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A MULTIVARIATE NOTION OF ASSOCIATION, WITH A RELIABILITY APPLICATION

by

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Abstract

Random variables $T_1, T_2, \ldots, T_n$ are, in this paper, *associated* if each pair of non-decreasing functions $F(T_1, T_2, \ldots, T_n)$, $G(T_1, T_2, \ldots, T_n)$ have a non-negative covariance. Association holds in cases ranging from $T_1, T_2, \ldots, T_n$ independent to $T_1, T_2, \ldots, T_n$ jointly restricted to a non-decreasing curve. Association is preserved under the standard multivariate operations of extracting subsets and pooling independent sets; and under the special operation of forming sets of non-decreasing functions. Suitable choices of F, G lead to various inequalities for associated random variables. The properties of association are studied in the simple, but representative, case that $T_1, T_2, \ldots, T_n$ are finitely discrete.

The notion of association is useful in extending the domain of validity of the minimal cut lower bound for the reliability of a coherent system [4,1], notably (here) to the case of repairable components with exponential times to failure and exponential times to repair.

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1. Introduction

It is customary to think that two random variables S,T are associated (in the sense of similar behavior) if $\mathrm{Cov}[S,T] = ES\cdot T - ES\cdot ET$ is non-negative. The same two random variables are more strongly associated if $\mathrm{Cov}[f(S),g(T)] \geq 0$ for all pairs of non-decreasing functions f,g. The same two random variables are still more strongly associated if $\mathrm{Cov}[F(S,T),G(S,T)] \geq 0$ for all pairs of functions F,G which are non-decreasing in each argument.

The strongest of the three criteria has a natural multivariate generalization. We consider the definition

(1.1) Random variables $T_1,T_2,\ldots,T_n$ are *associated* if $\mathrm{Cov}[F(\underset{\sim}{T}),G(\underset{\sim}{T})] \geq 0$, where $\underset{\sim}{T} = (T_1,T_2,\ldots,T_n)$, for all pairs F,G of non-decreasing functions.

Here, and later, we say that a function F is non-decreasing if it is non-decreasing in each argument, i.e. if $F(\underset{\sim}{s}) \leq F(\underset{\sim}{t})$ whenever $s_i \leq t_i$, $i=1,\ldots,n$.

This paper is intended to indicate an approach to the organization of the properties of (1.1), and to motivate it by a sample application.

We study (Section 2) the properties of (1.1) in the case that the joint probability distribution of $T_1,T_2,\ldots,T_n$ is finitely discrete, i.e. the vector $\underset{\sim}{T}$ takes a finite number of values. The finitely discrete case illustrates the general situation; *the properties of (1.1) obtained are true in general;* but by confining our attention we can defer discussion

of some equivalent versions of (1.1) which are of interest in the general theory.

Our application (Section 3) is a demonstration that the minimal cut lower bound for the reliability of a coherent system is valid in a greater variety of circumstances than previously known. This application reflects the origin of our interest in (1.1), and is suitable because it involves a number of the manipulations that our study of (1.1) is designed to facilitate. We do not attempt a complete discussion of this subject. Further we postpone consideration of certain non-reliability results, e.g. Kimball (1951) and Robbins (1954), which can be related to (1.1).

There is an extensive [see Goodman-Kruskal (1959) for a recent bibliography] and continuing literature on quantitative measures of association, primarily for bivariate distributions. By contrast, our interest is in the qualitative identification, and corresponding properties, of classes of associated distributions.*

The multivariate dependence described by (1.1) is one of several similar notions. When referring to it we should qualify the term "association", but for the present it is simpler to let it mean (1.1).

*At the time this paper was being typed, our attention was drawn to the manuscript "Some concepts of dependence" by E. L. Lehmann, which treats several concepts of association and their applications.

2. Properties of association for finitely discrete and binary random variables

For finitely discrete $T_1, T_2, \ldots, T_n$, functions $F(\underset{\sim}{T})$ have finite expectations $EF(\underset{\sim}{T})$. The covariances considered in (1.1) are finite.

2.1 General properties

Association has two properties desirable in any classification of multivariate distributions:

(P_1) *Any subset of a set of associated random variables is also a set of associated random variables.*

(P_2) *If two sets of associated random variables are independent of one another, then their union is a set of associated random variables.*

Property P_1 follows immediately from the definition of association. P_2 requires a short proof, which can be made by considering two independent, internally associated random vectors $\underset{\sim}{S} = (S_1, S_2, \ldots, S_n)$, $\underset{\sim}{T} = (T_1, T_2, \ldots, T_m)$ and writing

$$\begin{aligned}
\mathrm{Cov}[F(\underset{\sim}{S},\underset{\sim}{T}), G(\underset{\sim}{S},\underset{\sim}{T})] &= E_{\underset{\sim}{S},\underset{\sim}{T}}FG - E_{\underset{\sim}{S},\underset{\sim}{T}}F \cdot E_{\underset{\sim}{S},\underset{\sim}{T}}G = E_{\underset{\sim}{S}}E_{\underset{\sim}{T}}FG - E_{\underset{\sim}{S}}E_{\underset{\sim}{T}}F \cdot E_{\underset{\sim}{S}}E_{\underset{\sim}{T}}G \\
&= E_{\underset{\sim}{S}}E_{\underset{\sim}{T}}FG - E_{\underset{\sim}{S}}\{E_{\underset{\sim}{T}}F \cdot E_{\underset{\sim}{T}}G\} + E_{\underset{\sim}{S}}\{E_{\underset{\sim}{T}}F \cdot E_{\underset{\sim}{T}}G\} - E_{\underset{\sim}{S}}E_{\underset{\sim}{T}}F \cdot E_{\underset{\sim}{S}}E_{\underset{\sim}{T}}G \\
&= E_{\underset{\sim}{S}}\mathrm{Cov}_{\underset{\sim}{T}}[F,G] \quad + \quad \mathrm{Cov}_{\underset{\sim}{S}}[E_{\underset{\sim}{T}}F, E_{\underset{\sim}{T}}G],
\end{aligned}$$

where $E_{\underset{\sim}{S}}$ denotes expectation over the distribution of $\underset{\sim}{S}$. $E_{\underset{\sim}{T}}$ expectation over the distribution of $\underset{\sim}{T}$, and $E_{\underset{\sim}{S},\underset{\sim}{T}}$ expectation over the joint

distribution of $\underset{\sim}{S}$ and $\underset{\sim}{T}$. $E_{\underset{\sim}{S},\underset{\sim}{T}} = E_{\underset{\sim}{S}}E_{\underset{\sim}{T}}$ from the independence of $\underset{\sim}{S}$ and $\underset{\sim}{T}$. Since $\mathrm{Cov}_{\underset{\sim}{T}}[F(\underset{\sim}{s},\underset{\sim}{T}),G(\underset{\sim}{s},\underset{\sim}{T})] \geq 0$ for each fixed $\underset{\sim}{s}$, $E_{\underset{\sim}{S}}\mathrm{Cov}_{\underset{\sim}{T}}[F,G] \geq 0$ follows. Since $E_{\underset{\sim}{T}}F(\underset{\sim}{s},\underset{\sim}{T})$, $E_{\underset{\sim}{T}}G(\underset{\sim}{s},\underset{\sim}{T})$ are non-decreasing functions in $\underset{\sim}{s}$, $\mathrm{Cov}_{\underset{\sim}{S}}[E_{\underset{\sim}{T}}F,E_{\underset{\sim}{T}}G] \geq 0$.

Another standard multivariate property, valid for association, is

(P_3) *The set consisting of a single random variable is associated.*

To show P_3 consider a random variable T taking values $t_1 < t_2 < \cdots < t_k$. For non-decreasing F and G, $F(t_1) \leq F(t_2) \leq \cdots \leq F(t_k)$ and $G(t_1) \leq G(t_2) \leq \cdots \leq G(t_k)$. It follows from an inequality for similarly ordered sequences due to Chebyshev [Hardy, Littlewood, and Pólya (1934), §2.17] that $\mathrm{Cov}[F(T),G(T)] \geq 0$.

P_3, together with P_2, implies that *independent random variables are associated.* The example of independent random variables represents an "extreme" of association. Another extreme is represented by the example of random variables $\underset{\sim}{T} = (T_1,T_2,\ldots,T_n)$ taking values $\underset{\sim}{t}^{(1)} < \underset{\sim}{t}^{(2)} < \cdots < \underset{\sim}{t}^{(k}$ where $\underset{\sim}{s} < \underset{\sim}{t}$ means $s_i \leq t_i$, $i=1,\ldots,n$, and $s_i < t_i$ for some i. Then fo F,G non-decreasing we have again $F(\underset{\sim}{t}^{(1)}) \leq F(\underset{\sim}{t}^{(2)}) \leq \cdots \leq F(\underset{\sim}{t}^{(k)})$ and $G(\underset{\sim}{t}^{(1)}) \leq G(\underset{\sim}{t}^{(2)}) \leq \cdots \leq G(\underset{\sim}{t}^{(k)})$, so that by the Chebyshev inequality $\mathrm{Cov}[F(\underset{\sim}{T}),G(\underset{\sim}{T})] \geq 0$.

Properties P_1, P_2 and P_3 permit some standard multivariate manipulations with associated random variables. The applications of association, e.g. to reliability theory, are often founded on another manipulative property.

(P_4) *If* $T_1, T_2, \ldots, T_n$ *are associated, then any set of non-decreasing functions* $S_1(\underset{\sim}{T}), S_2(\underset{\sim}{T}), \ldots, S_m(\underset{\sim}{T})$ *are associated.*

To show P_4 we observe that $S_1(\underset{\sim}{T}), S_2(\underset{\sim}{T}), \ldots, S_m(\underset{\sim}{T})$ are finitely discrete; that if F, G are non-decreasing, then $F(\underset{\sim}{S})$, $G(\underset{\sim}{S})$ are non-decreasing; and that

$$\mathrm{Cov}_{\underset{\sim}{S}}[F(\underset{\sim}{S}), G(\underset{\sim}{S})] = \mathrm{Cov}_{\underset{\sim}{T}}[F\{\underset{\sim}{S}(\underset{\sim}{T})\}, G\{\underset{\sim}{S}(\underset{\sim}{T})\}] \geq 0.$$

2.2 A working definition of association

In the finitely discrete case there is one alternative definition of association, equivalent to (1.1), that is particularly useful.

Theorem 2.1. $T_1, T_2, \ldots, T_n$ *are associated if, and only if,*

$$\mathrm{Cov}[\Gamma(\underset{\sim}{T}), \Delta(\underset{\sim}{T})] \geq 0 \tag{2.2.1}$$

for all pairs Γ, Δ *of binary, non-decreasing functions.*

Proof. Condition (1.1) clearly implies (2.2.1). To show that (2.2.1) implies (1.1), suppose that $F(\underset{\sim}{T})$ takes the values $f_1 < f_2 < \cdots < f_k$ and write $F(\underset{\sim}{T}) = f_1 + (f_2 - f_1)\Gamma_2(\underset{\sim}{T}) + \cdots + (f_k - f_{k-1})\Gamma_k(\underset{\sim}{T})$, where $\Gamma_i(\underset{\sim}{T}) = 1$ if $F(\underset{\sim}{T}) \geq f_i$, $\Gamma_i(\underset{\sim}{T}) = 0$ otherwise. Similarly write $G(\underset{\sim}{T}) = g_1 + (g_2 - g_1)\Delta_2(\underset{\sim}{T}) + \cdots + (g_\ell - g_{\ell-1})\Delta_\ell(\underset{\sim}{T})$. Then

$$\mathrm{Cov}[F(\underset{\sim}{T}), G(\underset{\sim}{T})] = \sum_{i=2}^{k} \sum_{j=2}^{\ell} (f_i - f_{i-1})(g_j - g_{j-1}) \mathrm{Cov}[\Gamma_i(\underset{\sim}{T}), \Delta_j(\underset{\sim}{T})] \geq 0.$$

One application of (2.2.1) is the observation that for two binary random variables X, Y

(2.2.2) $\quad \mathrm{Cov}[X,Y] \geq 0$ implies X,Y are associated.

This result follows readily from (2.2.1) since the only possible binary, non-decreasing functions $\Gamma(X,Y)$ are

$$(\Gamma \equiv 0) \leq (\Gamma = XY) \leq \begin{matrix} (\Gamma = X) \\ (\Gamma = Y) \end{matrix} \leq (\Gamma = X + Y - XY) \leq (\Gamma \equiv 1).$$

The covariance between any pair of binary functions Γ,Δ such that $\Gamma \leq \Delta$ is automatically non-negative, and the remaining pair $\Gamma = X$, $\Delta = Y$ has non-negative covariance by hypothesis.

The utility of (2.2.1) partially rests on the well-known fact that for binary random variables X,Y, $\mathrm{Cov}[X,Y] \geq 0$ is equivalent to

(2.2.3) $\quad P[X=1,Y=1] \cdot P[X=0,Y=0] \geq P[X=1,Y=0] \cdot P[X=0,Y=1].$

In (2.2.1) each pair of binary, non-decreasing functions Γ,Δ defines a partition of the sample space of the vector $\underset{\sim}{T}$ into the four disjoint regions $\{\Gamma=j,\Delta=k\}$, $j,k = 0,1$. The condition (2.2.3), applied to the pairs of binary random variables $\Gamma(\underset{\sim}{T}),\Delta(\underset{\sim}{T})$, gives a good description of the joint probability distribution of associated $T_1,T_2,\ldots,T_n$ in terms of the geometry of the sample space.

In Section 2.1 we mentioned two "extremes" of association. Now (2.2.2) and (2.2.3) permit a reasonably complete discussion of a simple, non-extreme example. Let a vector of random variables $\underset{\sim}{T}$ take values

$$\underset{\sim}{t}^{00} < \begin{matrix} \underset{\sim}{t}^{01} \\ \underset{\sim}{t}^{10} \end{matrix} < \underset{\sim}{t}^{11}$$

with the possible additional orderings $\underset{\sim}{t}^{01} \leq \underset{\sim}{t}^{10}$ or $\underset{\sim}{t}^{10} \leq \underset{\sim}{t}^{01}$ excluded. In this case we can define $X_1(\underset{\sim}{t}^{jk}) = j$, $X_2(\underset{\sim}{t}^{jk}) = k$, $j,k = 0,1$, and have $X_1(\underset{\sim}{T})$, $X_2(\underset{\sim}{T})$ non-decreasing. We can also write $T_i = t_i^{jk}$ when $X_1 = j$, $X_2 = k$ and have $T_i(X_1,X_2)$ non-decreasing. It follows from P_4 that $T_1,T_2,\ldots,T_n$ are associated if, and only if, $X_1(\underset{\sim}{T})$, $X_2(\underset{\sim}{T})$ are associated. Then from (2.2.2) and (2.2.3) $X_1(\underset{\sim}{T})$, $X_2(\underset{\sim}{T})$ are associated if, and only if, $P_{11}P_{00} \geq P_{01}P_{10}$, where $p_{jk} = P[X_1(\underset{\sim}{T})=j, X_2(\underset{\sim}{T})=k] = P[\underset{\sim}{T}=\underset{\sim}{t}^{jk}]$.

2.3 Binary random variables

Certain properties of associated random variables have a special intuitive content if the random variables are binary. The most useful of these is

(BP_1) *If* $X_1,X_2,\ldots,X_n$ *are associated binary random variables, then* $1-X_1,1-X_2,\ldots,1-X_n$ *are associated binary random variables.*

To show BP_1, observe that if Γ is a binary, non-decreasing function, then the dual function $\Gamma^D(\underset{\sim}{x}) = 1 - \Gamma(\underset{\sim}{1}-\underset{\sim}{x})$, where $\underset{\sim}{1} - \underset{\sim}{x} = (1-x_1,1-x_2,\ldots,1-x_n)$, is also binary and non-decreasing. Let $\underset{\sim}{Y} = \underset{\sim}{1} - \underset{\sim}{X}$. Then

$$\mathrm{Cov}_{\underset{\sim}{Y}}[\Gamma(\underset{\sim}{Y}),\Delta(\underset{\sim}{Y})] = \mathrm{Cov}_{\underset{\sim}{X}}[\Gamma^D(\underset{\sim}{X}),\Delta^D(\underset{\sim}{X})] \geq 0,$$

and the property follows.

For $X_1,X_2,\ldots,X_n$ associated and $\Gamma(\underset{\sim}{X}) = X_1$, $\Delta(\underset{\sim}{X}) = X_2X_3\ldots X_n$, (2.2.1) gives $E(X_1X_2\ldots X_n) \geq E(X_1) \cdot E(X_2X_3\ldots X_n)$. Proceeding inductively we find

$E\prod_{i=1}^{n} X_i \geq \prod_{i=1}^{n} EX_i$. Since for a binary random variable X, $EX = P[X=1]$, we have

(2.3.1) $P[X_1=1, X_2=1, \ldots, X_n=1] \geq P[X_1=1] \cdot P[X_2=1] \cdot \ldots \cdot P[X_n=1]$.

From (2.3.1) and BP_1 follows

(2.3.2) $P[X_1=0, X_2=0, \ldots, X_n=0] \geq P[X_1=0] \cdot P[X_2=0] \cdot \ldots \cdot P[X_n=0]$.

3. A sample application of association, the minimal cut bound in reliability theory

The minimal cut lower bound for the reliability of a coherent system was obtained in Esary-Proschan (1963) in a simple, basic case. In this section we show that the bound is valid in some of the more complex situations considered in reliability theory. The derivation illustrates the application of a formal organization of the properties of association.

3.1 Association in time of performance processes

In the setting of reliability theory, the *performance process* of a device is a binary stochastic process $\{X(t), t \in \tau\}$, $\tau \subset [0,+\infty)$, (e.g. $\tau = [0,+\infty)$ or $\tau = \{0,1,2,\ldots\}$), where $X(t) = 1$ if the device is functioning at time t and $X(t) = 0$ if the device is failed at time t.

A performance process $\{X(t), t \in \tau\}$ is *associated in time* if for every finite set of times $\{t_1, t_2, \ldots, t_k\} \subset \tau$, the binary random variables $X(t_1), X(t_2), \ldots, X(t_k)$ are associated.

The basic example of a performance process that is associated in time arises when the device has a *life* [Esary-Marshall (1964)], i.e. when $P[X(s) \geq X(t)] = 1$ whenever $s > t$. In essence a device has a life if once failed, it stays failed. For $t_1 < t_2 < \cdots < t_k$ the vector $(X(t_1), X(t_2), \ldots, X(t_k))$ can, with probability one, take only the values $(0,0,\ldots,0) < (1,0,\ldots,0) < (1,1,0,\ldots,0) < \cdots < (1,1,\ldots,1)$ and thus $X(t_1), X(t_2), \ldots, X(t_k)$ are associated (cf. the cases of "extreme" association in Section 2).

3.2 The exponential-exponential process

Aside from the cases in which a device has a life, perhaps the most frequently studied performance process is the alternating renewal process that arises when a device fails, then revives (e.g. is repaired or replaced), then fails, and so on, where the time S from revival to failure is exponentially distributed, $P[S > s] = e^{-\lambda s}$, $\lambda > 0$, $s \geq 0$, and the time T from failure to revival is also exponentially distributed, $P[T > t] = e^{-\mu t}$, $\mu > 0$, $t \geq 0$. The corresponding (exponential-exponential) performance process $\{X(t),\ t \in [0,+\infty)\}$ is Markov with the transition law, for $s < t$,

$$
\begin{aligned}
P[X(t)=1 \mid X(s)=1] &= (\lambda+\mu)^{-1}\{\mu + \lambda \exp[-(\lambda+\mu)(t-s)]\} \\
P[X(t)=0 \mid X(s)=0] &= (\lambda+\mu)^{-1}\{\lambda + \mu \exp[-(\lambda+\mu)(t-s)]\}.
\end{aligned}
\tag{3.2.1}
$$

[See e.g. Cox (1962) or Barlow-Proschan (1965)].

<u>Theorem 3.1</u>. *For any initial distribution, i.e.* $P[X(0)=1]$, *the exponential-exponential performance process is associated in time.*

<u>Proof</u>. We consider $t_1, t_2, \ldots, t_k$ such that $0 < t_1 < t_2 < \cdots < t_k$, and let $X_j = X(t_j)$, $j = 1, \ldots, k$. We show that $X_1, X_2, \ldots, X_k$ are associated by induction. X_1 alone is associated by P_3. Assume that $X_1, X_2, \ldots, X_{j-1}$ are associated.

Let $p_j = P[X_j=1 | X_{j-1}=1]$ and $q_j = P[X_j=0 | X_{j-1}=0]$. From (3.2.1), $p_j + q_j \geq (\lambda+\mu)^{-1}(\lambda+\mu) = 1$. Construct binary random variables U_j, V_j, independent of $X_1, X_2, \ldots, X_{j-1}$, with the joint distribution $P[U_j=1, V_j=1] = 1 - q_j$, $P[U_j=1, V_j=0] = p_j + q_j - 1$, and $P[U_j=0, V_j=0] = 1 - p_j$. For this joint distribution $P[U_j=1] = p_j$ and $P[V_j=0] = q_j$.

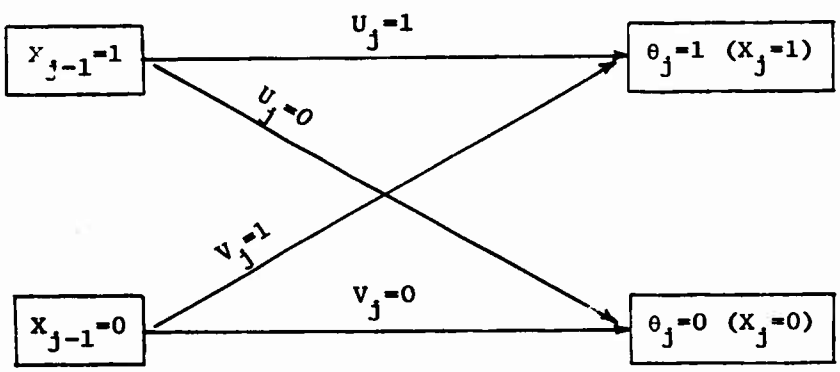

Figure 1

Let $\theta_j(X_{j-1}, U_j, V_j) = X_{j-1}U_j + (1-X_{j-1})V_j$. We can regard X_j as generated by the branching process shown in Figure 1 in the sense that $X_1, X_2, \ldots, X_{j-1}, \theta_j$ *have the same joint probability distribution as* $X_1, X_2, \ldots, X_{j-1}, X_j$, since e.g.

$$P[\theta_j=1, X_{j-1}=1, \underset{\sim}{X}=\underset{\sim}{x}] \qquad \text{where } \underset{\sim}{X} = (X_{j-2}, X_{j-3}, \ldots, X_1)$$

$$= P[U_j=1, X_{j-1}=1, \underset{\sim}{X}=\underset{\sim}{x}] = P[U_j=1] \cdot P[X_{j-1}=1, \underset{\sim}{X}=\underset{\sim}{x}]$$

$$= P[X_j=1 | X_{j-1}=1] \cdot P[X_{j-1}=1, \underset{\sim}{X}=\underset{\sim}{x}] = P[X_j=1, X_{j-1}=1, \underset{\sim}{X}=\underset{\sim}{x}].$$

Because $U_j \geq V_j$, $\mathrm{Cov}[U_j, V_j] \geq 0$, and by (2.2.2) U_j, V_j are associated. Also because $U_j \geq V_j$, the binary function θ_j is non-decreasing in X_{j-1}, U_j, and V_j.

Now $X_1, X_2, \ldots, X_{j-1}, U_j, V_j$ are associated by P_2, since $X_1, X_2, \ldots, X_{j-1}$ are associated, U_j, V_j are associated, and the two groups of variables are independent. Then $X_1, X_2, \ldots, X_{j-1}, \theta_j$ are associated by P_4, since these variables may be viewed as a set of non-decreasing functions of $X_1, X_2, \ldots, X_{j-1}, U_j, V_j$.

Thus $X_1, X_2, \ldots, X_{j-1}, X_j$ are associated, because their joint distribution agrees with that of $X_1, X_2, \ldots, X_{j-1}, \theta_j$.

The same method can be used to show association in time for the geometric-geometric performance process $\{X(t), t=0,1,\ldots\}$, i.e. the alternating renewal process where time to failure S has the geometric distribution $P[S > s] = p^s$, $s = 0,1,\ldots$, and time to revival T has the geometric distribution $P[T > t] = q^t$, $t = 0,1,\ldots$. In this case the condition $p + q \geq 1$ becomes a necessary hypothesis, but the result is again independent of the initial distribution.

Other performance processes of interest in reliability theory are also associated in time, e.g. the process that arises from scheduled replacement of a device whose intrinsic performance process is associated in time, and that of a device with several unloaded or lightly loaded spares (under exponential assumptions about times to failure and repair). For our present purposes, however, we want to focus attention on the elementary life and alternating processes.

3.3 Joint performance processes generated by systems

We consider systems whose performance at a time t is determined by the performance of their components at the same time t. If $\{\underset{\sim}{X}(t), t \in \tau\}$, where $\underset{\sim}{X}(t) = (X_1(t),\ldots,X_n(t))$, is the joint performance process for the components in a system, the performance process for the system is $\{\phi\underset{\sim}{X}(t), t \in \tau\}$, where $\phi\underset{\sim}{X}(t) \equiv \phi(\underset{\sim}{X}(t))$ is a binary function that takes the value one if the system is functioning at time t and takes the value zero if the system is failed at time t. A system is *coherent* [see Barlow-Proschan (1965) for a survey and bibliography] if ϕ is non-decreasing and $\phi(1,1,\ldots,1) = 1$, $\phi(0,0,\ldots,0) = 0$. The function ϕ is called the *structure function* of the system.

We write $\{\underset{\sim}{\phi X}(t), t \in \tau\}$, where $\underset{\sim}{\phi X}(t) = (\phi_1\underset{\sim}{X}(t), \phi_2\underset{\sim}{X}(t),\ldots,\phi_m\underset{\sim}{X}(t))$, for the joint performance process of a set of m systems, with structure functions $\phi_1,\phi_2,\ldots,\phi_m$, formed from a set of n components with the joint performance process $\{\underset{\sim}{X}(t), t \in \tau\}$. We admit the case in which a component participates in several systems.

The joint performance process $\{\underset{\sim}{X}(t), t \in \tau\}$ for a set of n devices is *associated in time if* for each finite set of times $\{t_1, t_2, \ldots, t_k\} \subset \tau$, the binary random variables in the array

$$\begin{array}{cccc} X_1(t_1) & X_1(t_2) & \ldots & X_1(t_k) \\ X_2(t_1) & X_2(t_2) & \ldots & X_2(t_k) \\ \vdots & & & \vdots \\ X_n(t_1) & X_n(t_2) & \ldots & X_n(t_k) \end{array} \tag{3.3.1}$$

are associated.

If n devices perform independently, and the performance process of each device is associated in time, then the rows in (3.3.1) are independent and each row is associated. From P_2 their joint performance process is associated in time.

<u>Theorem 3.2</u>. *If the joint performance process* $\{\underset{\sim}{X}(t), t \in \tau\}$ *of a set of components is associated in time, then the joint performance process* $\{\phi\underset{\sim}{X}(t), t \in \tau\}$ *of a set of coherent systems formed from those components is associated in time.*

<u>Proof</u>. For $\{t_1, t_2, \ldots, t_k\} \subset \tau$, each $\phi_\ell X(t_j)$, $\ell = 1, \ldots, m$, $j = 1, \ldots, k$, is a non-decreasing function of variables in the array $X_i(t_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, k$. The conclusion follows from P_4.

Using Theorem 3.2 one can show that the joint performance process corresponding to the multivariate exponential (life) distribution defined in Marshall-Olkin (1966) is associated in time.

3.4 The minimal cut lower bound for the reliability of a coherent system

The *reliability* at time t, $\bar{F}(t)$, of a device with the performance process $\{X(t), t \in \tau\}$ is defined as $\bar{F}(t) = P[X(s)=1, \text{ all } s \in \tau(t)]$, where $\tau(t) = \tau \cap [0,t]$. We make the convention that *the sample functions of any performance process are continuous from the right on* τ. The reliability of a device then becomes the survival probability or the time T until the first failure of the device occurs, i.e. $P[T > t] = \bar{F}(t)$, $t \geq 0$.

Every coherent system has a finite number of minimal cuts $C_1, C_2, \ldots, C_m$. A *minimal cut* is a minimal set of components such that if all are simultaneously failed, the system is then failed. The performance process $\{\phi\underset{\sim}{X}(t), t \in \tau\}$ of the system can be related to the performance processes $\{X_i(t), t \in \tau\}$, $i = 1, \ldots, n$, of its components, through the minimal cuts, by

$$(3.4.1) \quad \phi\underset{\sim}{X}(t) = \prod_{j=1}^{m} \eta_j\underset{\sim}{X}(t), \quad \text{where} \quad \eta_j\underset{\sim}{X}(t) = 1 - \Pi_{i \in C_j}(1-X_i(t)),$$

for all $t \in \tau$ [see e.g. Barlow-Proschan (1965)]. The performance process $\{\eta_j\underset{\sim}{X}(t), t \in \tau\}$ is that of the (coherent) system in which the components in the jth minimal cut perform in parallel. The representation (3.4.1) corresponds to connecting these *minimal cut parallel structures* in series.

The minimal cut lower bound was previously obtained in what amounts to the case of components that perform independently and have lives. The

following theorem extends the bound to the case in which the joint performance process of the components is associated in time.

Theorem 3.3. *Let* $\bar{F}(t)$ *be the reliability at time* t *of a coherent system. Let* $\bar{F}_j(t)$, $j = 1,\ldots,m$ *be the reliability at time* t *of its minimal cut parallel structures. If the components in the system have a joint performance process that is associated in time, then*

$$\bar{F}(t) \geq \prod_{j=1}^{m} \bar{F}_j(t), \quad \textit{for all} \quad t \geq 0. \tag{3.4.2}$$

Proof. By Theorem (3.2) the joint performance process of the minimal cut parallel structures is associated in time. Let $S_k = \{s_1, s_2, \ldots, s_k\} \subset \tau(t)$. From (3.4.1) and (2.3.1)

$$\begin{aligned} P[\phi\underset{\sim}{X}(s_\ell)=1,\ \ell=1,\ldots,k] &= P[\prod_{j=1}^{m} \eta_j \underset{\sim}{X}(s_\ell)=1,\ \ell=1,\ldots,k] \\ &= P[\prod_{\ell=1}^{k} \eta_j \underset{\sim}{X}(s_\ell)=1,\ j=1,\ldots,m] \geq \prod_{j=1}^{m} P[\eta_j \underset{\sim}{X}(s_\ell)=1,\ \ell=1,\ldots,k]. \end{aligned} \tag{3.4.3}$$

Now let $S_k \uparrow S$, where S is a set dense in $\tau(t)$. By monotone convergence (3.4.3) becomes

$$P[\phi(\underset{\sim}{X}(s)=1, \text{ all } s \in S] \geq \prod_{j=1}^{m} P[\eta_j \underset{\sim}{X}(s)=1, \text{ all } s \in S].$$

Then (3.4.2) follows from the right continuity of sample functions on $\tau(t)$.

3.5 The minimal cut lower bound as an approximation to system reliability, a numerical example

When the minimal cut lower bound is applicable, it is simpler to find than the actual system reliability, because the bound requires only the

calculation (or simulation) of the reliabilities of systems of components in parallel. Viewed as an approximation to system reliability the bound is conservative, as is usually desired in practice. The bound *can* be a very good approximation to actual system reliability.

A numerical example is given in Esary-Proschan (1963) in the case of components that perform independently and have lives. The approximation is good when component reliabilities are reasonably high, the most likely practical case.

We want to give another numerical example, in the case that components perform independently according to identical exponential-exponential processes. We consider the "two out of three" system, i.e. the system which functions whenever two or more of its three components are functioning. The system has three minimal cuts $C_1' = \{c_1,c_2\}$, $C_2 = \{c_2,c_3\}$, $C_3 = \{c_1,c_3\}$, where c_1, c_2, c_3 denote the components. In this case both the reliability of the system and the reliability of the minimal cut parallel structures can be calculated. For the "2 out of 3" system the reliability at time t is

$$(3.5.1) \qquad \bar{F}(t) = (k_2-k_1)^{-1}\{k_2 \exp[-k_1 a] - k_1 \exp[-k_2 a]\},$$

where $2k_1 = (b+5) - (b^2+10b+1)^{\frac{1}{2}}$, $2k_2 = (b+5) + (b^2+10b+1)^{\frac{1}{2}}$, and $a = \lambda t$, $b = \mu/\lambda$ [Halperin (1964)]. For the minimal cut parallel structures (which may be considered "1 out of 2" systems) the reliabilities at time t $\bar{F}_1(t) = \bar{F}_2(t) = \bar{F}_3(t)$ are given by (3.5.1) with $2k_1 = (b+3) - (b^2+6b+1)^{\frac{1}{2}}$, $2k_2 = (b+3) + (b^2+6b+1)^{\frac{1}{2}}$ [Epstein-Hosford (1960), Halperin (1964)]. It is assumed in deriving (3.5.1) that all the components are functioning at time 0.

From (3.5.1) $\bar{F}(t)$ and $\bar{F}_1(t)$, $\bar{F}_2(t)$, $\bar{F}_3(t)$ depend only on $a = t/\lambda^{-1}$, i.e. the ratio of mission duration to the mean time λ^{-1} to component failure, and $b = \lambda^{-1}/\mu^{-1}$, i.e. the ratio of the mean time λ^{-1} to component failure to the mean time μ^{-1} to component revival. In Table 1 we compare $\bar{F}$ with $\bar{F}_1 \cdot \bar{F}_2 \cdot \bar{F}_3$ over a range of values for a,b that are of likely practical interest, i.e. $t > \lambda^{-1}$ and $\lambda^{-1} >> \mu^{-1}$.

The case of a "2 out of 3" system, when the components have lives distributed according to the trivariate exponential distribution of Marshall-Okin (1966), furnishes an example in which the minimal cut lower bound is a poor approximation to system reliability--because there is a positive probability that two or more components will fail simultaneously.

b \ a	1	10	100	1000	10,000
2 a	.44 .36	.089 .074	.0536 .0520	.05016 .05001	.049824 .049809
5 a	.565 .510	.336 .323	.3048 .3034	.30156 .30141	.301230 .301216
10 a	.682 .650	.565 .559	.5505 .5498	.548976 .548910	.5488281 .5488215
50 a	.8982 .8948	.88799 .88758	.88703 .88698	.886931 .886927	.8869215 .8869211
10^2 a	.9449 .9439	.94205 .94194	.941793 .941782	.9417673 .9417662	.9417648 .9417647
10^3 a	.994053 .994041	.994021 .994020	.9940183 .9940181	.9940180 .9940180	.9940180 .9940180
10^4 a	.9940053 .9940041	.9994002 .9994002	.9994002 .9994002	.9994002 .9994002	.9994002 .9994002

Table 1

Comparison of system reliability and the minimal cut lower bound for the "2 out of 3" system.

The components perform independently with exponential times to failure and exponential times to revival.

a = duration of mission/mean time to failure

b = mean time [illegible] failure/mean time to revival

The bound is tabled [illegible] w the corresponding system reliability.

4. Remarks on some conditions for bivariate dependence

As we observed in the introduction, for two finitely discrete random variables S,T, each of the conditions in the list

(4.1) $\mathrm{Cov}[S,T] \geq 0$

(4.2) $\mathrm{Cov}[f(S),g(T)] \geq 0$, all non-decreasing f,g

(4.3) S,T associated

implies its predecessor. We can add to the list the condition

(4.4) T is stochastically increasing in S,

i.e. $P[T > t|S=s_j]$ is, for all t, non-decreasing in j, where $s_1 < s_2 < \cdots < s_k$ are the possible values of S. It is not difficult to show that (4.4) implies (4.3), by an adaptation of the proof of P_2.

We have seen in (2.2.2) that for two binary random variables X,Y, (4.1) implies (4.3). The construction used in the proof of Theorem 3.1 can be readily modified to show that for binary X,Y, (4.1) implies (4.4). Thus in the bivariate, binary case conditions (4.1) through (4.4) are equivalent.

For general finitely discrete random variables S,T none of the conditions are equivalent. It is easy to find S,T satisfying (4.1) but not (4.2). If S,T each take the values $a_1 < a_2 < a_3$ and have the joint probability distribution,

$T=a_3$	8/64	0	15/64
$T=a_2$	0	18/64	0
$T=a_1$	15/64	0	8/64
	$S=a_1$	$S=a_2$	$S=a_3$

then S,T satisfy (4.2) but not (4.3). If S,T take the same values with the joint distribution,

$T=a_3$	1/8	0	1/4
$T=a_2$	0	1/4	0
$T=a_1$	1/4	0	1/8
	$S=a_1$	$S=a_2$	$S=a_3$

then S,T satisfy (4.3) but not (4.4).

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