AD 645663

NOLTR 66-211

# REPRESENTATION OF A NONSPHERICAL UNDERWATER EXPLOSION BUBBLE BY A VORTEX SHEET AND SOURCE SHEET

DDC
RECEIVED
JAN 26 1967
B

22 NOVEMBER 1966

NOL

UNITED STATES NAVAL ORDNANCE LABORATORY, WHITE OAK, MARYLAND

NOLTR 66-211

Distribution of this document is unlimited.

ARCHIVE COPY

# REPRESENTATION OF A NONSPHERICAL UNDERWATER EXPLOSION BUBBLE BY A VORTEX SHEET AND SOURCE SHEET

by

Larry W. Bell

ABSTRACT: The equations for the velocity potential, partial time derivative of the velocity potential, tangential velocity, and normal velocity for a source sheet and vortex sheet representation of the nonspherical underwater explosion bubble have been derived and programmed for the IBM 7090 computer. These equations were tested for the known case of a spherical underwater explosion bubble and the results were sufficiently accurate for practical application. The conclusion is that the nonspherical underwater explosion bubble can be represented by a source sheet and vortex sheet. This representation will not interfere with the jet rising into the interior of the bubble, as is the case of the earlier models consisting of a single point source and a single dipole located on the axis of symmetry.

UNDERWATER EXPLOSIONS DIVISION
EXPLOSIONS RESEARCH DEPARTMENT
U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, SILVER SPRING, MARYLAND

REPRESENTATION OF A NONSPHERICAL UNDERWATER EXPLOSION BUBBLE BY A VORTEX SHEET AND SOURCE SHEET

This report is part of a continuing study of the underwater explosion bubble which, it is hoped, will eventually give a complete quantitative description of the nuclear explosion bubble. This paper is an important step forward in that it is the first adequate mathematical description of the migrating explosion bubble and lays the groundwork for further theoretical development.

The work reported here was carried out in partial fulfillment of the requirements for the degree of Master of Science in Physics at the University of Maryland. It was supported by the Defense Atomic Support Agency, under Ordtask No. ORD-035-001/F008-21-03(003), Nuclear Explosion Bubble Phenomena, DASA NWER Subtask 14.086.

E. F. SCHREITER
Captain, USN
Commander

C. J. ARONSON
By direction

## TABLE OF CONTENTS


| Chapter | | Page |
|---|---|---|
| ACKNOWLEDGEMENTS | | ii |
| INTRODUCTION | | 1 |
| I. | BUBBLE THEORY | 5 |
| II. | VORTEX THEOREM FOR MIGRATING BUBBLES | 17 |
| III. | VELOCITY POTENTIAL OF THE SOURCE SHEET AND ITS TIME DERIVATIVE | 22 |
| IV. | VELOCITY POTENTIAL OF THE VORTEX SHEET AND ITS TIME DERIVATIVE | 31 |
| V. | TANGENTIAL VELOCITY ON THE BOUNDARY OF A NONRIGID BODY OF REVOLUTION EXPANDING AND MIGRATING IN A LIQUID MEDIUM | 40 |
| VI. | THE BERNOULLI EQUATION | 45 |
| VII. | THE MIGRATING SPHERICAL BUBBLE | 49 |
| | CONCLUSIONS | 56 |
| LITERATURE CITED | | 68 |


## LIST OF TABLES


| Table | | Page |
|---|---|---|
| 1a. | Computed Values of $c_0$ and $c_2$ Obtained From Expressions 7.11 and 7.13 Evaluated on the Boundary of a Unit Sphere | 57 |
| 1b. | Computed Values of $c_1$ and $c_3$ Obtained From Expressions 7.12 and 7.14 Evaluated on the Boundary of a Unit Sphere | 58 |
| 2. | Computed Values of $c_T$ Obtained From Expression 7.29 Evaluated on the Boundary of a Unit Sphere | 59 |

## LIST OF FIGURES


| Figure | | Page |
|---|---|---|
| 1. | Formation of the Bubble Jet | 60 |
| 2. | Impingement of the Bubble Jet on the Upper Bubble Surface | 61 |
| 3. | The Integrands $G_o$, $\bar{G}_o$, and $(G_o - \bar{G}_o)$ In Expression 7.11 Plotted as a Function of the Integration Variable z | 62 |
| 4. | The Integrands $G_o$, $\bar{G}_o$, and $(G_o - \bar{G}_o)$ In Expression 7.11 Plotted as a Function of the Integration Variable z | 63 |
| 5. | The Integrands $G_3$, $\bar{G}_3$, and $(G_3 - \bar{G}_3)$ In Expression 7.14 Plotted as a Function of the Integration Variable z | 64 |
| 6. | The Integrands $G_3$, $\bar{G}_3$, and $(G_3 - \bar{G}_3)$ In Expression 7.14 Plotted as a Function of the Integration Variable z | 65 |
| 7. | The Integrand $G_2$ of Expression 7.18 Plotted as a Function of the Integration Variable z | 66 |
| 8. | The Coefficient $c_2$ Away From The Boundary of a Unit Sphere Plotted as a Function of the Field Point Coordinate $r_o$ for $z_o = 0$ | 67 |


## ACKNOWLEDGEMENTS

The author acknowledges a special debt of gratitude to Dr. Hans G. Snay of the Explosions Research Department at the U. S. Naval Ordnance Laboratory. His advice and guidance in the direction of this research program, together with his unwavering patience, understanding, and encouragement were the major factors in the successful completion of the subject program.

# REPRESENTATION OF A NONSPHERICAL UNDERWATER EXPLOSION BUBBLE BY A VORTEX SHEET AND SOURCE SHEET

## INTRODUCTION

After the detonation of an explosive charge under water and the emission of the shock wave, the gaseous products of the explosion, which have formed a gas bubble, continue to expand outward at a gradually decreasing rate. The result of this expansion is a radial displacement of water. When the radial velocity becomes zero, the bubble has reached its first maximum radius and the pressure in the bubble is far below the hydrostatic pressure. Subsequently, the bubble contracts and the bubble pressure increases. When the radial velocity again becomes zero, the bubble pressure is substantially greater than the hydrostatic pressure. The bubble has now reached its first minimum radius and begins another expansion phase. The gas bubble may pulsate in this manner several times depending on the depth at which the charge is detonated.

Throughout the first expansion phase of the bubble there is very little vertical motion, commonly called migration. However, near the minimum radius an appreciable upward displacement may occur. The migration is a result of the buoyancy of the bubble and depends on the depth of explosion and the charge weight. There is little migration if small charges are exploded at great depths, say one pound at 200 feet. However, the explosion of large charges at ordinary depths, charges of the order of 300 pounds detonated to 80 feet, would show a strong migration.

The classical treatment of the underwater explosion bubble is based on the assumption that the bubble remains spherical at all times. The

explosion of small charges at great depths essentially behave in this manner. However, it has been observed in the explosion of large charges at ordinary depths, where strong migration is present, that the bubble does not remain spherical but undergoes a very definite change in shape. Initially the bubble is spherical and remains spherical somewhat beyond the first maximum radius. However, as the bubble contracts to its first minimum, the bubble is flattened; and shortly before the minimum, the bottom begins to move upward into the interior of the bubble, figure 1, and eventually impacts with the upper surface of the bubble, figure 2. The tongue of water protruding into the interior of the bubble is called the bubble jet.

Several authors have given theories of the change of shape due to the effect of buoyancy, notably Penney-Price[1] and Keller-Kolodner[2]. These authors represent the velocity potential $\varphi$ as a series of the Legendre polynomials $P_n(\cos\theta)$ in the form:

$$\varphi = \sum_{n=0} a_n P_n(\cos\theta) R^{-(n+1)} \qquad (0.01)$$

where R is the distance between the coordinate origin and the point at which the velocity potential is to be evaluated, the field point. Further, $\theta$ is the angle between the vertical axis and R, and the $a_n$'s are coefficients that are functions of time. The velocity potential can be interpreted as the sum of a point source, a dipole and higher order poles all having the same origin. The coordinate origin is placed on the axis of symmetry in the interior of the bubble. Since the potential must be evaluated for points on the bubble surface, a solution is possible so long as the bubble surface does not come near the origin. However, the bubble jet will reach every point in the interior on the axis of symmetry

and will cause irremovable singularities as R goes to zero. Therefore, the representation of the velocity potential fails at the most interesting and important part of the motion of the underwater explosion bubble.

Initially, the bubble expansion is radial and this motion is known to be irrotational, thus, vortex free. However, Snay has shown that as buoyancy takes effect and the bubble begins to migrate, vorticity is generated.

In this paper, the velocity potential will be represented by a distribution of point sources and vortex rings on the surface of the bubble, eliminating singularities from the interior of bubble. An alternate approach, as suggested by Kelvin's extention of Green's theorem, is a distribution of point sources and dipoles over the bubble surface, but the distribution of vortex rings is simpler in the analysis if the vortex theorem for migrating bubbles is applied.

The common assumptions of bubble theory will also be used in this study. They are:

a) The bubble motion can be represented by a velocity potential for points outside of the bubble surface.

b) The gas in the bubble expands adiabatically and the pressure is uniform throughout the bubble.

c) The water is incompressible.

In this paper, the water is assumed to be unbounded.

The purpose of this paper is not to solve the general case of a migrating bubble which changes shape. Rather, the effort is directed toward deriving the equations and providing the mathematical tools for the treatment of the general case and the mathematical treatment of the singularities generated by the distribution of point sources and vortex

rings on the surface of the bubble. The resulting equations have been tested by applying them to the classical case of a migrating spherical bubble.

# CHAPTER I

## BUBBLE THEORY

It is assumed that an incompressible, unbounded fluid contains a gas bubble within it, and the velocity $\hat{v}_f$ of the fluid, as observed from a fixed frame of reference in the fluid, is assumed to be derivable from a potential function $\varphi_f$ which satisfies Laplace's equation so that:

$$\hat{v}_f = -\nabla_f \varphi_f \tag{1.01}$$

$$\nabla_f^2 \varphi_f = 0 \tag{1.02}$$

The pressure P in the water can be obtained by Bernoulli's equation:

$$\frac{\partial \varphi_f}{\partial t_f} - \frac{1}{2}(\nabla_f \varphi_f)^2 - g z_f - \frac{P}{\rho_o} = f(t) \tag{1.03}$$

where the subscript f references the spatial and time coordinates to the fixed reference frame, $\rho_o$ is the density of water, g is the acceleration due to gravity, and f(t) is the integration constant arising from the integration of the classical hydrodynamic equation at constant time. The positive z axis is assumed to point vertically upward. It is further assumed that:

$$\varphi_f \to 0 \quad \text{as} \quad R_f \to \infty \tag{1.04}$$

which implies that:

$$\frac{\partial \varphi_f}{\partial t_f} \to 0 \quad \text{and} \quad \nabla_f \varphi_f \to 0 \quad \text{as} \quad R_f \to \infty \tag{1.05}$$

where $R_f$ is the distance from the coordinate origin to any point in the fluid. Statements 1.04 and 1.05 are simply the mathematical expressions for the assumption that the fluid is undisturbed at infinity as seen from the fixed frame of reference, consequently:

$$P \rightarrow P_o \quad \text{as} \quad R_f \rightarrow \infty \tag{1.06}$$

where $P_o$ is the hydrostatic pressure of the fluid for $z = 0$ at infinity, thus:

$$f(t) = -\frac{P_o}{\rho_o} \tag{1.07}$$

The Bernoulli equation can now be written as:

$$\frac{\partial \varphi_f}{\partial t_f} - \frac{1}{2}\left(\nabla_f \varphi_f\right)^2 - gz_f - \frac{P-P_o}{\rho_o} = 0 \tag{1.08}$$

The pressure P is continuous across the boundary of the bubble. Therefore, the pressure in the bubble must be the same as that in the adjacent water. When the Bernoulli equation is evaluated at the bubble interface, an interrelationship between the pressure P in the bubble and the motion of the surface is obtained.

Since the bubble surface is a free surface, a surface that moves with the fluid, the normal velocity of the bubble surface must be the same as that of the adjacent water. According to Lamb[3], if $F = 0$ is the surface equation of the free surface, the surface equation must satisfy the kinematic condition:

$$\nabla F \cdot \hat{v} + \frac{\partial F}{\partial t} = 0 \tag{1.09}$$

The unit normal, $\hat{e}_n$, of any surface is:

$$\hat{e}_n = \frac{\nabla F}{|\nabla F|} \tag{1.10}$$

Therefore, it is seen that if expression 1.09 is divided by $|\nabla F|$, the normal velocity at the boundary of the bubble can be expressed as:

$$v_n = - \frac{\frac{\partial F}{\partial t}}{|\nabla F|} \tag{1.11}$$

Assume that the bubble translates in the positive z direction with a velocity $\dot{B}(t)$. If a frame of reference with its coordinate origin at the bubble center is rigidly attached to the bubble, an observer at the origin would see a stream of velocity $- \dot{B}$ moving past the bubble. The relationships between the variables in the fixed reference frame and those in the moving frame are:

$$z_f = z_m + B(t) \tag{1.12}$$

$$t_f = t_m \tag{1.13}$$

$$\nabla_f = \nabla_m \tag{1.14}$$

If the velocity potential in the fixed reference frame is expressed in terms of the variables in the moving reference frame, the Bernoulli equation can be written as:

$$\frac{\partial \varphi_f^{\,m}}{\partial t_m} - \dot{B} \frac{\partial \varphi_f^{\,m}}{\partial z_m} - \frac{1}{2} (\nabla_m \varphi_f^{\,m})^2 - g(z_m + B) - \frac{P - P_o}{\rho_o} = 0 \tag{1.15}$$

where the superscript m shows that the velocity potential in the fixed reference system is expressed in terms of the moving coordinate variables. The relationship between the velocity in the fluid as seen by an observer in the moving frame and that seen by an observer in the fixed frame is:

$$- \nabla_m \varphi_m = - \nabla_m \varphi_f^{\,m} - \dot{B}\hat{k} \tag{1.16}$$

From expression 1.16 it is easy to see that:

$$\varphi_m = \varphi_f^{\,m} + \dot{B} z_m \tag{1.17}$$

where the subscript m refers to the moving reference system. Upon making the appropriate substitutions, the Bernoulli equation expressed in terms of the velocity potential in the moving frame of reference and the moving frame of reference and the moving coordinates is:

$$\frac{\partial \varphi}{\partial t} - \ddot{B} z + \frac{1}{2} \dot{B}^2 - \frac{1}{2} (\nabla \varphi)^2 - g(z+B) - \frac{P-P_o}{\rho_o} = 0 \tag{1.18}$$

The Bernoulli equation can be written as:

$$\frac{\partial \varphi}{\partial t} - \ddot{B} z + \frac{1}{2} \dot{B}^2 - \frac{1}{2} v_n^{\,2} - \frac{1}{2} \; {}_T^{\,2} - g(z+B) - \frac{P-P_o}{\rho_o} = 0 \tag{1.19}$$

where $v_T$ and $v_n$ are the tangential and normal velocities as seen by an observer in the moving frame.

In order to solve the Bernoulli equation, an expression for the velocity potential must be found.

According to Lamb[4], if $\varphi$ and $\varphi'$ are any two single valued functions which satisfy Laplaces's equation throughout a given region, then using Kelvin's extension of Green's theorem:

$$\iint_S \varphi \frac{\partial \varphi'}{\partial n} dS = \iint_S \varphi' \frac{\partial \varphi}{\partial n} dS \tag{1.20}$$

where S is the surface of a boundary enclosing a region of the fluid and n is the coordinate normal to S. Let $\varphi$ be the velocity potential and $\varphi' = 1/R$, the reciprocal of the distance of any point of the fluid to a fixed point P in the space occupied by the fluid. It is necessary to exclude P from the surface of integration since $\varphi'$ can become infinite if P is included in the region of integration. This may be done by describing a small sphere about P, where the fixed point P is the center. If $\Sigma$ refers to the surface of the small sphere and S to the remaining boundary, expression 1.20 gives:

$$\iint_\Sigma \varphi \frac{\partial}{\partial n}\left(\frac{1}{R}\right) d\Sigma + \iint_S \varphi \frac{\partial}{\partial n}\left(\frac{1}{R}\right) dS = \iint_\Sigma \left(\frac{1}{R}\right)\frac{\partial \varphi}{\partial n} d\Sigma + \iint_S \left(\frac{1}{R}\right)\frac{\partial \varphi}{\partial n} dS \tag{1.21}$$

With $\partial(1/R)/\partial n = -1/R^2$ at the surface $\Sigma$ and $d\Sigma = R^2 dw$, expression 1.21 can be written as:

$$-\int_\Sigma \varphi \, dw + \iint_S \varphi \frac{\partial}{\partial n}\left(\frac{1}{R}\right) dS = \int_\Sigma R \frac{\partial \varphi}{\partial n} dw + \iint_S \left(\frac{1}{R}\right)\frac{\partial \varphi}{\partial n} dS \tag{1.22}$$

As $R \to 0$ the first integral on the right hand side of expression 1.22 vanishes, while the first integral on the left hand side is $-4\pi\varphi_P$, where $\varphi_P$ is the value of $\varphi$ at the fixed point P. Upon rearranging terms, the velocity potential at the fixed point P is:

$$\varphi_P = -\frac{1}{4\pi} \iint_S \left(\frac{\partial \varphi}{\partial n}\right)\frac{dS}{R} + \frac{1}{4\pi} \iint_S \varphi \frac{\partial}{\partial n}\left(\frac{1}{R}\right) dS \tag{1.23}$$

Expression 1.23 gives the value of $\varphi$ at any point in the fluid in terms of $\varphi$ and $\partial\varphi/\partial n$ at the boundary. Since the velocity $\hat{v}$ is defined as:

$$\hat{v} = -\nabla\varphi \tag{1.24}$$

the normal velocity at the boundary is:

$$v_n = -\frac{\partial\varphi}{\partial n} \tag{1.25}$$

and expression 1.23 can be written as:

$$\varphi_P = \frac{1}{4\pi}\iint_S \frac{v_n}{R}\,dS + \frac{1}{4\pi}\iint_S \varphi\,\frac{\partial}{\partial n}\left(\frac{1}{R}\right)dS \tag{1.26}$$

The first integral represents a distribution of sources and the second integral, a distribution of dipoles on the boundary S. If S is finite, both integrals vanish as $R \to \infty$, as required by statements 1.04 and 1.05 for the potential in the fixed system. However, an observer in the moving frame of reference will see a stream with velocity $-\dot{B}\hat{k}$ flow past the bubble. Therefore, the fluid has a velocity of $-\dot{B}\hat{k}$ at infinity relative to the moving frame of reference, and the velocity potential of the stream is:

$$\varphi_{stream} = \dot{B}z \tag{1.27}$$

The velocity potential of the bubble, with a distribution of sources and dipoles on the bubble surface, as seen by an observer in the moving frame is:

$$\varphi_P = \frac{1}{4\pi}\iint_S \frac{v_n}{R}\,dS + \frac{1}{4\pi}\iint_S \varphi\,\frac{\partial}{\partial n}\left(\frac{1}{R}\right)dS + \dot{B}z \tag{1.28}$$

Maxwell[5] has shown that any magnetic dipole distribution over a closed surface, with dipole axes directed along the normals of the surface, may be replaced by a system of current-carrying rings distributed over the surface. In the analogous case of hydrodynamics, the distribution of dipoles can be replaced by a system of vortex rings lying in the surface. The rings have a common axis of symmetry. Such a representation of $\varphi$ simplifies the analysis when the vortex theorem for migrating bubbles is applied.

It is well known that for such surface distributions, the potential internal to the boundary of the surface is zero. Therefore, the gradient of the potential is also zero. In the case of the velocity potential, the velocity internal to the boundary is zero, in the same way as the electric field internal to a charged sphere.

The velocity potential for a single vortex ring can be shown to be, see Milne-Thomson[6]:

$$\varphi_V = \frac{\Gamma\Omega}{4\pi} \tag{1.29}$$

where $\Gamma$ is the circulation and $\Omega$ is the solid angle subtended at the field point by the diaphragm which is enclosed by the vortex ring. If $\gamma(s)$ is the circulation density for a distribution of vortex rings on the boundary of the bubble, then:

$$\varphi_V = \frac{1}{4\pi} \int_s \gamma(s)\Omega ds \tag{1.30}$$

where the integration is taken over the external boundary of the bubble. The circulation is:

$$\Gamma = \int_s \gamma(s)ds \tag{1.31}$$

but:

$$\Gamma = \oint v_T(s)ds \tag{1.32}$$

where $v_T(s)$ is the tangential velocity.

During the initial expansion of the bubble, expressions 1.31 and 1.32 are zero for any closed curve since only radial flow takes place. However, as vorticity is being generated by the action of gravity, the bubble surface becomes a surface of tangential discontinuity since there is a translation of the bubble in addition to the radial expansion. If the closed path is totally outside the bubble, expression 1.32 is zero at all times since the fluid motion is assumed to be irrotational at all times. Let the path of integration be on the external and internal boundary, thus:

$$\Gamma = \int_{s_{ext}} v_T(s)ds + \int_{s_{int}} v_T(s)ds \tag{1.33}$$

The second integral must be zero at all times since there is no fluid motion internal to the bubble boundary, as was previously assumed. Upon setting expression 1.29 equal to 1.31, it is seen that:

$$\gamma(s) = v_T(s) \tag{1.34}$$

since the path of integration may be taken as small as desired. Therefore, the potential of a vortex sheet in terms of the tangential velocity at the boundary of the bubble is:

$$\varphi_V = \frac{1}{4\pi}\int_s v_T(s)\Omega ds \tag{1.35}$$

The total velocity potential due to the source sheet, vortex sheet and stream is:

$$\varphi_P = \frac{1}{4\pi}\iint_S\left(\frac{v_n}{R}\right)dS + \frac{1}{4\pi}\int_s v_T(s)\Omega ds + \dot{B}z \tag{1.36}$$

The time derivative of expression 1.36 at the boundary is needed for the Bernoulli equation.

Let the velocity potential of the source sheet be defined as:

$$\varphi_S = \frac{1}{4\pi}\iint_S\left(\frac{v_n}{R}\right)dS \tag{1.37}$$

thus:

$$\frac{\partial\varphi_S}{\partial t} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint_S\left(\frac{v_n}{R}\right)dS \tag{1.38}$$

Describe a small sphere about the point P on the boundary, where P is the center. If $\Sigma$ refers to the surface of the small sphere and S to the original boundary, expression 1.38 gives:

$$\frac{\partial\varphi_S}{\partial t} = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint_\Sigma\left(\frac{v_n}{R}\right)d\Sigma + \frac{1}{4\pi}\frac{\partial}{\partial t}\iint_{S-\Sigma}\left(\frac{v_n}{R}\right)dS \tag{1.39}$$

If $d\Sigma = R^2dw$, Expression 1.39 can be written as:

$$\frac{\partial\varphi_S}{\partial t} = \frac{1}{4\pi}\int_\Sigma\frac{\partial v_n}{\partial t}Rdw + \frac{1}{4\pi}\int_\Sigma v_n\frac{dR}{dt}dw + \frac{1}{4\pi}\frac{\partial}{\partial t}\iint_{S-\Sigma}\left(\frac{v_n}{R}\right)dS \tag{1.40}$$

The factor dR/dt in the second integral can be expressed in terms of the normal velocity as:

$$\frac{dR}{dt} = v_n \sec \gamma \tag{1.41}$$

where $\gamma$ is the angle between the normal to the surface S at the point P and the radius vector R. Expression 1.40 can now be written as:

$$\frac{\partial \varphi_S}{\partial t} = \frac{1}{4\pi} \int_{\Sigma} \frac{\partial v_n}{\partial t} R dw + \frac{1}{4\pi} \int_{\Sigma} v_n^2 \sec \gamma dw + \frac{1}{4\pi} \frac{\partial}{\partial t} \iint_{S-\Sigma} \left(\frac{v_n}{R}\right) dS \tag{1.42}$$

The term sec $\gamma$dw is the solid angle of the disk subtended at a point on the axis of symmetry coinciding with the normal to the surface. If $R \to 0$ and the normal is the outward drawn normal to the surface, the first integral vanishes and $\int_{\Sigma} \sec \gamma dw \to 2\pi$ and one obtains:

$$\frac{\partial \varphi_S}{\partial t} = \frac{1}{2} v_n^2 + \frac{1}{4\pi} \frac{\partial}{\partial t} \iint_S \left(\frac{v_n}{R}\right) dS \tag{1.43}$$

The time derivative of the velocity potential due to the vortex distribution will be treated in a similar manner as that of the source sheet. Describe a small circle in the surface about the point P, where P is the center of the circle. If $\sigma$ refers to this circle while s refers to the original boundary, the time derivative of the velocity potential for the vortex distribution can be written as:

$$\frac{\partial \varphi_V}{\partial t} = \frac{1}{4\pi} \int_{\sigma} \frac{\partial v_T}{\partial t} \Omega \, d\sigma + \frac{1}{4\pi} \int_{\sigma} v_T \frac{\partial \Omega}{\partial t} d\sigma + \frac{1}{4\pi} \frac{\partial}{\partial t} \int_{s-\sigma} v_T \Omega ds \tag{1.44}$$

The factor $\partial\Omega/\partial t$ can be written as:

$$\frac{\partial\Omega}{\partial t} = \frac{\partial\Omega}{\partial R}\frac{\partial R}{dt} \tag{1.45}$$

and dR/dt can be expressed in terms of the normal velocity. Expression 1.44 can now be written as:

$$\frac{\partial\varphi_V}{\partial t} = \frac{1}{4\pi}\int_\sigma \frac{\partial v_T}{\partial t}\,\Omega d\sigma + \frac{1}{4\pi}\int_\sigma v_T v_n \sec\gamma\,\frac{\partial\Omega}{\partial R}\,d\sigma + \frac{1}{4\pi}\frac{\partial}{\partial t}\int_{s=\sigma} v_T\Omega ds \tag{1.46}$$

where $\sec\gamma\ \partial\Omega/\partial R$ is the projection of $\partial\Omega/\partial R$ onto the normal, thus:

$$\sec\gamma\,\frac{\partial\Omega}{\partial R} = \frac{\partial\Omega}{\partial n} \tag{1.47}$$

where n is the coordinate in the direction of the normal. Expression 1.46 can now be written as:

$$\frac{\partial\varphi_V}{\partial t} = \frac{1}{4\pi}\int_\sigma \frac{\partial v_T}{\partial t}\,\Omega d\sigma + \frac{1}{4\pi}\int_\sigma v_T v_n\,\frac{\partial\Omega}{\partial n}\frac{ds}{dn}\,dn + \frac{1}{4\pi}\frac{\partial}{\partial t}\int_{s=\sigma} v_T\Omega ds \tag{1.48}$$

where ds/dn is the inverse of the slope at the point P. If $d\sigma = rd\theta$ and $r \to 0$, the first integral vanishes and the time derivative of the velocity potential for the vortex distribution is:

$$\frac{\partial\varphi_V}{\partial t} = \frac{1}{2}\,v_n v_T \cot\theta + \frac{1}{4\pi}\frac{\partial}{\partial t}\int_s v_T\Omega ds \tag{1.49}$$

where the angle $\theta$ is the angle between the normal of the surface and the vertical axis. The time derivative of the total velocity potential at the boundary can now be written as:

$$\frac{\partial\varphi_P}{\partial t} = \frac{1}{2}\,v_n^{\,2} + \frac{1}{4\pi}\frac{\partial}{\partial t}\iint_S\left(\frac{v_n}{R}\right)dS + \frac{1}{2}\,v_n v_T \cot\theta + \frac{1}{4\pi}\frac{\partial}{\partial t}\int_s v_T\Omega ds + \ddot{B}z \tag{1.50}$$

Expression 1.50 refers to a coordinate system moving with the bubble and having its origin at the bubble center. Some important considerations concerning the coordinate system to be used in solving the problem are:

a) The problem is to be solved in terms of cylindrical coordinates.

b) The coordinates of any point on the surface of the bubble are r, $\theta$, and z.

c) The field point is to be held fixed in the r-z plane and its coordinates are $r_o$ and $z_o$.

d) The bubble is assumed to be a body of revolution with its axis of symmetry being the z axis. This condition implies that r and z are independent of $\theta$.

# CHAPTER II

## VORTEX THEOREM FOR MIGRATING BUBBLES

The Vortex-Theorem for migrating bubbles as proposed by Snay is so vital to this paper that it will be derived before the velocity potential of the source sheet and vortex sheet.

The first classical hydrodynamic equation is defined as:

$$\frac{\partial \hat{v}}{\partial t} + (\hat{v}\cdot\nabla)\hat{v} = \alpha_e - \frac{1}{\rho}\nabla P \qquad (2.01)$$

where $\hat{v}$ is the velocity vector in the fluid, $\alpha_e$ is the acceleration due to any external forces acting on the bubble, $\rho$ is the density of the fluid, and P is the pressure in the fluid. To an observer in a fixed coordinate system where there is no stream present, the only external force acting on the bubble would be the acceleration due to gravity. However, the observer in a moving frame observes a stream. Therefore, the acceleration seen by an observer in the moving frame is that due to the stream and the effect of gravity. Expression 2.01 may now be written as:

$$\frac{\partial \hat{v}}{\partial t} + (\nabla x\hat{v})x\hat{v} = (g+\ddot{B})\hat{k} - \frac{1}{2}\nabla(\hat{v}\cdot\hat{v}) - \frac{1}{\rho}\nabla P \qquad (2.02)$$

where $(\hat{v}\cdot\nabla)\hat{v}$ has been expressed in terms of the grad $v^2$ and the curl $\hat{v}$. If expression 2.01 is integrated about a closed path, then:

$$\oint\left[\frac{\partial \hat{v}}{\partial t} + (\nabla x\hat{v})x\hat{v}\right]\cdot d\hat{s} = \oint(g+\ddot{B})\hat{k}\cdot d\hat{s} - \oint\left[\frac{1}{2}\nabla(\hat{v}\cdot\hat{v}) + \frac{1}{\rho}\nabla P\right]\cdot d\hat{s} \qquad (2.03)$$

Since the fluid is incompressible, the last integral on the r. h. s. can be transformed into a surface integral which vanishes:

$$\iint_S \left[\frac{1}{2}\nabla x \nabla(\hat{v}\cdot\hat{v}) + \frac{1}{\rho}\nabla x \nabla P\right]\cdot d\hat{S} = 0 \qquad (2.04)$$

Expression 2.03 becomes:

$$\oint \left[\frac{\partial \hat{v}}{\partial t} + (\nabla x \hat{v}) x \hat{v}\right]\cdot d\hat{s} = \oint (g+\ddot{B})\hat{k}\cdot d\hat{s} \qquad (2.05)$$

If the flow is irrotational everywhere, as in the case of radial flow only, expression 2.05 is zero no matter what path is chosen. However, if there exists a discontinuity in the tangential velocity and if the path of integration crosses the boundary of this discontinuity, expression 2.05 is different from zero. As was discussed in Chapter I, the velocity of the fluid internal to the boundary is zero. Therefore, $\ddot{B}$ is zero internal to the boundary. Uniformity of the pressure within the bubble requires gravity not to act on the internal fluid. Thus:

$$\oint (g+\ddot{B})\hat{k}\cdot d\hat{s} = \int_{s_1}^{s_2} (g+\ddot{B})\hat{k}\cdot\hat{e}_T ds \qquad (2.06)$$

where the integration is over the external boundary, $s_1$ and $s_2$ are the coordinates where the boundary is crossed, and $\hat{e}_T$ is the unit tangential vector at the boundary.

The vector $(\nabla x \hat{v}) x \hat{v}$ is zero everywhere except for the infinitely thin interior of the vortex sheet. It lies in the plane of the line integral and is normal to the boundary. Hence:

$$\oint (\nabla x \hat{v}) x \hat{v} \cdot d\hat{s} = \left[ \int_o^{\Delta\tau} (\nabla x \hat{v}) x \hat{v} \cdot e_n d\tau \right]_2 - \left[ \int_o^{\Delta\tau} (\nabla x \hat{v}) x \hat{v} \cdot \hat{e}_n d\tau \right]_1 \tag{2.07}$$

where $\hat{e}_n$ is the unit normal to the vortex sheet, $\Delta\tau$ is the thickness of the sheet, and subscripts 1, 2 refer to the points of crossing on the boundary. Since $|(\nabla x \hat{v}) x \hat{v}|$ is finite, the integrals vanish as $\Delta\tau \to 0$.

The integral over $\partial\hat{v}/\partial t$ for the path interior to the bubble is zero since the velocity is everywhere zero internal to the boundary. The integral of $\partial\hat{v}/\partial t$ over the paths of integration interior to the vortex sheet can be written as:

$$\lim_{\Delta\tau \to 0} \left[ \int_o^{\Delta\tau} \frac{\partial \hat{v}}{\partial t} \cdot \hat{e}_n d\tau \right]_1 - \lim_{\Delta\tau \to 0} \left[ \int_o^{\Delta\tau} \frac{\partial \hat{v}}{\partial t} \cdot \hat{e}_n d\tau \right]_2 = 0 \tag{2.08}$$

where the subscripts have been previously defined. Expression 2.05 becomes:

$$\int_{s_1}^{s_2} \frac{\partial \hat{v}}{\partial t} \cdot \hat{e}_T ds = \int_{s_1}^{s_2} (g + \ddot{B}) \hat{k} \cdot \hat{e}_T ds \tag{2.09}$$

Since the path of integration can be made as small as desired, the integrands in expression 2.09 are equal, thus:

$$\frac{\partial v_T}{\partial t} = \frac{g + \ddot{B}}{\sqrt{1 + (r')^2}} \tag{2.10}$$

where:

$$\frac{\partial \hat{v}}{\partial t} \cdot \hat{e}_T = \frac{\partial v_T}{\partial t} \tag{2.11}$$

$$\hat{k} \cdot \hat{e}_T = \frac{1}{\sqrt{1+(r')^2}} \tag{2.12}$$

$$r' = \left(\frac{dr}{dz}\right) \tag{2.13}$$

To illustrate expression 2.10, the Herring-Zoller[7] equation for $\dot{B}$ will be derived at this point. The time rate of change of the tangential velocity, for a spherical bubble, at the boundary is:

$$\frac{\partial v_T}{\partial t} = \frac{3}{2} \frac{\ddot{B}}{\sqrt{1+(r')^2}} + \frac{3}{2} \frac{\dot{B}\dot{a}}{a\sqrt{1+(r')^2}} \tag{2.14}$$

where $\dot{a}$ is the normal velocity at the boundary of the sphere and a is the radius of the sphere. Expression 2.10 for a spherical bubble can be written as:

$$\ddot{B} + 3\,\frac{\dot{B}\dot{a}}{a} = 2g \tag{2.15}$$

Upon multiplying both sides by $a^3$ and rearranging terms, expression 2.12 can be put into the form:

$$(\dot{B}a^3)^{\cdot} = 2ga^3 \tag{2.16}$$

Integrating both sides of expression 2.16 over time where $\dot{B} = 0$ at $t = 0$ gives:

$$\dot{B} = \frac{2g}{a^3} \int_0^t a^3 dt \tag{2.17}$$

Expression 2.17 is the Herring-Zoller[8] equation.

The relationship between the tangential velocity and the circulation density of the vortex rings, established in Chapter I, leads to the result:

$$\frac{\partial\gamma(s)}{dt} = \frac{\ddot{B} + g}{\sqrt{1+(r')^2}} \tag{2.15}$$

# CHAPTER III

## VELOCITY POTENTIAL OF THE SOURCE SHEET AND ITS TIME DERIVATIVE

The contribution to total velocity potential by the source sheet, expressed in cylindrical coordinates, can be written as:

$$\varphi_S(r_o,z_o,t) = \frac{1}{2\pi}\int_a^b\int_o^\pi \frac{v_n(r,z,t)r\sqrt{1+(r')^2}\,d\theta dz}{\sqrt{r^2+r_o^{\,2}+(z_o-z)^2-2rr_o\cos\theta}} \tag{3.01}$$

where a and b are functions of time and are the value of z at the top and bottom (poles) of the bubble where r = 0. Expression 3.01 can be written in the form:

$$\varphi_S(r_o,z_o,t) = \frac{1}{2\pi}\int_a^b v_n(r,z,t)K(r,z,r_o,z_o)\sqrt{1+(r')^2}\,dz \tag{3.02}$$

where the kernel is defined as:

$$K(r,z,r_o,z_o) = \int_o^\pi \frac{r\,d\theta}{\sqrt{\alpha-\beta\cos\theta}} \tag{3.03}$$

$$\alpha = r^2 + r_o^{\,2} + (z_o-z)^2 \tag{3.04}$$

$$\beta = 2rr_o \tag{3.05}$$

According to Byrd[9], the solution to an elliptic integral of the type in expression 3.03 is:

$$\mathcal{K}(r,z,r_o,z_o) = r\mu K(k) \tag{3.06}$$

where K(k) is the complete elliptic integral of the first kind. The modulus k and the variable $\mu$ are defined as:

$$k^2 = \frac{2\beta}{\alpha + \beta} \tag{3.07}$$

$$\mu = \frac{2}{\sqrt{\alpha+\beta}} \tag{3.08}$$

The kernel can be expressed in terms of k:

$$\mathcal{K}(r,z,r_o,z_o) = K(k)k\sqrt{\frac{r}{r_o}} \tag{3.09}$$

The variable r is defined as:

$$r = r(z,q_1,q_2,\ldots,q_n) \tag{3.10}$$

where the q's are parameters of the surface and are functions of time only. An example of a surface parameter would be the radius in the case of a sphere or the semi-major and semi-minor axis in the case of an ellipse, etc.

The partial time derivative of expression 3.02, where the field points are fixed, is:

$$\frac{\partial}{\partial t}\varphi_b(r_o,z_o,t) = \frac{1}{2\pi}\int_a^b \frac{\partial}{\partial t} v_n(r,z,t)\mathcal{K}(r,z,r_o,z_o)\sqrt{1+(r')^2}\,dz \tag{3.11}$$

$$+ \frac{1}{2\pi}\int_a^b v_n(r,z,t)\frac{\partial}{\partial t}\mathcal{K}(r,z,r_o,z_o)\sqrt{1+(r')^2}\,dz$$

$$+ \frac{1}{2\pi}\int_a^b v_n(r,z,t)K(r,z,r_o,z_o)\frac{\partial}{\partial t}\left(\sqrt{1+(r')^2}\right)dz$$

$$+ \frac{1}{2\pi}\left[\dot{b}\, v_n(r,z,t)K(r,z,r_o,z_o)\sqrt{1+(r')^2}\Big|_{z\,=\,b}\right.$$

$$\left. - \dot{a}\, v_n(r,z,t)K(r,z,r_o,z_o)\sqrt{1+(r')^2}\Big|_{z\,=\,a}\right]$$

where:

$$\frac{\partial}{\partial t}K(r,z,r_o,z_o) = \frac{1}{2r}\frac{\partial r}{\partial t}\left[E(k)\left\{1 + \frac{1}{2}\left(\frac{r_o-r}{r_o}\right)\left(\frac{k}{k'}\right)^2\right\} + K(k)\right]k\sqrt{\frac{r}{r_o}} \qquad (3.12)$$

$$(k')^2 = 1 - k^2 \qquad (3.13)$$

$$\dot{b} = \frac{db}{dt} \qquad (3.14)$$

$$\dot{a} = \frac{da}{dt} \qquad (3.15)$$

and k' is called the complementary modulus. If k and k' are evaluated using expressions 3.07 and 3.13, the positive square root is always used. Expression 3.11 is the partial time derivative of the velocity potential for the source sheet at any point in the fluid excluding points on the boundary of the bubble. As was shown in Chapter I, the evaluation of the partial time derivative of the velocity potential at the boundary includes a $1/2\ v_n^2(r_o,z_o,t)$. Therefore, this extra term must be added to the r. h. s. of expression 3.11 when the partial time derivative of the velocity potential is to be evaluated at the boundary. From this point on, the velocity potential of the source sheet and its time derivative are to be evaluated at the boundary.

Recall that the normal velocity is derived from the surface equation $F(r,z,t)$. The partial operator $\partial/\partial t$ operates only on the surface parameters. Therefore, the normal velocity expressed in terms of the time derivative of the surface parameters is:

$$v_n = \sum_i \dot{q}_i f_i \tag{3.16}$$

where:

$$f_i = - \frac{\frac{\partial}{\partial q_i} F(r,z,t)}{|\nabla F(r,z,t)|} \tag{3.17}$$

The partial operator $\partial/\partial t$ operating on r and $v_n(r,z,t)$ can be expressed as:

$$\frac{\partial r}{\partial t} = \sum_i \dot{q}_i \frac{\partial r}{\partial q_i} \tag{3.18}$$

$$\frac{\partial}{\partial t} v_n(r,z,t) = \sum_i \left[ \ddot{q}_i f_i + \dot{q}_i^{\,2} \frac{\partial f_i}{\partial q_i} \right] \tag{3.19}$$

It must be remembered that $\partial/\partial q_i$ operating on $f_i$ will operate on r keeping z fixed, but the **partial derivative operates on F(r,z,t) while both r and z are fixed.**

The velocity potential of the source sheet and its time derivative can now be expressed as:

$$\varphi_S(r_o,z_o,t) = \sum_i \dot{q}_i \sigma_{o_i} \tag{3.20}$$

$$\frac{\partial}{\partial t} \varphi_S(r_o,z_o,t) = \frac{1}{2}\left(\sum_i \dot{q}_i f_i\right)^2 + \sum_i \left[ \ddot{q}_i \sigma_{o_i} + \dot{q}_i^{\,2}(\sigma_{2_i} + \sigma_{3_i} + \sigma_{4_i}) + \dot{q}_i \sigma_{5_i} \right] \tag{3.21}$$

where:

$$\sigma_{o_i} = \frac{1}{2\pi}\int_a^b f_i K(z,z_o)\sqrt{1+(r')^2}\,dz \tag{3.22}$$

$$\sigma_{2_i} = \frac{1}{2\pi}\int_a^b \frac{\partial f_i}{\partial q_i} K(z,z_o)\sqrt{1+(r')^2}\,dz \tag{3.23}$$

$$\sigma_{3_i} = \frac{1}{2\pi}\int_a^b f_i K_1(z,z_o)\sqrt{1+(r')^2}\,dz \tag{3.24}$$

$$\sigma_{4_i} = \frac{1}{2\pi}\int_a^b f_i K(z,z_o)\frac{\partial}{\partial q_i}\left(\sqrt{1+(r')^2}\right)dz \tag{3.25}$$

$$\sigma_{5_i} = \frac{1}{2\pi}\left[\dot{b} f_i K(z,z_o)\sqrt{1+(r')^2}\Big|_{z=b} - \dot{a} f_i K(z,z_o)\sqrt{1+(r')^2}\Big|_{z=a}\right] \tag{3.26}$$

$$K_1(z,z_o) = \frac{1}{2r}\frac{\partial r}{\partial q_i}\left[E(k)\left\{1 + \frac{1}{2}\left(\frac{r_o - r}{r_o}\right)\left(\frac{k}{k'}\right)^2\right\} + K(k)\right]k\sqrt{\frac{r}{r_o}} \tag{3.27}$$

The new notation for the argument of the kernels shows that the potential and its time derivative are to be evaluated on the boundary where:

$$r_o = r_o(z_o, q_1, q_2, \ldots, q_n) \tag{3.28}$$

The kernels $K(z,z_o)$ and $K_1(z,z_o)$ have singularities at $z = z_o$. As $z \to z_o$, $k \to 1$ and $K(k) \to \infty$ while $E(k) \to 1$. If $k \approx 1$, the complete elliptic integrals $K(k)$ and $E(k)$ can be represented by a series in terms of $\ln(4/k')$, Byrd[10]. If only the first term of the series is used in each case, the complete elliptic integrals behave as:

$$K(k) \approx -\ln\left(\frac{k'}{4}\right) \tag{3.29}$$

$$E(k) \approx 1 \tag{3.30}$$

Expand r in terms of z about $z_o$ such that:

$$r = r_o - (r')_o (z_o - z) + \frac{1}{2} (r'')_o^2 (z_o - z) \tag{3.31}$$

where terms higher than second order have been ignored and:

$$(r')_o = \left(\frac{dr}{dz}\right)_{z=z_o} \tag{3.32}$$

$$(r'')_o = \left(\frac{d^2r}{dz^2}\right)_{z=z_o} \tag{3.33}$$

It is easily shown that for $z \approx z_o$, k' behaves as:

$$k' \approx \frac{|z_o - z|\sqrt{1+(r')_o^2}}{2r_o} \tag{3.34}$$

Upon substituting the r. h. s. of expression 3.34 for k' in expression 3.29, the complete elliptic integral K(k), **for $z \approx z_o$, can be written as:**

$$K(k) \approx - \ln\left[\frac{|z_o - z|\sqrt{1+(r')_o^2}}{8r_o}\right] \tag{3.35}$$

If the kernels $K(z,z_o)$ and $K_1(z,z_o)$ are expanded in terms of z about $z_o$ and terms higher than the zero order term are ignored, **these kernels behave as:**

$$\overline{K}(z,z_o) = - \ln\left[\frac{|z_o - z|\sqrt{1+(r')_o^2}}{8r_o}\right] \tag{3.36}$$

$$\bar{K}_1(z,z_o) = \frac{1}{2r_o}\left(\frac{\partial r}{\partial q_i}\right)_o\left[1 + \frac{2(rr')_o}{(z_o-z)[1+(r')_o^2]} - \ln\left\{\frac{|z_o-z|\sqrt{1+(r')_o^2}}{8r_o}\right\}\right] \quad (3.37)$$

where $\bar{K}(z,z_o)$ and $\bar{K}_1(z,z_o)$ are the approximate kernels that represent the behavior of the original kernels about the singularity and:

$$\left(\frac{\partial r}{\partial q_i}\right)_o = \left(\frac{\partial r}{\partial q_i}\right)_{z=z_o} \quad (3.38)$$

$$(rr')_o = \left(r\,\frac{dr}{dz}\right)_{z=z_o} \quad (3.39)$$

If the other factors in the integrals of expressions 3.22 - 3.25 are evaluated at $z = z_o$ and are multiplied by the appropriate approximate kernel, the integration of the approximate integrand over z is:

$$\bar{\sigma}_{o_i} = \frac{1}{2\pi}\int_a^b f_{i_o}\,\bar{K}(z,z_o)\sqrt{1+(r')_o^2}\,dz \quad (3.40)$$

$$\bar{\sigma}_{2_i} = \frac{1}{2\pi}\int_a^b \left(\frac{\partial f_i}{\partial q_i}\right)_o \bar{K}(z,z_o)\sqrt{1+(r')_o^2}\,dz \quad (3.41)$$

$$\bar{\sigma}_{3_i} = \frac{1}{2\pi}\int_a^b f_{i_o}\,\bar{K}_1(z,z_o)\sqrt{1+(r')_o^2}\,dz \quad (3.42)$$

$$\bar{\sigma}_{4_i} = \frac{1}{2\pi}\int_a^b f_{i_o}\,\bar{K}(z,z_o)\left[\frac{\partial}{\partial q_i}\left(\sqrt{1+(r')^2}\right)\right]_o dz \quad (3.43)$$

where $\bar{\sigma}_{o_i}$, $\bar{\sigma}_{2_i}$, $\bar{\sigma}_{3_i}$, and $\bar{\sigma}_{4_i}$ are approximate coefficients that represent the behavior of the original coefficients about the singularity and:

$$f_{i_o} = \left(f_i\right)_{z=z_o} \quad (3.44)$$

$$\left(\frac{\partial f_i}{\partial q_i}\right)_o = \left(\frac{\partial f_i}{\partial q_i}\right)_{z=z_o} \tag{3.45}$$

$$\left[\frac{\partial}{\partial q_i}\left(\sqrt{1+(r')^2}\right)\right]_o = \left[\frac{\partial}{\partial q_i}\left(\sqrt{1+(r')^2}\right)\right]_{z=z_o} \tag{3.46}$$

If the approximate coefficients are added to and subtracted from the orinal coefficients, the singularity is removed and the original coefficients can now be written as:

$$\sigma_{o_i} = \frac{1}{2\pi}\int_a^b \left[f_i K(z,z_o)\sqrt{1+(r')^2} - f_{i_o}\bar{K}(z,z_o)\sqrt{1+(r')_o^2}\right]dz + \bar{\sigma}_{o_i} \tag{3.27}$$

$$\sigma_{2_i} = \frac{1}{2\pi}\int_a^b \left[\frac{\partial f_i}{\partial q_i} K(z,z_o)\sqrt{1+(r')^2} - \left(\frac{\partial f_i}{\partial q_i}\right)_o \bar{K}(z,z_o)\sqrt{1+(r')_o^2}\right]dz + \bar{\sigma}_{2_i} \tag{3.48}$$

$$\sigma_{3_i} = \frac{1}{2\pi}\int_a^b \left[f_i K_1(z,z_o)\sqrt{1+(r')^2} - f_{i_o}\bar{K}_1(z,z_o)\sqrt{1+(r')_o^2}\right]dz + \bar{\sigma}_{3_i} \tag{3.49}$$

$$\sigma_{4_i} = \frac{1}{2\pi}\int_a^b \left[f_i K(z,z_o)\frac{\partial}{\partial q_i}\left(\sqrt{1+(r')^2}\right) - f_{i_o}\bar{K}(z,z_o)\left\{\frac{\partial}{\partial q_i}\left(\sqrt{1+(r')^2}\right)\right\}_o\right]dz + \bar{\sigma}_{4_i} \tag{3.50}$$

where $\bar{\sigma}_{o_i}$, $\bar{\sigma}_{2_i}$, $\bar{\sigma}_{3_i}$, and $\bar{\sigma}_{4_i}$ are defined as:

$$\bar{\sigma}_{o_i} = -\frac{1}{2\pi} f_{i_o}\sqrt{1+(r')_o^2}\left[(b-z_o)\ln\left|\frac{(b-z_o)\sqrt{1+(r')_o^2}}{8r_o}\right| + (z_o-a)\ln\left|\frac{(z_o-a)\sqrt{1+(r')_o^2}}{8r_o}\right| - (b-a)\right] \tag{3.51}$$

$$\bar{\sigma}_{2_i} = -\frac{1}{2\pi}\left(\frac{\partial f_i}{\partial q_i}\right)_o \sqrt{1+(r')^2}\left[(b-z_o)\ln\left|\frac{(b-z_o)\sqrt{1+(r')_o^2}}{8r_o}\right|\right. \tag{3.52}$$

$$\left. + (z_o-a)\ln\left|\frac{(z_o-a)\sqrt{1+(r')_o^2}}{8r_o}\right| - (b-a)\right]$$

$$\bar{\sigma}_{3_i} = \frac{1}{4\pi r_o} f_{i_o}\sqrt{1+(r')_o^2}\left(\frac{\partial r}{\partial q_i}\right)_o\left[2(b-a) + \frac{2(rr')_o}{1+(r')_o^2}\ln\left|\frac{z_o-a}{b-z_o}\right|\right. \tag{3.53}$$

$$\left. - (b-z_o)\ln\left|\frac{(b-z_o)\sqrt{1+(r')_o^2}}{8r_o}\right| - (z_o-a)\ln\left|\frac{(z_o-a)\sqrt{1+(r')_o^2}}{8r_o}\right|\right]$$

$$\bar{\sigma}_{4_i} = -\frac{1}{2\pi} f_{i_o}\left\{\frac{\partial}{\partial q_i}\left(\sqrt{1+(r')^2}\right)\right\}_o\left[(b-z_o)\ln\left|\frac{(b-z_o)\sqrt{1+(r')_o^2}}{8r_o}\right|\right. \tag{3.54}$$

$$\left. + (z_o-a)\ln\left|\frac{(z_o-a)\sqrt{1+(r')_o^2}}{8r_o}\right| - (b-a)\right]$$

The $\sigma_{1_i}$ coefficient has been reserved for the $\ddot{B}$ term that will be a result of the partial time derivative of the potential due to the distribution of the vortex rings on the surface of the bubble. The reason for this type of notation will become clearer in the chapter on the Bernoulli equation.

# CHAPTER IV

## VELOCITY POTENTIAL OF THE VORTEX SHEET AND ITS TIME DERIVATIVE

The contribution to the total velocity potential by the vortex sheet, expressed in cylindrical coordinates, can be written as:

$$\varphi_V(r_o,z_o,t) = \frac{1}{4\pi}\int_a^b v_T(r,z,t)\Omega(r,z,r_o,z_o)\sqrt{1+(r')^2}\,dz \qquad (4.01)$$

where $v_T(r,z,t)$ is the tangential velocity at the boundary in the moving frame of reference, $\Omega(r,z,r_o,z_o)$ is the solid angle of the vortex ring subtended at the field point, and the limits a and b are defined in the same manner as in Chapter III.

The partial time derivative of expression 4.01 is:

$$\frac{\partial}{\partial t}\varphi_V(r_o,z_o,t) = \frac{1}{4\pi}\int_a^b \frac{\partial}{\partial t} v_T(r,z,t)\Omega(r,z,r_o,z_o)\sqrt{1+(r')^2}\,dz \qquad (4.02)$$

$$+ \frac{1}{4\pi}\int_a^b v_T(r,z,t)\frac{\partial}{\partial t}\Omega(r,z,r_o,z_o)\sqrt{1+(r')^2}\,dz$$

$$+ \frac{1}{4\pi}\int_a^b v_T(r,z,t)\Omega(r,z,r_o,z_o)\frac{\partial}{\partial t}\left(\sqrt{1+(r')^2}\right)dz$$

$$+ \frac{1}{4\pi}\left[\dot{b}\, v_T(r,z,t)\Omega(r,z,r_o,z_o)\Big|_{z=b} - a\, v_T(r,z,t)\Omega(r,z,r_o,z_o)\Big|_{z=a}\right]$$

As was shown in Chapter I, the evaluation of the partial time derivative of the velocity potential at the boundary includes an additive term of

$1/2\ v_n(r_o,z_o,t)v_T(r_o,z_o,t)\cot\theta$. Therefore, this extra term must be added to the r. h. s. of expression 4.02 when the partial time derivative of the velocity potential is to be evaluated at the boundary. From this point on, the velocity potential of the vortex sheet and its time derivative are to be evaluated at the boundary.

Recall that the result of the vortex theorem for migrating bubbles is:

$$\frac{\partial}{\partial t} v_T(r,z,t) = \frac{\ddot{B} + g}{\sqrt{1+(r')^2}} \tag{4.03}$$

where g is the acceleration due to gravity. If the variable r is defined in the same manner as in Chapter III, expression 4.02 can be written as:

$$\frac{\partial}{\partial t} \varphi_V(r_o,z_o,t) = \sigma_1 + \Sigma\dot{q}_1(\sigma_{6_i} + \sigma_{7_i} + \frac{1}{2} f_i v_T(r_o,z_o t)\cot\theta) \tag{4.04}$$

where it will be seen later in the chapter that $\Omega(r,z,r_o,z_o) = 0$ for $r = 0$ and this makes the last term of expression 4.02 zero and:

$$\sigma_1 = \frac{1}{4\pi}\int_a^b \frac{\partial}{\partial t} v_T(r,z,t)\Omega(z,z_o)dz \tag{4.05}$$

$$\sigma_{6_i} = \frac{1}{4\pi}\int_a^b v_T(r,z,t)\frac{\partial r}{\partial q_i}\frac{\partial}{\partial r}\Omega(z,z_o)\sqrt{1+(r')^2}\,dz \tag{4.06}$$

$$\sigma_{7_i} = \frac{1}{4\pi}\int_a^b v_T(r,z,t)\Omega(z,z_o)\frac{\partial}{\partial q_i}\left(\sqrt{1+(r')^2}\right)dz \tag{4.07}$$

The solid angle $\Omega(z,z_o)$ is the same as in expression 4.02, but the new notation for its arguement shows that the coefficients are to be evaluated on the boundary where:

$$r_o = r_o(z_o, q_1, q_2, \ldots, q_n) \tag{4.08}$$

The solid angle of a vortex ring subtended at any field point in the fluid is:

$$\Omega = \int_{S*} \frac{\partial}{\partial n}\left(\frac{1}{R*}\right) dS* \tag{4.09}$$

where R* is the distance from the field point to any point in the plane enclosed by the vortex ring and S* is the area of the plane enclosed by the vortex ring. The outward drawn normal of the plane is the unit vector $\hat{k}$ in the positive z direction. It is easily shown that:

$$R* = \sqrt{e^2 + r_o^{\,2} + (z_o - z)^2 - 2r_o e \cos\theta} \tag{4.10}$$

$$\frac{\partial}{\partial n}\left(\frac{1}{R*}\right) = \frac{(z_o - z)}{\left[e^2 + r_o^{\,2} + (z_o - z)^2 - 2r_o e \cos\theta\right]^{3/2}} \tag{4.11}$$

where e is the variable of integration. The expression for the solid angle is seen to be:

$$\Omega(r, z, r_o, z_o) = 2\int_0^{\pi}\int_0^{r} \frac{(z_o - z)e\,de\,d\theta}{\left[e^2 + r_o^{\,2} + (z_o - z)^2 - 2er_o \cos\theta\right]^{3/2}} \tag{4.12}$$

where r is the radius of the vortex ring enclosing the plane. Since the integration over e can be done in closed form, the solid angle can be expressed as:

$$\Omega(r,z,r_o,z_o) = 2(z_o-z)\int_o^{\pi} \frac{\sqrt{r_o^2 + (z_o-z)^2}\, d\theta}{r_o^2 + (z_o-z)^2 - r_o^2 \cos^2\theta} \tag{4.13}$$

$$- 2(z_o-z)\int_o^{\pi} \frac{\left[r_o^2 + (z_o-z)^2 - r_o r \cos\theta\right] d\theta}{\left[r_o^2 + (z_o-z)^2 - r_o^2 \cos^2\theta\right]\sqrt{r^2 + r_o^2 + (z_o-z)^2 - 2rr_o \cos\theta}}$$

The first integral can be integrated over $\theta$ in closed form for a body of revolution. Therefore, the solid angle can be written as:

$$\Omega(r,z,r_o,z_o) = 2\pi \frac{(z_o-z)}{|z_o-z|} - 2(z_o-z)\int_o^{\pi} \tag{4.14}$$

$$\frac{\left[r_o^2 + (z_o-z)^2 - r_o r \cos\theta\right] d\theta}{\left[r_o^2 + (z_o-z)^2 - r_o^2 \cos^2\theta\right]\sqrt{r_o^2 + r^2 + (z_o-z)^2 - 2rr_o \cos\theta}}$$

After using the method of partial fractions and rearranging terms, the solid angle can be put into the convient form:

$$\Omega(r,z,r_o,z_o) = 2\pi\left[\pm 1 - \frac{(z_o-z)}{2\pi}\int_o^{\pi} \frac{\dfrac{\lambda + r}{\lambda + r_o \cos\theta} + \dfrac{\lambda - r}{\lambda - r_o \cos\theta}}{\sqrt{\alpha - \beta \cos\theta}}\, d\theta\right] \tag{4.15}$$

where $\alpha$ and $\beta$ are the same as in expressions 3.04 and 3.05 and $\lambda$ is defined as:

$$\lambda = \sqrt{r_o^2 + (z_o-z)^2} \tag{4.16}$$

The integral in expression 4.16 can be broken up into two integrals of the type:

$$\int_0^{\pi} \frac{R(\cos\theta)}{\sqrt{\alpha - \beta\cos\theta}}\, d\theta \tag{4.17}$$

where $R(\cos\theta)$ is a rational function of $\cos\theta$. According to Byrd[9], the integrals can be reduced to Jacobian elliptic functions of the form:

$$\mu\int_0^{K(k)} R\left[\frac{1}{\beta}\,(\beta - \alpha k^2 sn^2 u)nd^2 u\right]du \tag{4.18}$$

where $\mu$ and k are defined in expressions 3.07 and 3.08. If the substitution for $\cos\theta$ in expression 4.18 is made in the two integrals resulting from the integral in 4.15 and the indentities:

$$nd^2 u = \frac{1}{dn^2 u} \tag{4.19}$$

$$dn^2 u = 1 - k^2 sn^2 u \tag{4.20}$$

are used, the solid angle can be written as:

$$\Omega(r,z,r_o,z_o) = 2\pi\left[\pm 1 - \frac{(z_o - z)k}{2\pi\sqrt{rr_o}}\left\{\left(\frac{\lambda+r}{\lambda+r_o}\right)\int_0^{K(k)} \frac{dn^2 u\,du}{1-\alpha_1^2 sn^2 u}\right.\right. \tag{4.21}$$

$$\left.\left. + \left(\frac{\lambda-r}{\lambda-r_o}\right)\int_0^{K(k)} \frac{dn^2 u\,du}{1-\alpha_2^2 sn^2 u}\right\}\right]$$

where:

$$\alpha_1^2 = \frac{[(\lambda+r)k]^2}{2r(\lambda+r_o)} \tag{4.22}$$

$$-\alpha_2^2 = \frac{[(\lambda-r)k]^2}{2r(\lambda-r_o)} \tag{4.23}$$

The integrals in expression 4.22 are complete elliptic integrals of the third kind and have five possible solutions depending on the value of $\alpha_1^2$ and $\alpha_2^2$. The condition for the five possible solutions are:

| | |
|---|---|
| I | $0 < -\alpha_1^2 < \infty$ |
| II | $k^2 < \alpha_1^2 < 1$ |
| III | $0 < \alpha_1^2 < k^2$ |
| IV | $1 < \alpha_1^2 < \infty$ |
| V | complex parameter |

If $\alpha_1^2$ is replaced by $\alpha_2^2$, the same five cases exist for the second integral. Case V is excluded since $\alpha_1^2$ and $\alpha_2^2$ are always positive by definition. It is easily shown that $\alpha_2^2$ always satisfies case I and $\alpha_1^2$ will satisfy either case II, III, or IV. Therefore, the three possible solutions for the solid angle are:

$0 < \alpha_1^2 < k^2$

$$\Omega(r,z,r_o,z_o) = 2\pi\left[\pm 1 - \frac{(z_o-z)k}{2\pi\sqrt{rr_o}}\left\{\left(\frac{\lambda+r}{\lambda+r_o}\right)\right[K(k) \tag{4.24}$$

$$+ \frac{(\alpha_1^2-k^2)K(k)Z(\beta,k)}{\sqrt{\alpha_1^2(1-\alpha_1^2)(k^2-\alpha_1^2)}}\Bigg] + \left(\frac{\lambda-r}{\lambda-r_o}\right)\frac{\pi(k^2-\alpha_2^2)\Lambda_o(\psi,k)}{2\sqrt{\alpha_2^2(1-\alpha_2^2)(\alpha_2^2-k^2)}}$$

$$\beta = \sin^{-1}\left(\frac{\alpha_1}{k}\right) \tag{4.25}$$

$$\psi = \sin^{-1}\sqrt{\frac{\alpha_2^2}{\alpha_2^2-k^2}} \tag{4.26}$$

$k^2 < \alpha_1^2 < 1$

$$\Omega(r,z,r_o,z_o) = 2\pi\left[\pm 1 - \frac{(z_o-z)k}{2\pi\sqrt{r_o r}}\left\{\left(\frac{\lambda+r}{\lambda+r_o}\right)\left[\left(\frac{k}{\alpha_1}\right)^2 K(k)\right.\right.\right. \tag{4.27}$$

$$\left.\left.\left. + \frac{\pi(\alpha_1^2-k^2)\Lambda_o(\xi,k)}{2\sqrt{\alpha_1^2(\alpha_1^2-k^2)(1-\alpha_1^2)}}\right] + \left(\frac{\lambda-r}{\lambda-r_o}\right)\frac{\pi(k^2-\alpha_2^2)\Lambda_o(\psi,k)}{2\sqrt{\alpha_2^2(1-\alpha_2^2)(\alpha_2^2-k^2)}}\right\}\right]$$

$$\xi = \sin^{-1}\sqrt{\frac{\alpha_1^2-k^2}{\alpha_1^2(k')^2}} \tag{4.28}$$

$1 < \alpha_1^2 < \infty$

$$\Omega(r,z,r_o,z_o) = 2\pi\left[\pm 1 - \frac{(z_o-z)k}{2\pi\sqrt{r_o r}}\left\{\left(\frac{\lambda+r}{\lambda+r_o}\right)\left[\left(\frac{k}{\alpha_1}\right)^2 K(k)\right.\right.\right. \tag{4.29}$$

$$\left.\left.\left. + \frac{\alpha_1(k^2-\alpha_1^2)K(k)Z(\eta,k)}{\sqrt{(\alpha_1^2-1)(\alpha_1^2-k^2)}}\right] + \left(\frac{\lambda-r}{\lambda-r_o}\right)\frac{\pi(k^2-\alpha_2^2)\Lambda_o(\psi,k)}{2\sqrt{\alpha_2^2(1-\alpha_2^2)(\alpha_2^2-k^2)}}\right\}\right]$$

$$\eta = \sin^{-1}\left(\frac{1}{\alpha_1}\right) \tag{4.30}$$

If the partial operator $\partial/\partial r$ operates on the solid angle in expression 4.14, the derivative of the solid angle is easily shown to be:

$$\frac{\partial}{\partial r}\,\Omega(r,z,r_o,z_o) = 2(z_o-z)\int_o^{\pi}\frac{r\,d\theta}{\left[\varepsilon-\beta\cos\theta\right]^{3/2}} \tag{4.31}$$

The solution to the integral of the type in expression 4.32 has been shown by Byrd[9] to be:

$$\int_0^{\pi} \frac{d\theta}{\left[\alpha-\beta \cos \theta\right]^{3/2}} = \frac{\mu}{\alpha-\beta} E(k) \tag{4.32}$$

Upon rearranging terms, expression 4.32 can be written as:

$$\frac{\partial}{\partial r} \Omega(r,z,r_o,z_o) = \frac{1}{2} \frac{E(k)}{\sqrt{rr_o}} \frac{k^3}{(k')^2}\left(\frac{z_o-z}{r_o}\right) \tag{4.33}$$

The solid angle does not have a singularity, but the derivative of the solid angle has a singularity at $z = z_o$. The derivative of the solid angle can be shown to behave as:

$$\overline{\frac{\partial}{\partial r} \Omega(z,z_o)} = \frac{2}{(z_o-z)\left[1+(r')_o^2\right]} \tag{4.34}$$

where the same method was employed here as in Chapter III in evaluating the behavior of the kernel for $z \approx z_o$. The approximate coefficient for $\sigma_{6_i}$ is:

$$\bar{\sigma}_{6_i} = \frac{1}{4\pi}\int_a^b v_T(r_o,z_o,t)\left(\frac{\partial r}{\partial q_i}\right)_o \overline{\frac{\partial}{\partial r} \Omega(z,z_o)}\sqrt{1+(r')_o^2}\, dz \tag{4.35}$$

If the approximate coefficient is added to and subtracted from the original coefficient, the singularity is removed and the coefficient can be written as:

$$\sigma_{6_i} = \frac{1}{4\pi}\int_a^b \left[v_T(r,z,t)\left(\frac{\partial r}{\partial q_i}\right)\frac{\partial}{\partial r} \Omega(z,z_o)\sqrt{1+(r')^2}\right. \tag{4.36}$$

$$\left. - v_T(r_o,z_o,t)\left(\frac{\partial r}{\partial q_i}\right)_o \overline{\frac{\partial}{\partial r} \Omega(z,z_o)}\sqrt{1+(r')_o^2}\right] dz + \bar{\sigma}_{6_i}$$

where:

$$\bar{\sigma}_{6_i} = \frac{v_T(r_o, z_o, t)}{2\pi\sqrt{1+(r')_o^2}} \left(\frac{\partial r}{\partial q_i}\right)_o \ln\left|\frac{z_o - a}{b - z_o}\right| \tag{4.37}$$

# CHAPTER V

## TANGENTIAL VELOCITY ON THE BOUNDARY OF A NONRIGID BODY OF REVOLUTION EXPANDING AND MIGRATING IN A LIQUID MEDIUM

The tangential velocity **at the boundary of a migrating and oscillating body of revolution will be derived in the moving coordinate system.**

Assume a body of revolution is immersed in a liquid medium and the surface S of the body divides the whole space into two regions and that both regions are occupied by the fluid. If the fluid external to the body moves past the body with a velocity $\dot{B}$ in the positive z direction and the fluid internal to the body is at rest, the surface S can be regarded as a surface of discontinuity of the velocity. The tangential component of the velocity can be produced by a layer of vortex rings on the surface of the body.

If a velocity of $-\dot{B}$ is superposed onto the whole flow, so that the fluid external to the body is at rest at infinity, there will be a uniform parallel flow of velocity $-\dot{B}$ internal to the body, where the fluid was at rest before. This flow is produced by the vortex layer on S. The problem is to determine a distribution of vortex rings on the surface S of the body which produces a uniform parallel flow internal to the body such that the tangential velocity at the field point $P(r_o, z_o)$ on the boundary is $-\dot{B}/\sqrt{1+(r')_o^2}$, where:

$$(r')_o = \left(\frac{dr}{dz}\right)_{r=r_o} \tag{5.01}$$

It has been shown, by Prandtl[11], that the contribution to the tangential velocity by the vortex on which the field point is located is $-1/2\ v_T(r_o, z_o, t)$.

The induced tangential velocity at $P(r_o, z_o)$ due to all the vortex rings, except the one located at the field point, can be derived through the use of the Biot-Survart Law:

$$v_{T_I}(r_o, z_o, t) = \frac{1}{4\pi}\int_{s_1}\int_{s_2} v_T(r,z,t)\,\frac{\hat{e}_1 \times \vec{R}}{|\vec{R}|^3}\cdot e_2 ds_2 ds_1 \qquad (5.02)$$

where:

$$\vec{R} = \hat{i}\,(r_o - r\cos\theta) - \hat{j}\,r\sin\theta + \hat{k}\,(z_o - z) \qquad (5.03)$$

$$\hat{e}_1 = -\hat{i}\sin\theta + j\cos\theta \qquad (5.04)$$

$$\hat{e}_2 = \frac{\hat{i}\,(r')_o + \hat{k}}{\sqrt{1+(r')_o^{\,2}}} \qquad (5.05)$$

$$ds_1 = \sqrt{1+(r')^2}\,dz \qquad (5.06)$$

$$ds_2 = r d\theta \qquad (5.07)$$

Upon substituting expressions 5.03 - 5.07 into expression 5.02 and rearranging terms, the induced tangential velocity is found to be:

$$v_{T_I}(r_o, z_o, t) = -\frac{1}{\pi\sqrt{1+(r')_o^{\,2}}}\int_a^b v_T(r,z,t)K_T(r,z,r_o,z_o)\sqrt{1+(r')^2}\,dz \qquad (5.08)$$

where a and b are defined in the same way as in Chapter III and:

$$K_T(r,z,r_o,z_o) = \frac{1}{2}\int_o^{\pi} \frac{r - r_o \cos\theta \left[1 - \frac{z_o - z}{r_o}(r')_o\right]}{\left[r^2 + r_o^2 + (z_o - z)^2 - 2rr_o \cos\theta\right]^{3/2}} r d\theta \qquad (5.09)$$

Expression 5.09 can be put into the form:

$$K_T(r,z,r_o,z_o) = \frac{1}{2}\left[r^2 \int_o^{\pi} \frac{d\theta}{(\alpha - \beta\cos\theta)^{3/2}} - r_o r\left\{1 - \frac{z_o - z}{r_o}(r')_o\right\}\int_o^{\pi} \frac{\cos\theta}{(\alpha - \beta\cos\theta)^{3/2}} d\theta\right. \qquad (5.10)$$

where:

$$\alpha = r^2 + r_o^2 + (z_o - z)^2 \qquad (5.11)$$

and:

$$\beta = 2rr_o \qquad (5.12)$$

According to Byrd[9], **the solution to elliptic integrals of the type in expression 5.10 leads to the solution:**

$$K_T(r,z,r_o,z_o) = \frac{1}{2}\left[\frac{r^2\mu}{\alpha - \beta} E(k) + r_o r\left\{1 - \frac{z_o - z}{r_o}(r')_o\right\}\right. \qquad (5.13)$$

$$\left.\left\{-\frac{\mu}{\alpha - \beta} E(k) + \frac{\mu}{\beta} K(k) - \frac{\mu}{\beta} E(k)\right\}\right]$$

where:

$$\mu = \frac{2}{\sqrt{\alpha + \beta}} \qquad (5.14)$$

Upon expressing $\alpha$, $\beta$, and $\mu$ in terms of r, z, $r_o$, and $z_o$, expression 5.13 can be expressed as:

$$K_T(r,z,r_o,z_o) = \frac{1}{\sqrt{(r_o+r)^2+(z_o-z)^2}}\left[K(k)\left\{1 - \frac{z_o-z}{r_o}(r')_o\right\}\right. \tag{5.15}$$

$$\left. - E(k)\left\{1 + \frac{2r(r_o-r)}{(r_o-r)^2+(z_o-z)^2} - \frac{z_o-z}{r_o}(r')_o\left(1 + \frac{2rr_o}{(r_o-r)^2+(z_o-z)^2}\right)\right\}\right]$$

where the 1/2 factor is now taken outside of the integral sign. Expression 5.15 can be simplified further to give:

$$K_T(r,z,r_o,z_o) = \frac{1}{2r}\sqrt{\frac{r}{r_o}}\,k\left[K(k)\left\{1 - \frac{z_o-z}{r_o}(r')_o\right\}\right. \tag{5.16}$$

$$\left. - E(k)\left\{1 + \frac{1}{2}\left(\frac{r_o-r}{r_o}\right)\left(\frac{k}{k'}\right)^2 - \frac{z_o-z}{r_o}(r')_o\left[1 + \frac{1}{2}\left(\frac{k}{k'}\right)^2\right]\right\}\right]$$

The induced tangential velocity is:

$$v_{T_I}(r_o,z_o,t) = \frac{1}{2\pi\sqrt{1+(r')_o^2}}\int_a^b v_T(r,z,t)K_T(r,z,r_o,z_o)\sqrt{1+(r')^2}\,dz \tag{5.17}$$

where the kernel $K_T(r,z,r_o,z_o)$ is that of expression 5.16.

If the tangential velocity at $P(r_o,z_o)$ due to the fluid flow internal to the boundary is set equal to the sum of $-1/2\ v_T(r_o,z_o,t)$ and the induced tangential velocity, the tangential velocity $v_T(r_o,z_o,t)$ is found to be:

$$v_T(r_o,z_o,t) = \frac{2\dot{B} - \frac{1}{\pi}\int_a^b v_T(r,z,t)K_T(r,z,r_o,z_o)\sqrt{1+(r')^2}\,dz}{\sqrt{1+(r')_o^2}} \tag{5.18}$$

The kernel in the integral of expression 5.18 has a singular point at $z = z_o$. If the same method is used here, as was employed in Chapter III, to remove the singularity, the approximate kernel is found to be:

$$\bar{K}_T(r,z,r_o,z_o) = -\frac{1}{2r_o}\left[\ln\left\{\frac{|z_o-z|\sqrt{1+(r')_o^2}}{8r_o}\right\} + 1 - \frac{r_o(r'')_o}{1+(r')_o^2}\right] \tag{5.19}$$

where:

$$(r'')_o = \left(\frac{d^2r}{dz^2}\right)_{r=r_o} \tag{5.20}$$

The tangential velocity at the boundary with the singularity removed is:

$$v_T(r_o,z_o,t) = \left[2\dot{B} - \frac{1}{\pi}\int_a^b \left\{v_T(r,z,t)K(r,z,r_o,z_o)\sqrt{1+(r')^2}\right.\right. \tag{5.21}$$

$$\left.\left. - v_T(r_o,z_o,t)\bar{K}_T(r,a,r_o,z_o)\sqrt{1+(r')_o^2}\,dz\right]\frac{1}{\sqrt{1+(r')_o^2}} + \frac{v_T(r_o,z_o t)}{2\pi r_o}\right.$$

$$\left[(z_o-a)\ln\left\{\frac{|z_o-a|\sqrt{1+(r')_o^2}}{8r_o}\right\} + (b-z_o)\ln\left\{\frac{|b-z_o|\sqrt{1+(r')_o^2}}{8r_o}\right\}\right.$$

$$\left. - (b-a)\left\{1 - \frac{r_o(r'')_o}{1+(r')_o^2}\right\}\right]$$

# CHAPTER VI

## THE BERNOULLI EQUATION

Recall that the Bernoulli equation in the moving coordinate system was written as:

$$\frac{\partial}{\partial t}\varphi(r_o,z_o,t) - \ddot{B}z_o + \frac{1}{2}\dot{B}^2 - \frac{1}{2}\left[v_n^{\,2}(r_o,z_o,t) + v_T^{\,2}(\bar{r}_o,z_o,t)\right] \quad (6.01)$$

$$- g\left[z_o + B(t)\right] - \frac{P-P_o}{\rho_o} = 0$$

If the time derivative of the velocity potentials for the source sheet and the vortex sheet are added together to give the time rate of change of the total velocity potential and $v_n(r_o,z_o t)$ is expressed as a function of the $\dot{q}_i$, the Bernoulli equation can be written as:

$$\sum_i\Big[\ddot{q}_i\sigma_{o_i} + \dot{q}_i^{\,2}(\sigma_{2_i} + \sigma_{3_i} + \sigma_{4_i}) + \dot{q}_i(\sigma_{5_i} + \sigma_{6_i} + \sigma_{7_i} + \quad (6.02)$$

$$\frac{1}{2} f_{i_o} v_T(r_o,z_o,t)\cot\theta_o)\Big]$$

$$+ \frac{1}{2}\dot{B}^2 - \frac{1}{2} v_T^{\,2}(r_o,z_o,t) + \sigma_1 - g\left[z_o + B(t)\right] - \frac{P-P_o}{\rho_o} = 0$$

The Bernoulli equation must be evaluated for i field points on the surface of the bubble since a solution must be found for each $\ddot{q}_i$. The system of equations can be put into the form:

$$\sum_j\left\{\sum_i \ddot{q}_i\sigma_{o_{ji}}\right\} + Q_j = 0 \quad (6.03)$$

where:

$$Q_j = \Sigma\Big[\dot{q}_i^{\,2}(\sigma_{2_{ji}} + \sigma_{3_{ji}} + \sigma_{4_{ji}}) + \dot{q}_i(\sigma_{5_{ji}} + \sigma_{6_{ji}}\ \sigma_{7_{ji}} \tag{6.04}$$

$$\frac{1}{2}\, f_{ji_o} v_T(r_{o_j}, z_{o_j}, t)\cot\theta_{o_j})\Big] + \frac{1}{2}\,\dot{B}^2$$

$$-\frac{1}{2}\, v_T^{\,2}(r_{o_j}, z_{o_j}, t) + \sigma_{1_j} - g\Big[z_{o_j} + B(t)\Big] - \left(\frac{P-P_o}{\rho_o}\right) = 0$$

and the subscript j is for the evaluation of the equation at the j'th field point. Let X be the variable matrix, C the coefficient matrix and Q the matrix whose elements are the $Q_j$, the system of equations can now be written as:

$$CX + Q = 0 \tag{6.05}$$

where:

$$C = \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1i} \\ \vdots & \vdots & \cdots & \vdots \\ c_{j_i} & c_{j_2} & \cdots & c_{ji} \end{Vmatrix} \tag{6.06}$$

$$c_{ji} = \sigma_{o_{ji}} \tag{6.07}$$

$$X = \begin{Vmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_i \end{Vmatrix} \tag{6.08}$$

$$Q = \begin{Vmatrix} Q_1 \\ \vdots \\ Q_j \end{Vmatrix} \tag{6.09}$$

If the inverse of the coefficient operates on expression 6.08, the equation is transformed into:

$$X = - C^{-1}Q \tag{6.10}$$

Since:

$$C^{-1}C = I \tag{6.11}$$

and:

$$IX = X \tag{6.12}$$

where I is the unitary matrix. The resulting matrix on the r. h. s. of expression 6.14 is the solution matrix where each element in the matrix is the solution to the corresponding element in the X matrix. Expression 6.14 can now be written as:

$$X = S \tag{6.13}$$

where:

$$S = - C^{-1}Q \tag{6.14}$$

$$S = \begin{Vmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_i \end{Vmatrix} \tag{6.15}$$

and:

$$\begin{Vmatrix} \ddot{q}_1 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{q}_i \end{Vmatrix} = \begin{Vmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_i \end{Vmatrix} \tag{6.16}$$

Therefore, the solution for the $\ddot{q}$'s can be arrived at in the same manner as solving a set of linear algebraic equations. Once the solution of the $\ddot{q}$'s is known, the $\dot{q}_i$'s and $q_i$'s can be solved for by integrating the variables over time.

Now that the $q_i$'s and $\dot{q}_i$'s are known, the shape of the bubble, the normal velocity, and the tangential velocity at the boundary can be determined. Therefore, the velocity potential, the time derivative of the velocity potential, and the spatial derivatives of the velocity potential can be found at any point in the fluid. The Bernoulli equation can now be solved for the pressure at any point in the fluid.

# CHAPTER VII

## THE MIGRATING SPHERICAL BUBBLE

The surface equation for a sphere in the moving reference frame is:

$$F(r,z,t) = r^2 + z^2 - a^2 = 0 \tag{7.01}$$

where $a = a(t)$ is the radius of the sphere. Since there is only one surface parameter $q_1 = a$ (compare with expression 3.10), the normal velocity at the boundary of the sphere is:

$$v_n = \dot{a} \tag{7.02}$$

and the function $f_i$ defined by expression 3.17 becomes:

$$f_1 = 1 \tag{7.03}$$

The velocity potential for a migrating spherical bubble in cylindrical coordinates is[12]:

$$\varphi(r_o,z_o,t) = \dot{a}\,\frac{a^2}{\sqrt{r_o^{\,2}+z_o^{\,2}}} + \frac{1}{2}\,\dot{B}\,\frac{a^3 z_o}{(r_o^{\,2}+z_o^{\,2})^{3/2}} + \dot{B}z_o \tag{7.04}$$

where $r_o$ and $z_o$ are the field point coordinates and $\dot{B} = \dot{B}(t)$ is the velocity of migration. The tangential velocity at the boundary of a moving sphere is:

$$v_T = \frac{3}{2}\,\dot{B}\left(\frac{r}{a}\right) \tag{7.05}$$

The time derivative of the velocity potential due to the source and vortex sheets is obtained from expressions 3.21 and 4.04:

$$\ddot{a}ac_o + \ddot{B}ac_1 + \dot{a}^2c_2 + \dot{B}\dot{a}c_3 + \ddot{B}z_o \tag{7.06}$$

where:*

$$c_o = \sigma_o \tag{7.07}$$

$$c_1 = \sigma_1 + \sigma_7 \tag{7.08}$$

$$c_2 = \sigma_3 + \sigma_4 + \sigma_5 + \frac{1}{2} \tag{7.09}$$

$$c_3 = \sigma_6 + \frac{3}{4}\left(\frac{z_o}{a}\right) \tag{7.10}$$

The $\sigma_i$'s are given by the expressions 3.22 through 3.25 and 4.05 through 4.07. The 1/2 term in $c_2$ comes from the additive term included in the time derivative of the velocity potential of the source sheet and the $3/4(z_o/a)$ term is from the additive term included in the time derivative of the velocity potential of the vortex sheet. The normal and tangential velocities have been substituted into the $\sigma_i$ coefficients and the $\dot{a}$, $\ddot{B}$, and $\dot{B}$ variables have been factored out to give $c_o$, $c_1$, $c_2$, and $c_3$. If the integrands of $\sigma_1$ and $\sigma_7$ are combined as well as those of $\sigma_3$ and $\sigma_4$, the coefficients in expression 7.07 can be written as:

$$c_o = \frac{1}{2\pi a}\left[\int_{-a}^{z_o}(G_o-\bar{G}_o)dz + \int_{z_o}^{a}(G_o-\bar{G}_o)dz + I_o\right] \tag{7.11}$$

* The $c_i$ coefficients correspond to the $a_n$ and $\dot{a}_n$ coefficients resulting from the partial time derivative of expression 0.01 as follows: $c_o = a_o$, $c_1 = a_1$, $c_2 = \dot{a}_o$, and $c_3 = \dot{a}_1$.

$$c_1 = \frac{3}{8\pi a}\left[\int_{-a}^{z_o} \Omega(z_o,z)dz + \int_{z_o}^{a} \Omega(z_o,z)dz\right] \tag{7.12}$$

$$c_2 = \frac{1}{2\pi}\left[\int_{-a}^{z_o} (G_2-\bar{G}_2)dz + \int_{z_o}^{a} (G_2-\bar{G}_2)dz + I_2\right] \tag{7.13}$$

$$+ \frac{a}{2}\left[\frac{1}{\sqrt{r_o^{\,2}+(z_o-a)^2}} + \frac{1}{\sqrt{r_o^{\,2}+(z_o+a)^2}}\right] + \frac{1}{2}$$

$$c_3 = \frac{3}{8\pi}\left[\int_{-a}^{z_o} (G_3-\bar{G}_3)dz + \int_{z_o}^{a} (G_3-\bar{G}_3)dz + I_3\right] + \frac{3}{4}\left(\frac{z_o}{a}\right) \tag{7.14}$$

where $G_o$, $G_2$, and $G_3$ are the original integrands with singularities at $z = z_o$; $\bar{G}_o$, $\bar{G}_2$, and $\bar{G}_3$ are the approximate integrands for $z \approx z_o$; $I_o$, $I_2$, and $I_3$ are the analytic integrals and $\Omega(z_o,z)$ is the solid angle of a vortex ring at z subtended at the field point $z_o$. The difference between the original integrand and the approximate integrand is called the compensated integrand. The original integrands, approximate integrands, and the analytic integrals for a spherical migrating bubble are:

$$G_o = K(k)k\sqrt{\frac{r}{r_o}}\ \frac{a}{r} \tag{7.15}$$

$$\bar{G}_o = \left(\frac{a}{r_o}\right)\ln\left(\frac{|z_o-z|a}{8r_o^{\,2}}\right) \tag{7.16}$$

$$I_o = -\left(\frac{a}{r_o}\right)\left[(a-z_o)\ln\left(\frac{|z_o-z|a}{8r_o^{\,2}}\right) + (z_o+a)\ln\left(\frac{|z_o+a|a}{8r_o^{\,2}}\right) - 2a\right] \tag{7.17}$$

$$G_2 = \left\{\left(\frac{a^2}{2r^3}\right)\left[E(k)\left\{1 + \frac{1}{2}\left(\frac{r_o-r}{r_o}\right)\left(\frac{k}{k'}\right)^2\right\} + K(k)\right] - \left(\frac{z^2}{r^3}\right)K(k)\right\}k\sqrt{\frac{r}{r_o}} \quad (7.18)$$

$$\bar{G}_2 = \left(\frac{a^2}{2r_o^3}\right)\left[1 - \frac{2z_o r_o^2}{a(z_o-z)} - \ln\left(\frac{|z_o-z|a}{8r_o^2}\right)\right] + \left(\frac{z_o^2}{r_o^3}\right)\ln\left(\frac{|z_o-z|a}{8r_o^2}\right) \quad (7.19)$$

$$I_2 = \left(\frac{a^2}{2r_o^3}\right)\left[4a - \frac{2z_o r_o^2}{a^2}\ln\left|\frac{z_o+a}{a-z_o}\right| - (a-z_o)\ln\left(\frac{|a-z_o|a}{8r_o^2}\right)\right. \quad (7.20)$$

$$- (z_o+a)\ln\left(\frac{|z_o+a|a}{8r_o^2}\right)\Bigg] + \left(\frac{z_o^2}{r_o^3}\right)\left[(a-z_o)\ln\left(\frac{|a-z_o|a}{8r_o^2}\right)\right.$$

$$\left. + (z_o+a)\ln\left(\frac{|z_o+a|a}{8r_o^2}\right) - 2a\right]$$

$$G_3 = \left(\frac{a}{4r^2}\right)E(k)\left(\frac{k}{k'}\right)^2\left(\frac{z_o-z}{r_o}\right)k\sqrt{\frac{r}{r_o}} \quad (7.21)$$

$$\bar{G}_3 = \frac{r}{a(z_o-z)} \quad (7.22)$$

$$I_3 = \left(\frac{r_o}{a}\right)\ln\left|\frac{z_o+a}{a-z_o}\right| \quad (7.23)$$

If expression 7.04 is differentiated with respect to time, keeping the field point variables fixed, and evaluated on the boundary, the coefficients $c_o$, $c_1$, $c_2$, and $c_3$ become:

$$c_o = 1 \quad (7.24)$$

$$c_1 = \frac{1}{2}\left(\frac{z_o}{a}\right) \tag{7.25}$$

$$c_2 = 2 \tag{7.26}$$

$$c_3 = \frac{3}{2}\left(\frac{z_o}{a}\right) \tag{7.27}$$

The tangential velocity at the boundary of a spherical migrating bubble is:

$$v_T = \dot{B}c_T \tag{7.28}$$

where the tangential velocity in expression 7.05 has been substituted into expression 5.21 and $\dot{B}$ has been factored out. The coefficient $c_T$ is written as:

$$c_T = \left(\frac{r_o}{a}\right)\left[2 - \frac{3}{4\pi}\left\{\int_{-a}^{z_o}(G_T-\bar{G}_T)dz + \int_{z_o}^{a}(G_T-\bar{G}_T)dz\right\} + I_T\right] \tag{7.29}$$

where:

$$G_T = \left(\frac{1}{r}\right)\left[K(k)\left\{1 + \left(\frac{z_o-z}{r_o}\right)\left(\frac{z_o}{r_o}\right)\right\} - E(k)\left\{1 + \frac{1}{2}\left(\frac{r_o-r}{r_o}\right)\left(\frac{k}{k'}\right)^2 \right.\right. \tag{7.30}$$

$$\left.\left. + \left(\frac{z_o-z}{r_o}\right)\left(\frac{z_o}{r_o}\right)\left[1 + \frac{1}{2}\left(\frac{k}{k'}\right)^2\right]\right\}\right]k\sqrt{\frac{r}{r_o}}$$

$$\bar{G}_T = \left(\frac{1}{r_o}\right)\left[\ln\left(\frac{|z_o-z|a}{8r_o^2}\right) + 2\right] \tag{7.31}$$

$$I_T = \left(\frac{1}{r_o}\right)\left[(a-z_o)\ln\left(\frac{|a-z_o|a}{8r_o^2}\right) + (z_o+a)\ln\left(\frac{|z_o+a|a}{8r_o^2}\right) + 2a\right] \tag{7.32}$$

The tangential velocity obtained from the velocity potential in expression 7.04 evaluated at the boundary shows the coefficient to be:

$$c_T = \frac{3}{2}\left(\frac{r_o}{a}\right) \tag{7.33}$$

The coefficients in expressions 7.11 through 7.15 and expression 7.29 were programmed by this writter for the IBM 7090 computer and evaluated for several field points on the boundary of a unit sphere. Since all $(G - \bar{G})$ integrands are continuous functions of z such that they are zero or finite at $z = z_o$, the integral can be obtained by a numerical quadrature; e.g., Gauss' formula. The Gaussian formula has an advantage in that the integrand is not evaluated at the limits of integration. Using $z_o$ for one of the limits, the integrand is computed for values of $z \neq z_o$, thus eliminating the need of determining the integrand for $z = z_o$. If the integrand is evaluated at n points, Gauss' formula integrates a $(2n - 1)$'th polynomial exactly.

Table la is a list of computed values for $c_o$ and $c_2$, where an 8-point Gaussian quadrature was used for each integration range. The accuracy of the computed coefficients $c_o$ and $c_2$ decreases somewhat as the field point moves away from the equator of the sphere where $z_o = 0$. The reason for this loss of accuracy can be qualitatively seen in figures 3 and 4. These are plots of the original integrand $G_o$, approximate integrand $\bar{G}_o$, and compensated integrand $(G_o - \bar{G}_o)$ as a function of the integration variable z. The curve of the compensated integrand in figure 4 $(z_o = 0)$ appears to have a parabolic shape which the Gaussian formula would integrate exactly. However, the curve of the compensated integrand in figure 3 $(z_o = 0.5)$ appears to have higher inclinations in the region of $z_o$ than are commensurate with a polynomial and a lower accuracy of integration is to be expected.

Table 1b is a list of the computed values of $c_1$ and $c_3$ as compared with the true values calculated using expressions 7.25 and 7.27. Figures 5 and 6 are plots of the integrands $G_3$, $\bar{G}_3$, and $(G_3 - \bar{G}_3)$. Judging from the shape of the curves for the compensated integrands, the accuracy of $c_3$ would be expected to be less for $z_o = 0.5$ than for $z_o = 0$. The numerical results for $c_1$, $c_2$, and $c_T$ show the same trend of accuracy.

Although the accuracy of the coefficients is sufficient for practical purposes, it could be improved as follows: Figures 3-6 suggest employment of four regions of integration, thus:

$$c_i = \alpha_i \left[ \int_{-a}^{z_o-\epsilon} G_i dz + \int_{z_o-\epsilon}^{z_o} (G_i - \bar{G}_i) dz \right. \tag{7.34}$$

$$\left. + \int_{z_o}^{z_o+\epsilon} (G_i - \bar{G}_i) dz + \int_{z_o+\epsilon}^{a} G_i dz + I_{o_i} \right] + \beta_i$$

where an $\epsilon$ is chosen such that the region of high inclination is covered by the two inner integrals. The compensated integrand is not necessary for the two outer regions of integration since there are no singular points. This method has not been tried yet.

The $c_2$ coefficient, like the $c_3$ coefficient, includes an additive term when evaluated on the boundary of the bubble (1.43). However, if the coefficient is evaluated at points off the boundary, the additive term is not needed. Also, the method of compensating integrands is not needed since there are no singularities for field points off the boundary. Figure 7 is a plot of the integrand $G_2$ as a function of the integration variable z, for $z_o = 0$. The distance of the field point from the boundary is $r_o - 1$. Since $G_2$ is symmetrical about the $G_2$ axis, only values for

$z \geq 0$ are shown. The value of $G_2$ is finite for $z = 0$ and $r_o > 1$. The integral of $G_2$ yields the value of $c_2$ shown in figure 8. The value of $c_2$ is extrapolated to 2 at $r_o = 1$ which is the correct value of $c_2$ at the boundary. As $r_o \to 1$, the $G_2$ curve becomes more and more pointed and the integral of $G_2$ increases so that: $\lim_{\epsilon \to 0} c_2 \to 2$ for $r_o = 1 + \epsilon$. However, for $r_o = 1$, the $G_2$ curve lies very close to the axes in figure 7 and the integral of $G_2$ decreases in value such that $c_2 = 3/2$. The additive term of 1/2 gives the correct value of 2.

## CONCLUSIONS

The above numerical analysis pertaining to a spherical bubble demonstrates that an underwater explosion bubble can be represented by a distribution of sources and vortex rings on the surface. Since the equations were derived without regard to the shape of the bubble, they should be applicable to the case of a nonspherical bubble. This method would not encounter the problems of the model consisting of a single point source and a single dipole located on the axis of symmetry which interferes with the jet rising into the interior of the bubble.

It is planned in the near future to apply the theory and mathematical tools developed in this paper to the case of a nonspherical bubble with a jet. The differential equations for the $\ddot{q}$'s are to be solved using the Adams-Moulton predictor-corrector method or that of Runge-Kutta . A generalized program using these two methods has been developed at the Naval Ordnance Laboratory and has been used successfully in the past.

The model of a jet-forming bubble would allow for a detailed and complete study of bubble phenomena associated with underwater explosions. A study of this type would greatly enhance the present knowledge.

TABLE 1a

COMPUTED VALUES OF $c_o$ AND $c_2$ OBTAINED FROM EXPRESSIONS 7.11 AND 7.13 EVALUATED ON THE BOUNDARY OF A UNIT SPHERE

These coefficients are needed for the time derivative of the velocity potential of a migrating spherical bubble and are associated with the potential due to a source sheet. The true values of $c_o$ and $c_2$ are 1 and 2 as obtained from expressions 7.24 and 7.26.

| Field Point Coordinate $z_o$ | Computed Values Of: $c_o$ | $c_2$ |
|---|---|---|
| 0.9 | 0.99980 | 2.00251 |
| 0.8 | 0.99995 | 2.00058 |
| 0.7 | 0.99998 | 2.00036 |
| 0.6 | 0.99999 | 2.00010 |
| 0.5 | 0.99999 | 2.00001 |
| 0.4 | 0.99999 | 1.99998 |
| 0.3 | 0.99999 | 2.00000 |
| 0.2 | 0.99999 | 2.00000 |
| 0.1 | 0.99999 | 2.00000 |
| 0.0 | 1.00000 | 2.00000 |
| -0.1 | 0.99999 | 2.00000 |
| -0.2 | 0.99999 | 2.00000 |
| -0.3 | 0.99999 | 2.00000 |
| -0.4 | 0.99999 | 1.99998 |
| -0.5 | 0.99999 | 2.00001 |
| -0.6 | 0.99999 | 2.00010 |
| -0.7 | 0.99998 | 2.00036 |
| -0.8 | 0.99995 | 2.00058 |
| -0.9 | 0.99980 | 2.00251 |

TABLE 1b

COMPUTED VALUES OF $c_1$ AND $c_3$ OBTAINED FROM EXPRESSIONS 7.12 AND 7.14 EVALUATED ON THE BOUNDARY OF A UNIT SPHERE

These coefficients are needed for the time derivative of the velocity potential of a migrating spherical bubble and are associated with the velocity potential due to the vortex sheet. The true values of $c_1$ and $c_3$ are obtained from expressions 7.25 and 7.27.

| Field Point Coordinate $z_o$ | Computed Value of $c_1$ | True Value of $c_1$ | Computed Value of $c_3$ | True Value of $c_3$ |
|---|---|---|---|---|
| 0.9 | 0.44648 | 0.45 | 1.35006 | 1.35 |
| 0.8 | 0.39382 | 0.40 | 1.19985 | 1.20 |
| 0.7 | 0.34142 | 0.35 | 1.04972 | 1.05 |
| 0.6 | 0.29440 | 0.30 | 0.89989 | 0.90 |
| 0.5 | 0.24353 | 0.25 | 0.75000 | 0.75 |
| 0.4 | 0.19555 | 0.20 | 0.60009 | 0.60 |
| 0.3 | 0.14701 | 0.15 | 0.44997 | 0.45 |
| 0.2 | 0.09941 | 0.10 | 0.29995 | 0.30 |
| 0.1 | 0.05008 | 0.05 | 0.14998 | 0.15 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| -0.1 | -0.05008 | -0.05 | -0.14998 | -0.15 |
| -0.2 | -0.09941 | -0.10 | -0.29995 | -0.30 |
| -0.3 | -0.14701 | -0.15 | -0.44997 | -0.45 |
| -0.4 | -0.19555 | -0.20 | -0.60009 | -0.60 |
| -0.5 | -0.24353 | -0.25 | -0.75000 | -0.75 |
| -0.6 | -0.29440 | -0.30 | -0.89989 | -0.90 |
| -0.7 | -0.34142 | -0.35 | -1.04972 | -1.05 |
| -0.8 | -0.39382 | -0.40 | -1.19985 | -1.20 |
| -0.9 | -0.44648 | -0.45 | -1.35006 | -1.35 |

TABLE 2

COMPUTED VALUES OF $c_T$ OBTAINED FROM EXPRESSION 7.29

EVALUATED ON THE BOUNDARY OF A UNIT SPHERE

This coefficient is needed for the tangential velocity of a migrating spherical bubble. The true values of $c_T$ are obtained from expression 7.33.

| Field Point Coordinate $z_o$ | Computed Value of $c_T$ | True Value of $c_T$ |
|---|---|---|
| 0.9 | 0.65373 | 0.65383 |
| 0.8 | 0.89996 | 0.90000 |
| 0.7 | 1.07119 | 1.07121 |
| 0.6 | 1.19999 | 1.20000 |
| 0.5 | 1.29903 | 1.29904 |
| 0.4 | 1.37477 | 1.37477 |
| 0.3 | 1.43091 | 1.43091 |
| 0.2 | 1.46969 | 1.46969 |
| 0.1 | 1.49248 | 1.49248 |
| 0.0 | 1.50000 | 1.50000 |
| -0.1 | 1.49248 | 1.49248 |
| -0.2 | 1.46969 | 1.46969 |
| -0.3 | 1.43091 | 1.43091 |
| -0.4 | 1.37477 | 1.37477 |
| -0.5 | 1.29903 | 1.29904 |
| -0.6 | 1.19999 | 1.20000 |
| -0.7 | 1.07119 | 1.07121 |
| -0.8 | 0.89996 | 0.90000 |
| -0.9 | 0.65373 | 0.65383 |

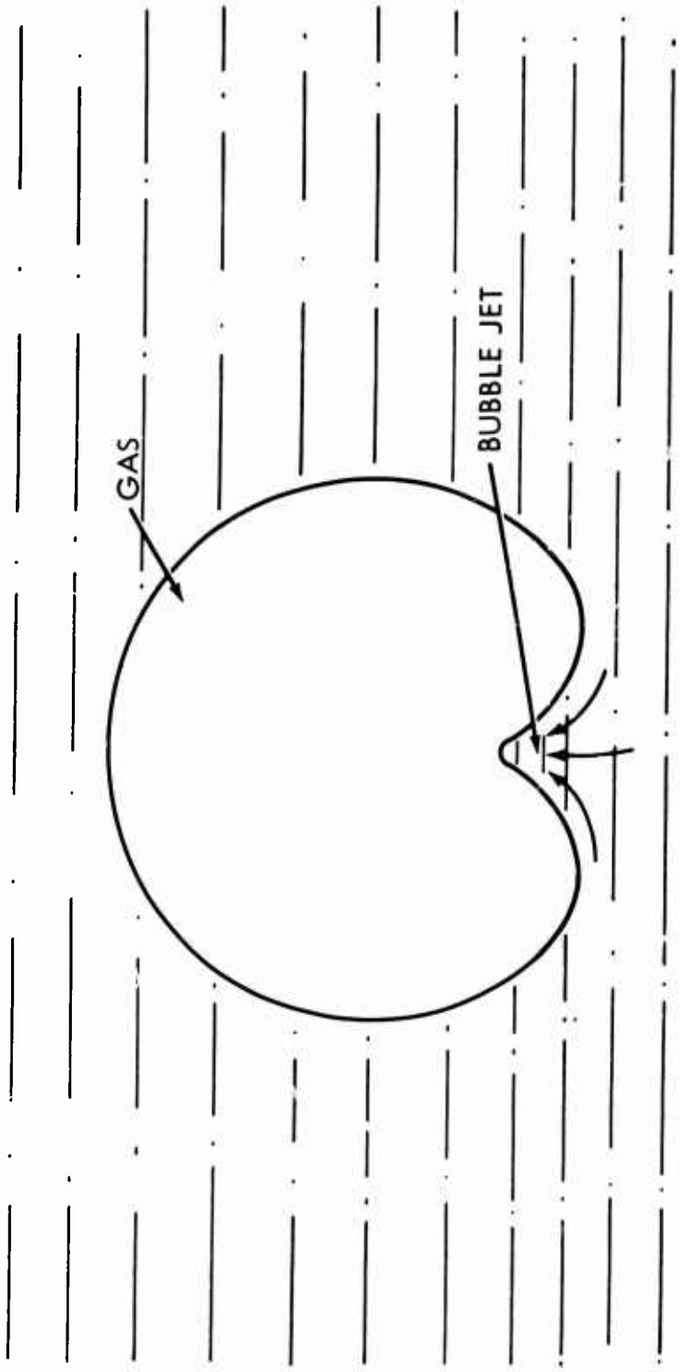


Figure 1 FORMATION OF THE BUBBLE JET

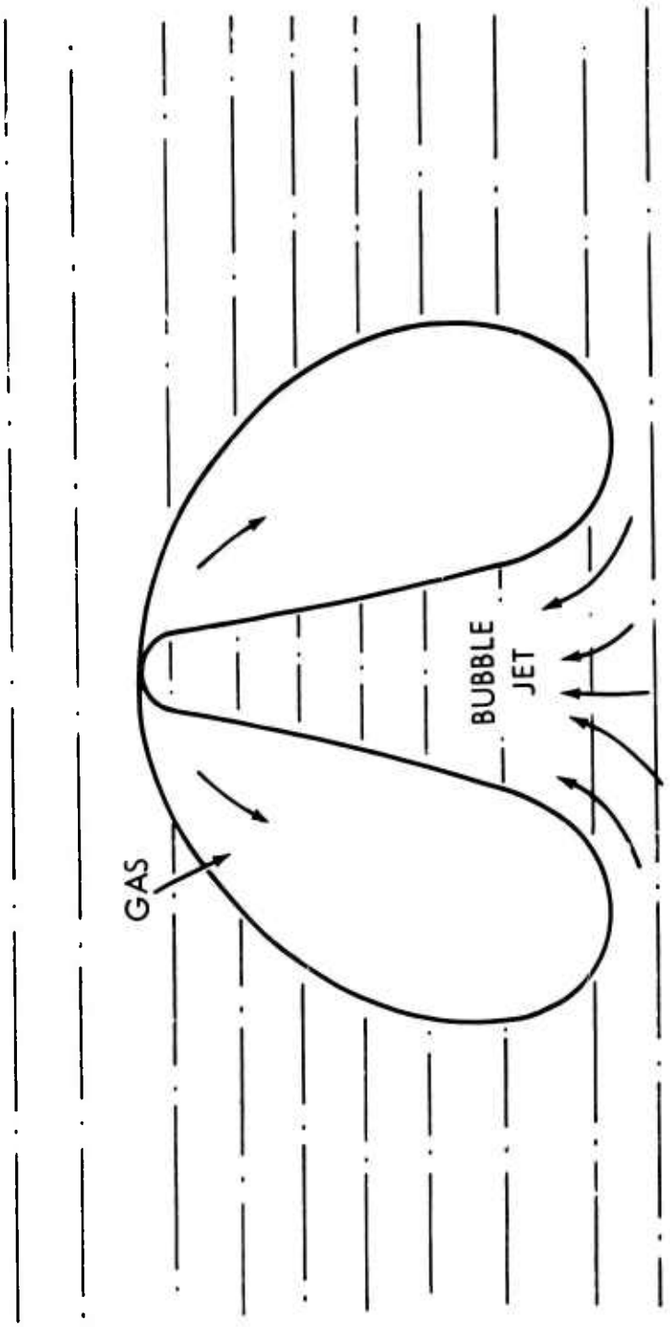


Figure 2 IMPINGEMENT OF THE BUBBLE JET ON THE UPPER BUBBLE SURFACE

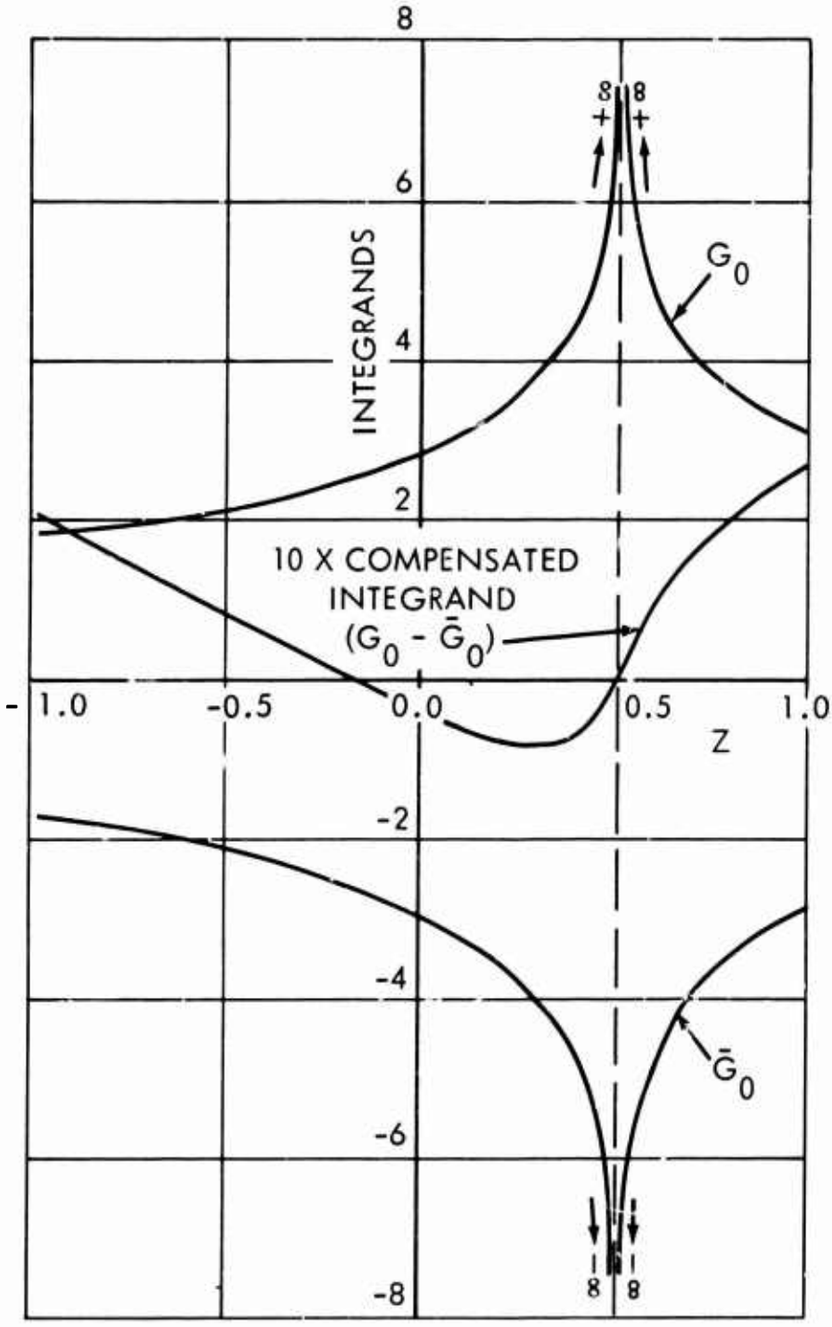


Figure 3 THE INTEGRANDS $G_o$, $\bar{G}_o$, AND $(G_o - \bar{G}_o)$ IN EXPRESSION 7.11 PLOTTED AS A FUNCTION OF THE INTEGRATION VARIABLE z

The integrands are evaluated at the boundary of unit sphere for $z_o = 0.5$.

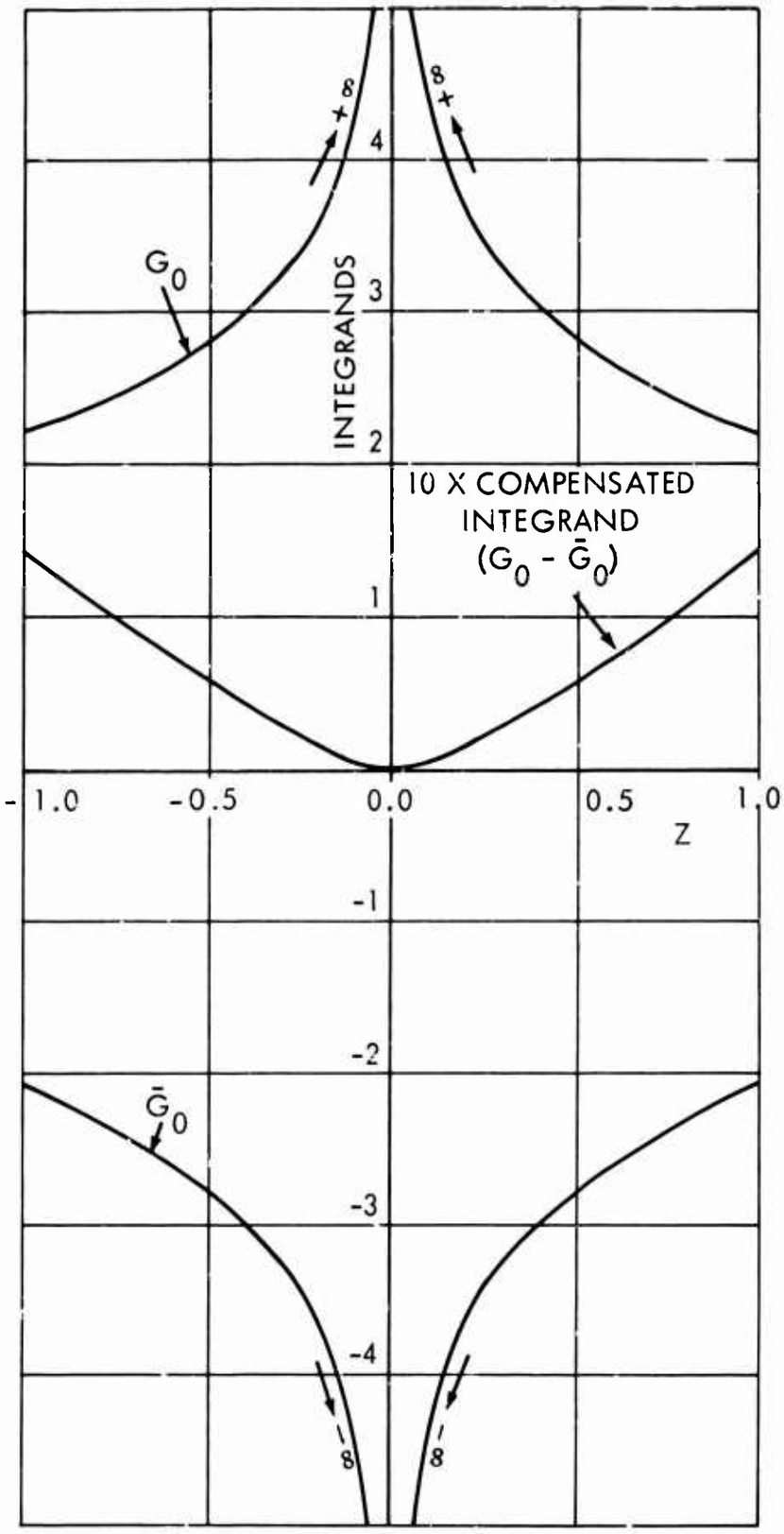


Figure 4

Same as figure 3 for $z_o = 0$.

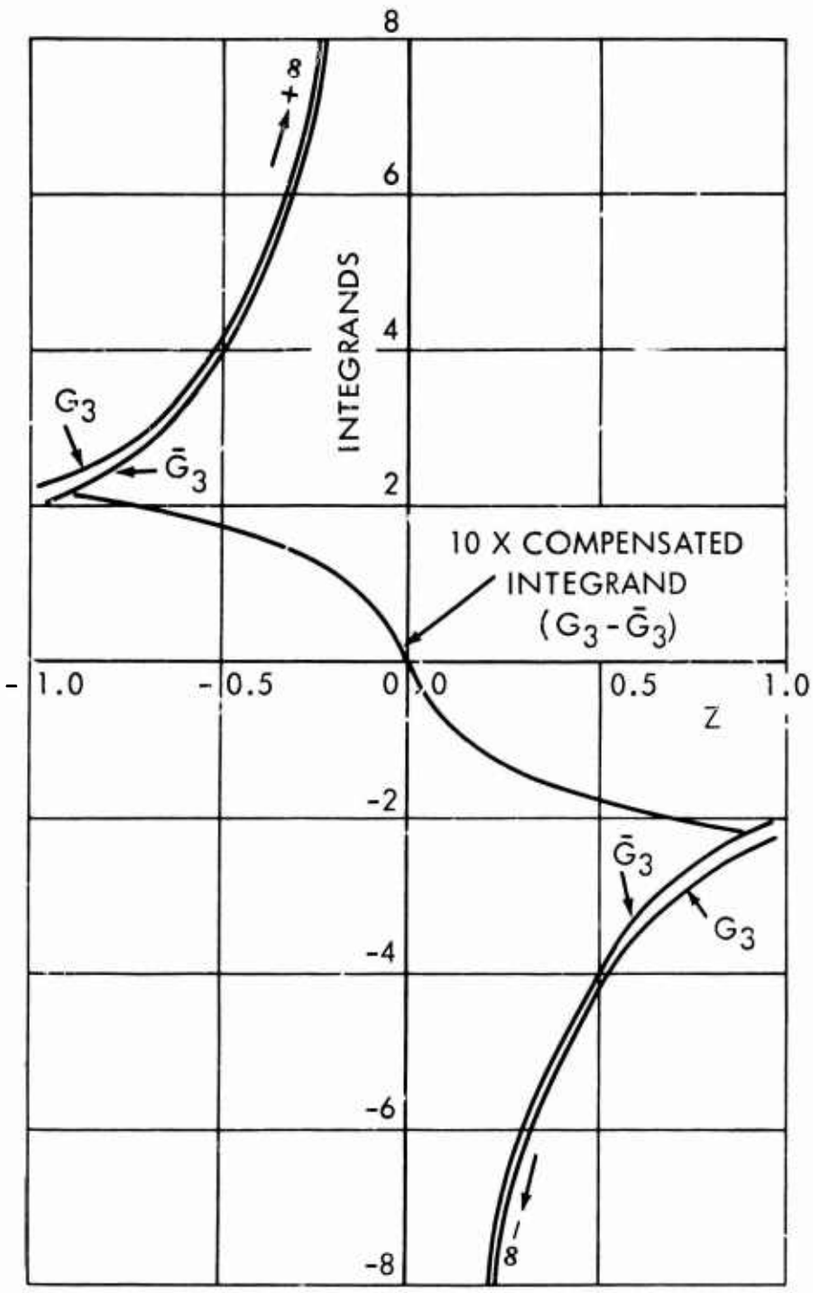


Figure 5 THE INTEGRANDS $G_3$, $\bar{G}_3$, AND $(G_3 - \bar{G}_3)$ IN EXPRESSION 7.14 PLOTTED AS A FUNCTION OF THE INTEGRATION VARIABLE z

The integrands are evaluated at the boundary of a unit sphere for $z_o = 0$.

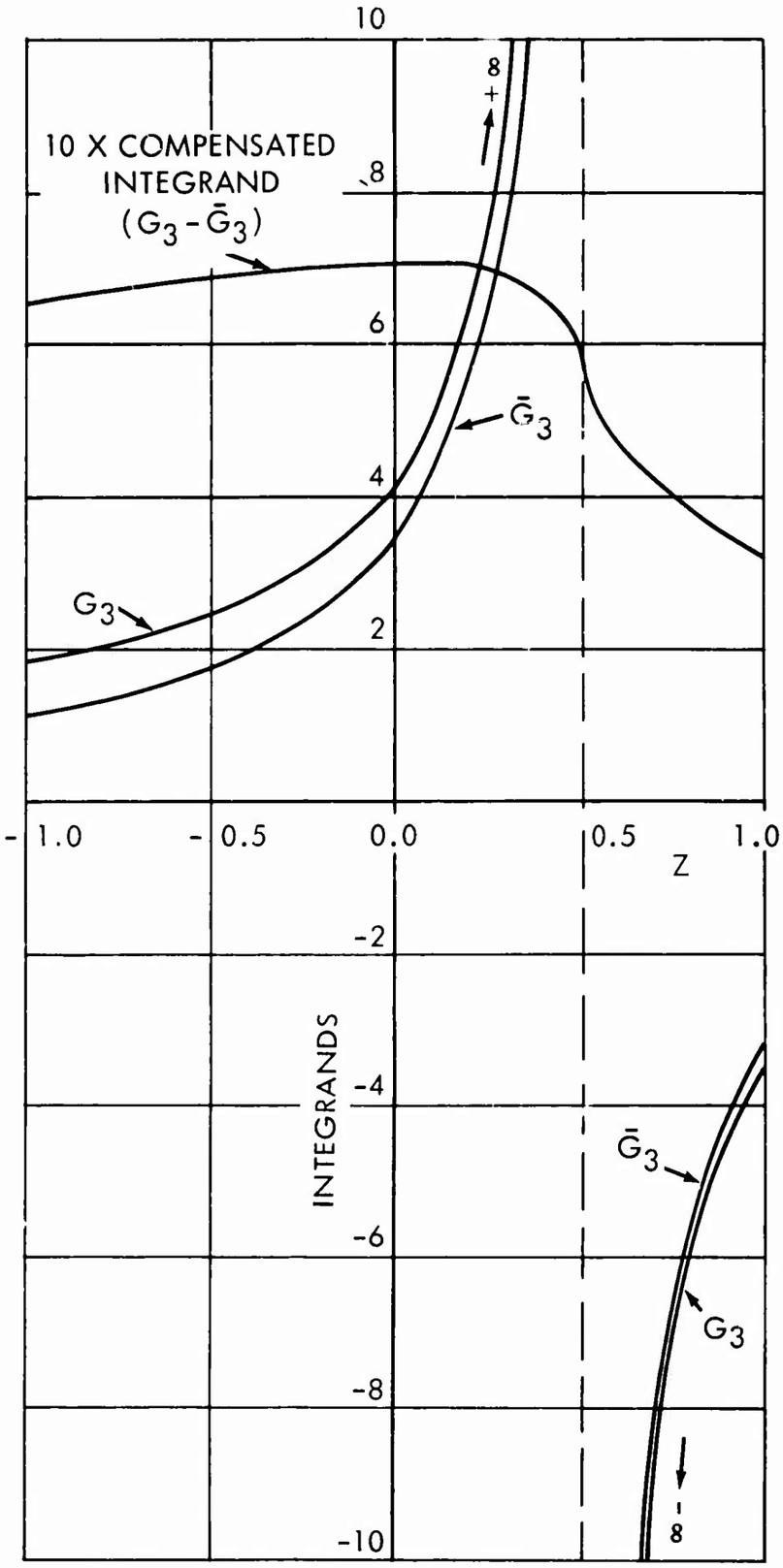


Figure 6

Same as figure 5 for $z_o = 0.5$.

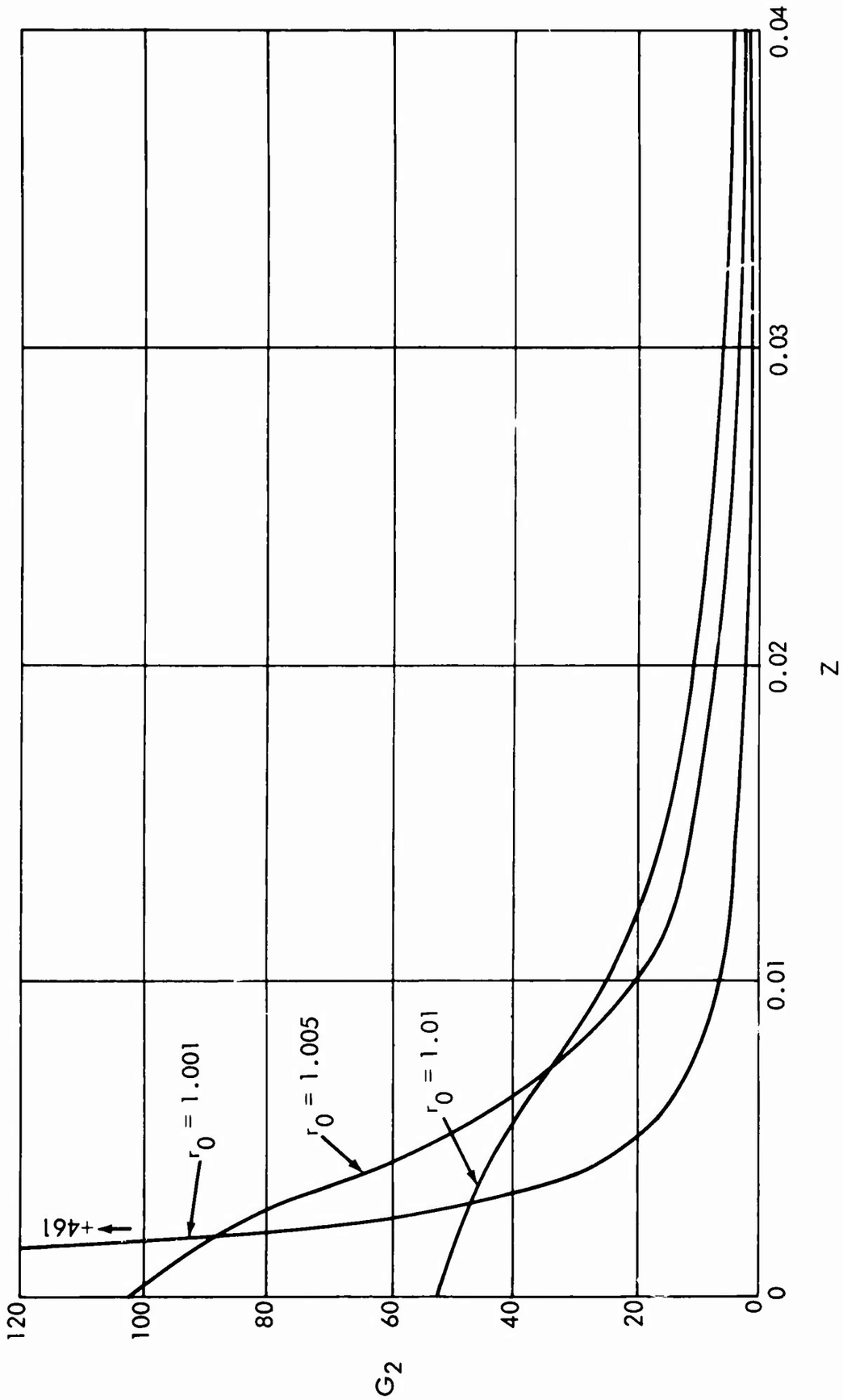


**Figure 7 THE INTEGRAND $G_2$ OF EXPRESSION 7.18 PLOTTED AS A FUNCTION OF THE INTEGRATION VARIABLE z**

The integral of $G_2$ is evaluated away from the boundary of a unit sphere for $z_o = 0$. The distance away from the boundary is $r_o - 1$.

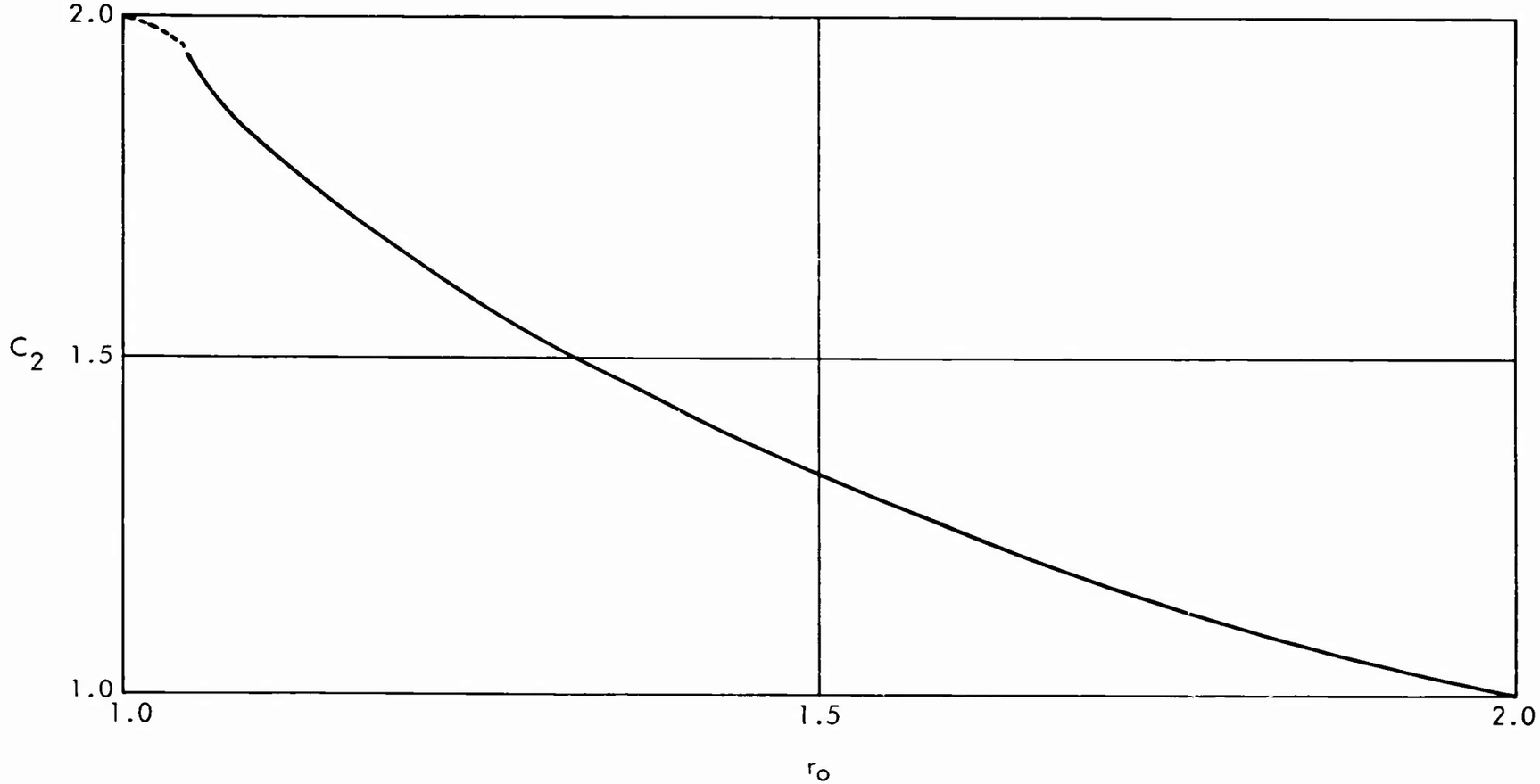


Figure 8 THE COEFFICIENT $c_2$ AWAY FROM THE BOUNDARY OF A UNIT SPHERE PLOTTED AS A FUNCTION OF THE FIELD POINT COORDINATE $r_o$ FOR $z_o = 0$

The dashed line is the extrapolated values of $c_2$ from the last computed value away from the boundary to the value of $c_2$ on the boundary, expression 7.26. The distance away from the boundary is $r_o - 1$.

## LITERATURE CITED


1 Penney, W. G. and Price, A. T., "On the Changing Form of a Nearly-Spherical Submarine Bubble", Underwater Explosion Research, Vol II, Office of Naval Research--Department of the Navy, 1950.

2 Kolodner, I. I. and Keller, J. B., "The Effect of Gravity and the Change of Shape", Underwater Explosion Bubbles II, Institute for Mathematics and Mechanics, New York University.

3 Lamb, H., "Hydrodynamics", 6th Edition, Cambridge University Press, pp. 6-8, 1932.

4 Lamb, H., "Hydrodynamics", 6th Edition, Cambridge University Press, pp. 54-55, 1932.

5 Maxwell, J. C., "A Treatise on Electricity and Magnetism", 3rd Edition, Oxford University Press, Vol 2, pp. 286-293, 1892.

6 Milne-Thompson, L. M., "Theoretical Hydrodynamics", 2nd Edition, The Macmillan Company, pp. 486-490, 1950.

7 Cole, R. H., "Underwater Explosions", Princeton University Press, p. 308, 1948.

8 Zoller, R., "Aufsteigen der Pulsierenden Schwadenblase", Forschungsanstalt Graf Zeppelin, Stuttgart, Germany, 1944.

9 Byrd, P. F. and Friedman, M. D., "Handbook of Elliptic Integrals for Engineers and Physicists", Lange, Maxwell and Springer Ltd., pp. 176-177, 1954.

10 Byrd, P. F. and Friedman, M. D., "Handbook of Elliptic Integrals for Engineers and Physicists", Lange, Maxwell and Springer Ltd., pp. 297-298, 1954.

11 Prandtl, L. and Betz, A., "Tragflugeltheorie, 1. Mitterlung", Vier Abhandlugen zur Hydrodynamik u. Aerodynamik, Gottingen, 1927.

12 Cole, R. H., "Underwater Explosions", Princeton University Press", p. 289, 1948.

Unclassified

Security Classification

## DOCUMENT CONTROL DATA - R & D

*Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

1. ORIGINATING ACTIVITY *(Corporate author)*

U. S. Naval Ordnance Laboratory
White Oak, Silver Spring, Maryland 20910

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

Representation of a Nonspherical Underwater Explosion Bubble By a Vortex Sheet and Source Sheet

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5. AUTHOR(S) *(First name, middle initial, last name)*

Larry W. Bell

6. REPORT DATE

22 November 1966

7a. TOTAL NO. OF PAGES

iv & 68

7b. NO. OF REFS

12

8a. CONTRACT OR GRANT NO

b. PROJECT NO Task ORD-035-001/F008-21-03(003)

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

NOLTR 66-211

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Defense Atomic Support Agency

13. ABSTRACT

The non-spherical explosion bubble can be represented by a source sheet and vortex sheet. The equations for velocity potential, tangential velocity, and normal velocity for a source sheet and vortex sheet representation have been derived and programmed for the IBM 7090. The results are numerically satisfactory; moreover, jet formation can be included.

DD FORM 1 NOV 65 1473 (PAGE 1)

Unclassified

Security Classification

Unclassified
Security Classification

| 14 KEY WORDS | LINK A ROLE | LINK A WT | LINK B ROLE | LINK B WT | LINK C ROLE | LINK C WT |
|---|---|---|---|---|---|---|
| Explosions | | | | | | |
| Underwater Explosion Bubble | | | | | | |
| Non-spherical Bubble | | | | | | |
| Vortex Rings | | | | | | |

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

Unclassified
Security Classification