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ON THE DETERMINATION OF A SAFE LIFE FOR CLASSES OF DISTRIBUTIONS CLASSIFED BY FAILURE RATE

by

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ABSTRACT

Using a lower tolerance bound as the derated life provides a guaranteed service period with the confidence level as a measure of safety in situations where mass production is contemplated. However, when only a limited number of items are to be produced, the probability of no failures among the fleet of specified size provides a better measure of assurance. Assuming that the life distribution is one of a specified set of subclasses of those distributions which have increasing failure rates we find those derating functions which can be used to provide lower tolerance bounds of given confidence or a safe service life with specified fleet assurance.

A method of finding such derating functions is exhibited and the calculation of a lower bound for the probability of no failure in the fleet is carried out when such derating functions are used. The confidence in the tolerance bound and the assurance of no fleet failure are compared when using bounds obtained from these derating functions.

1. INTRODUCTION

Suppose we have a structural component in an airplane which is subject to failure from fatigue, perhaps due to the cyclic loading of the ground-air-ground cycle, to acoustic loading or to some other well-defined and reconstructible phenomenon. We ask what estimate can we make of a safe service period, or more specifically, the lower quantiles of the distribution of the time until failure, from the testing of a small number of specimens under simulated conditions within the laboratory.

The first question with which we concern ourselves is, what is the probability that the weakest component within a fleet of airplanes will fail before the time established by derating the life estimated from the simulated testing. The problem is to determine the derating procedure, by which we mean a derating function depending upon the test sample, so that there is little probability of failure within the fleet.

We let $X_{i,m}$ be the ith ordered observation of m independent observations resulting from the simulated testing within the laboratory and we let $Y_{i,n}$ be the ith ordered observation of the n components in service within the fleet. We assume that the lives simulated in the laboratory have the same distribution is the actual service lives. This means that the ordered observations obtained are from independent and identically distributed random variables with the same distribution, which we call F.

From the laboratory testing we have the data $\underset{\sim}{X} = (X_{1,m},\ldots,X_{m,m})$ which, together with a specified derating function d, we can use to obtain the derated (or guaranteed) life $d(\underset{\sim}{X})$. The probability that no more than $k-1$

out of n failures occur in the fleet before the derated life, $k = 1,\ldots,n$, we call the fleet assurance, labeled $\alpha(k)$ and it is given by

$$\alpha(k) = P[Y_{k,n} > d(\underset{\sim}{X})].$$

Since

$$P[Y_{k,n} > y] = \sum_{j=0}^{k-1} \binom{n}{j} F^j(y)[1-F(y)]^{n-j}$$

$$= k\binom{n}{k} \int_{Q(y)}^{\infty} (1-e^{-z})^{k-1} e^{-(n-k+1)z} dz$$

where $Q = -\ln(1-F)$,

we have, letting G be the joint distribution of the sample $\underset{\sim}{X}$

$$\alpha(k) = k\binom{n}{k} \int_{\underset{\sim}{x}} \int_{Qd(\underset{\sim}{x})}^{\infty} (1-e^{-z})^{k-1} e^{-(n-k+1)z} dz \, dG(\underset{\sim}{x}).$$

For the applications that we consider here, we shall only be interested in the case $k = 1$. So that we omit k and write

$$\alpha = \int_{\underset{\sim}{x}} e^{-nQd(\underset{\sim}{x})} dG(\underset{\sim}{x}). \tag{1.1}$$

The event, no failure in the entire fleet before the derated life, is the only event which seems to provide assurance in the ordinary sense to customers.

A usual method of quality assessment, applicable to the diameter of machined parts, etc., is that of a *tolerance bound*. We include this for purposes of comparison even though it is not of primary interest to us.

The probability that a proportion of at least β of all future observations will not fail before the derated life we call the *confidence*, in order to be distinguished from the other measure of safety. We write this as

$$\gamma = P[1-F(d\underset{\sim}{X}) > \beta]. \tag{1.2}$$

where we here establish the convention that juxtaposition of functions indicates composition.

In this interpretation the derated life is a lower tolerance bound for the population described by the distribution F, and the confidence level γ of the lower tolerance bound $d(\underset{\sim}{X})$ in our designation does not depend upon the fleet size.

In order to introduce the fleet size, one might ask that the proportion of all future fleets of size n which have their first failure occurring after the derated life $d(\underset{\sim}{X})$ be at least β. But this we see becomes merely

$$P[[1-F(d\underset{\sim}{X})]^n > \beta]$$

which is (1.2) with β replaced by $\beta^{1/n}$.

Our problem is to choose the function d so that we can be sure the measures of safety which we have chosen, namely α and γ, are sufficiently near one, keeping in mind the economic desirability of using as few of the first ordered observations as possible.

2. THE MINIMUM OBSERVATION

For a nonnegative random variable X with density f and distribution F, the *hazard rate* (failure rate) q is defined by

$$q(t) = \frac{f(t)}{1-F(t)} \quad \text{for} \quad F(t) < 1.$$

Hence

$$Q(t) = \int_0^t q(x)dx = -\ln[1-F(t)]$$

is called the *cumulative hazard*, or more simply, the *hazard*.

Some nonparametric classes which have been studied previously are hazard rate increasing: labeled

$$\text{IHR} = \{q\colon q(t) \text{ is increasing on } 0 < t < \infty\}$$

and hazard rate increasing on the average: labeled

$$\text{IHRA} = \{q\colon \frac{Q(t)}{t} \text{ is increasing on } 0 < t < \infty\},$$

where we use increasing in the weak sense i.e. $q(t_1) \leq q(t_2)$ for $t_1 \leq t_2$. In this connection, see [2], [3], and the bibliography given there. These two classes are thought to represent the appropriate classes of life lengths for which wearout should be considered.

It has been pointed out, see [6], that the log-normal distribution has a hazard rate which does not increase for all $t > 0$. As a matter of fact, a simple calculation shows that it is not in the IHRA class either. In many people's opinion this is sufficient reason to discredit it as a representation for life length. However, it is still used in fatigue studies [7].

Another distribution which has recently received much attention and has been applied to fatigue problems is the Weibull distribution: this has a hazard of the form

$$(2.1) \qquad Q(t) = \left(\frac{t-a}{b-a}\right)^{\kappa+1} \qquad t > a$$

$$= 0 \qquad t \leq a$$

where a is the certain life (time before which no failure can occur) and b is a measure of the central tendency, and κ is the shape parameter. See [6], [7].

It has been said that the range of κ which is applicable to fatigue lives is $2 \leq \kappa \leq 5$ (see [5]).

To indicate the direction in which our investigation will proceed, we set out for later comparison the special case in which the derating function is proportional to the minimum observation,

$$(2.2) \qquad d(\underset{\sim}{X}) = p\, X_{1,m} \qquad 0 < p < 1.$$

Hence from (1.1) we have

$$(2.3.1) \quad \alpha = P[Y_{1,n} > pX_{1,m}] = m \int_0^\infty \exp\{-nQ(px)-mQ(x)\}q(x)dx$$

and from (1.2)

$$(2.3.2) \qquad \gamma = P\left[e^{-Q(pX_{1,m})} > \beta\right].$$

Notice that if we make the IHRA assumption, namely,

$\frac{Q(t)}{t}$ is increasing, then $Q(pt) \leq pQ(t)$ and we obtain an immediate lower bound for γ. However, we can obtain another bound by strengthening our assumption. If $\frac{q(t)}{t}$ is increasing, then $q(pt) \leq pq(t)$ and by integrating we obtain

$$Q(pt) \leq p^2 Q(t).$$

While if $q'(0) = 0$ and, say,

$\frac{q(t)}{t}$ is convex increasing, then $q(pt) \leq p^2 q(t)$ from which follows

$$Q(pt) \leq p^3 Q(t).$$

(Note that we use only the fact that $q(t)/t^2$ is increasing.)

Let us introduce the notation: for $\kappa \geq 0$

$q \in S_\kappa$ iff $\frac{q(x)}{x^\kappa}$ is increasing for $0 < x < \infty$

$q \in C_\kappa$ iff $\frac{q(x)}{x^\kappa}$ is convex increasing and $\lim_{x \downarrow 0} \frac{q(x)}{x^\kappa} = 0$.

Then we have almost immediately

<u>THEOREM 1</u>. If $q \in C_{\kappa-2}$ or $q \in S_{\kappa-1}$ for some $\kappa \geq 2$, then using the derating function of (2.2) we have

$$\alpha \geq \frac{1}{1 + \frac{n}{m} p^\kappa}$$

$$\gamma \geq 1 - \beta^{(m/p^\kappa)}$$

and there exist failure rates in $C_{\kappa-2} \cup S_{\kappa-1}$ for which these bounds are attained.

<u>Proof</u>. Since for $q \in S_{\kappa-1} \cup C_{\kappa-2}$ it follows that

$$Q(px) \le p^{\kappa} Q(x)$$

by simply substituting into (2.3.1) and (2.3.2) and integrating we obtain the bounds given. Equality is obtained when $Q(x) = ax^{\kappa}$ for some $a > 0$.||

3. THE GENERAL DERATING FUNCTION

Let q be the unknown failure rate of the distribution from which we obtain our sample and let r be a known failure rate. Sometimes it is reasonable to assume that q has at least as strong a behavior as r in that the ratio q/r has some specified behavior. Because the hazard rate has the intuitive interpretation of the force of mortality, it seems to carry a certain feeling of understanding. The classification of random variables by the behavior of the ratio of their failure rate with that of a known function is the point of view that we adopt.

For any positive functions f and r defined on $(0,\infty)$ we say

(3.1) $f \in S_r$ iff (f/r) is increasing

(3.2) $f \in C_r$ iff (f/r) is convex increasing and $\lim_{0+} f/r = 0$.

In the special case where $r(x) = x^{\kappa}$ for some $\kappa \ge 0$, we shall write S_{κ} and C_{κ} respectively so as to agree with our previous notation.

Note that by definition

$$Q \in C_0 \quad \text{iff} \quad q \quad \text{is IHR}$$

and also by definition

$$Q \in S_1 \quad \text{iff} \quad q \quad \text{is IHRA.}$$

Thus our comments show that the classifications which we introduce are particular subclasses of the IHR and IHRA distributions. We shall show this relationship more fully in the following remarks.

First note that if $f \in S_1$, then f is said to be *star shaped*, see [4]. Now this property is intermediate to increasing and convex increasing: that is, $S_0 \supset S_1 \supset C_0$. One might think that this would allow us to introduce another classification, namely

$$f \in I_r \quad \text{iff} \quad f/r \in S_1,$$

but, in fact, letting e denote the identity function, we see

$$I_r = S_{e \cdot r}, \tag{3.3}$$

since

$$\frac{f}{r} \in S_1 \quad \text{iff} \quad \frac{f}{e \cdot r} \text{ is increasing iff } f \in S_{e \cdot r}.$$

We will now derive some relationships between these classes for the case we consider to be of prime importance, namely, when r is a known failure rate such that $r(0) \geq 0$, $r(x) > 0$ for $x > 0$ and $\int_0^\infty r(t)dt = \infty$. Now define

$$(3.4) \qquad R(x) = \int_0^x r(t)dt, \qquad R^{\#}(x) = \int_0^x t \cdot r(t)dt.$$

We now have

<u>Remark 1</u>. $\quad Q \in S_R$ iff $QR^{-1} \in S_1$.

<u>Proof</u>. Since $y = R^{-1}(x)$ is an increasing transformation,

$\frac{QR^{-1}(x)}{x}$ is increasing in x iff $\frac{Q(y)}{R(y)}$ is increasing in y.||

<u>Remark 2</u>. $\quad q \in S_r$ iff $QR^{-1} \in C_0$.

<u>Proof</u>. Let $\Phi = QR^{-1}$. Now consider the right-hand derivative $\Phi' = q(R^{-1})/r(R^{-1})$; again set $y = R^{-1}(x)$ and $\frac{q(y)}{r(y)}$ is increasing in y iff $\Phi'(x)$ is increasing in x iff Φ is convex.||

This leaves the reader with the obvious question: what about the relationship between $q \in C_r$ and $Q \in C_R$? We can only say, since $C_r \subset I_r = S_{e \cdot r}$ that Remarks 1, 2 provide necessary conditions.

We now state a result due to A. W. Marshall.

<u>Lemma 1</u>. For a signed measure μ a necessary and sufficient condition that $\int_0^\infty f d\mu \geq 0$ for all f in one of the given classes C_r, or S_r is, respectively,

$$(3.4.0.1) \qquad f \in C_r \ : \ \text{for all } t > 0 \quad \int_t^\infty (x-t) r(x) d\mu(x) \geq 0$$

$$(3.4.0.2) \qquad f \in S_r \ : \ \text{for all } t > 0 \quad \int_t^\infty r d\mu \geq 0.$$

<u>Proof</u>. We sketch the ideas only. We first introduce the notation $\{\Omega(x,t)\}$ for the indicator function of the relation Ω, being one if true and zero otherwise. Consider S_0; the function of $x \geq 0$ defined by $\{x \geq t\}$ is in S_0 for any $t > 0$. Hence we must have $\int_0^\infty \{x \geq t\} d\mu(x) = \int_t^\infty d\mu \geq 0$. So the condition is necessary. On the other hand, any $f \in S_0$ can be approximated by an increasing sequence of linear combinations of such step functions. By using the monotone convergence theorem, the condition is sufficient, also. To obtain the proof for S_r we multiply and divide under the integral by r and we apply the result just shown to $f/r \in S_0$, changing the measure appropriately. In the same way, linear combinations of functions of $x > 0$ of the form $(x-t)\ \{x \geq t\}$ for any $t > 0$ are dense in C_0, and can be used to obtain the results quoted.||

We wish to discover a derating function d such that

$$Q[d(\underset{\sim}{x})] \leq \omega[Q(x_1),\ldots,Q(x_m)].$$

For reasons of simplicity we restrict our attention to functions ω which are linear combinations of its arguments. Hence for some real a_i we have $\omega(y_1,\ldots,y_m) = \sum_1^m a_i y_i$.

We comment that the next theorem and its corollaries have much in common with some of the results stated in [1] which are to be proved in a forthcoming report by the same authors. We give the proofs since they follow easily from the lemma of Marshall.

<u>THEOREM 2</u>. Let $0 < x_1 < \cdots < x_m < \infty$ be arbitrary. Let $a_j = A_j - A_{j+1}$ for $j=1,\ldots,m-1$, $a_m = A_m$ where

$$0 \leq A_j \leq 1 \quad \text{for} \quad j=1,\ldots,m. \tag{3.4.1}$$

Then among all nonnegative functions d such that

$$Q[d(\underset{\sim}{x})] \leq \sum_{i=1}^m a_i Q(x_i) \quad \text{for all} \quad q \in S_r, \tag{3.4.2}$$

the largest is

$$d(\underset{\sim}{x}) = R^{-1}\left[\sum_{i=1}^m a_i R(x_i)\right]. \tag{3.4.3}$$

A necessary and sufficient condition on the a_i that d of the form (3.4.3) satisfy (3.4.2) is that (3.4.1) hold.

<u>Proof</u>. We use the result of Marshall. Let $\mu = \nu - \sigma$ be the difference of two positive measures. We choose ν so that $\int_0^\infty q d\nu = \Sigma a_i Q(x_i)$, by taking ν_i as the unique measure such that the ν_i measure of the interval $(0,x)$ is $x\{x < x_i\}$ and set $\nu = \Sigma a_i \nu_i$. (We make the convention that a summation without limits is over the range 1 to m.) We also choose σ so that $\int_0^\infty q d\sigma = Q[d(\underset{\sim}{x})]$ by letting the σ measure of the interval $(0,x)$ be $x\{x < d(\underset{\sim}{x})\}$. Thus from Lemma 1 we have that (3.4.2) holds iff $h(t) \geq 0$ for all $t > 0$ where

$$h(t) = \Sigma a_i\{x_i > t\}[R(x_i)-R(t)] - \{t < d(\underset{\sim}{x})\}[R(d\underset{\sim}{x})-R(t)].$$

It is clear that we must have

$$0 \leq d(\underset{\sim}{x}) \leq x_m, \tag{3.5.1}$$

otherwise we would have $h(t) < 0$ for $x_m < t < d(\underset{\sim}{x})$. Now we examine the right-hand derivative

$$h'(t) = -r(t)d\mu(t) = r(t)[\{t < d(\underset{\sim}{x})\} - \Sigma a_i\{x_i > t\}].$$

$$h'(t) = -r(t)\Sigma a_i\{x_i > t\}, \qquad d(\underset{\sim}{x}) \leq t \leq x_m$$

$$h'(t) = r(t)[1 - \Sigma a_i\{x_i > t\}] \qquad 0 \leq t \leq d(\underset{\sim}{x}).$$

Clearly, since $A_j = \sum_{i=j}^{m} a_i$, $0 \leq A_j \leq 1$ for $j=1,\ldots,m$ implies for $t > 0$ that

(3.5.2) $h(t)$ is increasing for $t < d(\underset{\sim}{x})$, $h(t)$ is decreasing for $t > d(\underset{\sim}{x})$.

Since $h(t) = 0$ for $t > x_m$ we must only check that

$$h(0) = \Sigma a_i R(x_i) - R[d\underset{\sim}{x}] \geq 0.$$

Since R^{-1} exists and is order preserving, it follows that

$$d(\underset{\sim}{x}) \leq R^{-1}[\Sigma a_i R(x_i)]$$

and the largest value of d is obtained at equality.

Now if (3.4.3) holds we claim (3.4.1) is a necessary and sufficient condition for (3.4.2). That it is sufficient we have shown above. To show necessity we realize by Lemma 1 that (3.5.1) must follow and hence that

$$0 \leq \Sigma a_i R(x_i) \leq R(x_m)$$

must hold for arbitrary $0 < x_1 < \cdots < x_m < \infty$. But by proper choice of the x_i we see it is necessary that

$$0 \leq \sum_{i=j}^{m} a_i \leq 1 \text{ must hold for } j=1,\ldots,m. \|$$

We now have the

<u>Corollary 1</u>. If in the hypothesis of Theorem 2 we replace S_r by either I_r or C_r we merely replace the R in the conclusion (3.4.2) by $R^{\#}$.

<u>Proof.</u> Choose μ, σ, ν exactly as before. First consider $q \in I_r$. We must find the largest d such that for all $t \geq 0$

$$h_1(t) = \int_t^\infty xr(x)d\mu(x) \geq 0.$$

Now clearly the argument goes through exactly as before except that we replace $r(t)$ by $t \cdot r(t)$ and hence R by $R^{\#}$.

Now consider $q \in C_r$. We now must find the largest d such that the function

$$h_2(t) = \int_t^\infty (x-t)r(x)d\mu(x) \geq 0 \quad \text{for all } t > 0.$$

But we see that

$$h_2'(t) = -\int_t^\infty r(x)d\mu(x) = -h(t).$$

For any given $d(\underset{\sim}{x}) < x_m$, we have previously seen from (3.5.2) that $h(t)$ is unimodal with mode at $d(\underset{\sim}{x})$ and decreases to zero for $t > x_m$. Therefore, since $h(0) < 0$ is yet possible $h(t)$ can cross zero at most once. Thus h_2 must also be either unimodal or monotone. We note that also h_2 ultimately decreases to zero and so we must assure ourselves only that

$$h_2(0) = h_1(0) = \sum_1^m a_i R^{\#}(x_i) - R^{\#}[d(\underset{\sim}{x})] \geq 0.$$

So our best choice of d must be the one specified.||

<u>Corollary 2</u>. If in the hypothesis of Theorem 2 we replace $q \in S_r$ by $Q \in S_R$, then we replace (3.4.1) by the condition: for some $k=1,\ldots,m$

$$(3.5.4) \qquad 0 \leq A_1 \leq \cdots \leq A_k \leq 1, \quad A_{k+1} = \cdots = A_m = 0, \quad \text{and} \quad 0 < A_k.$$

<u>Proof</u>. Making the proper identification of the measures ν,σ as before, we obtain

$$h_3(t) = \int_t^\infty R d\mu \geq 0 \quad \text{for all} \quad t \geq 0,$$

as a necessary and sufficient condition that

$$Q(d\underset{\sim}{x}) \leq \Sigma a_i Q(x_i) \quad \text{for all} \quad Q \in S_R.$$

Recall $A_j = \sum_{i=j}^m a_i$ for $j=1,\ldots,m$, now fix $a_1,\ldots,a_m$ so as to satisfy (3.5.4), which is equivalent with

$$a_1 \leq 0,\ldots,a_{k-1} \leq 0 \qquad a_{k+1} = \cdots = a_m = 0$$

where

$$0 < a_k \leq 1 \quad \text{and} \quad 0 \leq \sum_1^k a_i \leq 1.$$

Integration shows

$$h_3(t) = \Sigma\{t < x_i\}a_i R(x_i) - \{t < d(x)\}R(d\underset{\sim}{x}).$$

We see that we must have $d(\underset{\sim}{x}) \leq x_k$, the index being the same as that above, otherwise for $x_k \leq t < d(\underset{\sim}{x})$ we would have $h_3(t) = -R(d\underset{\sim}{x}) < 0$. Clearly, for $t \geq x_k$, $h_3(t) = 0$, while for $d(\underset{\sim}{x}) \leq t < x_k$ $h_3(t) = \Sigma\{t < x_i\}a_i R(x_i) > 0$ which is an increasing function of t. But since $h_3(t)$ is also increasing for $0 \leq t < d(\underset{\sim}{x})$ we see that $h_3(t) \geq 0$ for all $t \geq 0$ iff $h_3(0) = \Sigma a_i R(x) - R(d\underset{\sim}{x}) \geq 0$ and the largest value of d that accomplishes this is (3.4.3).

For the second part of the proof we assume that

$$d(\underset{\sim}{x}) = R^{-1}[\Sigma a_i R(x_i)], \quad h_3(t) \geq 0 \quad \text{for all} \quad t \geq 0.$$

If we let $y_i = R(x_i)$, $s = R(t)$, the above is equivalent with

$$(3.5.5) \quad \Sigma\{s < y_i\}a_i y_i \geq \{s < \Sigma a_i y_i\}\Sigma a_i y_i \quad \text{for all} \quad s \geq 0.$$

We must show that if $(a_1,\ldots,a_m)$ is such that (3.5.5) holds for all $0 \leq y_1 \leq \cdots \leq y_m < \infty$, then for some $k=1,\ldots,m$ we have

$$0 \leq A_1 \leq \cdots \leq A_k \leq 1, \quad A_{k+1} = \cdots = A_m = 0, \quad 0 < A_k.$$

Since clearly we must have $0 \leq \Sigma a_i y_i \leq y_m$, by fixing y_m we see the linear form $\Sigma a_i y_i$ attains a maximum as a function of $y_1,\ldots,y_{m-1}$ on the simplex $0 \leq y_1 \leq \cdots \leq y_m$. Let $k \leq m$ be the least index such that

$$(3.5.6) \qquad 0 \leq \sum_{i=1}^{m} a_i y_i \leq y_k$$

for all $(y_1,\ldots,y_{m-1})$ in that region. It follows immediately letting all $y_i \to y_m$ in (3.5.6) and cancelling that $A_1 \geq 0$. In the same manner letting $y_k \to y_m$, $y_{k-1} \to 0$ we see $A_k \leq 1$. Again letting $y_j \to 0$ and $y_{j+1} \to y_m$ for $j=k+1,\ldots,m$ we obtain $A_{k+1} = \cdots = A_m = 0$.

Now we pick $0 < y_1 < \cdots < y_m$ such that

$$(3.5.7) \qquad y_{k-1} < \Sigma a_i y_i \leq y_k.$$

From (3.5.5) consider $s < \Sigma a_i y_i$ and take $y_1 < s < y_2$ and we obtain $-a_1 y_1 \geq 0$ and therefore $a_1 \leq 0$. But $\Sigma a_i y_i$ increases for y_1 decreasing to zero so the index k defined by (3.5.7) does not decrease if we take $y_1 = 0$. Again from (3.5.5), consider $s < \Sigma a_i y_i$ and take $y_2 < s < y_3$. We conclude as before $a_2 \leq 0$. Continuing in this fashion we obtain $a_3 \leq 0,\ldots,a_{k-1} \leq 0$, from which we conclude $A_1 \leq \cdots \leq A_k$. To see that $a_k > 0$ we suppose otherwise and show that (3.5.5) is violated for $s > \Sigma a_i y_i$.||

We now state the

<u>THEOREM 3</u>. If $Q(d\underset{\sim}{x}) \leq \sum_1^m a_i Q(x_i)$ for some real a_i, and all $\underset{\sim}{x}$ then we have α_0 and γ_0 as sharp bounds for the assurance α and the confidence γ,

$$(3.6.1) \qquad \alpha \geq \frac{1}{\prod_{j=1}^{m} (1+nb_j)} = \alpha_0 \qquad \gamma \geq P\left[\sum_1^m b_j Z_j < \ell n \frac{1}{\beta}\right] = \gamma_0$$

where $b_j = A_j/(m-j+1)$, $A_j = \sum_{i=j}^{m} a_i$ and Z_i are independent exponential random variables with mean 1.

<u>Proof</u>. By definition from (1.1)

$$\alpha = \int_{0<x_1<\cdots<x_m<\infty} \int m! \exp\left\{-nQ[d\underset{\sim}{x}] - \sum_1^m Q(x_i)\right\} \prod_1^m dQ(x_i).$$

By using the result (3.4.1) and setting $y_i = Q(x_i)$

$$\alpha \geq \alpha_0 = m! \int_{0<y_1<\cdots<y_m<\infty} \int \exp\left\{-\sum_1^m (1+na_i)y_i\right\} \Pi dy_i$$

$$\alpha_0 = \frac{m!}{(1+nA_m)} \int_{0<y_1<\cdots<y_{m-1}<\infty} \int \exp\left\{-\sum_{i=1}^{m-2} (1+na_i)y_i - (2+nA_{m-1})y_{m-1}\right\} \prod_1^{m-1} dy_i.$$

Repeating this integration we finally obtain

$$\alpha_o = \frac{m!}{\prod_{j=1}^{m}(m+1-j+nA_j)}$$

which is the result claimed for α_o. It follows from (1.2) that

$$\gamma = P[e^{-Q(d\underset{\sim}{x})} \geq \beta] \geq P\left[\sum_1^m a_i Y_{i,m} \leq \ln \frac{1}{\beta}\right] = \gamma_o$$

by letting $Y_{i,m} = Q(X_{i,m})$ be the ordered observation from the exponential distribution with mean 1. But since

$$\sum_1^m a_i Y_{i,m} = \sum_1^m b_j (m-j+1)(Y_{j,m} - Y_{j-1,m})$$

where $Y_{0,m} = 0$ and

$$(m-j+1)(Y_{j,m} - Y_{j-1,m}) = Z_j \qquad j = 1,\ldots,m$$

are independent exponentially distributed with mean 1 we have the result.||

We now derive a formula which is useful in computing the exact confidence bound for small values of m. By definition of the distribution of the Z_i we have

$$P[b_1 Z_1 \leq t] = 1 - e^{-t/b_1} \qquad t > 0 \tag{3.6.2}$$

and by straightforward integration we find

$$(3.6.3) \qquad P[b_1Z_1+b_2Z_2 \le t] = 1 + \frac{b_2}{b_1-b_2} e^{-t/b_2} - \frac{b_1}{b_1-b_2} e^{-t/b_1}.$$

Define

$$B_1^{(2)} = \frac{b_1}{b_1-b_2}, \qquad B_2^{(2)} = \frac{-b_2}{b_1-b_2}.$$

<u>Lemma 2</u>. If $Z_1,\ldots,Z_k$ are independent exponential random variables with unit mean then for $b_i > 0$

$$P\left[\sum_1^k b_iZ_i \le t\right] = 1 - \sum_{j=1}^k B_j^{(k)} e^{-t/b_j}$$

where we have the recursion relations holding:

$$B_1^1 = 1 \quad \text{and for} \quad k \ge 1$$

$$(3.6.4) \qquad B_j^{(k)} = B_j^{(k-1)} b_j/(b_j-b_k) \qquad j=1,\ldots,k-1$$

$$B_k^{(k)} = 1 - \sum_{j=1}^{k-1} B_j^{(k)}.$$

<u>Proof</u>. We shall give an induction on k. Clearly the result is true for $k = 1,2$. Assume the assertion true for $(k-1)$. Now

$$P\left[\sum_1^k b_i Z_i \leq t\right] = \int_0^{t/b_k} e^{-z_k - \sum_{j=1}^{k-1} B_j^{(k-1)} \exp\left[-\frac{(t-b_k z_k)}{b_j} - z_k\right]} dz_k$$

$$= 1 - e^{-t/b_k}\left[1 + \sum_{j=1}^{k-1} B_j^{(k-1)} \int_0^{t/b_k} e^{-z_k\left(1-\frac{b_k}{b_j}\right)} dz_k\right]$$

$$= 1 - e^{-t/b_k} - \sum_{j=1}^{k-1} B_j^{(k-1)} \frac{b_j}{b_j - b_k}\left[e^{-t/b_j} - e^{-t/b_k}\right]$$

$$= 1 - \sum_{j=1}^{k-1}\left(\frac{B_j^{(k-1)} b_j}{b_j - b_k}\right) e^{-t/b_j} - \left(1 - \sum_{j=1}^{k-1} \frac{B_j^{(k-1)} b_j}{b_j - b_k}\right) e^{-t/b_k}$$

which completes the proof.||

We also set out the special case as

<u>Corollary 3</u>. Let $q \in S_r$ then fix $k=1,\ldots,m$ and $0 < p < 1$, now define

$$a_i = p/m \qquad i = 1,\ldots,k-1$$

$$a_k = (p/m)(m-k+1)$$

$$a_i = 0 \qquad i = k+1,\ldots,m$$

and we have

$$\alpha_o = \frac{1}{(1 + \frac{n}{m} p)^k}$$

$$\gamma_o = H_{2k}(\frac{-2m\ln\beta}{p})$$

where H_{2k} is the Chi-square distribution with $2k$ degrees of freedom. See Reference [1] in this connection.

<u>Proof.</u> By the preceding result we see that

$$A_j = \sum_{i=j}^{m} a_i = \begin{cases} (p/m)(m-j+1) & j=1,\ldots,k \\ 0 & j=k+1,\ldots,m \end{cases}$$

hence A_j satisfy (3.4.3). Now we see that

$$b_j = p/m \qquad j=1,\ldots,k$$

$$= 0 \qquad j=k+1,\ldots,m$$

and by direct substitution into the results of Theorem 3 we have the result as claimed.||

We remark that the results of Theorem 1 are a specialization of Theorem 3, with a slight change of notation. Also, we can use the normal approximation to $\sum_1^k b_i Z_i$ for k of moderate size if the b_i are not too different in value.

4. SOME NUMERICAL COMPARISONS

In order to appreciate the effect of the various assumptions concerning the failure rate on the derating function we shall examine several simple cases.

Case I $\qquad Q \in S_1$

We take $0 < p < 1$ and set

$$a_1 = -p, \quad a_2 = p \quad a_j = 0 \quad \text{for } j = 3,\ldots,m$$

then

$$A_2 = p \qquad A_j = 0 \quad \text{for } j \neq 2.$$

$$d(\underset{\sim}{x}) = p(x_2 - x_1).$$

Clearly these A_j satisfy (3.5.4) and we have by (3.6.1) and (3.6.3), letting $t = -\ln\beta$,

$$\frac{1}{1 + \frac{n}{m-1}p} = \alpha_o \qquad 1 - e^{-(m-1)t/p} = \gamma_o.$$

Thus if we prescribe α_o near unity, then we must choose

$$p_{\alpha_o} = \frac{m-1}{n}\left(\frac{1}{\alpha_o} - 1\right). \tag{4.1}$$

Thus for example, for $m = 10$, $n = 200$ and $\alpha_0 = .99$, we have from (4.1)

$$p = \frac{1}{2200}.$$

We see that a high assurance for a moderate fleet size must have an extremely small multiplication factor.

If we prescribe β and γ_0, then determine p we have

$$p_{\gamma_0} = \frac{(m-1)\ln\beta}{\ln(1-\gamma_0)}. \tag{4.2}$$

If we set $\beta = .99 = \gamma_0$ for $m = 10$, then from (4.2) we obtain

$$p = \frac{9(.01)}{n\ 100} = \frac{1}{51.1}\ .$$

Thus the assurance requires a factor which is of the order $\frac{1}{n}$ smaller than the confidence does.

Case II $\qquad q \in S_{\kappa-1}$

take $0 < p < (\frac{1}{2})^{\kappa}$ and set

$$a_1 = a_2 = p^{\kappa} \qquad a_2 = 0 \qquad i = 3,\ldots,m.$$

Then

$$A_1 = 2p^{\kappa},\quad A_2 = p^{\kappa},\quad A_j = 0 \qquad \text{for } j = 3,\ldots,m.$$

These A_j satisfy (3.4.3) and

$$d(\underset{\sim}{x}) = p(x_1^{\kappa} + x_2^{\kappa})^{\frac{1}{\kappa}}.$$

Now, by (3.6.1)

$$\alpha_0 = \frac{1}{(1 + \frac{n}{m}\,2p^{\kappa})(1 + \frac{n}{m-1}\,p^{\kappa})} \tag{4.3}$$

and using (3.6.3) with $t = -\ln\beta$, $b_1 = \frac{2p^{\kappa}}{m}$, $b_2 = \frac{p^{\kappa}}{m-1}$,

$$\gamma_0 = 1 + \frac{m}{m-2}\exp\left|-\frac{(m-1)t}{p^{\kappa}}\right| - \frac{2(m-1)}{m-2}\exp\left|-\frac{mt}{2p^{\kappa}}\right|. \tag{4.4}$$

If we prescribe α_0 near 1 and set $\rho = \frac{1}{\alpha_0} - 1$ then solving (4.3) for p^κ yields

$$p^\kappa = \frac{-3m+2 + \sqrt{(9+8\rho)m^2-(12+8\rho) + 4}}{4n} . \tag{4.5}$$

If we expand the right-hand side of (4.5) in a Maclaurin's series in ρ and retain only the first two terms, we have the approximation

$$p_{\alpha_0} = \left[\frac{m(m-1)\rho}{n(3m-2)}\right]^{\frac{1}{\kappa}} . \tag{4.6}$$

For $n = 200$, $m = 10$, $\alpha_0 = .99$ we have from (4.6)

$$p = \left[\frac{1}{560(11)}\right]^{\frac{1}{\kappa}} = \begin{cases} \frac{1}{8.8} & \text{if } \kappa = 4 \\ \frac{1}{4.3} & \text{if } \kappa = 6 \\ \frac{1}{3} & \text{if } \kappa = 8. \end{cases}$$

The reciprocal of p is sometimes called the *scatter factor* and is usually taken between 2 and 4. Here we can see how strong an assumption is necessary to justify scatter factors of such magnitude.

If we prescribe γ_0 near 1, then from (4.4) we have

$$\ln\left[\frac{(m-2)(1-\gamma_0)}{2(m-1)}\right] = -\frac{mt}{2p^\kappa} + \ln\left[1 - \frac{m}{2(m-1)}\exp\left\{-\frac{(3m-2)t}{2p^\kappa}\right\}\right]$$

and by neglecting the second term on the right-hand side of the equation we have the approximation

$$(4.7) \qquad p_{\gamma_0} = \left[\frac{m \ln \beta}{2 \ln \left[\frac{(m-2)(1-\gamma_0)}{2m-2} \right]} \right]^{\frac{1}{\kappa}} .$$

For $m = 10$, $\gamma_0 = .99$ and β near 1 we have

$$p = \left[\frac{1-\beta}{1.083} \right]^{\frac{1}{\kappa}} ,$$

β so that if we set $\beta = .99$,

$$p = \begin{cases} \frac{1}{3.2} & \text{if } \kappa = 4 \\ \frac{1}{2.2} & \text{if } \kappa = 6 \end{cases}$$

but if we set $\beta = .999$

$$p = \begin{cases} \frac{1}{5.7} & \text{if } \kappa = 4 \\ \frac{1}{3.2} & \text{if } \kappa = 6 . \end{cases}$$

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