THE RATIO OF THE EXCHANGE COEFFICIENTS
FOR HEAT AND MOMENTUM IN A HOMOGENEOUS,
THERMALLY STRATIFIED ATMOSPHERE

By

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ABSTRACT

A hypothesis concerning the ratio of the exchange coefficients for heat and momentum is formulated. From the basic assumptions of the Monin-Obukhov similarity theory and the theory of free convection, it is shown that the ratio of the coefficients is a two-part function of the nondimensional logarithmic wind shear and the gradient Richardson number. Experimental data tend to corroborate the theoretical values derived in this study within the limits of experimental error.
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INTRODUCTION

Contemporary hypotheses pertaining to the turbulent processes of the surface boundary layer are based upon the principles of dynamic similarity of flows. True dynamic similarity requires that the measured quantities of one system go identically or by an exact ratio into those of another, implying a constant ratio of forces in the two systems. For the surface boundary layer, this requires that the wind and temperature profiles be similar, as well as the vertical fluxes of heat, momentum, and water vapor.

The structure of turbulence in the boundary layer is dependent upon the thermal stratification. Richardson (1920, 1925) found that the generation or suppression of turbulence could be expressed as the ratio of work done by buoyancy forces against gravity forces where turbulence increased as stability decreased. The converse was also found to be true. Richardson's criterion, as originally derived, assumed that the exchange coefficients for heat, momentum, and water vapor were equal, allowing the Richardson number to be expressed in generalized form. Current thinking indicates that the exchange coefficients are not equal, so that the Richardson number may be represented by either its flux or gradient form. The two Richardson numbers are related by

\[ R_f = R_i \frac{K_H}{K_M} \]  

where \( R_f \) is the flux form, \( R_i \) is the gradient form and \( \frac{K_H}{K_M} \) is the ratio of the exchange coefficients for heat and momentum.

Batchelor (1953) demonstrated that the Richardson number possessed the characteristics of true dynamic similarity, which allows it to be used as a basic parameter for the investigation of turbulent processes in the surface boundary layer. This leads to the premise that the ratio of the exchange coefficients is also a similarity ratio of major importance owing to the nature of its relationship to the Richardson numbers. If dynamic similarity of flows indeed exists in the planetary boundary layer, it should be possible from the basic definitions to write an expression for the ratio of the exchange coefficients for heat and momentum.

The purposes of this paper then, are to: (1) examine the role that \( \frac{K_H}{K_M} \) assumes in the analysis of wind and temperature profiles observed in the planetary boundary layer, (2) examine the dependency of diabatic boundary layer hypotheses upon \( \frac{K_H}{K_M} \), and (3) to present an expression for computing the ratio in terms of similarity parameters. In addition, values of \( \frac{K_H}{K_M} \) as determined from experimental data are presented.

SIMILARITY CONCEPTS

The application of dynamic similarity concepts to surface boundary layer flow is usually attributed to Monin and Obukhov (1954), even though the earlier work of Sverdrup, Rossby and Montgomery, and Holtzman (see Deacon, 1955 and Lettau, 1949) was based upon this postulate of aerodynamics. The conjugate law for the wind profile in the surface boundary layer may be written as
where $\bar{V}$ is the wind speed, $z$ is height, $u_*$ is the friction velocity, $k$ is Karman's constant and $S$ is a stability correction factor known as the diabatic influence function. Monin and Obukhov (1954) determined that dynamic similarity of flows was controlled by the unique existence near the ground of the friction velocity $u_*$, a scaling temperature $T^*$ and a scaling length $L$. All the equivalent measured values could then be expressed as functions of these parameters. With the aid of the dimensional constants $g/\theta$ and $H/c_p\rho$, $L$ and $T^*$ were defined as

$$L = -\frac{u_*^3}{k(\sigma/\theta)H/c_p\rho} = -\frac{u_*^3 c_p\rho \theta}{kgH} \quad (3)$$

and

$$T^* = \frac{1}{ku_*} \frac{H}{c_p\rho} \quad (4)$$

where $g$ is the acceleration of gravity, $\theta$ is potential temperature, $H$ the vertical heat flux, $c_p$ the specific heat of air at constant pressure, and $\rho$ the mean density.

From the above definitions, an exact statement describing the shape of the wind and temperature profiles in a diabatic surface boundary layer may be written as

$$\frac{Z}{L} = SR_f \quad (5)$$

where $z/L$ is a scaling ratio, and the flux Richardson number is given by

$$R_f = -\frac{\frac{\partial T}{\partial z}}{c_p \rho \frac{\partial \bar{V}}{\partial z}} \quad (6)$$

The wind profile may then be determined by integration to be

$$\bar{V} = \frac{u_*}{k} \left[ \ln \frac{z}{z_0} + \alpha \left( \frac{z}{L} \right) \right] \quad (7)$$

where $z_0$ is a constant of integration known as the roughness length and $\alpha$ is a universal constant.
Since the expressions for both $L$ and $R_f$ contain the heat flux $H$, a parameter difficult to obtain by inference or direct measurement, Eq. (5) may be restated

$$\frac{z}{L'} = S H_i$$  \hspace{1cm} (8)

where $R_i$ is the gradient Richardson number defined as

$$R_i = \frac{\frac{\partial \theta}{\partial z}}{K \frac{\partial \theta}{\partial z}}$$  \hspace{1cm} (9)

and $L'$ is a gradient length (Panofsky, Blackadar, and McVehil, 1960) given by

$$L' = \frac{V}{\frac{\partial \theta}{\partial z}}$$  \hspace{1cm} (10)

and related to $L$ by $L' = L \frac{K_H}{K_M}$. The wind profile may now be stated as

$$V = \frac{u_0}{k} \left[ \ln \frac{z}{z_o} - \psi \left( \frac{z}{L'} \right) \right]$$  \hspace{1cm} (11)

where $\psi$ is a universal function. A similar expression may be written for the temperature profile.

Thus, from easily determined parameters which are functions of the vertical gradients of wind and temperature, the turbulent processes in the surface boundary layer may be investigated. Accurate evaluation of the vertical fluxes from profile measurements is entirely dependent upon the ratio of the exchange coefficients.

Another factor influencing the dynamic similarity of flows in a diabatic surface boundary layer, particularly as the atmosphere becomes less stable, is the effect of convection upon the turbulent processes. It is generally accepted that there exists a transition zone between the forced and free convection regimes at $-0.02 < R_i > -0.05$. In fully forced convection ($R_i > -0.02$), Priestley (1960) has shown that the potential temperature gradient can be stated as
Beyond the transition, the profile no longer follows the $z^{-1}$ law of forced convection, but, as shown by Priestley (1954), becomes proportional to height to the minus four-thirds. The potential temperature gradient in free convection then becomes

$$\frac{\partial \theta}{\partial z} = -h \left( \frac{H}{c_p \rho} \right)^{2/3} \left( \frac{\zeta}{\theta} \right)^{-1/3} \left( \frac{z}{h} \right)^{-4/3} \tag{13}$$

where $h$ is a constant whose value remains to be determined. In the evaluation of $h$, Priestley (1955) introduced a reduced or dimensionless heat flux $H$ in the form

$$H = \frac{H}{c_p \rho (\zeta/\theta)^{1/3} (\partial \theta/\partial z) |z^2|} \tag{14}$$

allowing the flux-gradient reciprocal dependence between free and forced convection to be given by

$$h = h \quad \text{(free)} \tag{15}$$

$$\hat{H} = \kappa \left| Ri \right|^{-1/4} \quad \text{(forced)} \tag{16}$$

where $\hat{H}$ is a constant with height in the free convection regime and a variable in the forced regime.

Eqs. (14) through (16) were evaluated using Swinbank's (1955) data in terms of $\hat{H}$ and $Ri$. An extension of this inquiry by Taylor (1956) resulted in establishing a value for $h$ such that

$$h = h = 0.79 \pm 0.04$$

According to Priestley (1960), measurements of the heat flux were some 10 percent low and a tentative adjusted value of $H=H = 0.9$ was adopted for the free convection regime.

Heat flux measurements by Priestley (1955) and Taylor (1956) and temperature profiles by Webb (1958) indicate the height to the minus four-thirds law for free convection is valid for $z/l < 0.03$ through a range up to about 30 times this value. Eqs. (12) and (13) then represent the behavior of the temperature structure in the surface boundary layer with the scale height of the
transition from forced to free convection proportional to the Monin-Obukhov scaling length \( L \).

The transition between forced and free convection can be considered smooth since moving turbulent elements can traverse the junction. With no smoothing, it can be shown from the diabatic similarity theory that a junction height is given by

\[
\frac{z}{L} = -k^2 h^{-2} \left( \frac{K_H}{K_M} \right).
\] (17)

Eq. (17) yields an arbitrary junction height of \( z/L = -0.0317 \) if it is assumed that \( k = 0.4, h = 0.9 \) and \( K_H/K_M \) is set equal to 1. The value of the gradient Richardson number is also found to be \(-0.0317\) at the unsmoothed junction.

THE RATIO OF THE EXCHANGE COEFFICIENTS FOR HEAT AND MOMENTUM

Now, from the related diabatic similarity and free convection theories, we can formulate an expression for the exchange coefficients for heat and momentum. The exchange coefficient hypothesis states that the mean flux per unit area of a conservative quantity such as heat, momentum, or water vapor is proportional to the gradient of the mean value of the quantity, that is

\[
\text{mean flux per unit area} = -K \frac{d\overline{E}}{dn}
\]

where \( K \) is the exchange coefficient, \( d\overline{E}/dn \) is the gradient of the mean, and \( n \) is the direction normal to the surface. For turbulent flow in the boundary layer, \( K \) is dependent upon time and location. A differentiation between the various coefficients is necessary to adequately describe turbulent processes in the boundary layer. These can be stated as

\[
E = -pK_w \frac{\partial \overline{\theta}}{\partial z}
\] (18)

\[
H = -p\frac{K_H}{p} \frac{\partial \overline{V}}{\partial z}
\] (19)

\[
Y = -pK_M \frac{\partial \overline{V}}{\partial z}
\] (20)

where \( K_w, K_H, \) and \( K_M \) are the coefficients for water vapor, heat, and momentum, respectively.

Considering only heat and momentum, the exchange coefficients can be shown to be
\[ K_H = \frac{\nu' \theta'}{\frac{\partial \theta}{\partial z}} \]  

(21)

and

\[ K_M = \frac{u'w'}{\frac{\partial \nu}{\partial z}} \]  

(22)

which can be verified by direct observation. Since independent observations of the terms necessary to provide a solution for Eqs. (19) through (22) are difficult, if not sometimes impossible, to obtain by direct measurement, an indirect determination of the ratio of \( K_H \) to \( K_M \) can be derived from the basic framework of the similarity and free convection theories.

From Eqs. (1) and (9) it is seen that

\[ R_f = \frac{K_H}{K_M} = \frac{K_H \frac{\partial \theta}{\partial z}^2}{K_M \left( \frac{\partial \nu}{\partial z} \right)^2} \]  

(23)

Substitution of Eq. (19) and (20) yields

\[ R_f = -\frac{Hg}{c_p \nu' \frac{\partial \theta}{\partial z}} \]  

(24)

By definition \( u^2 = \tau/\rho \), so that Eq. (19) may be restated as

\[ R_f = \frac{Hg}{\rho c u^2 \theta' \frac{\partial \nu}{\partial z}} \]  

(25)

Introducing Priestley's (1955) reduced heat flux \( \hat{H} \), it is seen that

\[ R_f = \frac{Hg \left( \frac{\rho}{\theta} \right)^{1/2} \left| \frac{\partial \theta}{\partial z} \right|^{3/2}}{\theta u^* \left( \frac{\partial \nu}{\partial z} \right)} \]  

(26)
Multiplying and dividing by $k^2 (\partial V/\partial z)^2$, introducing the diabatic influence function $S$, and rearranging terms, Eq. (26) becomes

$$R_f = - \frac{S R H (\partial V/\partial z)^2}{k \partial (\partial V/\partial z)^2} \frac{\partial \theta}{\partial z}^{3/2}.$$  

(27)

Simplifying

$$R_f = - \frac{|R_l|^k S^2 \theta}{k^2} R_l.$$  

Since by definition $R_f = R_l (K_H/K_M)$, then

$$\frac{K_H}{K_M} = \frac{H S^2}{k^2 |R_l|^k}.$$  

(28)

If the junction between forced and free convection is assumed to occur at $K_l = -0.0317$, then

(a) for forced convection where $|R_l| < 0.0317$ and $H = |R_l|^{-k^2}$

$$\frac{K_H}{K_M} = \frac{k^2 S^2 |R_l|^k}{|R_l|^k} = S^2.$$  

(29)

(b) for free convection where $|R_l| > 0.0317$ and $H = 0.9$

$$\frac{K_H}{K_M} = \frac{0.9 S^2}{k^2} |R_l|^k.$$  

(30)

It is seen that the ratio $K_H/K_M$ may be calculated from two of the basic parameters of similarity theory, the Richardson number and the diabatic influence function $S$.

NUMERICAL EVALUATION OF THE RATIO OF THE EXCHANGE COEFFICIENTS

The ratio of the exchange coefficients may be evaluated numerically by use of any of the contemporary hypotheses for the diabatic boundary layer. Since
none of the dozen or so hypotheses are universally accepted, the choice is rather arbitrary. Application of Eqs. (29) and (30) is dependent upon the definition of \( S \) assumed as a stability parameter for solution of the log-linear profile. By definition

\[
S = \frac{k_S}{u_s} \frac{\partial \overline{v}}{\partial z}
\]

(31)

the solution to which is available only with independent measurements of stress. The Monin-Obukhov (1954) hypothesis assumed that

\[
S = (1 + a \frac{z}{L})
\]

(32)

which is a first approximation for small \( z/L \) only and consequently is not valid in the free convection regime. Interpolation formulæ to span the transition zone between forced and free convection and fit at both small and large \( z/L \) have been formulated, those of Businger (1959, 1961), Priestley (1960), Webb (1960) and the KEYPS function of Panofsky (1963) being the most notable.

Depending upon the choice of an interpolation scheme, slightly different values of \( S \) are obtained for the same value of \( \text{Ri} \) or \( z/L' \), causing the stress determination as obtained from Eqs. (7) or (9) to differ. The same holds true for the determination of \( \frac{K_H}{K_M} \) from Eqs. (29) and (30). The differences among the expressions for the diabatic influence functions as proposed by various investigators are currently unresolved.

In this particular study, the KEYPS function was chosen to evaluate \( \frac{K_H}{K_M} \). Panofsky (1963) has shown that for a diabatic surface boundary layer, the diabatic influence function from the KEYPS expression is given by

\[
S^4 - \gamma \frac{z}{L} S^3 = 1
\]

(33)

where \( \gamma \) is a universal function. In terms of the gradient length \( L' \), this may be expressed as

\[
S^n - \gamma' \frac{z}{L'} S^3 = 1
\]

(34)

where \( \gamma' \) is another universal function. From similarity theory it can be shown that

\[
\frac{z}{L'} = \frac{\text{Ri}}{(1 - \gamma' \text{Ri})^2}
\]

(35)

and

\[
S = (1 - \gamma' \text{Ri})^{-\frac{3}{2}}
\]

(36)
allowing determination of the various profile parameters if the vertical gradients of wind speed and temperature are measured.

Numerical evaluation of $K_H/K_M$ was accomplished using high-speed digital computer techniques. Eqs. (29), (30), (35), and (36) were programmed to provide a solution in terms of $S$, $-R_i$, $-z/L'$ and the equivalent values of $K_H/K_M$ in the stability range $0 > Ri > -10.0$. $K_H/K_M$ as a function of $-R_i$ and $-z/L'$ is shown in Figures 1 and 2.

**DISCUSSION**

The solution of many problems of boundary layer flow depends upon the exact numerical value of the ratio of the exchange coefficients for heat and momentum. Early experimenters such as Richardson (1920, 1925) assumed that $K_H/K_M = 1$, an assumption not justified by current hypotheses. The evolving similarity theory of Monin-Obukhov utilized this assumption. Extension of the Monin-Obukhov hypothesis by the interpolation formulae tended to indicate that $K_H/K_M$ was closer to 1.3 in the free convection regime.

Experimental values of the ratio of the exchange coefficients have been published by a number of investigators, notably Rider (1954), Swinbank (1955), Deacon (1958), and Senderikhina (1961). Published values range from 1.08 to 1.67, although Lumley and Panofsky (1964) quote unpublished data by Priestley showing that values as large as 3.0 have been observed in the boundary layer. Brooks (1963) found values as high as 2.5 during extremely unstable periods over an irrigated field, but according to Dyer and Pruitt (1962) and Dyer (1963), the process in operation when Brook's data were collected was heterogeneous, probably accounting for the large $K_H/K_M$ ratio.

Lettau (1957) analyzed two-dimensional wind and temperature data obtained during the Great Plains Turbulence Field Program and determined the ratio of the exchange coefficients from the Bowen ratio and a similarity assumption referred to as the "Ground-flux Ratio." It was assumed that conditions valid for the neutral case held true for the forced and free convection regimes also. This led to biasing the data to the extent that computed values of $K_H/K_M$ in the range $R_i < -0.0317$ are abnormally high as shown in Figure 3.

All evidence, both empirical and experimental, indicate that the ratio of the exchange coefficients for heat and momentum varies between unity and 1.3 from the neutral case to the upper limit of the free convection regime which lies in the stability range $-0.1 > R_i > -1.0$ according to Webb (1958) and Townsend (1962). This is considered to be a second transition zone and separates the free and "windless" or natural convection regimes. The temperature profile no longer follows the minus one-third law of free convection but obeys a $z^{-2}$ law according to Lumley and Panofsky (1964). This is borne out by the failure of any of the diabatic models to predict stress and roughness lengths accurately at large negative values of $R_i$, since the interpolation formulae are tailored to free convection theory and the minus one-third potential temperature profile of that stability regime. The effect on predicted or measured values of $K_H/K_M$ is not known.
FIGURE 1. THE RELATION BETWEEN THE RATIO OF THE EXCHANGE COEFFICIENTS FOR HEAT AND MOMENTUM AND THE GRADIENT RICHARDSON NUMBER.
Figure 2. The relation between the ratio of the exchange coefficients for heat and momentum and the Keyps ratio.
CONCLUSIONS

The empirical solution for the ratio of the exchange coefficients for heat and momentum yields numerical values in good agreement with the published results derived from experimental data. It must be emphasized that the solution is valid only for a homogeneous and stationary diabatic boundary layer. Other assumptions used for the solution are that: (1) the basic similarity and free convection theories are valid; (2) that the transition from forced to free convection occurs in the range $-0.02 < R_i < -0.05$ and more particularly at $R_i = -0.0317$; and (3) that the interpolation formulae for the diabatic influence function span the transition from forced to free convection. Since none of the interpolation schemes are universally accepted and verification of the solution for $K_H/K_M$ by direct observation is extremely difficult, the empirical results of this study perhaps do not represent the actual ratio of the exchange coefficients.
REFERENCES


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A hypothesis concerning the ratio of the exchange coefficients for heat and momentum is formulated. From the basic assumptions of the Monin-Obukhov similarity theory and the theory of free convection, it is shown that the ratio of the coefficients is a two-part function of the nondimensional logarithmic wind shear and the gradient Richardson number. Experimental data tend to corroborate the theoretical values derived in this study within the limits of experimental error.
1. Boundary Layer.
2. Stability.
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