STUDY NUMBER 6

Circular Error Probability of a Quantity Affected by a Bias

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CIRCULAR ERROR PROBABILITY OF A QUANTITY AFFECTED BY A BIAS

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ABSTRACT

A procedure for determining the radius of the 39.35, 50, and 90% probability circles for a biased distribution is presented. Both circular and elliptical normal bivariate distributions are considered. The elliptical distribution is replaced by an equivalent circular distribution and an approximate radius obtained using the circular distribution procedure. Tables giving an indication of the error resulting from this replacement are included.

1. Purpose and Scope

In error analysis it is assumed that all systematic errors have been removed from the observational data. This is seldom true for it is virtually impossible to eliminate all systematic errors. In the analysis it is hoped that those systematic errors of consequence have been eliminated leaving only numerous small systematic errors whose combinations cannot be distinguished from random errors. When a systematic error of consequence has not been removed, the value obtained is said to be biased. That is, the value deviates from its true or accepted value by some known or undetermined amount. Systematic errors, their origin, form, and removal are discussed in ACIC RP-2 and ACIC TR-96.

This report is concerned with the effect these undetected, biased quantities have on the probability interpretation applied in error analysis. That is, it is desired to determine the radius of a circle which will include a certain portion of an error distribution effected by a biased quantity. This may best be explained in the following manner: consider a missile, with circular standard error $\sigma_c = r$, aimed at a point T, (see Fig. 1). The missile is biased (or the point misidentified) by the amount d; therefore the distribution of impacts are centered around the point A not T. It is desired to know how large the R_c circle (centered on T) must be to include a certain proportion of the impacts distributed around A. Or, given limiting values for R_c and r, how large a value of d can be allowed? These are the questions with which this report is concerned. Only the normal bivariate error distribution will be considered.

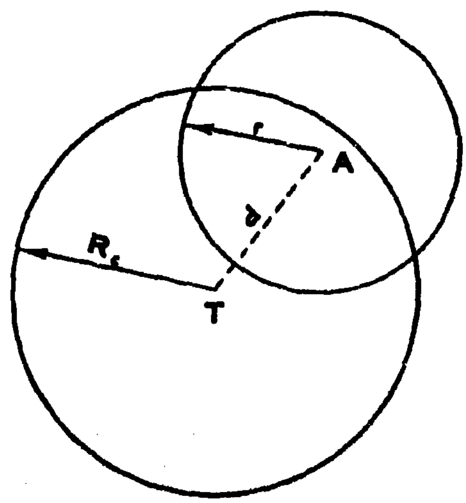


Fig. 1

In the preceding paragraph the error distribution is a special case, that is, the distribution is considered circular. To have a circular distribution the standard errors in the x and y directions must be equal ($\sigma_x = \sigma_y$). When circular, the probability function of a bivariate normal distribution is simplified by reducing the two parameters σ_x and σ_y to one parameter σ_c. Section 2 is concerned with the determination of R_c and probability values under the circular condition.

Unfortunately it is unusual in bivariate analysis to obtain a circular distribution. The general case is an elliptical distribution where $\sigma_x \neq \sigma_y$. Since this complicates the solutions, the elliptical distribution is replaced by an equivalent circular distribution in Section 3 to obtain an approximate solution by the method purposed in Section 2.

2. <u>Circular Error</u>. If no bias error exists (d = 0), the center of the error circle σ_c coincides with the center of the probability circle R_c. Equation (1) expresses the probability of a circle of radius R_c with a circular standard error σ_c.

$$Pr\,[x^2 + y^2 \leq R_c^2] = \int\int \frac{1}{2\pi\,\sigma_c^2} \exp\left[-\frac{1}{2\sigma_c^2}(x^2 + y^2)\right] dxdy \qquad (1)$$

Fig. 2 illustrates equation (1):

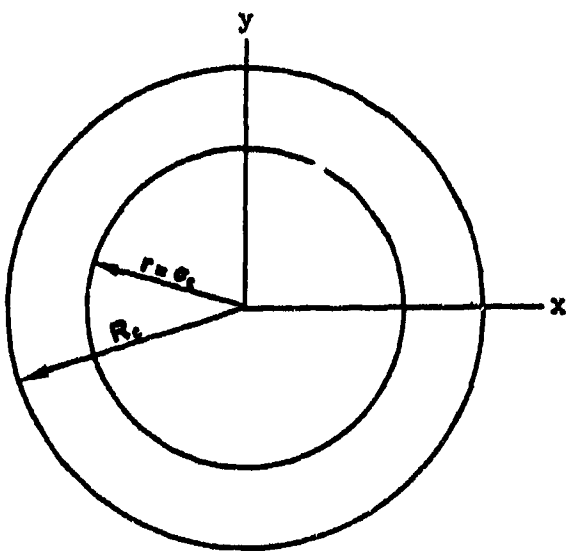


Fig. 2

When the center of the σ_c circle does not coincide with the R_c circle, equation (1) takes the form:

$$Pr\,[x^2 + y^2 \leq R^2] = \int\int \frac{1}{2\pi\,\sigma_c^2} \exp\left[-\frac{1}{2\sigma_c^2}[(x - \bar{x})^2 + (y - \bar{y})^2]\right] dxdy \quad (2a)$$

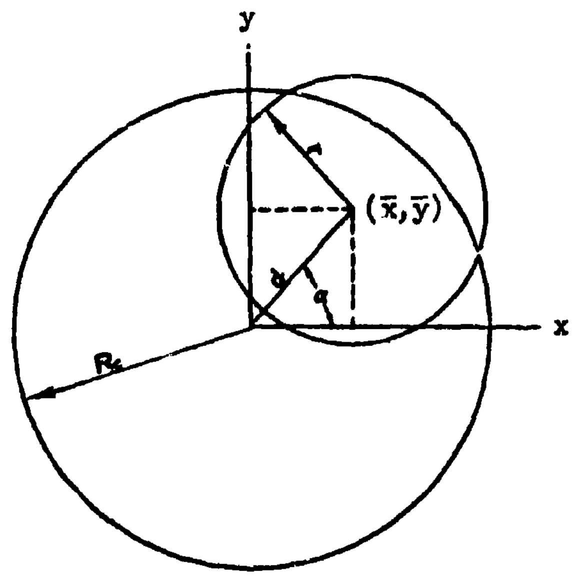


Fig. 3

From Fig. 3 it can be seen that, if the coordinate axes are rotated by the angle α so that d lies along the x-axis, equation (2a) can be expressed:

$$Pr\,[x^2 + y^2 \le R_c^2] = \int \int \frac{1}{2\pi\,\sigma_c^2} \exp\left[- \frac{1}{2\sigma_c^2}\,[(x - d)^2 + y^2]\right] dxdy \qquad (2b)$$

$$\text{where: } d = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2} \text{ for all cases.}$$

Fig. 4 illustrates equation (2b).

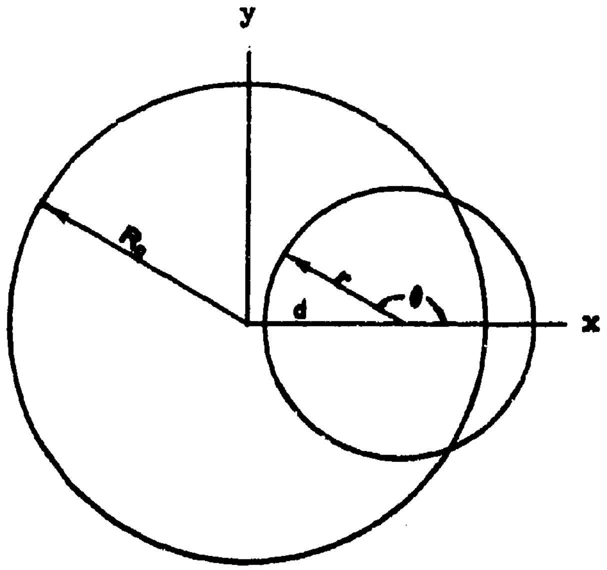


Fig. 4

Converting equation (2b) into polar coordinates:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$Pr\,[R_c] = \int_{r=0}^{R_c} \int_{\theta=0}^{2\pi} \frac{1}{2\pi\,\sigma_c^2} \exp\left[-\frac{1}{2\sigma_c^2}\,[r^2 - 2rd\cos\theta + d^2]\right] r\,dr\,d\theta$$

$$Pr\,[R_c] = \exp\left[-\frac{d^2}{2\sigma_c^2}\right] \frac{1}{2\pi\,\sigma_c^2} \int_{r=0}^{R_c} \int_{0=0}^{2\pi} \exp\left[-\frac{1}{2\sigma_c^2}\,[r^2 - 2rd\cos\theta]\right] r\,dr\,d\theta \quad (3a)$$

when d = 0 equation (3a) has the form:

$$\Pr[R_c] = \frac{1}{2\pi\,\sigma_c^2}\int_{r=0}^{R_c}\int_{\theta=0}^{2\pi} \exp\left[-\frac{r^2}{2\sigma_c^2}\right] r\,dr\,d\theta$$

$$= 1 - \exp\left[-\frac{R_c^2}{2\sigma_c^2}\right]$$

When d ≠ 0, equation (3a) cannot be integrated in closed form. Vitalis (1956) computed his table of Circular Normal Probabilities by the polynomial:

$$\Pr = \exp\left[-\frac{d^2}{2\sigma_c^2}\right]\sum_{n=0}^{\infty}\frac{\left(\frac{d^2}{2\sigma_c^2}\right)^n}{n!}\left[1 - \exp\left[-\frac{R_c^2}{2\sigma_c^2}\right]\sum_{m=0}^{n}\frac{\left(\frac{R_c^2}{2\sigma_c^2}\right)^m}{m!}\right] \quad (3b).$$

Tabulated values of probability are given for $\frac{R_c}{\sigma_c}$ and $\frac{d}{\sigma_c}$ values. The computation of Pr for various values of $\frac{R_c}{\sigma_c}$ and $\frac{d}{\sigma_c}$ is relatively easy with a machine calculator but very time consuming. The desired degree of accuracy in Pr is obtained by summing

$$\frac{\left(\frac{d^2}{2\sigma_c^2}\right)^n}{n!}$$ until the desired correct digit is unaffected.

Example 1:

Let $\frac{d}{\sigma_c} = 1, \frac{R_c}{\sigma_c} = 1$

$$\exp\left[-\frac{1}{2}\right] = 0.60653$$

$$\therefore \ Pr = 0.60653\Big[(1)(1 - 0.60653) + 0.5\,(1 - 0.60653 \cdot 1.5)$$
$$+ 0.125\,(1 - 0.60653 \cdot 1.625) + 0.020833\,(1 - 0.60653 \cdot 1.645833)$$
$$+ 0.002604\,(1 - 0.60653 \cdot 1.648437) + 0.000260\,(1 - 0.60653 \cdot 1.648697)$$
$$+ 0.000022\,(1 - 0.60653 \cdot 1.648719)\Big]$$

$$Pr = 0.60653\ [0.440406] = 0.26712 \text{ or } 26.712\%$$

The table was utilized to develop a nomogram and graph, Figs. 5 and 6, as an uncomplicated method for obtaining values of R_c and probability. (The probability curves of 39.35, 50, and 90 per cent were graphed because they are the probability levels of circular standard, CEP, and CMAS errors.) The following steps should be followed in using the nomogram and probability curves: enter Fig. 5 with $\frac{d}{\sigma_c}$ value; place a straight edge at the origin 0 so that it passes over the $\frac{d}{\sigma_c}$ value; obtain numerical value from the nomogram A. With the A value, enter Fig. 6 and determine an $\frac{R_c}{\sigma_c}$ value from the intersection of the A value with the desired probability curve. The final R_c is the radius of the probability circle resulting from a normal circular distribution displaced from the origin by distance d.

Example 2: Given a missile weapon system with a circular standard error (σ_c) of 150m. The missile is aimed at a point 200m in the x direction, and 400m in the y direction from the target. What is the radius of the 50% probability circle centered on the target? Assume the origin of coordinate system; (x, y) = (0, 0) as the target.

Solution:

$$d = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2} = \sqrt{(200)^2 + (400)^2}$$

$$= 200{,}000 = 448\text{m}$$

$$\frac{d}{\sigma_c} = \frac{448}{150} = 2.99$$

enter Fig. 5 with $\frac{d}{\sigma_c}$

$$A = 9.96$$

enter Fig. 6 with A = 9.96

at 50%, $\frac{R_c}{\sigma_c} = 3.15$

$$R_c = 3.15 \cdot 150$$

$$= 472\text{m}$$

The result (472m) is the radius of a target centered circle which has a 50% probability of containing the missile impact.

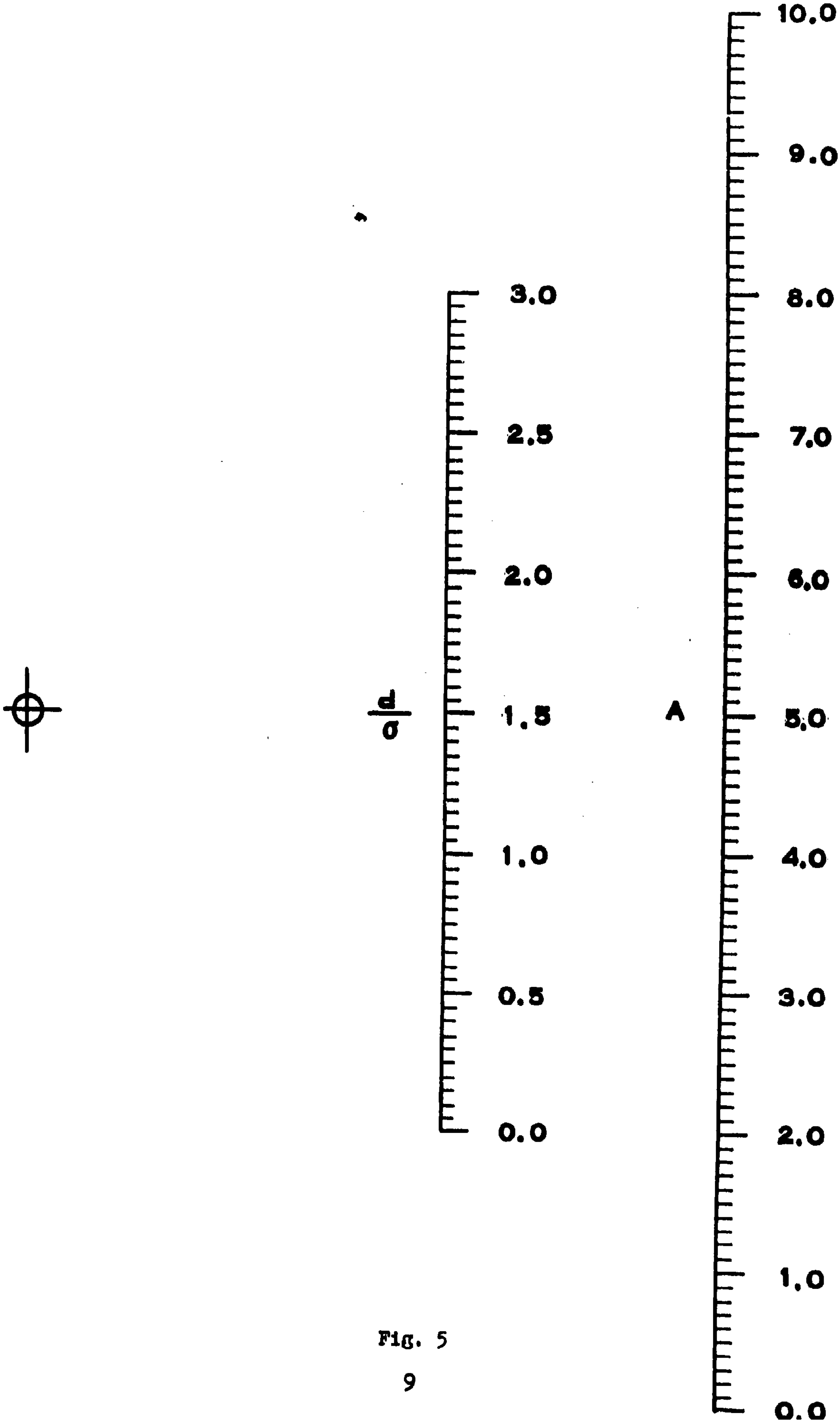


Fig. 5

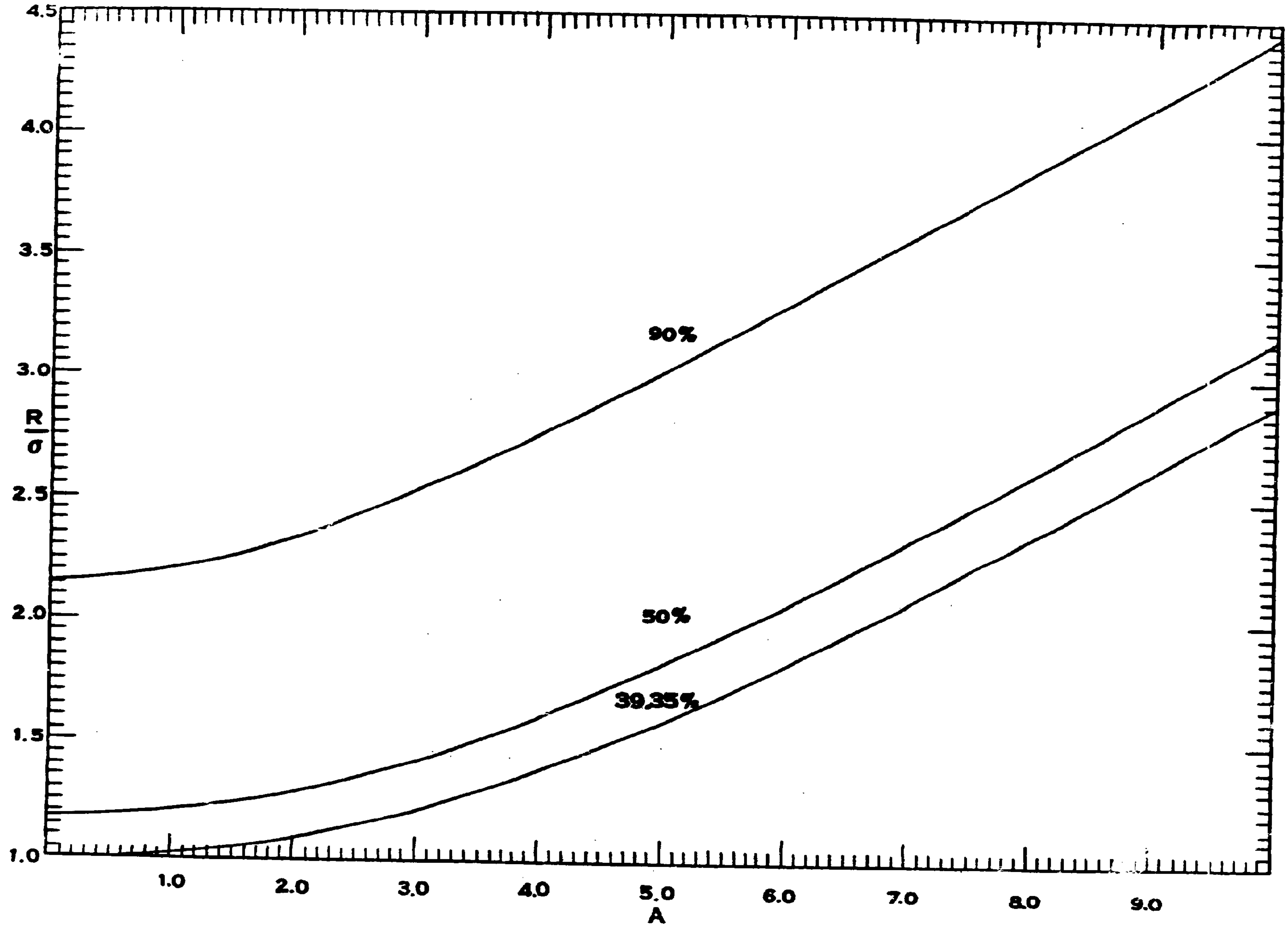


Fig. 6

Third order approximation equations for 39.35, 50, and 90% probability levels are:

$$(\frac{R}{\sigma})_{39.35} = -0.0498 (\frac{d}{\sigma_c})^3 + 0.3885 (\frac{d}{\sigma_c})^2 - 0.0876 (\frac{d}{\sigma_c}) + 1.010$$

$$\sigma_{39.35} = \pm 0.003$$

$$(\frac{R}{\sigma})_{50} = -0.0535 (\frac{d}{\sigma_c})^3 + 0.3952 (\frac{d}{\sigma_c})^2 -0.0453 (\frac{d}{\sigma_c}) + 1.1813$$

$$\sigma_{50} = \pm 0.004$$

$$(\frac{R}{\sigma})_{90} = -0.0550 (\frac{d}{\sigma_c})^3 + 0.3623 (\frac{d}{\sigma_c})^2 + 0.1674 (\frac{d}{\sigma_c}) + 2.1272$$

$$\sigma_{90} = \pm 0.011$$

The $(\frac{R}{\sigma})_{39.35}$ values have a standard deviation of ±0.003 with maximum deviation of 0.010 occurring at $\frac{d}{\sigma} = 0$. Values of $(\frac{R}{\sigma})_{50}$ have a standard deviation of ±0.004 and maximum deviation of 0.008 occurring at $\frac{d}{\sigma_c} = 3.00$, and at 90% the values have standard deviation of ±0.011 with a maximum deviation of 0.020 occurring at $\frac{d}{\sigma_c} = 0.45$.

Considering example 2 with $\frac{d}{\sigma_c} = 2.99$, the $(\frac{R}{\sigma})_{50}$ value will be

$$(\frac{R}{\sigma})_{50} = -0.0535 (26.7306) + 0.3952 (8.9400) -0.0453 (2.9900 + 1.1813)$$

$$= 3.149 \pm 0.004$$

3. <u>Elliptical Error</u>. As stated in section 1, a bivariate error analysis generally results in an elliptical error distribution. With $\sigma_x \neq \sigma_y$, the probability integral takes the form of equation (4).

$$\Pr\left[x^2 + y^2 \leq R_E^2\right] = \int\int \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{(x - \bar{x})^2}{\sigma_x^2} + \frac{(y - \bar{y})^2}{\sigma_y^2}\right)\right] dxdy \quad (4)$$

The geometrical expression of equation (4) is shown by Fig. 7.

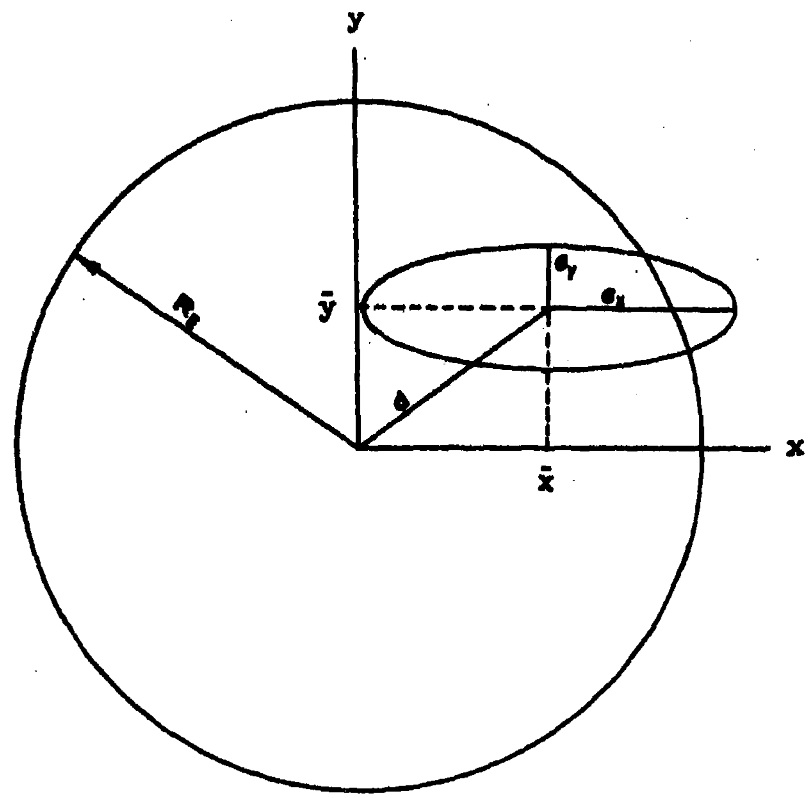


Fig. 7

Equation (4) is not integrable in closed form, but a solution is obtained by a method developed by Rosenthall and Rodden (1961)[1]. Evaluation of (4), for all possible values of $(x - \bar{x})$, $(y - \bar{y})$, $\frac{\sigma_{min}}{\sigma_{max}}$, and $\frac{R_E}{\sigma_x}$, would result in a table of monstrous proportions, therefore, a method of approximation will be introduced so that, in many instances, the table will not be required.

If d is zero, $\bar{x}$ and $\bar{y}$ are zero and equation 4 is expressed:

$$Pr\,[x^2 + y^2 \leq R_E^2] = \int\int \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x{}^2} + \frac{y^2}{\sigma_y{}^2}\right)\right] dxdy \quad (5)$$

Equation (5) determines the radius of a circle, $(x^2 + y^2 \leq R_E^2)$, which has a probability density equivalent to that of the normal elliptical distribution. ACIC TR-96 discusses the use of equation (5) for replacing a normal standard elliptical distribution with a more convenient circular distribution. TR-96 suggests that:

$$\sigma_c = 0.500\,(\sigma_x + \sigma_y) \text{ for } 0.2 \leq \frac{\sigma_{min}}{\sigma_{max}} \leq 1.0$$

will produce a standard error circle (σ_c) which will effectively replace the standard error ellipse. By replacing the standard error ellipse, the evaluation of equation (4) is reduced to an approximate solution obtained through the nomogram and probability curves or equations of section 2.

1. Ref. 1

Example 3: Given a missile weapon system with a crossrange error (σ_x) of 1.0 units and a downrange error (σ_y) of 0.6 units. The missile is aimed at a point 2.0 units in the x-direction and 1.0 unit in y-direction from the true position of the target. What is the radius of the 39.35%, 50% and 90% probability circle (R_c) centered on the target? Fig. 8.

$$(x - \bar{x}) = 2.0; \quad (y - \bar{y}) = 1.0; \quad \frac{\sigma_y}{\sigma_x} = \frac{0.6}{1.0} = 0.6$$

$$\sigma_c = 0.5\,(1.0 + 0.6) = 0.8$$

$$d = \sqrt{(2.0)^2 + (1.0)^2} = 2.24$$

$$\frac{d}{\sigma_c} = \frac{2.24}{0.8} = 2.80$$

enter Fig. 5 with A = 9.33

enter Fig. 6 $\frac{R_c}{\sigma_c}$ for 39.35% = 2.72 $\qquad R_c = 2.18$

$\frac{R_c}{\sigma_c}$ for 50% = 2.98 $\qquad R_c = 2.38$

$\frac{R_c}{\sigma_c}$ for 90% = 4.23 $\qquad R_c = 3.38$

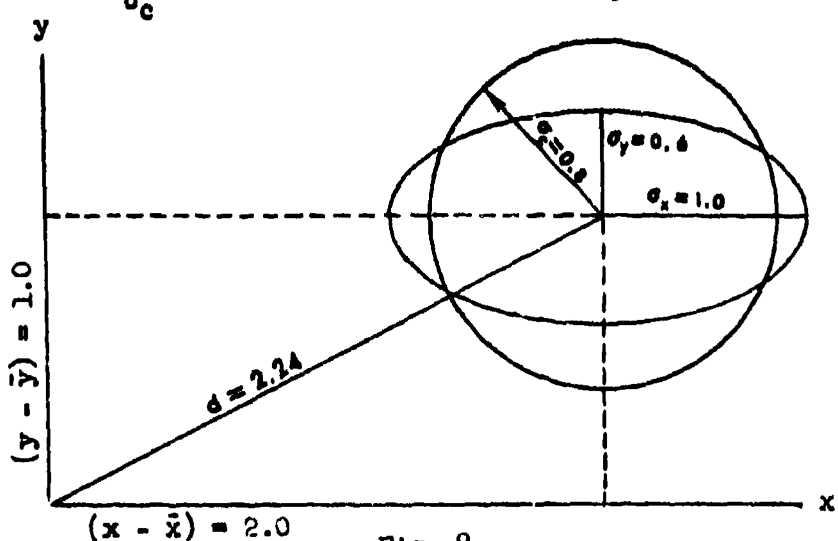


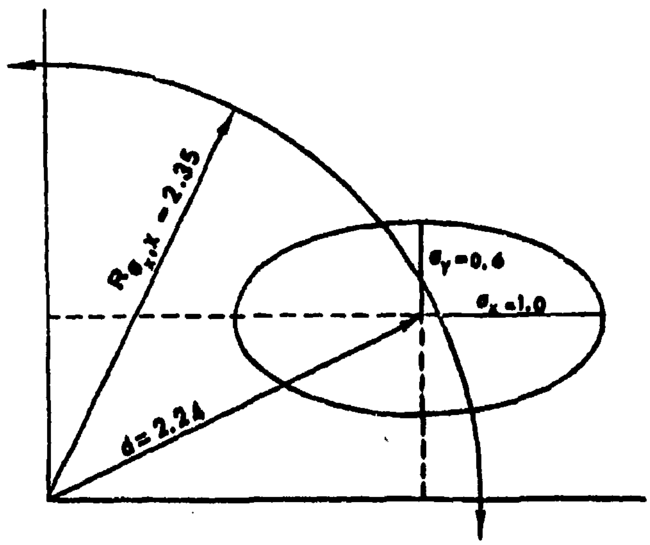
Fig. 8

4. R_E - R_c Differences. An error in either probability or $\frac{R}{\sigma}$ results from replacing the error ellipse by the corresponding circular form. This error is the difference between the true value of the elliptical error distribution minus the approximation value. A brief discussion of these differences is presented in order to qualify the approximation method proposed in section 3. The notation in this discussion will be as follows: The R_c value is the radius obtained from a circular distribution by equation (3b). The R_E value refers to the radius obtained from an elliptical distribution. R_E will be used in general cases where the orientation of the ellipse is not in question. $R_{\sigma_x,x}$ is used to signify the radius obtained by equation (4) in place of R_E when orientation is considered. The first subscript denotes the direction of the larger standard error value and the second the larger biased component, or $R_{\sigma_x,x}$ means that $\sigma_x > \sigma_y$ and $(x - \bar{x}) > (y - \bar{y})$. The values of R_c are those obtained from Vitalis' table and the R_E ($R_{\sigma_x,x}$, etc) values are obtained from the tables of Rosenthal and Rodden.

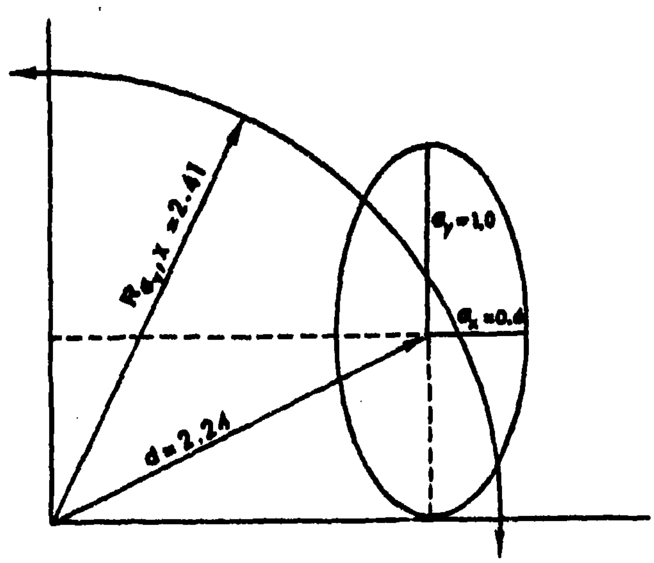
For the 39.35 and 50% probability levels it was found that if the σ_{max} direction and the direction of maximum bias, $(x - \bar{x})$ or $(y - \bar{y})$, coincide, the R_c value is too large. This problem is shown in Fig. 9a where $\sigma_x > \sigma_y$ and $(x - \bar{x}) > (y - \bar{y})$. In this particular instance, see ex. 3, for 50% probability R_c = 2.38 while $R_{\sigma_x,x}$ = 2.35. If the σ_{max} direction and the direction of maximum bias do not coincide (Fig. 9b), the R_c value is too small. The $R_{\sigma_y,x}$ in this situation is 2.41 units and the R_c value is the same for both cases 2.38 units. However, for some of the 90% probability level values $R_{\sigma_x,y} < R_c < R_{\sigma_x,x}$.

The next question: with the same error ellipse as Fig. 9a and 9b, what are the R and probability values when $(x - \bar{x}) = 1$, and $(y - \bar{y}) = 2$? Since d is the same as ex. 3, the R_c value for 50% probability level is again 2.38 units, Fig. 10. By rotating Fig. 10a through an angle of 90° to the left it is a reflection of Fig. 9a, thus $R_{\sigma_y,y} = R_{\sigma_x,x}$. Similarly, Fig. 10b rotated to the left 90° is a reflection of 9b and $R_{\sigma_x,y} = R_{\sigma_y,x}$.

If d makes an angle of 45° with the coordinate axis (Fig. 11a and b) and $(x - \bar{x}) = (y - \bar{y})$, the direction of σ_{max} is immaterial. Both Fig. 11a and 11b have equal R and probability values. By rotating 11a through an angle of 90° to the left it becomes a reflection of 11b. Computing R_c values at $\alpha = 45°$ produces the values of minimum difference $(R_E - R_c)$. As d progresses in either direction away from $\alpha = 45°$ the difference can be expected to increase to maximums at 0° and 90°. Tables 1 and 2 give values of $(R_{\sigma_x,y} - R_c)$ and $(R_c - R_{\sigma_x,x})$ respectively, to indicate a possible maximum magnitude for the differences. The tables are computed for the $\frac{\sigma_{min}}{\sigma_{max}}$ ratios of 0.8, 0.6, 0.4 and 0.2 and for probability levels of 39.35, 50 and 90%. Tables 3 and 4 give the probability difference for each of the $(R_E - R_c)$ terms in tables 1 and 2.

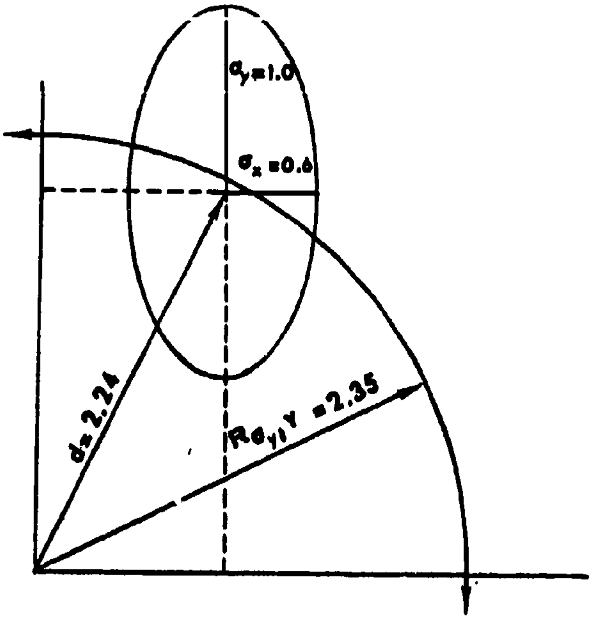


9a

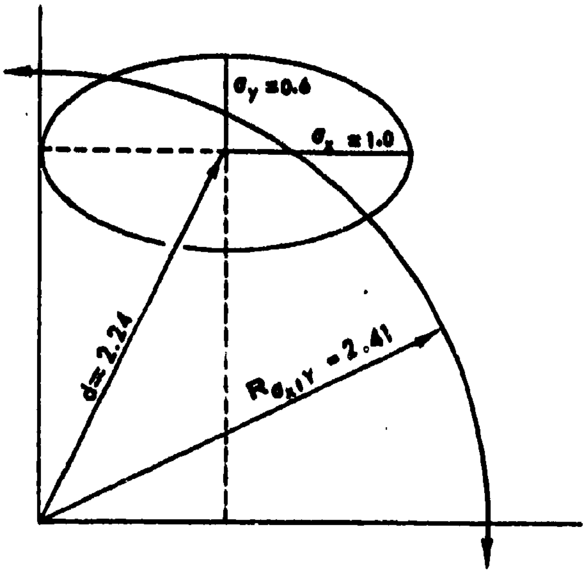


9b

Fig. 9

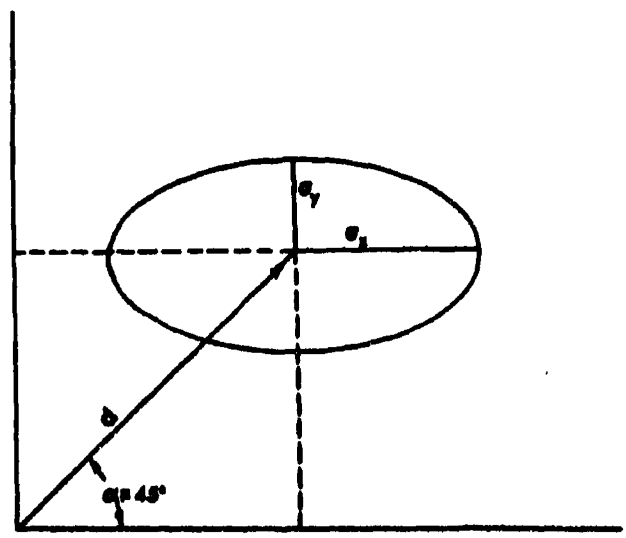


10a

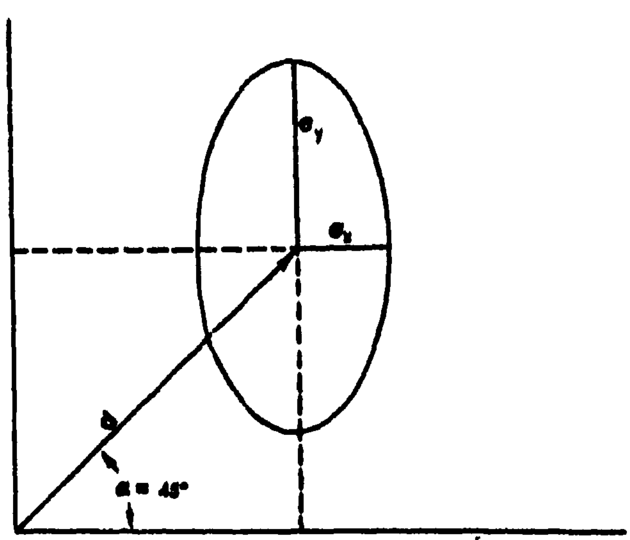


10b

Fig. 10



11a



11b

Fig. 11

The R values for example (3) are:

R_c 39.35% = 2.18 $\quad R_{\sigma_{x},x}$ = 2.09

R_c 50 = 2.38 $\quad R_{\sigma_{x},x}$ = 2.33

R_c 90 = 3.38 $\quad R_{\sigma_{x},x}$ = 3.51

From table 2a, b, and c for $(x - \bar{x}) = 2.0$ and $(y - \bar{y}) = 0$, the maximum differences to be expected are 0.118, 0.66, and -0.189 respectively. The values in the example all fall within the expected differences.

5. Conclusions. The simplified computation suggested is feasible in most cases. Where the error in the R value created by the transformation from elliptical to circular forms is great enough, such treatment cannot be used. Tables 1, 2, 3, and 4 are designed to assist in determining the magnitude of such errors. Whether to apply the less complicated method or not is the choice of the user. The Rosenthal and Rodden method must be used where the error is considered decisive.

$R_{\sigma_{x,y}} - R_c$

For 39.35% Probability

$(x - \bar{x}),(y - \bar{y})$ / $\frac{\sigma_{min}}{\sigma_{max}}$	(0.0, 0.5)	(0.0, 1.0)	(0.0, 1.5)	(0.0, 2.0)
0.8	0.012	0.039	0.053	0.059
0.6	0.026	0.086	0.111	0.107
0.4	0.057	0.128	0.150	0.146
0.2	0.067	0.147	0.166	-

(a)

For 50% Probability

$(x - \bar{x}),(y - \bar{y})$ / $\frac{\sigma_{min}}{\sigma_{max}}$	(0.0, 0.5)	(0.0, 1.0)	(0.0, 1.5)	(0.0, 2.0)
0.8	0.012	0.033	0.037	0.038
0.6	0.026	0.068	0.077	0.066
0.4	0.044	0.090	0.094	0.082
0.2	0.058	0.090	0.089	-

(b)

For 90% Probability

$(x - \bar{x}),(y - \bar{y})$ / $\frac{\sigma_{min}}{\sigma_{max}}$	(0.0, 0.5)	(0.0, 1.0)	(0.0, 1.5)	(0.0, 2.0)
0.8	0.009	-0.021	-0.052	-0.074
0.6	0.041	-0.034	-0.088	-0.129
0.4	0.117	-0.025	-0.120	-0.185
0.2	0.250	-0.047	-0.099	-

(c)

Table 1

$R_c - R_{\sigma_x, x}$

For 39.35% Probability

$(x - \bar{x}),(y - \bar{y})$ / $\frac{\sigma_{min}}{\sigma_{max}}$	(0.5, 0.0)	(1.0, 0.0)	(1.5, 0.0)	(2.0, 0.0)
0.8	-	0.039	0.062	0.059
0.6	-	0.089	0.119	0.118
0.4	-	0.149	0.176	0.164
0.2	-	0.187	0.209	-

(a)

For 50% Probability

$(x - \bar{x}),(y - \bar{y})$ / $\frac{\sigma_{min}}{\sigma_{max}}$	(0.5, 0.0)	(1.0, 0.0)	(1.5, 0.0)	(2.0, 0.0)
0.8	-	0.033	0.045	0.036
0.6	-	0.072	0.076	0.066
0.4	-	0.107	0.101	0.082
0.2	-	0.109	0.103	-

(b)

For 90% Probability

$(x - \bar{x}),(y - \bar{y})$ / $\frac{\sigma_{min}}{\sigma_{max}}$	(0.5, 0.0)	(1.0, 0.0)	(1.5, 0.0)	(2.0, 0.0)
0.8	-	-0.045	-0.066	-0.076
0.6	-	-0.131	-0.170	-0.189
0.4	-	-0.255	-0.288	-0.342
0.2	-	-0.388	-0.423	-

(c)

Table 2

Probability Differences

$$R_{\sigma_{x,y}} - R_c$$

For 39.35% Probability

$(x - \bar{x}),(y - \bar{y})$ / $\frac{\sigma_{min}}{\sigma_{max}}$	(0.0, 0.5)	(0.0, 1.0)	(0.0, 1.5)	(0.0, 2.0)
0.8	-0.73%	-2.14%	-2.68%	-2.91%
0.6	-1.79	-5.43	-6.86	-6.61
0.4	-3.76	-10.20	-12.19	-14.52
0.2	-7.46	-17.58	-21.07	-

(a)

For 50% Probability

$(x - \bar{x}),(y - \bar{y})$ / $\frac{\sigma_{min}}{\sigma_{max}}$	(0.0, 0.5)	(0.0, 1.0)	(0.0, 1.5)	(0.0, 2.0)
0.8	-0.77%	-1.84%	-1.93%	-1.96%
0.6	-1.74	-4.39	-4.98	-4.27
0.4	-3.43	-7.49	-8.17	-7.[illegible]
0.2	-5.23	-9.07	-11.99	-

(b)

For 90% Probability

$(x - \bar{x}),(y - \bar{y})$ / $\frac{\sigma_{min}}{\sigma_{max}}$	(0.0, 0.5)	(0.0, 1.0)	(0.0, 1.5)	(0.0, 2.0)
0.8	-0.21%	+0.49%	+1.18%	+1.90%
0.6	-1.01	+0.83	+2.14	+3.10
0.4	-3.01	+0.61	+3.09	+4.80
0.2	-2.85	-1.24	+2.51	-

(c)

Table 3

Probability Differences

$R_c - R_{\sigma_{x},x}$

For 39.35% Probability

$(x - \bar{x}),(y - \bar{y})$	(0.5, 0.0)	(1.0, 0.0)	(1.5, 0.0)	(2.0, 0.0)
$\frac{\sigma_{min}}{\sigma_{max}}$				
0.8	-	1.60%	2.74%	2.43%
0.6	-	4.83	5.23	2.71
0.4	-	7.73	7.39	6.56
0.2	-	8.84	8.48	-

(a)

For 50% Probability

$(x - \bar{x}),(y - \bar{y})$	(0.5, 0.0)	(1.0, 0.0)	(1.5, 0.0)	(2.0, 0.0)
$\frac{\sigma_{min}}{\sigma_{max}}$				
0.8	-	1.65%	1.99%	1.51%
0.6	-	3.65	3.31	2.75
0.4	-	5.10	4.21	3.35
0.2	-	4.86	4.15	-

(b)

For 90% Probability

$(x - \bar{x}),(y - \bar{y})$	(0.5, 0.0)	(1.0, 0.0)	(1.5, 0.0)	(2.0, 0.0)
$\frac{\sigma_{min}}{\sigma_{max}}$				
0.8	-	-0.90%	-1.24%	-1.44%
0.6	-	-2.63	-3.42	-3.79
0.4	-	-5.26	-6.13	-6.54
0.2	-	8.47	-9.59	

(c)

Table 4

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5. AUTHOR(S) *(Last name, first name, initial)*

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A procedure for determining the radius of the 39, 35, 50, and 90 per cent probability circles for a biased distribution is presented. Both circular and elliptical normal bivariate distributions are considered. The elliptical distribution is replaced by an equivalent circular distribution and an approximate radius obtained, using the circular distribution procedure. Tables giving an indication of the error resulting from this displacement are included.

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