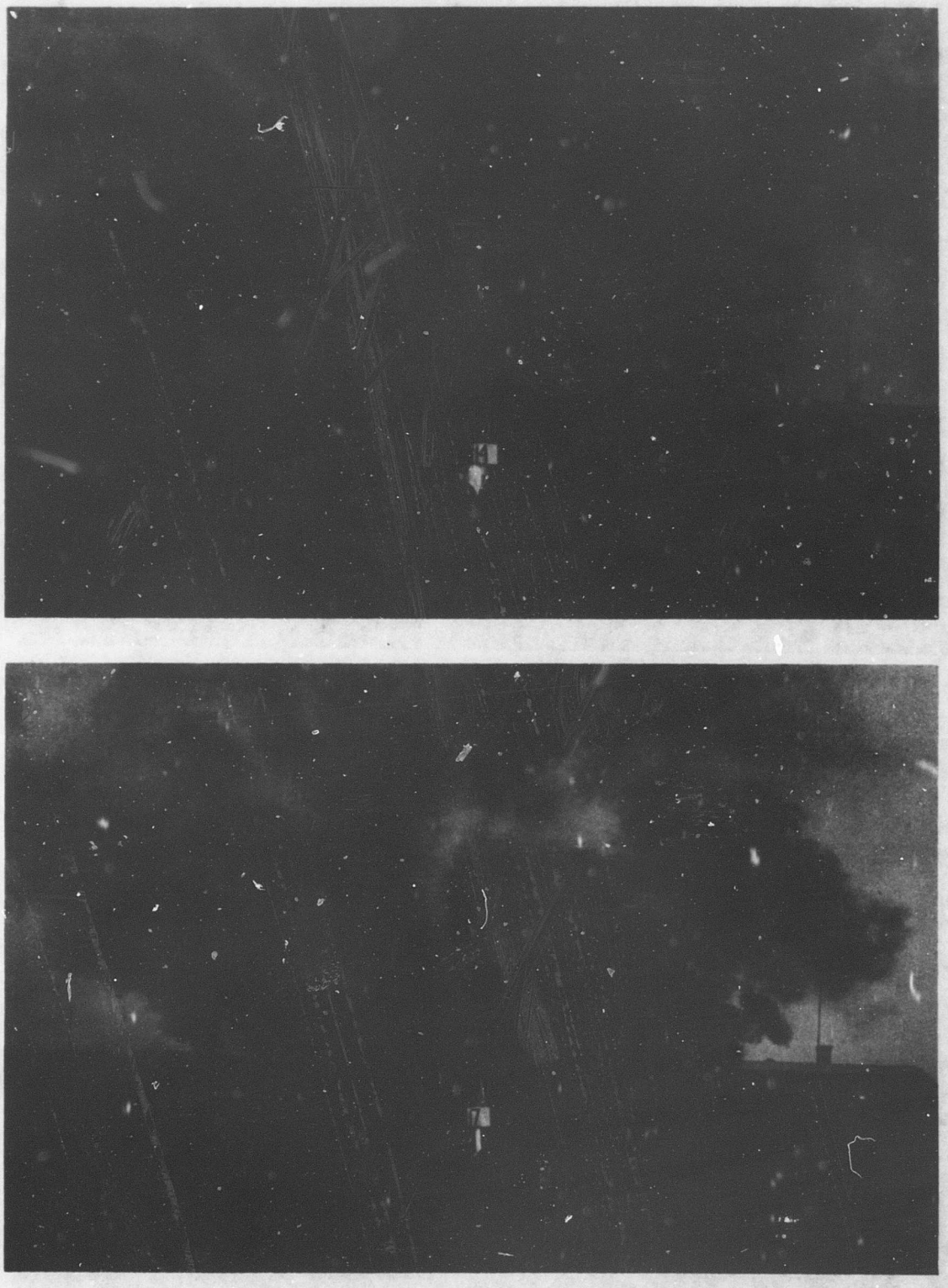

Figure 3. Tip Vortex in Ground Proximity.

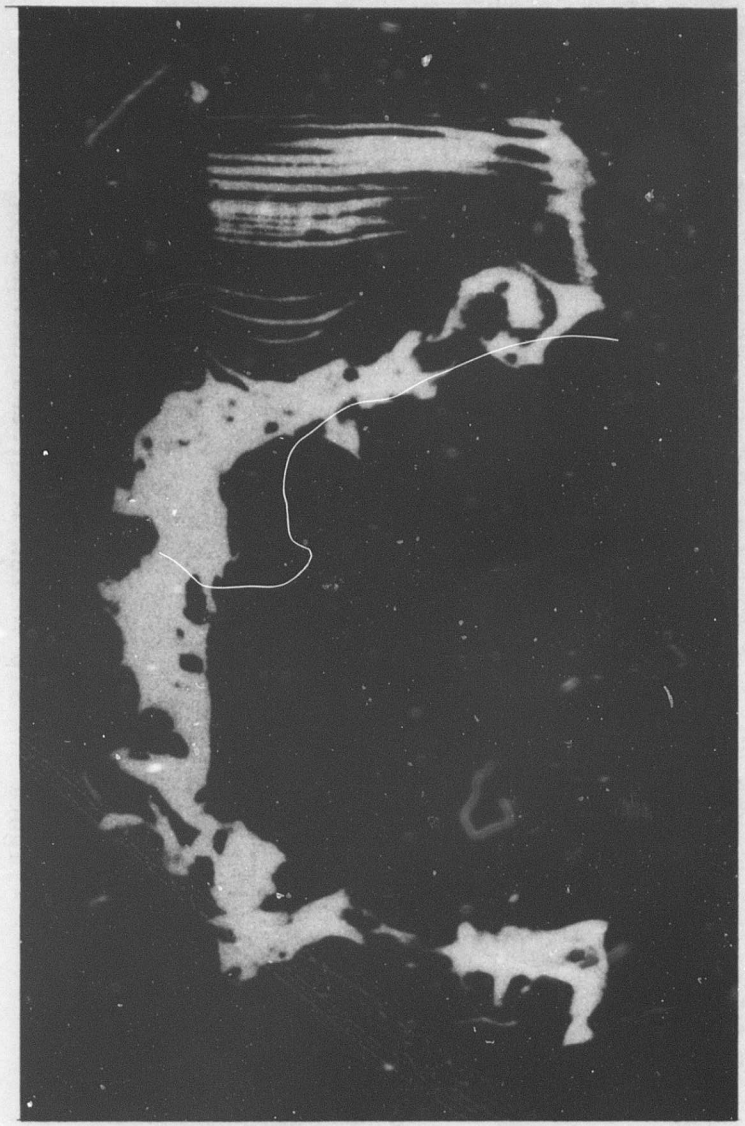

Figure 4. Wake of a Cylinder.

Figure 5. Wake of a Half-Sphere.

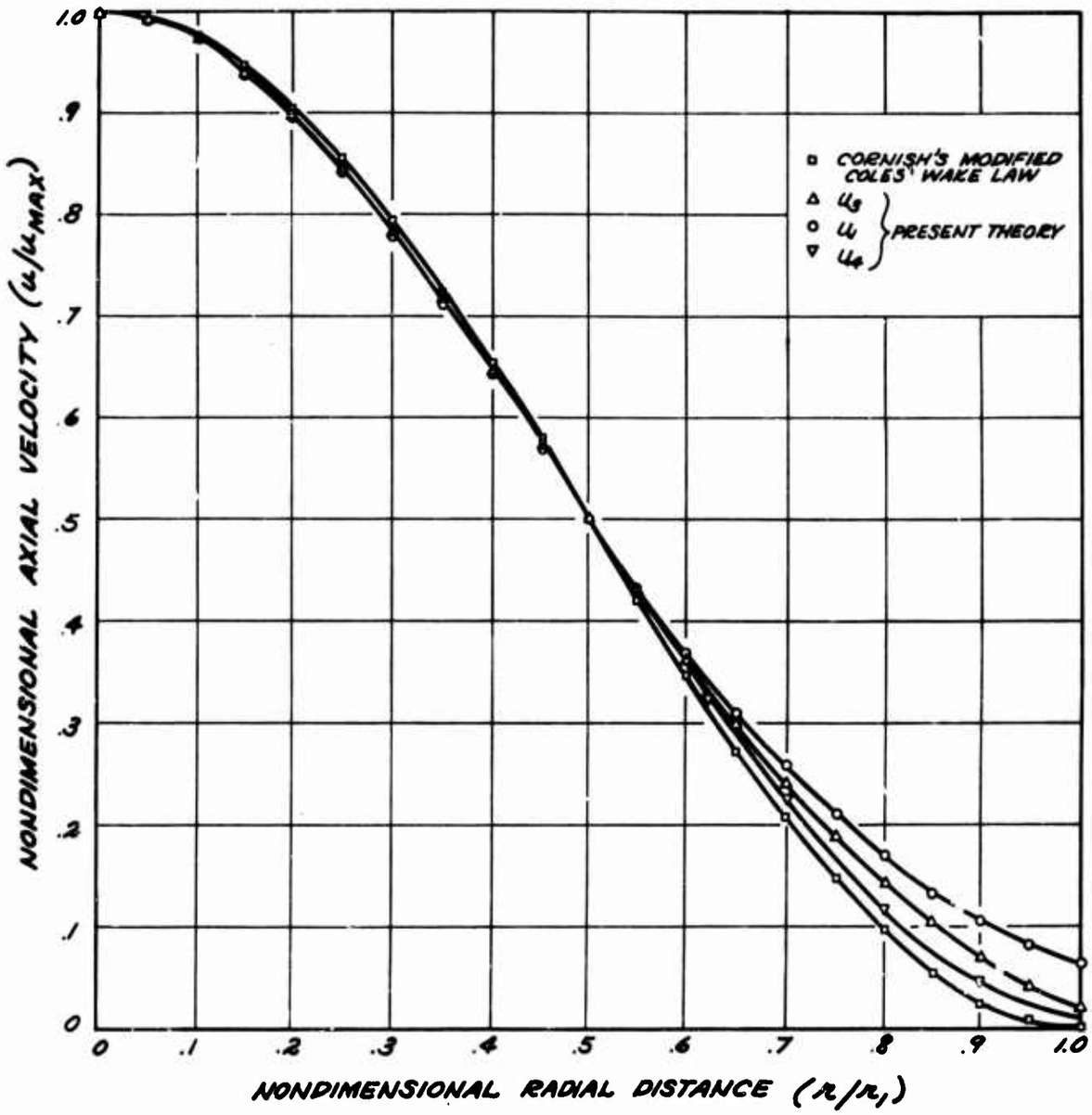

Figure 6. Cornish's Modified Coles' Wake Law and Present Theory.

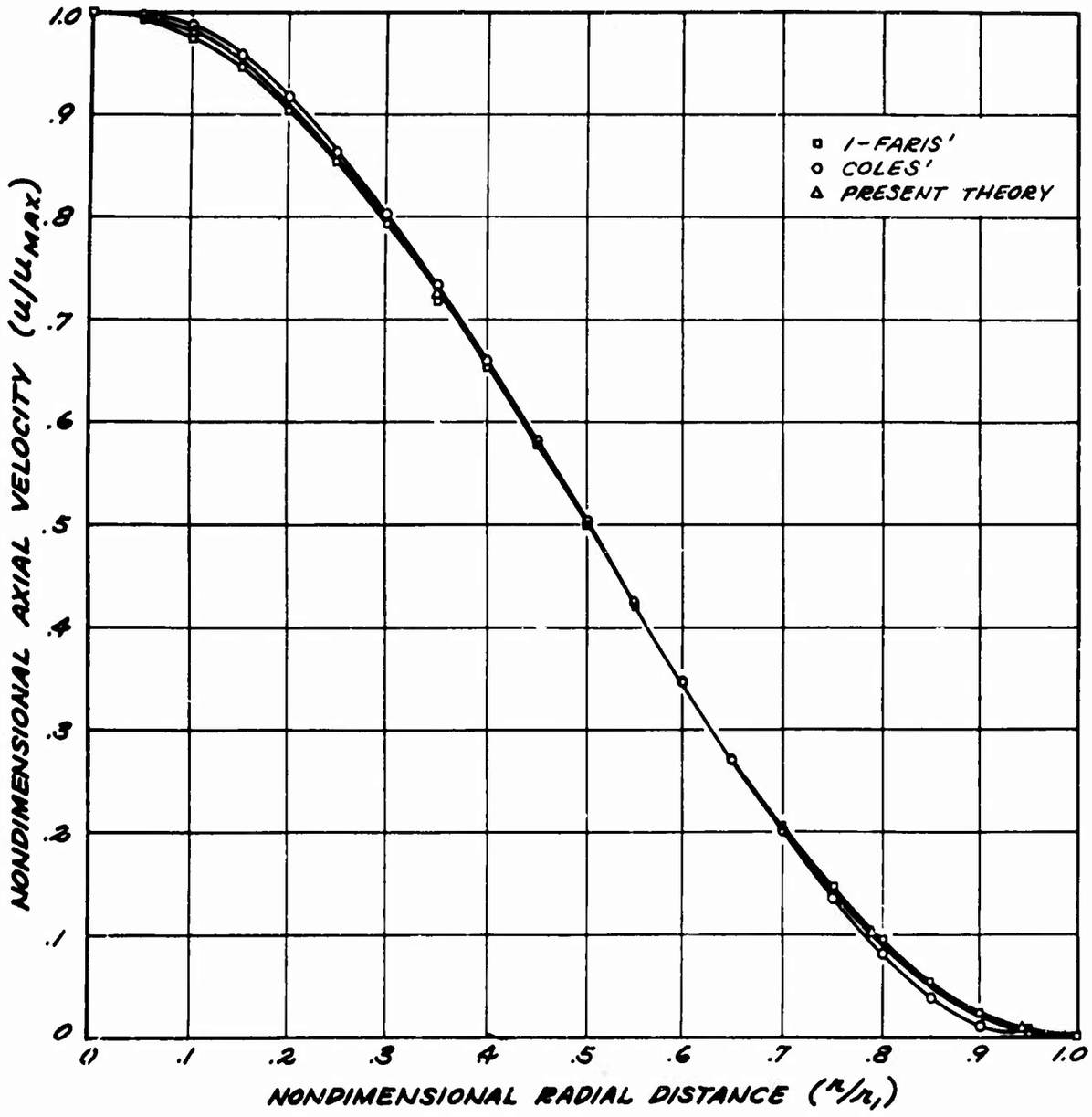

Figure 7. Faris' and Coles' Wake Law and Modified Present Theory.

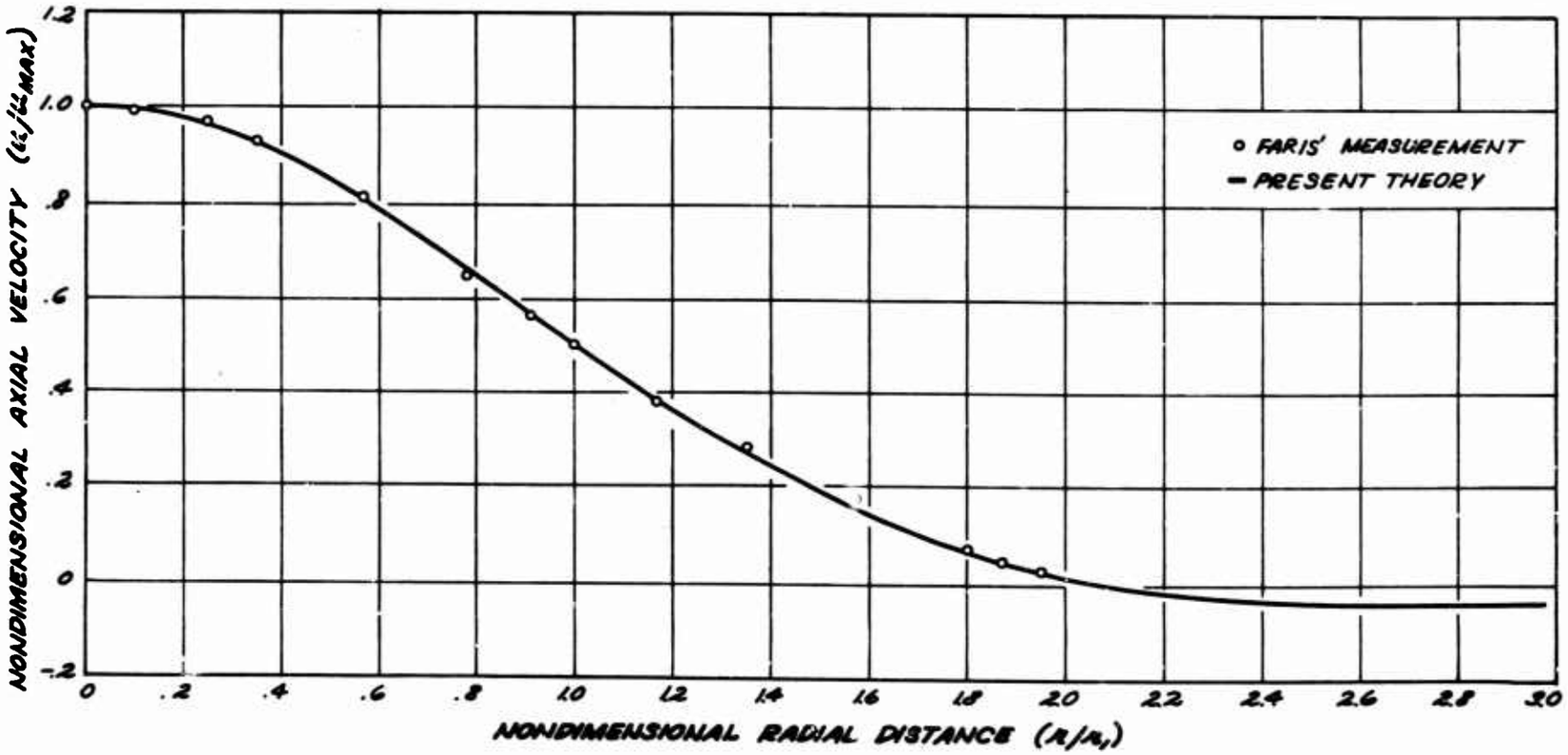

Figure 8. Faris' Experimental Value and Present Theory.

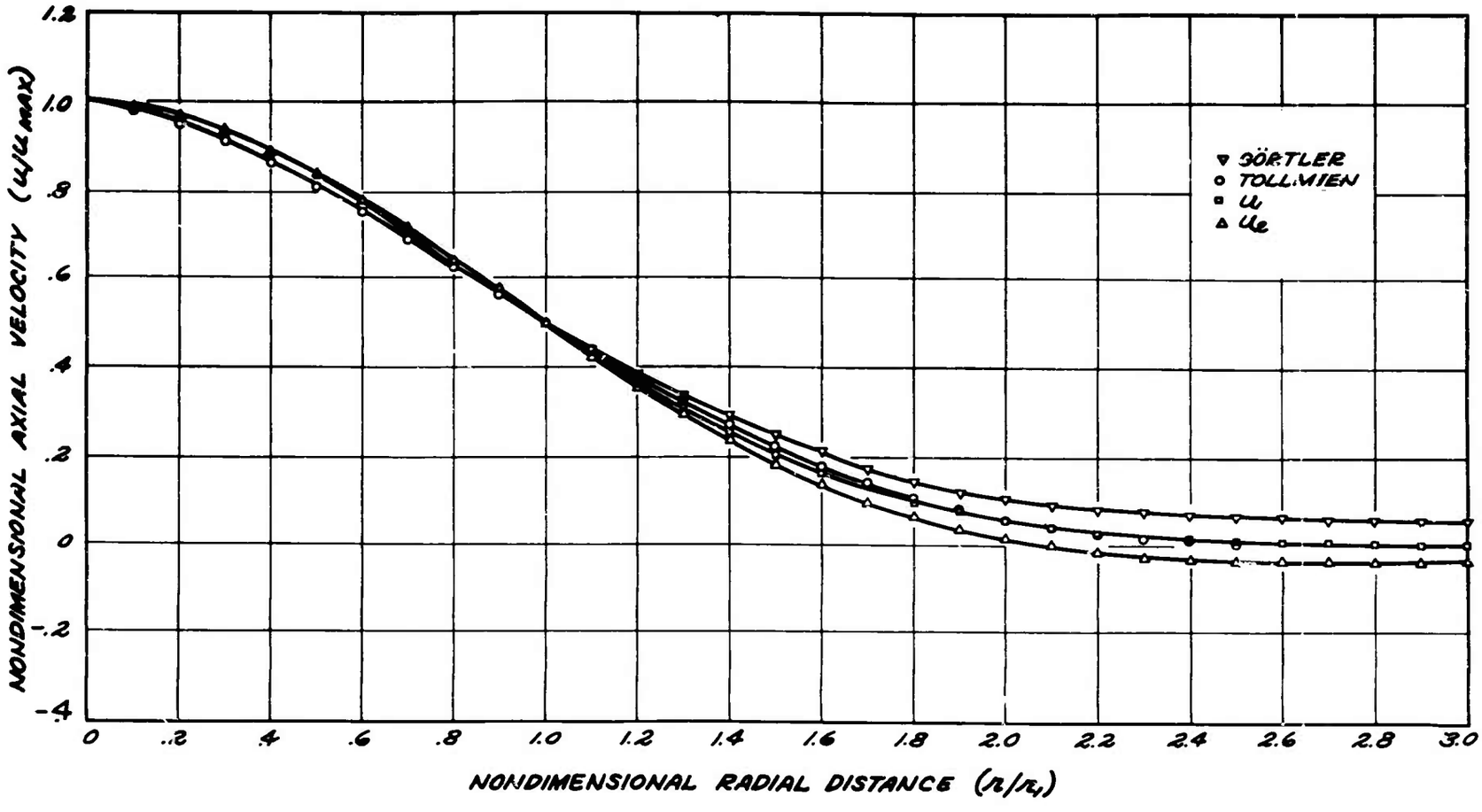

Figure 9. Comparison of Görtler, Tollmien, and Present Theories.

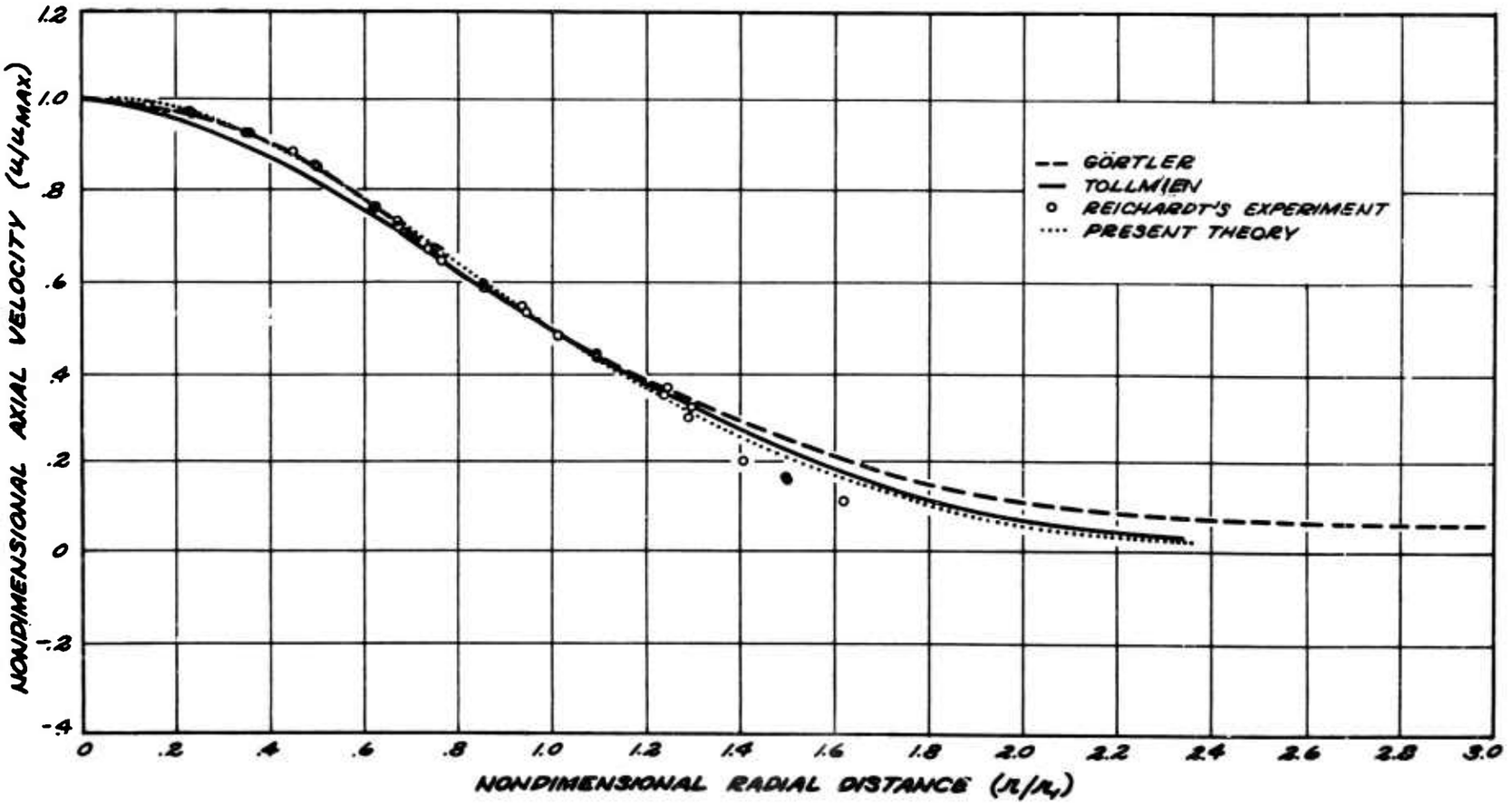

Figure 10. Comparison of Different Theoretical Velocity Profiles to Reichardt's Experiments.

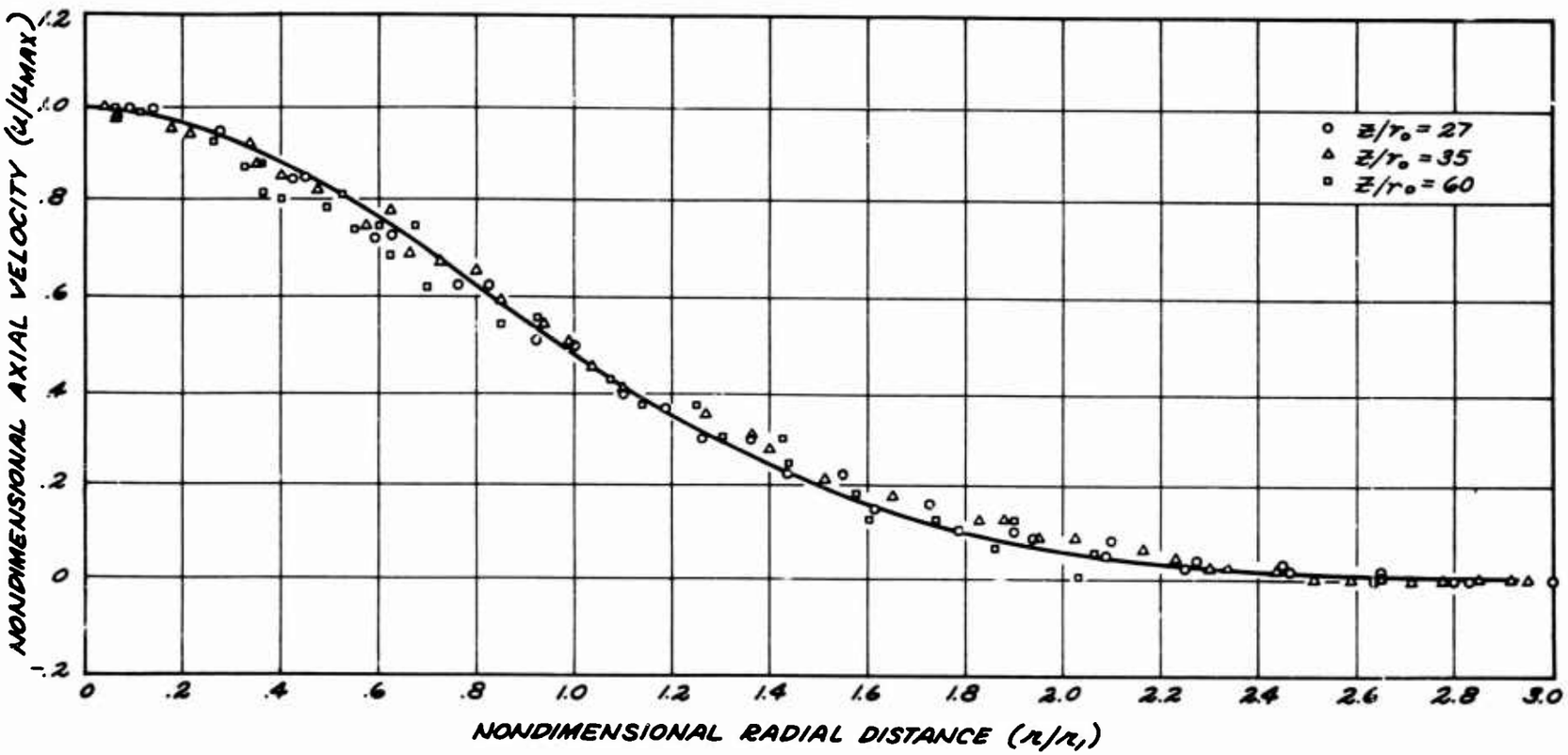

Figure 11. Reichardt's Experimental Results for Axisymmetric Jet and Present Theory.

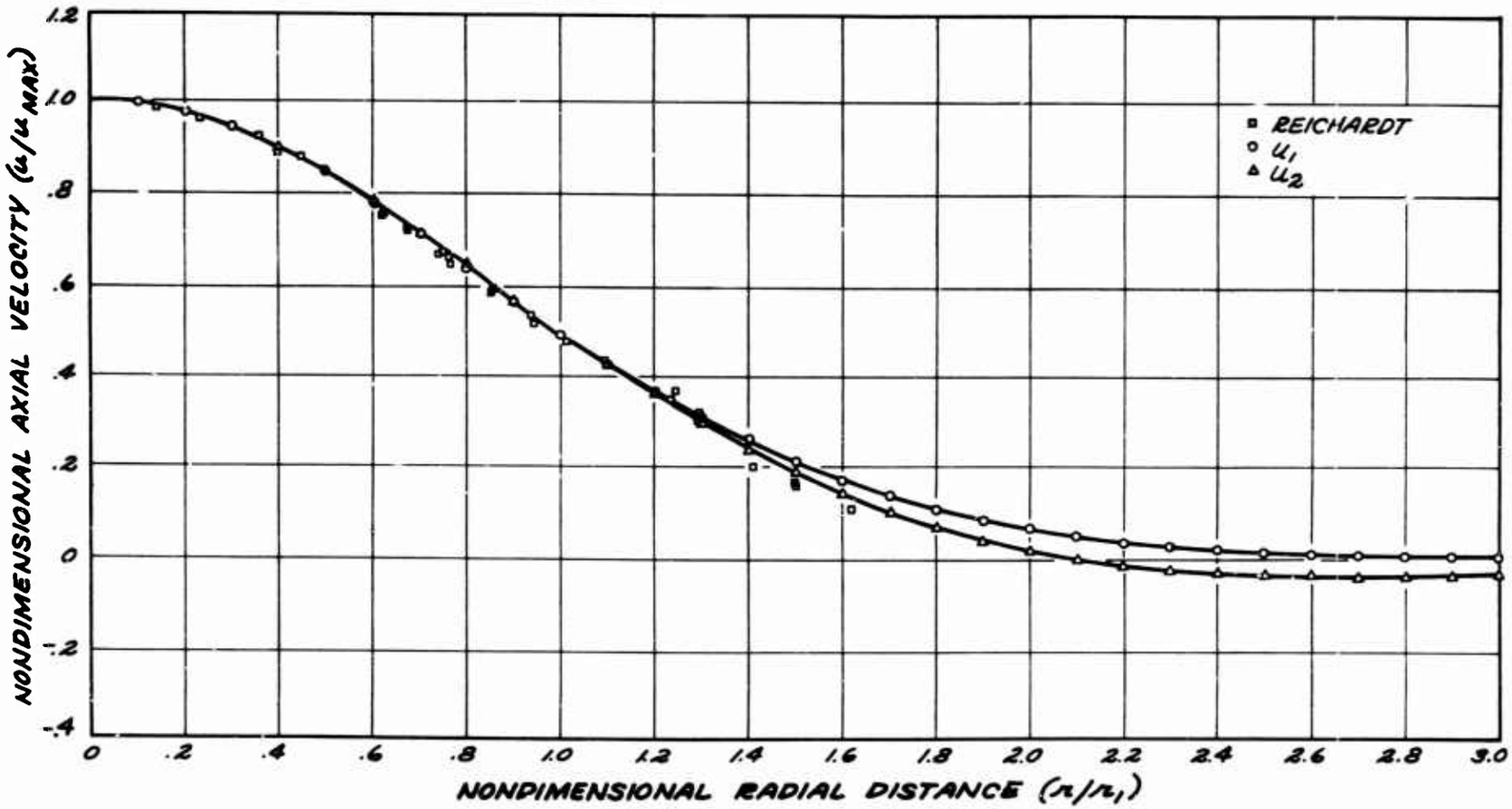

Figure 12. Reichardt's Experimental Results for Two-Dimensional Jets.

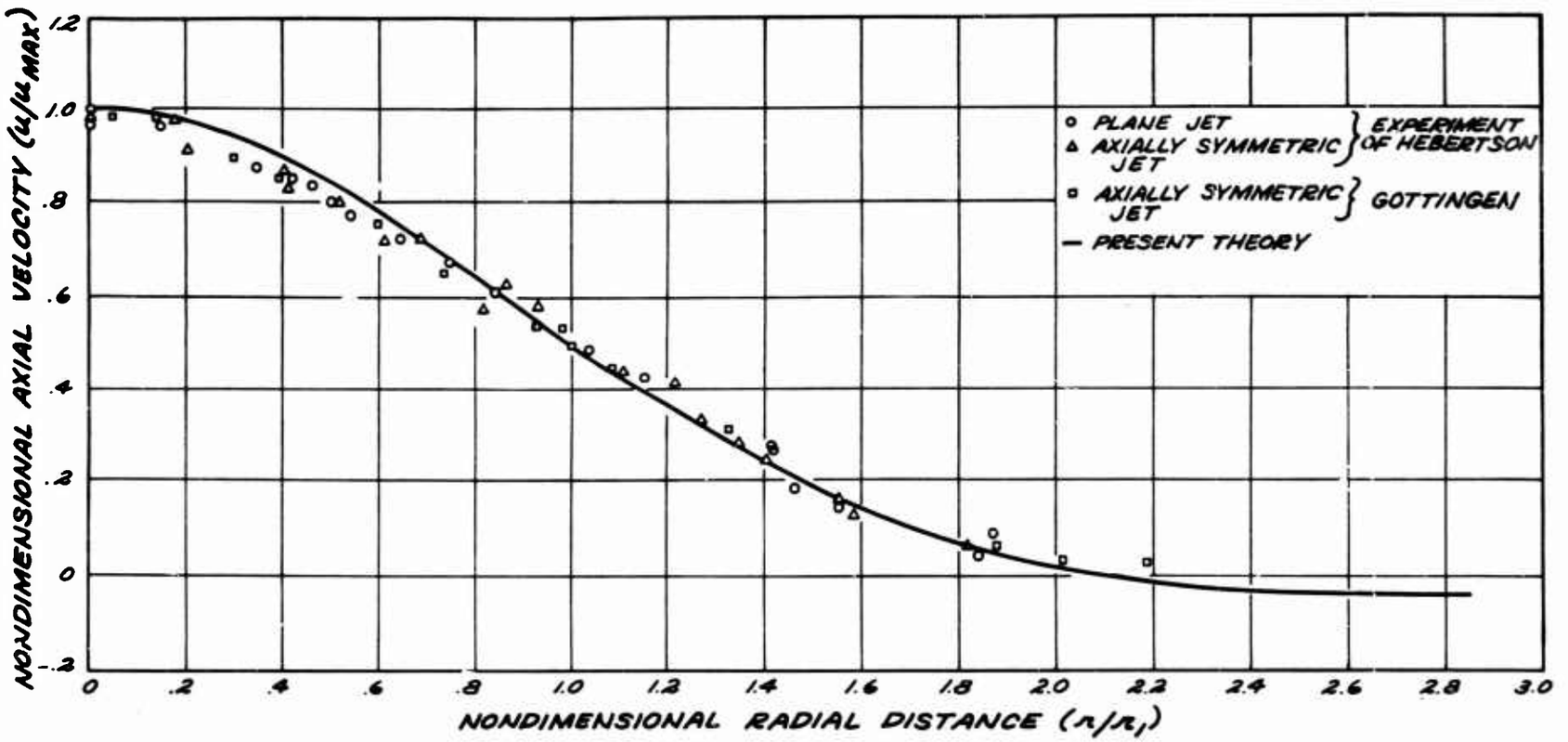

Figure 13. Comparison of Excess Velocity Experimental Profiles and Present Theory.

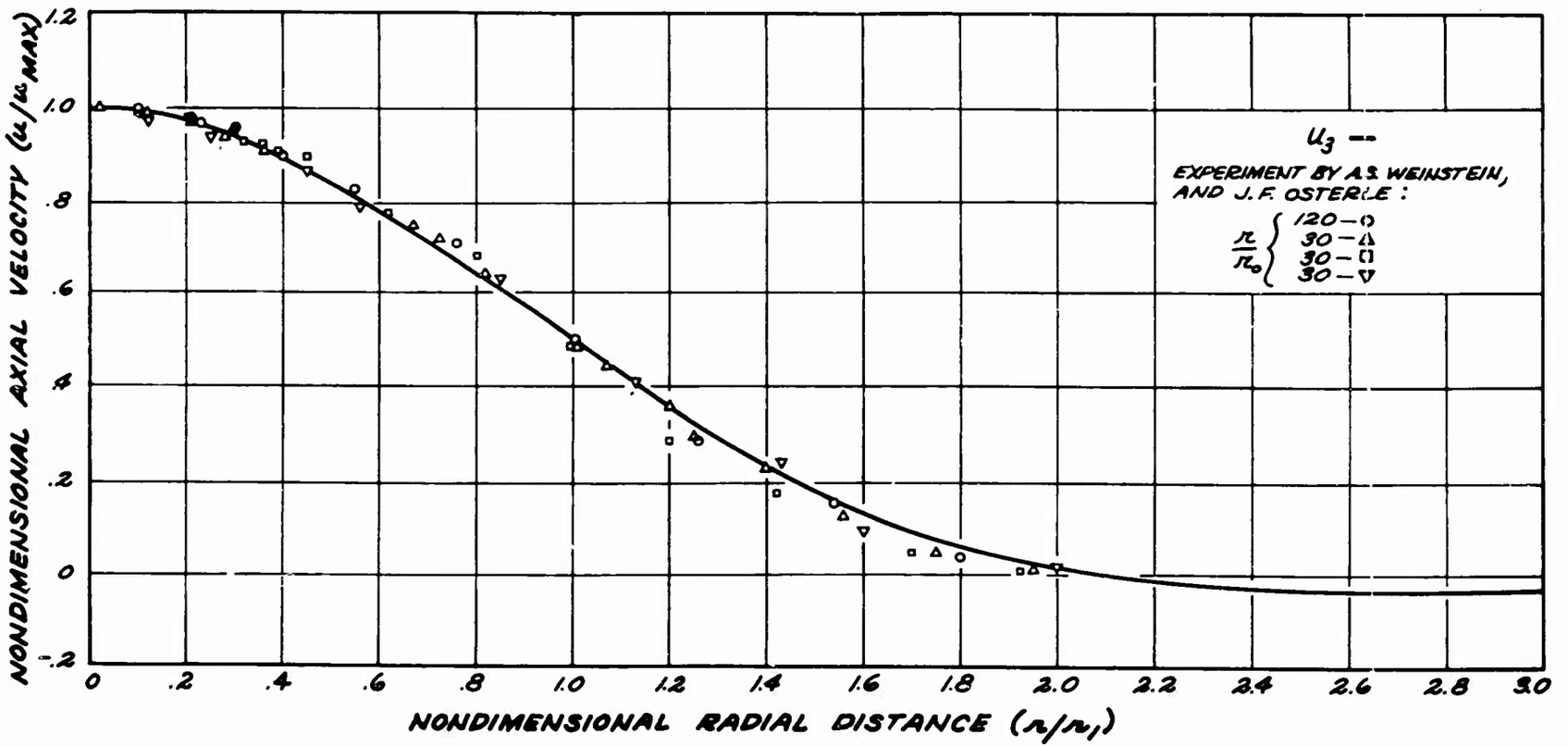

Figure 14. Experimental Two-Dimensional Excess Velocity Profile and Present Theory.

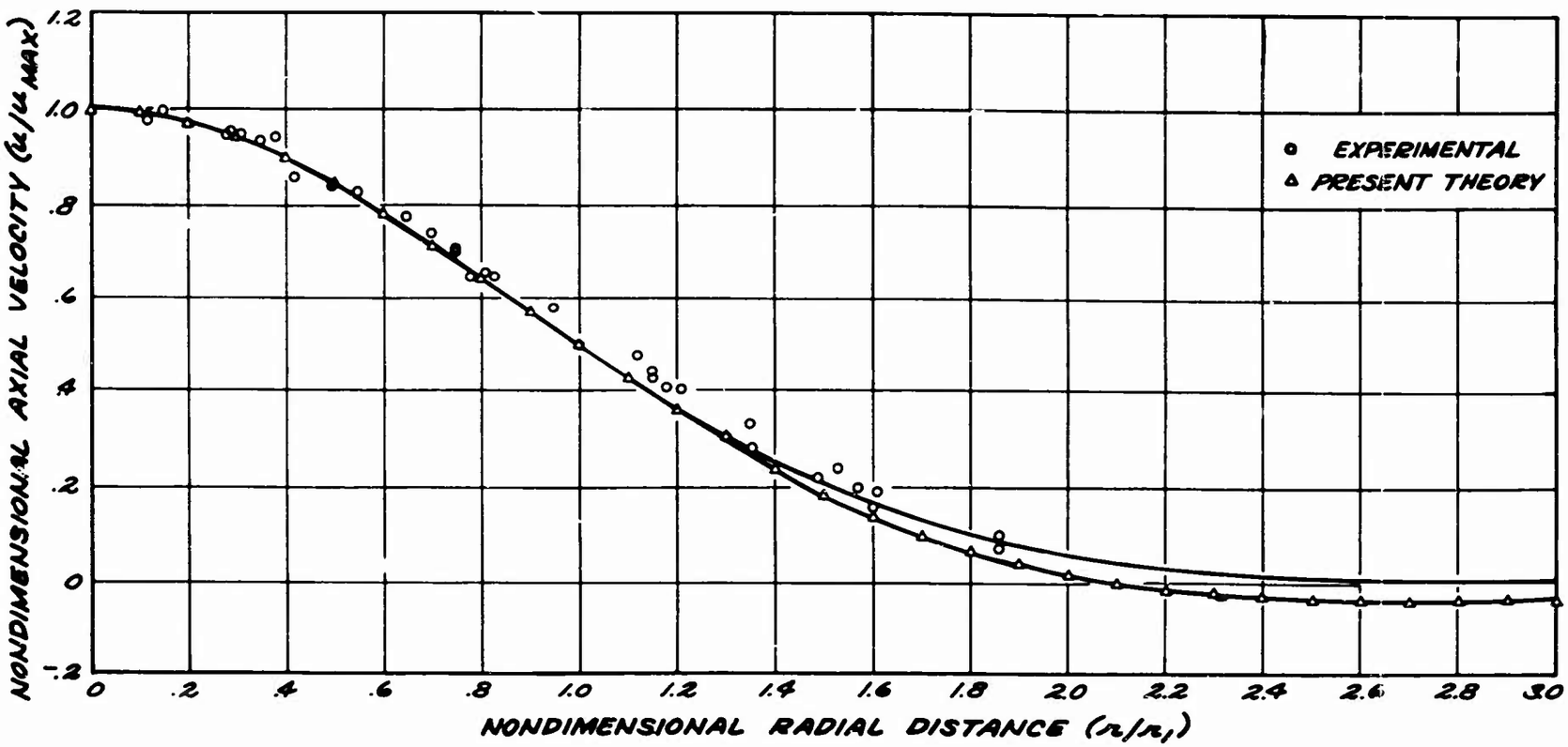

Figure 15. Velocity Distribution in a Two-Dimensional Turbulent Jet Due to Forthmann.

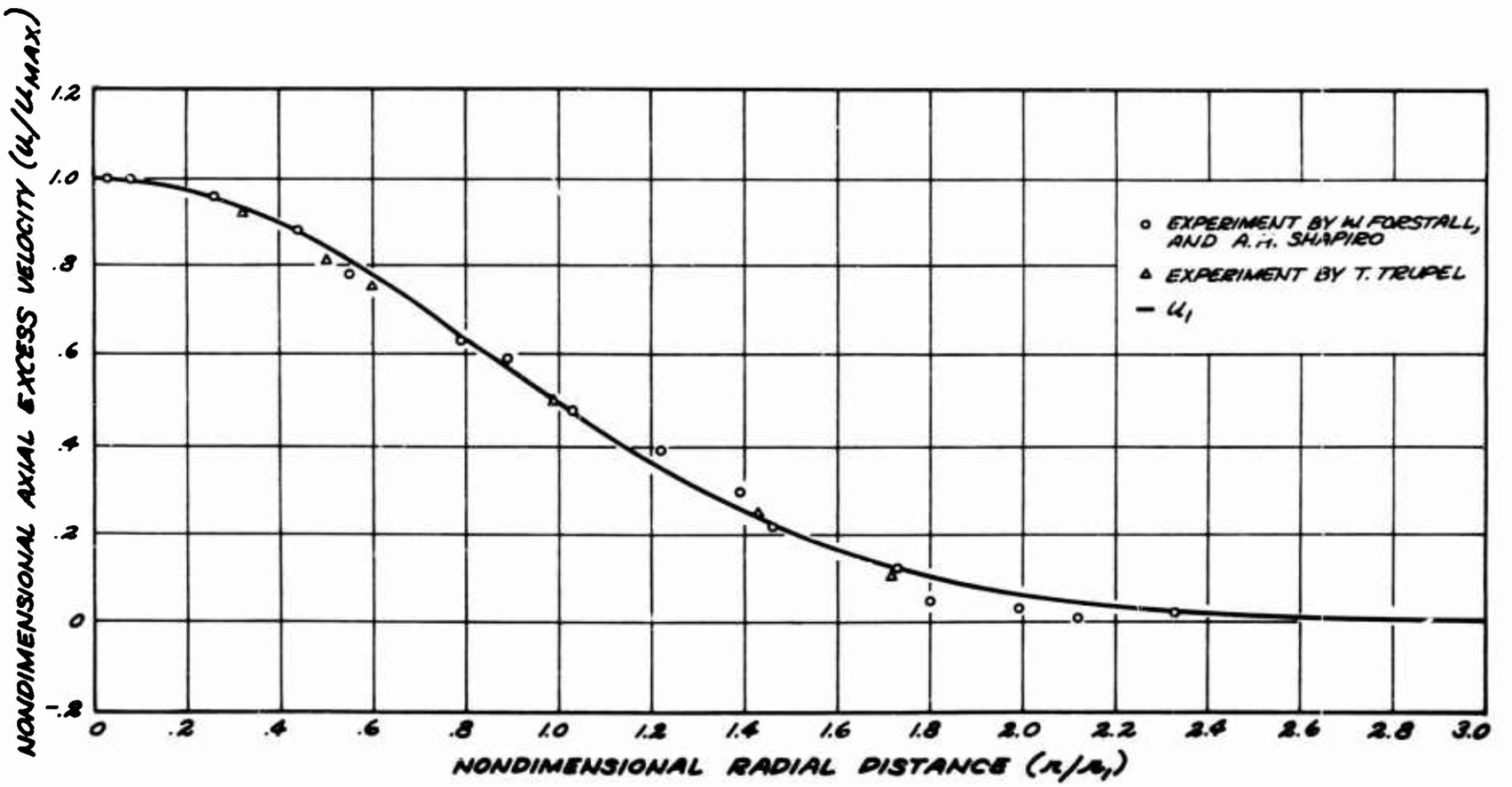

Figure 16. Present Theory Excess Axisymmetric Velocity Profile and Experimental Results of References 58 and 66.

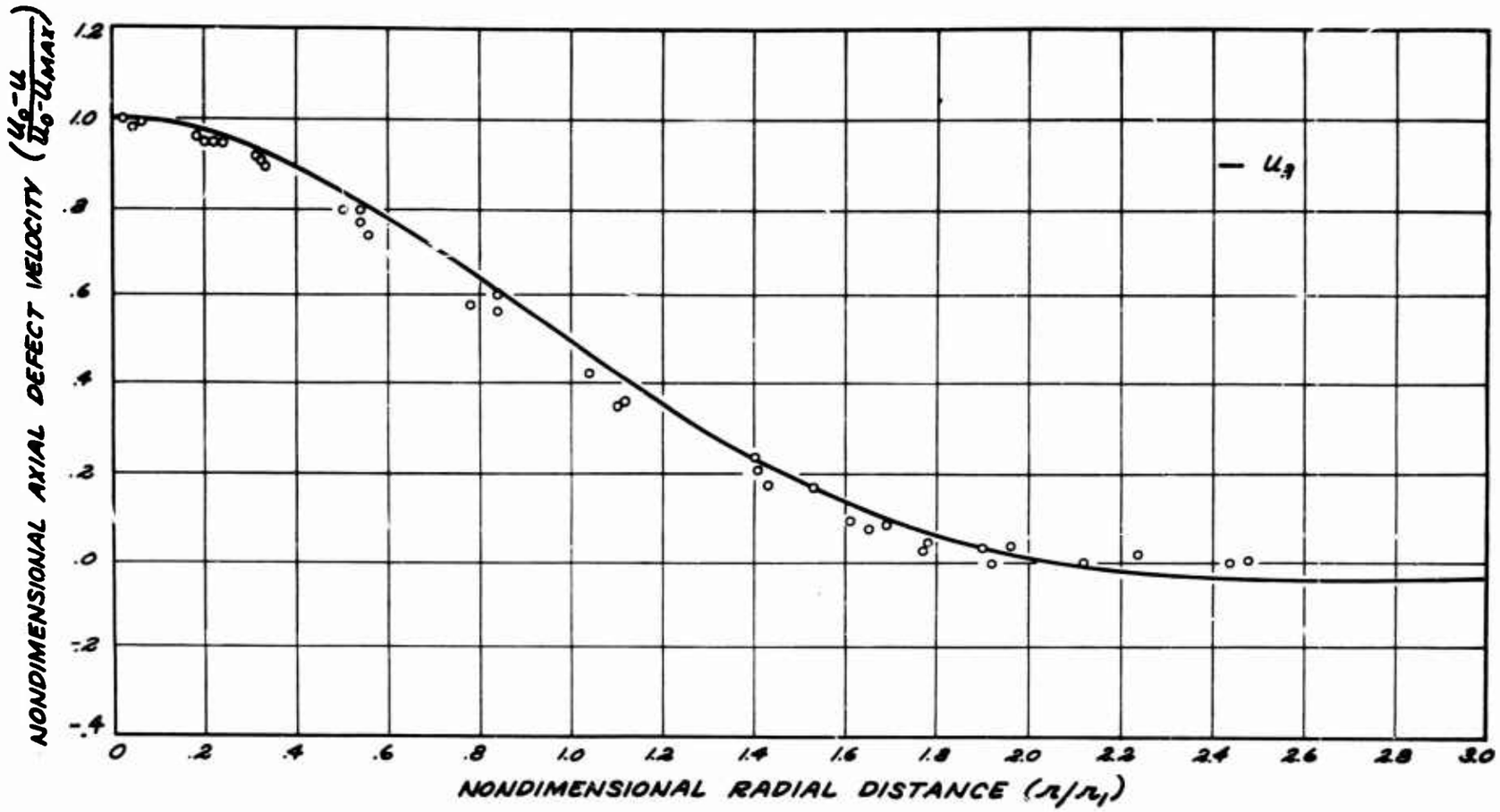

Figure 17. Velocity Distribution in a Two-Dimensional Wake Behind Circular Cylinders, Present Theory-Measurement Due to Schlichting.

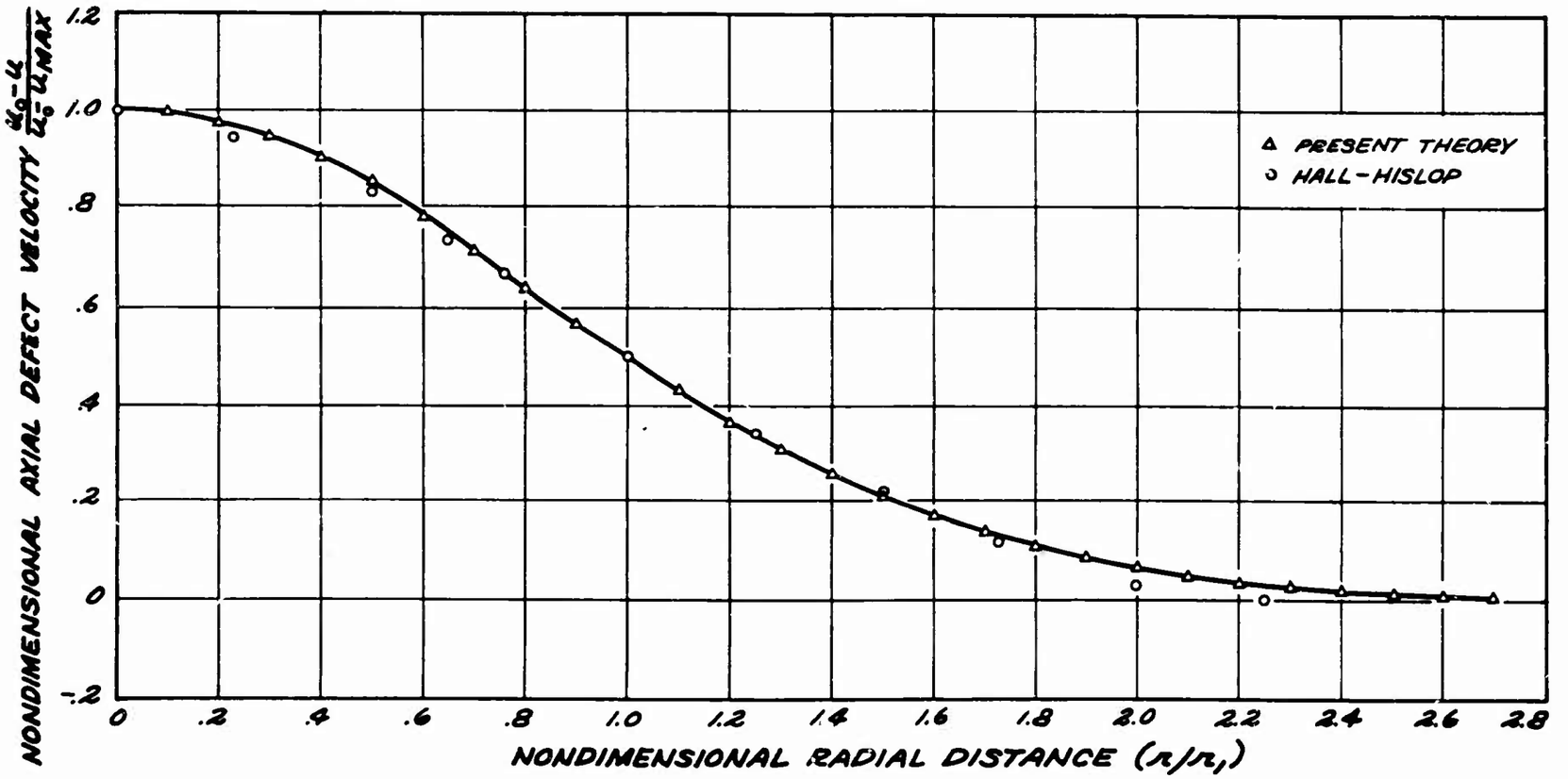

Figure 18. Comparison of Axisymmetric Wake Experimental Results of Hall-Hislop to Present Theory.

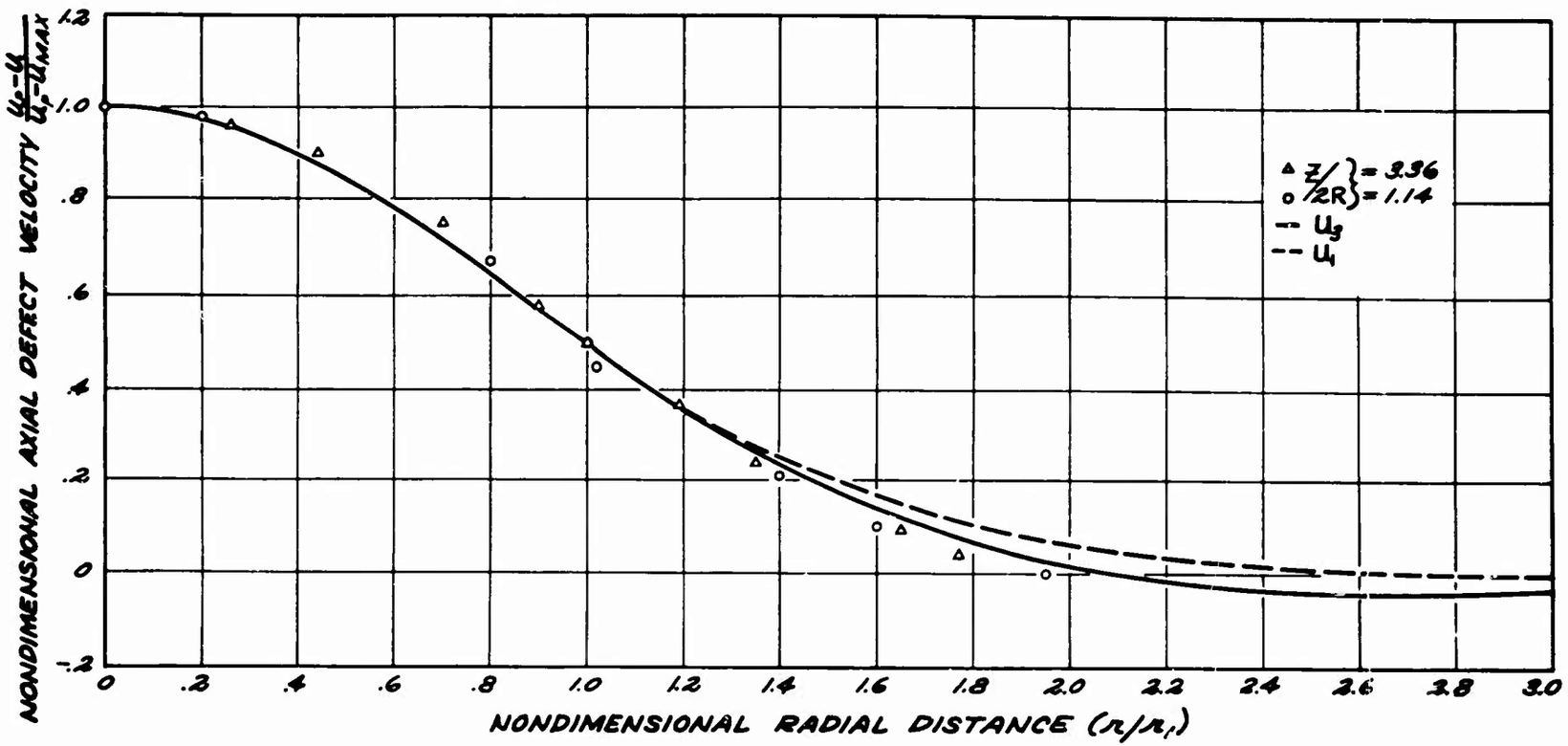

Figure 19. Theoretical and Experimental Velocity Profiles in the Inlet Length of an Ejector.

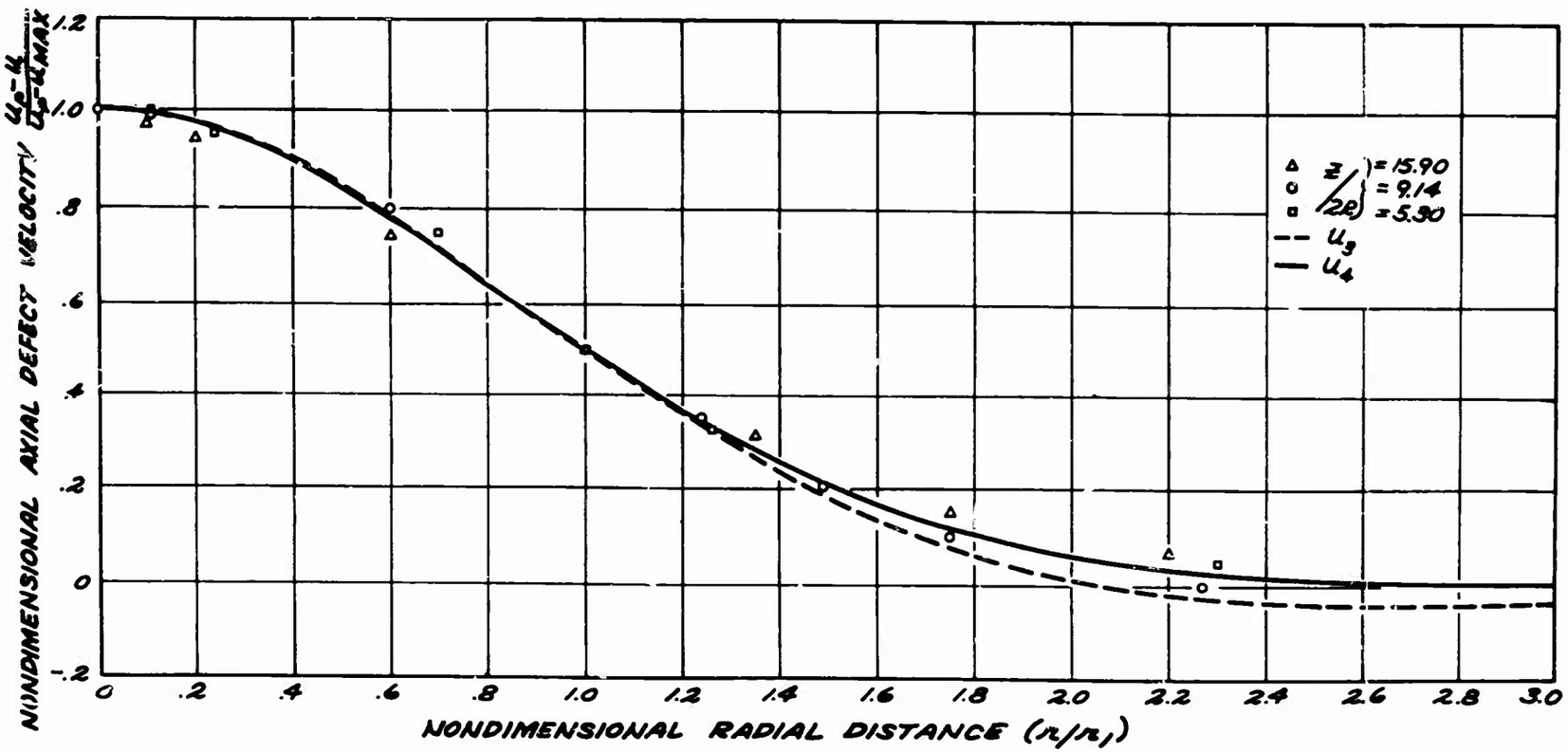

Figure 20. Theoretical and Experimental Velocity Profiles in the Inlet Length of an Ejector.

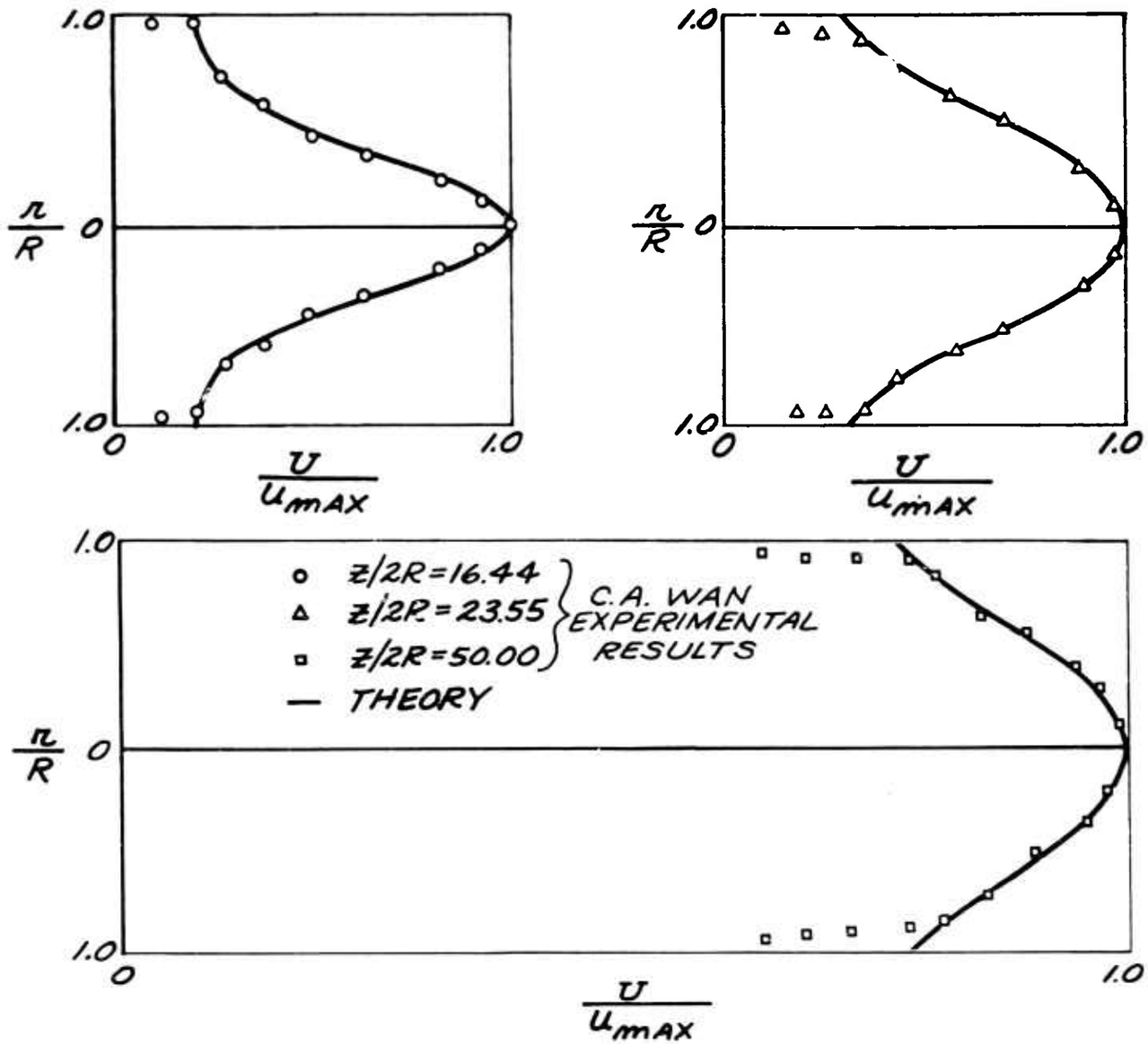

Figure 21. Comparison of Free-Jet Theoretical Profiles to Experimental Confined Profile.

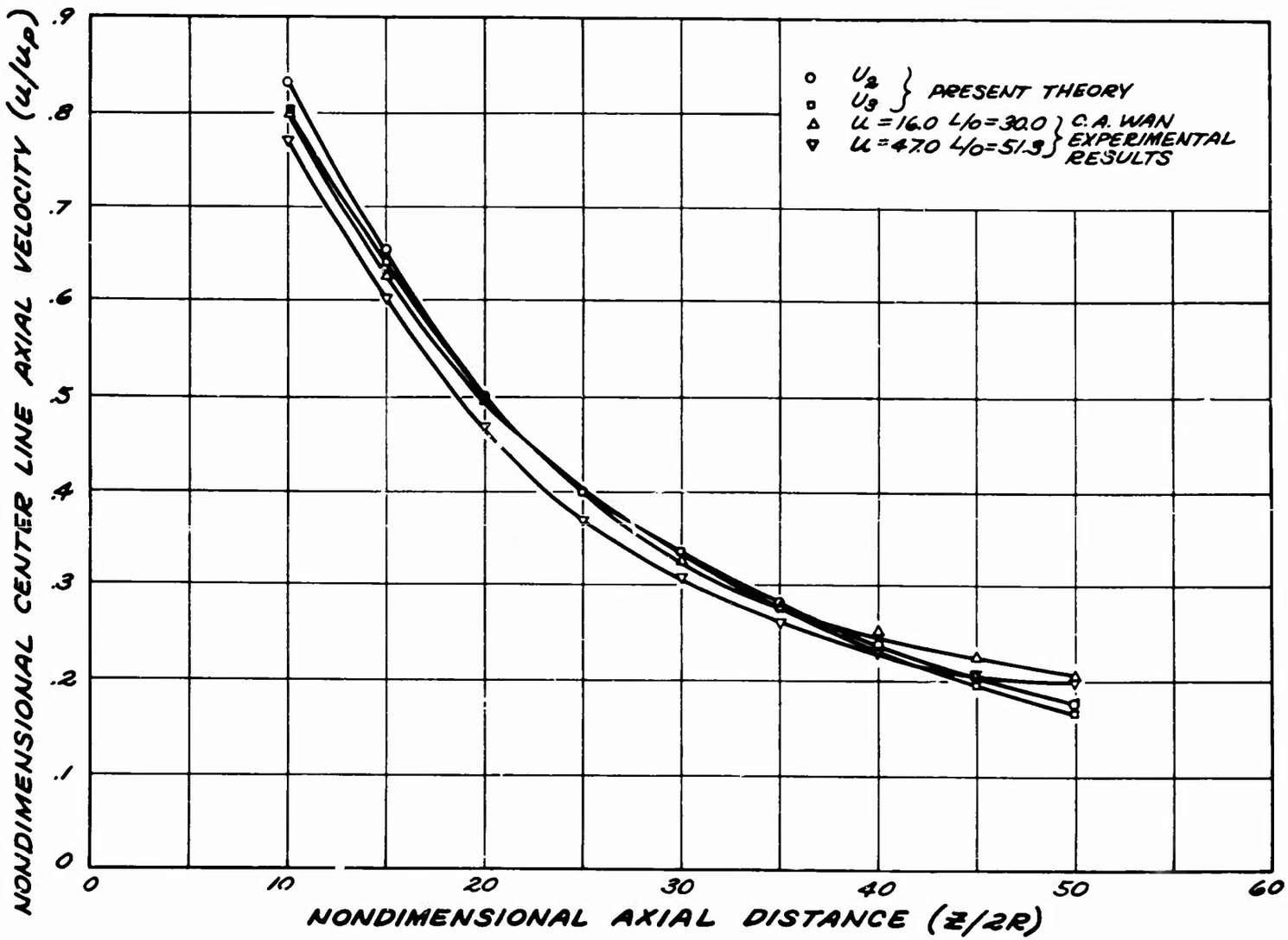

Figure 22. Comparison of Theoretical and Experimental Center Line Velocity Distribution in an Ejector Mixing Chamber.

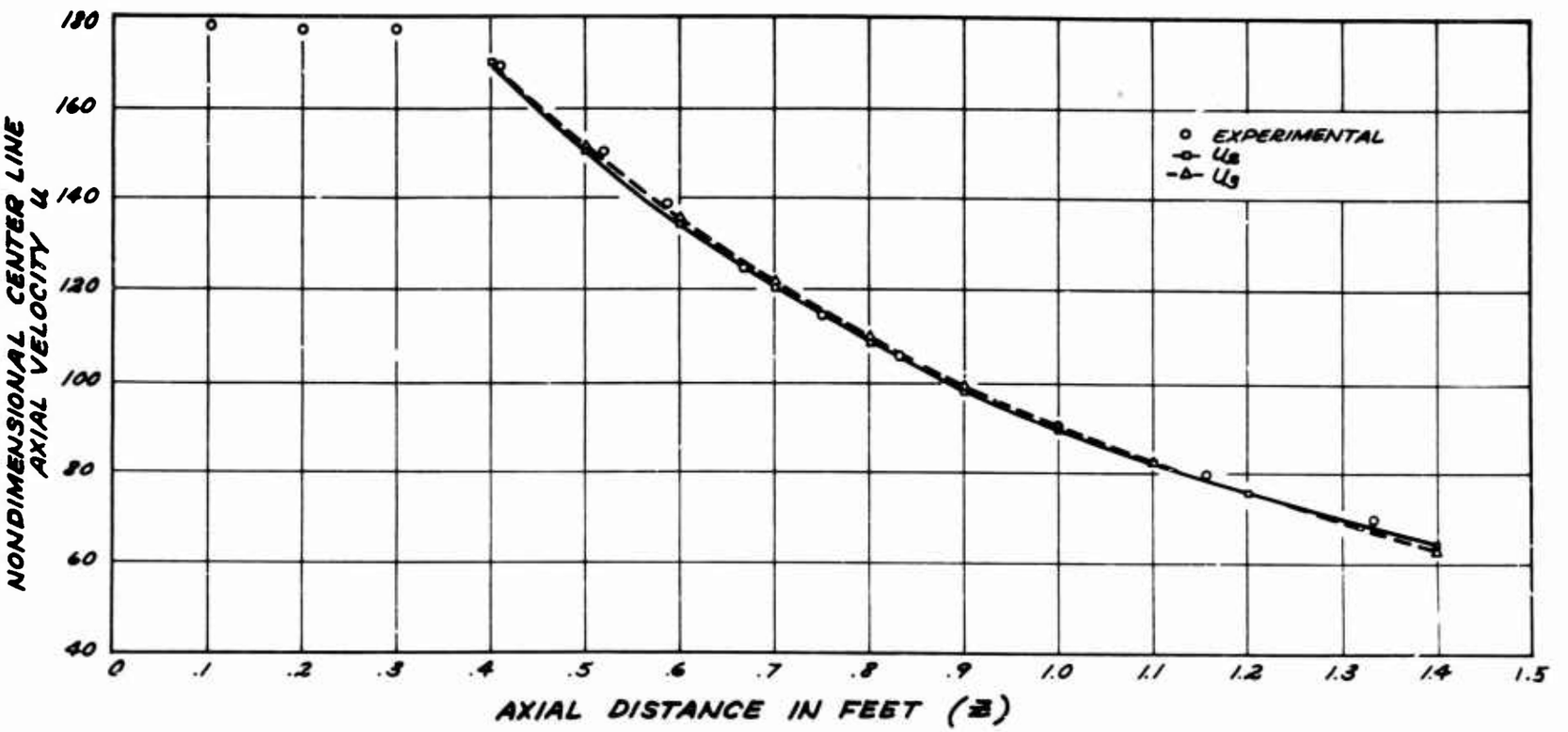

Figure 23. Comparison of Theoretical and Experimental Center Line Velocity Distribution for a Free Jet.

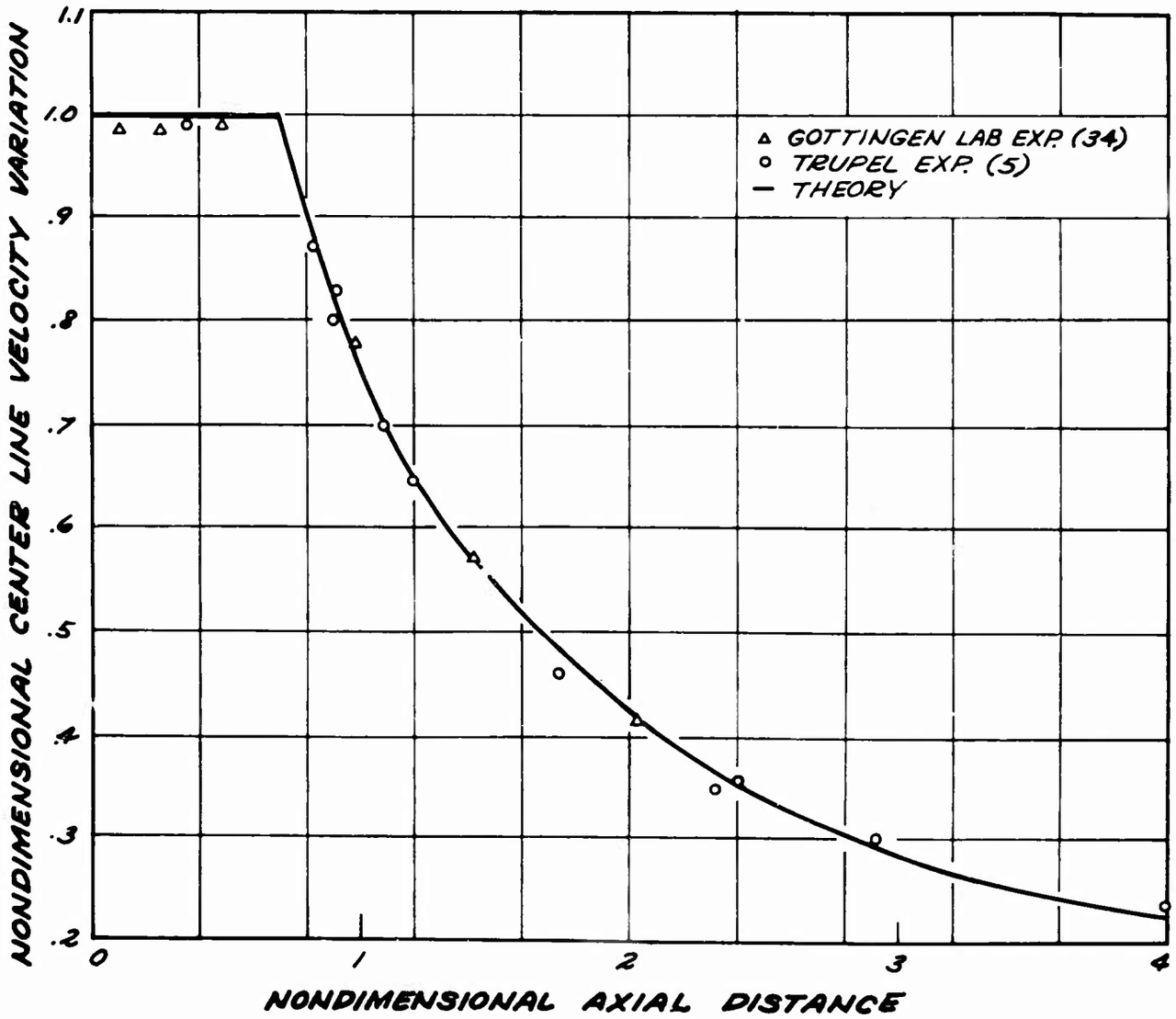

Figure 24. Comparison of Theoretical and Experimental Center Line Velocity Distribution.

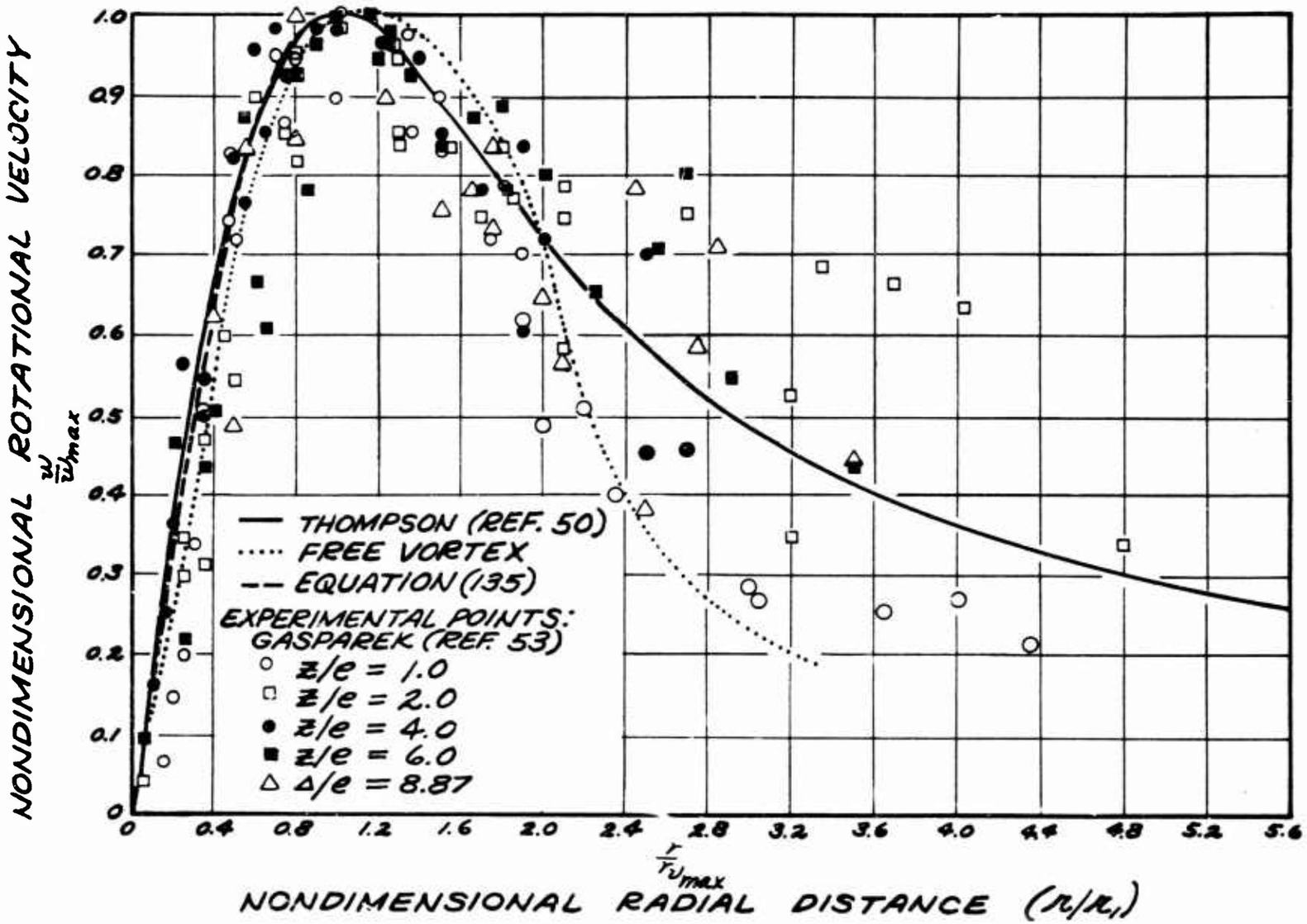

Figure 25. Comparison of Theoretical and Experimental Rotational Velocity in a Wing-Tip Vortex.

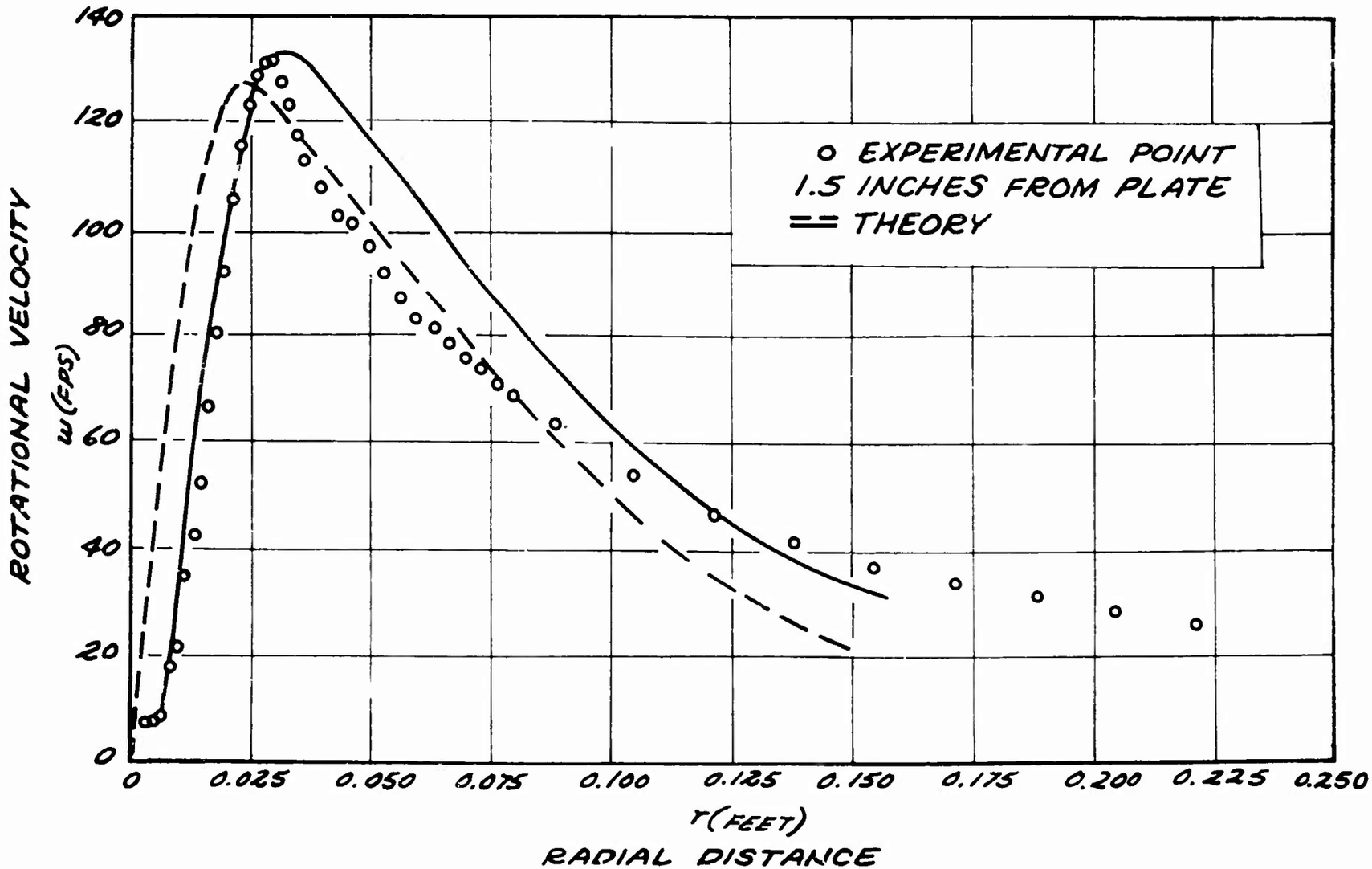

Figure 26. Confined Vortex Experimental and Theoretical Rotational Velocity Profiles.

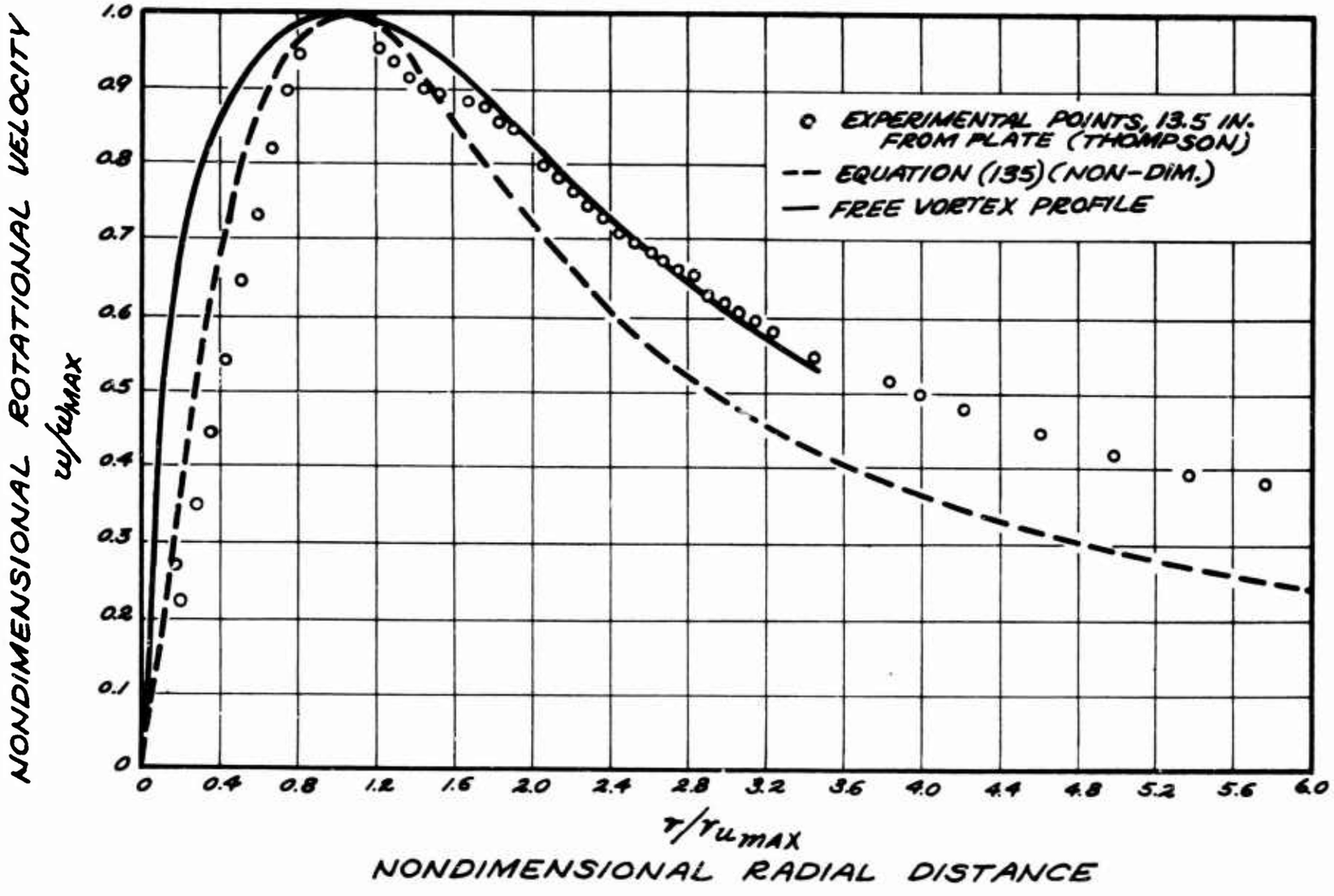

Figure 27. Comparison of Theoretical and Experimental Rotational Velocity Distribution for Confined Vortex.

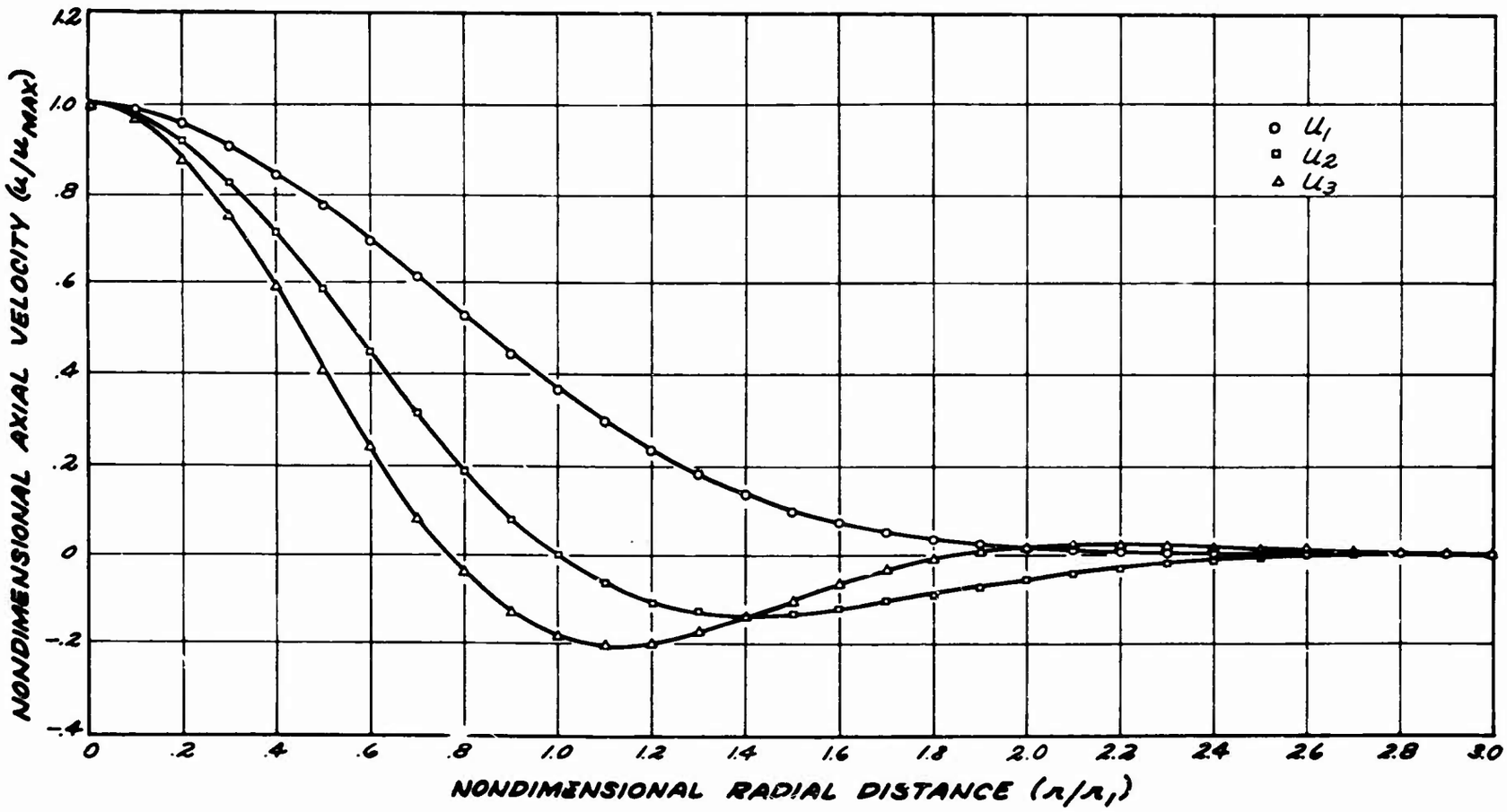

Figure 28. Axial Velocity Characteristic Form.

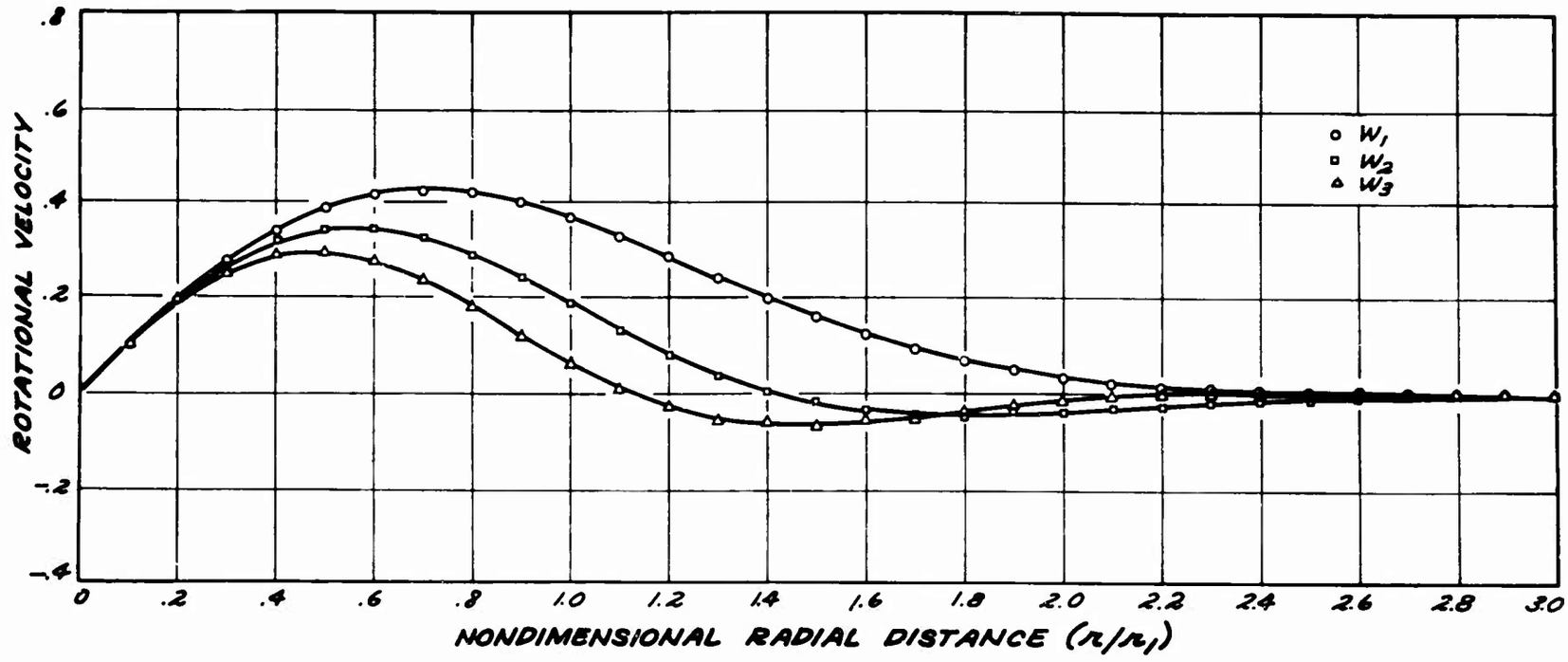

Figure 29. Rotational Velocity Characteristic Form.

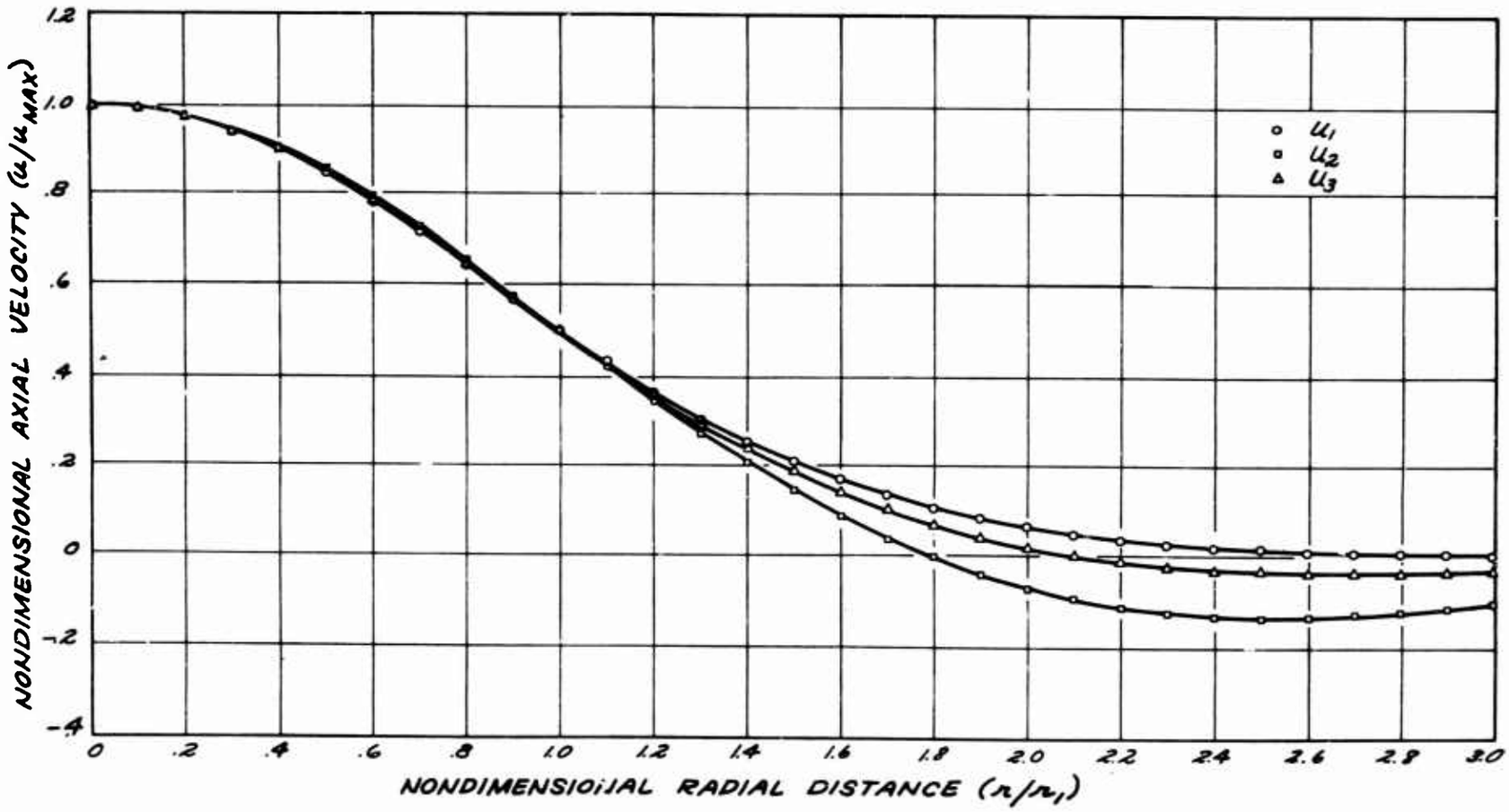

Figure 30. Modified Characteristic Axial Velocity Form.

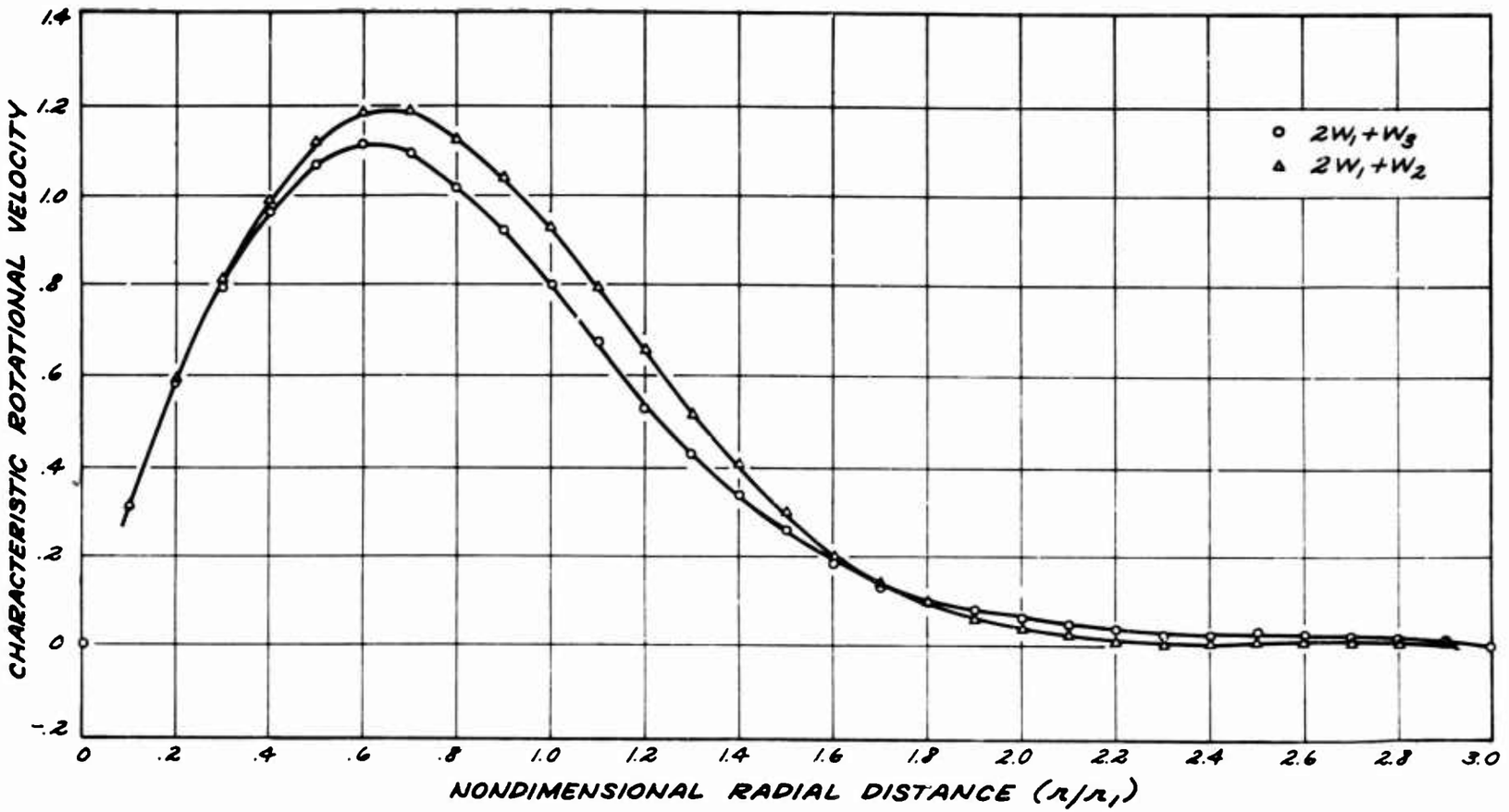

Figure 31. Modified Characteristic Rotational Velocities.

REFERENCES

1. Schlichting, H., Boundary Layer Theory, Fourth Edition, McGraw-Hill Book Company Inc., New York, New York, 1960.

2. Schlichting, H., "Laminare Strahlausbreitung", ZAMM 13 260, 1933.

3. Görtler, H., "Decay of Swirl in an Axially Symmetrical Jet", Publicado en la Revista Mathematica Hispano Americana, 4ª Serie Tomo XIV Nums 4y5.

4. Pai, S., Fluid Dynamics of Jets, D. van Nostrand Company Inc., New York, 1954.

5. Schlichting, H., "Über Das Ebene Windscha Henproblem", Thesis Gottingen Ing-Arch 1, 533-571, 1930.

6. Pai, S., "Viscous Flow Theory", I-Laminar Flow, D. van Nostrand Company Inc., New York, 1956.

7. Newman, B. G., "Flow in a Viscous Trailing Vortex", Aeronautical Quarterly, Volumn X, May 1959.

8. Blasius, H., Grenzschichten in Flüssigkeiten Mit Kleiner Reibung", Z. Math. u. Phys. 56, 1, 1908, Engl. transl. in NACA T.M. No. 1256.

9. Taylor, G. I., On the Dissipation of Eddies, No. 598, A.C.A. - R & M, 1918.

10. Squire, H. B., The Growth of a Vortex in Turbulent Flow, 16,666, Aeronautical Research Council, 1954.

11. Boussinesq, J., "Essai sur la Theorie des Eaux Courantes", Mem. Acad. Science, 23, 1-680, Imprimerie National Paris, 1877, pp. 24-46.

12. Görtler, H., "Berechnung von Aufgaben dez frein Turbulenz auf Grund eines neuen Näh erungsansatzes", ZAMM 22, 244-254, 1942.

13. Prandtl, L., "Über die Ausgebildete Turbulenz", ZAMM 5, 136, 1925.

14. Prandtl, L., "Bemerkung zur der Theories der freien Turbulenz", ZAMM 22, 5, 1942.

15. Taylor, G. I., "The Transport of Vorticity and Heat Through Fluids in Turbulent Motion", Proe. Roy. Soc., London A, 135, 828, 685, 1932.

16. Reichardt, H., "Gesetzmässigkei ten der freien Turbulenz", VDI-Forschungsh 414, 1951.

17. Mattioli, G. D., "Theoria Dinamical dei Regimi Fluidi Turbolenti", XV Gedam, Padova, 1937.

18. von Kármán, Th., Mechanische Ahnlichkeit und Turbulenz, TM 611, Transl. N.A.C.A., 1931.

19. von Kármán, Th., "Progress in the Statistical Theory of Turbulence", Proc. Nat. Acad. Sci., S30-539, Washington 34, 1948.

20. Broszko, M. Ann., Acad. Sci. Pol. Sci Techniques, 1945, p. 145, and Bull Acad. Polon. Sci. et LeHers-I, 1951, p. 243.

21. Squire, W., "Some Aspects of Turbulent Flow", Proc. Sem. Aer. Scien., Volumn I, N.A.L. Bangalore, 1961, p. 179-195.

22. Townsend, A. A., The Structure of Turbulent Shear Flow, Cambridge University Press, Massachusetts, 1956.

23. Goldstein, S., ed., Modern Developments in Fluid Dynamics, Oxford at the Clarondon Press, England, 1938.

24. Reichardt, H., "Uber eine Neue Theorie de Freien Turbulenz", ZAMM 21, 257, 1941.

25. Mowforth, E., "Details of Measurements in a Turbulent Trailing Vortex", Appendix in Reference 7.

26. Tollmien, W., "Die von Karmansche Ähnlichkeitshypothese in der Turbulenz - Theorie und das ebene Windschatten problem", Ing. Arch 4, 1, 1933.

27. Helmbold, H. B., Contribution to Jet Pump Theory, Report No. 294, University of Wichita, Kansas, September 1957.

28. Helmbold, H. B., Luessen, G., and Heinrich, A. M., An Experimental Comparison of Constant-Pressure and Constant-Diameter Jet Pumps, Report No. 147, University of Wichita, Kansas, July 1954.

29. Helmbold, H. B., Energy Transfer by Turbulent Mixing Under a Longitudinal Pressure Gradient, E. S. 182, University of Wichita, Kansas, August 1955.

30. Cruse, E. R., Tontini, R., "Research on Coaxial Jet Mixing", Convair/G. D. - G. D./C 62-354A, November 1962.

31. Squire, H. B., and Trouncer, J., "Round Jets in a General Stream", British A.R.C. R & M 1974, 1944.

32. Rabenneck, G. L., Shumpert, P. K., and Sutton, J. F., Steady Flow Ejector Research Program, Lockheed Georgia Company, December 1960.

33. Johnson, J. K., Shumpert, P. K, and Sutton, J. F., Steady Flow Ejector Research Program, Lockheed Georgia Company, September 1961.

34. Flugel, G., The Design of Jet Pumps, N.A.C.A. - T.M. No. 982, July 1941.

35. Fox, N. L., Analytical Solution for Gross Thrust Change, Report No. SM-13881, Douglas Company, December 1950.

36. von Kármán, T., "Theoretical Remarks of Thrust Augmentation", Reissner Anniv. Vol., J. W. Edwards, Ann Arbor, Michigan, 1949.

37. Lockwood, R. M., Investigation of the Process of Energy Transfer from an Intermittent Jet to an Ambient Fluid, Report No. ARD-238, Hiller Aircraft Company, June 1959.

38. Wells, W. G., Theoretical and Experimental Investigation of a High Performance Jet Pump Utilizing Boundary Layer Control, Research Report No. 30, Aerophysics Department, Mississippi State University, State College, Mississippi, June 1960.

39. Wan, Chia-an, A Study of Jet Ejector Phenomena, Report No. 57, Aerophysics Department, Mississippi State University, State College, Mississippi, November 1964.

40. Knox, R. M., A Study of Optimized Constant Pressure Jet Pumps, Report 5827, Marquardt Company, December 1960.

41. Yakovlevskiy, O. V., The Mixing of Jets in a Channel with Variable Cross-Section, A.F.S.C., FID-TT, 62-1571, Translated by Foreign Technical Division, 1962.

42. Abramovich, G. N., Applied Gas Dynamics, Gostekhnizdat, Moscow, 1953.

43. Carslaw, H. S., and Jaeger, J. C., Conduction of Heat in Solids, Oxford at the Clareston Press, England, 1948.

44. Pai, S. I., "Axially Symmetrical Jet Mixing of a Compressible Fluid", Quarterly of Applied Mathematics, Volume 10, No. 2, July 1952, pp. 141-148.

45. Rosenzweig, M. L., Lewellen, W. S., and Ross, D. H., Confined Vortex Flow with Boundary Layer Interaction, Report No. ATN-64 (9227) -2, Aerospace Corp., California, 1964.

46. Lewellen, W. S., and Grabowsky, W. R., "Nuclear Space Power Systems Using Magnetohydrodynamics Vortices", ARS J., May 1962.

47. Long, R. R., A Vortex in an Infinite Viscous Fluid, TR. 11 (ONR) or TR 14, John Hopkins University, Department of Mechanics, Maryland, 1961.

48. Anderson, O., Theoretical Solutions for the Secondary Flow on the End Wall of a Vortex Tube, UAC Research Laboratories, Report R-2494-1, United Alc Corporation.

49. Ross, D. H., An Experimental Study of Secondary Flow in Jet-Driven Vortex Chambers, Report No. ATN-64(9227)-1, Aero and Propolsion Research Laboratory, Aero. Corporation, Florida, January 1964.

50. Thompson, J. F., Jr., The Structure of Free and Confined Turbulent Vortices, Research Report No. 4[illegible], Aerophysics Department, Mississippi State University, State College, Mississippi, May 1963.

51. Faris, G. N., Some Entrainment Properties of a Turbulent Axi-Symmetric Jet, Research Report No. 39, Aerophysics Department, Mississippi State University, State College, Mississippi, January 1963.

52. Gasparek, E. P., and Dosanjh, D. C., Decay of Viscous Trailing Vortex.

53. Cornish, J. J., III, A Universal Description of Turbulent Boundary Layer Profiles With or Without Transpiration, Research Report No. 29, Aerophysics Department, Mississippi State University, State College, Mississippi, June 1960.

54. Coles, D., "The Law of the Wake in the Turbulent Boundary Layer", Journal of Fluid Mechanics, Volumn 1, London, 1956, p. 191.

55. Forthmann, E., "Uber Turbulente Strahlausbreitung", Ing. Arch 5, 42 1934, NACA T.M 789, 1936.

56. Hislop, G. S., and Hall, A. A., "Velocity and Temperature Distribution in the Turbulent Wake Behind a Heated Body of Revolution", Proc. Cambridge Phil. Soc., 34, 345, 1938.

57. Shigemitsu, Yutaka, "Statistical Theory of Turbulence I. Hypothesis of Vortex Chaos Motion", Journal of the Physical Society of Japan, Volumn 10, No. 6, June 1955, pp. 472-482.

58. Forstall, W., and Shapiro, A. H., "Momentum and Mass Transfer in Coaxial Gas Jets", Journal of Applied Mechanics, 17, 4, 399, 1950.

59. Forstall, W., Osterle, J. F., and Weinstein, A. S., "Momentum Diffusion From a Slot Jet into a Moving Secondary", Journal of Applied Mechanics, 23 9 437, 1956.

60. Abramovich, G. N., "Turbulent Jets in a Stream", Trudy Soveshch po. Prike. Gass. Dinam. Izd. AN Kaz. SSR. Alma-Ata., 1956.

61. Albertson, M. L., Dai, Y. B., Jensen, R. A., and Rouse, H., "Diffusion of Submerged Jets", Proclamation American Society Civil Engineers, 74, 1751, 1948.

62. Tollmien, W., Calculation of Turbulent Expansion Processes, Trans. N.A.C.A. - T.M. 1085, September 1945.

63. Terazawa, K., On the Decay of Vortical Motion in Viscous Fluid, Report No. 4, Aeronautical Research Institute, Tokyo Imp. University, Japan, 1922.

64. Pierce, D., Photographic Evidence of the Formation and Growth of Vorticity Behind Plates Accelerated From Rest in Still Air, R.A.E. T.N. Aero 2745, March 1961.

65. Smith, M. R., "Personal Communication", Aerophysics Department, Mississippi State University, State College, Mississippi, May 1965.

66. Trüpel, T., "Über die Einwirk ung eines Luftstrahles auf die Umgebende Luft", Zeitschrift fur das gesammte Turbinenwexy 5-6, 1915.

67. Cornell, D., Smoke Generation for Flow Visualization, Research Report No. 54, Aerophysics Department, Mississippi State University, State College, Mississippi, November 1964.

68. Cornish, J. J., III, "Personal Communication", Aerophysics Department, Mississippi State University, State College, Mississippi, December 1964.

DISTRIBUTION

US Army Materiel Command	5
US Army Aviation Materiel Command	6
Chief of R&D, DA	2
Director of Defense Research and Engineering	1
US Army R&D Group (Europe)	2
US Army Aviation Materiel Laboratories	18
US Army Engineer R&D Laboratories	2
US Army Limited War Laboratory	1
US Army Human Engineering Laboratories	1
US Army Ballistic Research Laboratories	1
US Army Research Office-Durham	1
Plastics Technical Evaluation Center	1
US Army Engineer Waterways Experiment Station	1
US Army Test and Evaluation Command	1
US Army Electronics Command	2
US Army Combat Developments Command, Fort Belvoir	2
US Army Combat Developments Command Experimentation Command	3
US Army War College	1
US Army Command and General Staff College	1
US Army Aviation School	1
US Army Tank-Automotive Center	2
US Army Armor and Engineer Board	1
US Army Aviation Test Activity, Edwards AFB	2
Air Force Flight Test Center, Edwards AFB	2
Air Proving Ground Center, Eglin AFB	1
US Army Field Office, AFSC, Andrews AFB	1
Air Force Flight Dynamics Laboratory, Wright-Patterson AFB	1
Systems Engineering Group, Wright-Patterson AFB	3
Naval Ship Engineering Center	1
Naval Air Systems Command, DN	8
Office of Naval Research	5
Marine Corps Liaison Officer, US Army Transportation School	1
Lewis Research Center, NASA	1
Manned Spacecraft Center, NASA	1
NASA Scientific and Technical Information Facility	2
NAFEC Library (FAA)	2
National Tillage Machinery Laboratory	1
US Army Aviation Human Research Unit	2
US Army Board for Aviation Accident Research	1
Bureau of Safety, Civil Aeronautics Board	2
US Naval Aviation Safety Center, Norfolk	1
Federal Aviation Agency, Washington, D. C.	1

Civil Aeromedical Research Institute, FAA	2
US Army Medical R&D Command	1
Defense Documentation Center	20

APPENDIX I

TWO-DIMENSIONAL WAKES AND JETS

For the case of steady two-dimensional motions of a jet or wake, the Navier-Stokes equations of motion may be simplified by using boundary layer approximations to give (in Cartesian coordinates)

$$\mathcal{U}\frac{\partial \mathcal{U}}{\partial z} + \mathcal{V}\frac{\partial \mathcal{U}}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\frac{\partial^2 \mathcal{U}}{\partial r^2}. \tag{139}$$

Also, the continuity is

$$\frac{\partial \mathcal{U}}{\partial z} + \frac{\partial \mathcal{V}}{\partial r} = 0 \tag{140}$$

where z denotes distance measured along the direction of motion and r the distance from the axis of symmetry z.

For the case of a co-flowing surrounding fluid, the axial velocity $\mathcal{U}$ may be equated to the algebraic sum of the free-stream velocity $\mathcal{U}_o$ and the defect velocity u.

$$\mathcal{U} = \mathcal{U}_o + \alpha u, \tag{141}$$

where α has the same value as for axially symmetric flow.

Then, substituting the velocity in equation (141) into equation (139) and using an order-of-magnitude evaluation, we have

$$\mathcal{U}_o\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial r^2}. \tag{142}$$

The solution of the above equation may describe the laminar velocity profile of a jet or wake at sufficient distance downstream of some initial plane, and was first given in that form (reference 1) to describe the wake of a flat-plate at zero incidence.

But, also, if we replace the kinematic viscosity by the virtual kinematic viscosity (references 3, 4, and 5), or if some other "viscosity function" is used to relate stress to rate of strain, equation (94), the velocity profile of a turbulent flow may be obtained.

As for the axisymmetric case, the above partial differential equation is transformed to an ordinary differential equation by introducing the transformation

$$u = r^a z^b f(\theta), \tag{143}$$

where

$$\theta = r^\gamma z^\epsilon.$$

This gives

$$\theta^2 f'' + \theta f' \left[\frac{2a+\gamma-1}{\gamma} - \frac{\epsilon}{\gamma^2}\frac{r^2 u_0}{zV}\right] + f\left[\frac{a(a-1)}{\gamma^2} - \frac{b}{\gamma^2}\frac{r^2 u_0}{zV}\right] = 0. \tag{144}$$

If it is now assumed that this equation represents a self-preserving velocity field, then

$$\theta = r^\gamma z^\epsilon \sim \frac{\varepsilon}{\gamma^2}\frac{r^2 u_0}{zV} = x,$$

where

$$|\epsilon| = -\epsilon = \varepsilon.$$

For a laminar case where kinematic viscosity is constant, as previously explained,

$$\gamma = 2 \text{ and } \epsilon = -1;$$

but, in general, for a turbulent flow, a relation of equation (84) may be used.

Then, by applying a tranformation

$$f = x^{-\frac{a}{\gamma}} \phi(x),$$

the result is

$$\phi'' x + \phi'\left[\frac{\gamma-1}{\gamma} + x\right] - \phi\left[\frac{a}{\gamma} + \frac{b}{\varepsilon}\right] = 0. \tag{145}$$

The corresponding equation in a Laplacian coordinate system is

$$\bar{\phi}''(s+1)s + \bar{\phi}\left[s\frac{\gamma+1}{\gamma} + \frac{a}{\gamma} + \frac{b}{\varepsilon} + 1\right] = \frac{1}{\gamma}\phi(0), \tag{146}$$

because $\phi(0) \neq 0.$

The solution of equation (146) is

$$\bar{\phi} = \frac{(S+1)^{\frac{\alpha}{\gamma}+\frac{\beta}{\varepsilon}-\frac{1}{\gamma}}}{S^{\frac{\alpha}{\gamma}+\frac{\beta}{\varepsilon}+1}} \int \frac{\phi(0)}{\gamma} \frac{S^{\frac{a}{\gamma}+\frac{\beta}{\varepsilon}}}{(S+1)^{\frac{\alpha}{\gamma}+\frac{\beta}{\gamma}-\frac{1}{\gamma}+1}} dS, \tag{147}$$

where, in view of the boundary conditions, the constant of integration is taken as zero.

The boundary conditions in the r, z coordinates system are given by equations (20), (21), and (22).

Also, the axial momentum integral has to be constant as expressed by

$$M = \rho \int_0^\infty u_0 (u_0 + \alpha u)\, dr,$$

or

$$M_\alpha = \alpha \rho u_0 \int_0^\infty u\, dr.$$

The velocity function that must satisfy the above conditions is given by

$$u = z^{\beta + \frac{\varepsilon a}{\gamma}} \phi\left(\frac{r\gamma}{z\varepsilon}\right). \tag{148}$$

Using elementary processes, it may be deduced that for the laminar case there is one solution that satisfies all the above conditions:

$$\beta + \frac{\varepsilon a}{\gamma} = -\frac{1}{2}.$$

Any other solution of the form

$$\beta + \frac{\varepsilon a}{\gamma} = -\frac{1}{2} - |m|,$$

where m = 1, 2, 3, ...,
satisfies the boundary conditions and produces a zero momentum integral as can be expected in view of the mathematical form of equation (142). The elementary solution m = 0 as given by Reference 1 is

$$u = A_0 z^{-\frac{1}{2}} e^{-x}. \tag{149}$$

For $m = 1$,

$$u = A_1 z^{-\frac{3}{2}} (1-2x)e^{-x}. \tag{149-a}$$

For $m = 2$,

$$u = A_2 z^{-\frac{5}{2}}\left(2-8x+\frac{8}{3}x^2\right)e^{-x}. \tag{149-b}$$

Again, any linear combination of the elementary solution with any of equation (148) for which $m = 1, 2, \ldots\ldots$, satisfies the equation of motion (142), the boundary conditions, and the momentum considerations. A_0 can be calculated from momentum considerations and A_1, A_2, ... may be found experimentally. Finally, all other solutions of equation (142) for

$$\beta + \frac{\varepsilon a}{\gamma} = -\frac{1}{2} + |m|$$

or for nonintegral m do not satisfy the constancy of the momentum.

It may be noted here that the solution

$$u_0 + \alpha u = A_0 z^{-\frac{1}{2}} e^{-x} \tag{150}$$

is also a solution that satisfies all boundary conditions and the constancy of the momentum.

Thus, the final form of the analytic expression of the velocity profiles may be selected only on the basis of experimental results.

Turbulent Flow

In Reference 1, the process of extending the solution of the laminar velocity profile of the wake of a flat-plate at zero incidence to describe the turbulent motion of a wake is discussed. Prandtl's virtual kinematic viscosity is used, which for this case is

$$\varepsilon_v = K u_{1max} b \quad = \text{a constant},$$

and it is indicated that the velocity profile, equation (149), is expected to describe turbulent flow far from the origin.

Also, using "mixing length hypothesis", Schlichting (reference 5) has presented a solution in his thesis for the case of a two-dimensional wake behind a single body, where the velocity profile is given by

$$\frac{u}{u_0} = C z^{-\frac{1}{2}} \left[1-\left(\frac{r}{b}\right)^{\frac{3}{2}}\right]^2. \tag{151}$$

Here, b is the width of the wake

$$b = C_1 z^{1/2},$$

and C and C_1 are constants that can be calculated from the drag of a cylinder.

Reichardt (reference 16), using his "momentum transfer length" hypothesis, has presented a solution similar to the integral form of the one given by equation (66) for an axially symmetric jet.

Finally, Tollmein (reference 26), using von Kármán's hypothesis (reference 18), and Görtler (reference 12), using Prandtl's "mixing length theory", have presented solutions in series form for the same problem.

As has already been indicated under an assumption of "viscosity function" and an accurate expression of characteristic width of the layer b with axial distance, it may be expected that the solution of the form of equation (147) may be extended to describe the turbulent velocity profile in dimensional or nondimensional form.

More specifically, the condition

$$\beta + \frac{\varepsilon a}{\gamma} = -\frac{\varepsilon}{\gamma},$$

due to the constancy of the momentum, may be rewritten as

$$\frac{\beta}{\varepsilon} + \frac{a}{\gamma} = -\frac{1}{\gamma}.$$

Then, substituting into equation (147), we have

$$\bar{\phi} = \frac{(S+1)^{-\frac{2}{\gamma}}}{S^{1-1/\gamma}} \int \frac{\phi(0)}{\gamma} \frac{S^{-\frac{1}{\gamma}}}{(S+1)^{1-\frac{2}{\gamma}}}. \tag{152}$$

Now, if $\gamma = 2$, as for the Reichardt solution where the nondimensional parameter is of the form $X = (r/z)^K$, an infinite number of solutions of exponential or polynomial form can be obtained. The elementary solution satisfying all conditions of the problem, with

$$\gamma = \varepsilon = \frac{3}{2}$$

in the (r, z) coordinate system, is

$$u = A_1 z^{-\frac{1}{2}} e^{\left(\frac{r}{z}\right)^{3/2} \cdot C_1} . \tag{153}$$

A and C_1 are constants that can be defined using given conditions of the problem.

Also, if the nondimensional parameter is of the form

$$X = \left(\frac{r^2}{z}\right)^K$$

as for laminar flow or for turbulence (reference 5), then the elementary solution may be identical to equation (149).

Using relations corresponding to equations for two-dimensional flow, we may transform the laminar velocity profile, equation (149), to describe turbulent motion. Then, the relation established is in nondimensional form:

$$\frac{u}{u_{MAX}} \sim e^{-\frac{r^2}{z^2} \cdot C} .$$

Finally, the same nondimensional turbulent velocity profiles have been obtained using equation (147) for the values $\varepsilon = \gamma = 2$ and $a + b = -1$, for which

$$\frac{u}{u_{MAX}} \quad \text{or} \quad \frac{u_0 + \alpha u}{u_{MAX}} = e^{-\frac{r^2}{z^2} \cdot C_2} . \tag{154}$$

This corresponds to the derivative of the elementary solution obtained by Reichardt (reference 16).

APPENDIX II

AXIALLY SYMMETRIC JET UNDER PRESSURE GRADIENT

For the case of a steady, axially symmetric jet, the Navier-Stokes equations of motion using Prandtl's boundary layer approximation may be simplified to give equations (1), (2), and (3), and the continuity equation is given by equation (4).

In the previous chapters, it has been possible to obtain solutions by various techniques that, under the concept of "virtual origin" or "viscosity function", may approximate with sufficient accuracy the existing experimental results. However, the solutions that have been obtained are valid for isobaric motion for almost every case.

Some possible techniques that can be used to simplify equation (1) in an integrable form will now be discussed. It will also be shown,by assuming different forms of the "viscosity funtion" or pressure gradient along the z axis, how the modified form of the final differential equation of motion can be affected.

However, the final solution is highly affected by the existing different boundary and physical conditions of the problem; in this respect, reference is made to the work of Tollmein (reference 26), Görtler (reference 12), and others who, by using different "free turbulence theories", have obtained equations that give very similar results under the proper conditions.

The complete solution of the problem of an axisymmetric jet under a pressure gradient will be given in a separate paper, and will include the especially interesting case of confined motion (jet ejectors).

The motion that will be considered in this appendix is axisymmetric, which the rotational velocity is zero. This condition reduces the number of previously required equations, (1), (2), (3), and (4), to only two equations, (1) and (4).

As has already been indicated in Chapter Three, this system of two equations contains three unknown functions and can be solved only if some additional relation exists between them.

In order to transform the partial differential equation to an ordinary one, the same process as for equation (1-b) may be used. Thus, for the present problem, a stream function similar to the one used by Schlichting (reference 2) for the case of a circular isobaric jet is assumed.

Then, letting

$$\psi = z^{a} F(\theta) \tag{155}$$

where

$$\theta = r^{\beta} z^{\gamma},$$

the continuity equation (4) gives

$$u = \frac{1}{r} \frac{\partial \psi}{\partial r} \tag{156}$$

and

$$v = -\frac{1}{r} \frac{\partial \psi}{\partial z} . \tag{157}$$

In terms of (1

$$u = \beta \frac{z^{a}}{r^{2}} \theta F'(\theta) \tag{158}$$

and

$$v = -\frac{z^{a}}{z r} \left(a F(\theta) + \gamma \theta F'(\theta) \right) \tag{159}$$

Substituting into equation (1), we have, after some simplifications,

$$\begin{aligned} \Phi \left[\beta^{2} \theta^{2} F''' + \beta \theta F''(3\beta - 4) + F'(\beta^{2} - 4\beta + 4) \right] + \\ \theta F'' F (a\beta) - \theta (F')^{2} (2\gamma + a\beta) + \\ F' F \alpha (\beta - 2) = \frac{1}{\theta} \frac{r^{4} z}{z^{2a} \beta} \cdot \frac{1}{\rho} \frac{\partial P}{\partial z} , \end{aligned} \tag{160}$$

where

$$\Phi = \frac{\nu}{z^{a-1}} .$$

The "viscosity function" is given by equation (94). To further simplify the above equation, some form for the viscosity and pressure function must be assumed.

In the following cases, some possible processes are indicated that can be used to lead to a simple modified equation of motion.

Case 1:

First assume that $\alpha = 1$. This implies that $\Phi = \nu$, and, from dimensional considerations, we have $\nu \sim$ constant. Then, if $\beta = -\gamma = 1$, the equation of motion takes the form

$$\nu\left[\theta^2 F''' - F''\theta + F'\right] + \theta F'' F + \theta (F')^2 - F'F = \frac{r^3}{\rho}\frac{\partial P}{\partial z}, \tag{161}$$

or, by proper transformation and rearrangement, gives

$$\frac{d}{d\theta}\left[\theta\frac{d}{d\theta}\left(\frac{F'}{\theta}\right) + \frac{FF'}{\theta}\right] = \frac{z^2 r}{\rho}\frac{\partial P}{\partial z}. \tag{162}$$

The right side of this equation is similar to the one given by Schlichting (reference 2), and may be integrated twice to give

$$F'\theta - 2F + \frac{F^2}{2} = \int\theta\int\frac{r z^2}{\rho}\frac{\partial P}{\partial z}\,d\theta\,d\theta. \tag{163}$$

This equation is of Reccati's form, and by transforming

$$F = 2\theta\sigma,$$

then

$$2\theta^2\sigma' - 2\theta\sigma + 2\theta^2\sigma^2 = \int\theta\int\frac{r z^2}{\rho}\frac{\partial P}{\partial z}\,d\theta\,d\theta. \tag{164}$$

If $\sigma = f'/f$,

$$\theta^2 f'' - \theta f' - f\,\frac{1}{2\theta}\left[\int\theta\int\frac{r z^2}{\rho}\frac{\partial P}{\partial z}\,d\theta\,d\theta\right] = 0. \tag{165}$$

The final form of the equation (165) depends on the pressure distribution along the longitudinal axis. In general, the pressure may be regarded as independent of the radial distance.

This assumption, along with dimensional considerations, leads to the pressure function

$$P \sim \frac{A}{z^2}$$

that will transform the above equation to a Bessel or Laguerre type of equation.

Case 2:

Alternatively, it is observed that for $\beta = 2$, equation (160) may be simplified to give

$$\phi\left[4\theta^2 F''' + 4\theta F''\right] + \theta F'' F 2a - \theta (F')^2 2(\gamma + a) = \frac{r^3 z^2}{z^{2a} \cdot 2} \cdot \frac{1}{\rho} \frac{\partial P}{\partial z} . \quad (166)$$

For further simplification, assume that ϕ = constant, a condition that implies:

$$V \sim C \qquad a = 1$$
$$V \sim z \qquad a = 2$$
$$V \sim z^2 \qquad a = 3$$

Then, for a = $-\gamma$, we have

$$2\Phi\left[\theta F''' + F''\right] + F'' F a = \frac{r^2 z^3}{z^{2a} \cdot 4} \cdot \frac{1}{\rho} \frac{\partial P}{\partial z} . \quad (167)$$

Under the above assumption, the axial velocity function takes the form

$$u = 2F' .$$

Again, the final equation of motion will be highly affected by the decision made relative to the pressure functions. If

$$\frac{\partial P}{\partial z} \sim \frac{1}{z^{\nu}} , \quad (168)$$

and if the motion is of self-preserving or universal character, then

$$\frac{\partial P}{\partial z} \sim \frac{1}{z^{3-a}} .$$

Finally, if an axial velocity function is desired,

$$u = \frac{\beta}{z^{\gamma}} F',$$

that corresponds to a radial velocity

$$U = \gamma \frac{\theta}{z} F',$$

then the differential equation of motion may take the form

$$2\Phi\left[\theta F''' + F''\right] + \gamma (F')^2 = \frac{r^2 z^3}{4} \cdot \frac{1}{\rho} \frac{\partial P}{\partial z}. \tag{169}$$

For different values of the constants α, β, γ, and on the basis of dimensionless considerations, different differential equations of motion are obtained.

An indication, relative to some algebraic relation existing between the induced constants, may be obtained by the use of the momentum and continuity integrals.

However, at the present stage of this research, no definite decision will be made for the value of the induced constants or the pressure distribution along the longitudinal axis. Instead, it will only be noted that, in an "arbitrary pressure gradient mixing tube", the pressure distribution along the axis of symmetry may be adjusted using boundary layer suction or blowing techniques and a proper mixing tube profile.

The solutions of equation (169), as they may be adjusted to satisfy the physical and boundary conditions of the problem (under a given pressure function - and the integro-differential relations - equation (130)) of maximum augmentation factor, will be presented in a separate paper.

It is believed that the solution of the outlined problem, when necessary experimental information is availalbe, will prove to be a very useful tool for the design of a high-thrust augmentation factor ejector or high-performance jet ejector pump.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Mississippi State University Aerophysics Department State College, Mississippi 39762	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A Contribution to the Theory of Jet-Wakes and Vortices in Free and Confined Surroundings

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5. AUTHOR(S) *(Last name, first name, initial)*

Fragoyannis, George Basil

6. REPORT DATE November 1966	7a. TOTAL NO. OF PAGES 118	7b. NO. OF REFS 68

8a. CONTRACT OR GRANT NO. DA 44-177-AMC-264(T) b. PROJECT NO. Task 1P125901A14203 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) USAAVLABS Technical Report 66-69 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* Aerophysics Research Report No. 67

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY U.S. Army Aviation Materiel Laboratories Fort Eustis, Virginia

13. ABSTRACT The report presents a study of the flow field of steady, viscous, incompressible jet-wakes and vortices submerged in free and confined surroundings. The general principles of the motion of a free laminar jet have been analyzed and reviewed. From these principles, an understanding of the very complex behavior of a turbulent jet has been obtained.

Using a linearized form of the Navier-Stokes equations of motion, a set of solutions capable of describing the laminar axial, rotational, and radial velocity profiles of a jet was deduced.

It was indicated that by introducing a "viscosity function" to the simplified equations of laminar motion, solutions can be obtained to describe the turbulent motion. For the confined jets and vortices, two different approaches have been used to simplify and solve the equations of motion. Analytic expressions for the different characteristics involved in the design of a jet ejector were calculated. Also, a process for the estimation of the different parameters for the optimum design of a jet pump or a thrust augmenter was indicated.

The comparison of the calculated general analytic expressions of this study to existing experimental results indicates that the assumptions and processes used to predict the different velocity profiles of free motions are reasonable. It was concluded that the laws describing jet-wakes and vortices for laminar and turbulent motion are of the same general nature.

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Idenfiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.

Unclassified

Security Classification

7954-66