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MATHEMATICAL ANALYSIS OF THE LAMINATED ELASTOMERIC BEARING

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By

R. Clyde Herrick

May 1966

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

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THE FRANKLIN INSTITUTE
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The purpose of the effort reported herein was to develop a mathematical model to assist in the design of laminated elastomeric bearings. The approach of the study was based upon the application of large-deformation elasticity and viscoelastic mathematical theories. However, the complexity of the large-deformation elasticity theory and the problems associated with computer solutions of equations of linear or classical elasticity for the particular bearing geometry and for materials that are almost incompressible were such that this analytical work was discontinued.

Although the effort reported herein did not result in attainment of the program objectives, the effort did result in the development of analytical approximations to the application of linear elastic theory and yielded results that are readily applicable and very close to exact solutions for bearings for light loads.

Future efforts by this command toward the development of design formulae for laminated elastomeric bearings will be limited to an experimental investigation approach.

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USAAVLABS Technical Report 66-21
May 1966

MATHEMATICAL ANALYSIS OF ~~THE~~ THE LAMINATED ELASTOMERIC BEARING

Final Report F-B2140-1

by

R. Clyde Herrick

Prepared by

The Franklin Institute
Laboratories for Research and Development
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for

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FORT EUSTIS, VIRGINIA

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ABSTRACT

The purpose of this study was the establishment of analytical design procedures for laminated elastomeric bearings. This was approached with the application of the linear mathematical theory of elasticity and later with nonlinear large-deformation elasticity theory.

The linear theory yielded analytical approximations that are close to exact solutions and which are easily applied and evaluated. This analysis of one typical lamination yields the distribution of stress and deformation in the elastomer between "rigid" metal lamina. However, the limits of the linear elasticity theory are exceeded for greater than small bearing loads, indicating the need for the application of the more comprehensive large-deformation elasticity theory.

The large-deformation theory was stated and the equilibrium equations were derived, but the solution of these equations was not carried out.

PREFACE

This report was prepared by The Franklin Institute Laboratories, Philadelphia, Pennsylvania, for the U. S. Army Aviation Materiel Laboratories under Contract DA 44-177-AMC-110(T).

The mathematical studies of the laminated elastomeric bearing presented herein, which began in September 1963 and were concluded in December 1964, represent the sole effort of The Franklin Institute, for which Mr. R. Clyde Herrick, Senior Research Engineer, Applied Mechanics Laboratory, was the principal investigator and author of this report. Acknowledgment is made to Mr. T. Y. Chu for the considerable contribution of the analytical approximations and computer solutions for the linear theory of elasticity shown in Appendixes A, B, and C, and for the brief presentation of pure torsion in large deformation elasticity shown in Appendix D. Acknowledgment is also made to Dr. Barry Wolf for the many formulations in large deformation elasticity leading to that presented in Appendix E and for his contributions to the work in linear theory done in conjunction with Mr. Chu. The contributions of Dr. Kishor D. Doshi and Zenons Zudans are also noted.

This constitutes the final report covering the first application of the mathematical theory of elasticity to the problem of design and laminated elastomeric bearings.

CONTENTS

	Page
ABSTRACT	iii
PREFACE	v
LIST OF ILLUSTRATIONS	viii
LIST OF SYMBOLS	ix
SUMMARY	1
CONCLUSIONS	2
INTRODUCTION	3
REVIEW OF EFFORT	4
RESULTS OF LINEAR THEORY	7
Solutions for Incompressible Elastomers	7
Approximate Solution for Compressible Elastomers	9
BIBLIOGRAPHY	15
DISTRIBUTION	16
APPENDIXES	
A - Computer Solution for a Compressible Elastomer	17
B - Analytical Approximation and Computer Solution for an Incompressible Elastomer	30
C - Approximation for a Compressible Elastomer	53
D - Pure Torsion in Large-Deformation Elasticity of an Incompressible Elastomer	88
E - Large-Deformation Elasticity Theory for Axial Compression	93

ILLUSTRATIONS

Figure		Page
1	Average Pressure Versus Axial Compression (Incompressible Elasticity Theory)	11
2	Shear Strain Versus Axial Compression (Incompressible Elasticity Theory)	12
3	Average Pressure Versus Shear Strain (Incompressible Elasticity Theory)	13
4	Comparative Data for Compressible and Incompressible Theory	14
5	Sketch of Coordinates for One Lamination	17
6	Main Flow Chart	24
7	Subroutine Solut	25
8	Main Flow Chart	42
9	Solution Subroutine Flow Chart	43
10	Geometry and Key to Computer Output	49
11	Main Flow Chart	78
12	Subroutine Int.	79
13	Subroutine Coe.	80
14	Subroutine Stress	81
15	Sketch of Coordinates for One Lamination	96

SYMBOLS

Coordinate System

r, θ, z = cylindrical coordinates

u, v, w = displacement in the radial, tangential and axial directions, respectively

r_i = inside radius

r_o = outside radius

h = one-half the thickness of elastomer per lamination

$\pm w_o$ = axial displacement at $z = \pm h$

Stress

σ_{ij} = general component of stress

σ_{rr} = normal stress in r direction

σ_{rz} = shear stress parallel to rz plane on a surface whose surface normal is a radial line

Strain

ϵ_{ij} = general component of strain

ϵ_{rr} = normal strain, denoted by repeated subscript of coordinate system symbols

ϵ_{rz} = shear strain

ϵ_{kk} = dilation, $\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}$, denoted by a repeated subscript with other than the coordinate symbols, r, θ, z

Material Constants

G = shear modulus

ν = Poisson's ratio

k = bulk modulus

λ, μ = Lamé material constants

<u>Nomenclature Unique to Appendixes D and E</u>

τ^{ij} = tensor components of stress

σ_{ij} = physical components of stress

$G_{ij}, G^{ij}, g_{ij}, g^{ij}$ = metric tensors

(r, θ', z') = undeformed coordinates

(ρ, θ, z) = deformed coordinates

Φ, Ψ = constants from constitutive relationship

Γ_i^{jm} = Christoffel symbols

P = pressure

SUMMARY

The purpose of this work was to establish a design procedure for laminated elastomeric bearings based upon the application of large deformation elasticity and viscoelastic mathematical theories. However, the complexity of the large deformation elasticity theory and certain problems associated with computer solutions of the equations of linear or classical elasticity for the particular bearing geometry and for materials that are almost incompressible were such that this work could not be completed.

In the course of this effort, analytical approximations to the application of linear elasticity theory were developed that not only yield results that are very close to exact solutions but are easily applied and evaluated. However, these approximations are for linear elasticity theory, the limits of which are exceeded for greater than very light bearing loads. While not quantitatively applicable for bearings under full loads, they do give much information concerning elastomer behavior under light loads.

The equilibrium equations for large deformation elasticity theory in cylindrical coordinates were stated for combined axial compression and torsion, but were fully derived for the case of compression only.

CONCLUSIONS

While the objectives of the study were not reached, much useful knowledge about the behavior of the elastomer in each lamination under load was generated. The approximate theories, (a) for incompressible materials and (b) for almost incompressible materials, are useful to indicate the relative effect upon bearing performance of bulk modulus (compressibility), ratio of inside diameter to outside diameter, and width-to-thickness ratio. More important, much was learned with respect to techniques for obtaining a useful solution of the nonlinear large-deformation equations that is necessary to predict adequately the distribution of displacement and stress within the bearing, especially along the bonded surface under realistic loading conditions. The original beliefs that large deformations are present within the elastomer even for very light compressive loads were confirmed. But, although much is known as to the behavior of the elastomer under load from the application of linear elasticity theory, this must be conditioned with the statement that this is qualitative and not quantitative for full loads.

INTRODUCTION

During the experimental development of the laminated elastomeric bearing for low-temperature applications (Technical Documentary Report No. ASD-TDR-63-769, prepared for Wright-Patterson Air Force Base, Ohio, November 1963), the need for a mathematical analysis of the bearing which would form the basis of a design method was recognized. Of primary interest were the magnitude and distribution of shear and normal stresses between the elastomer and the metal lamina. This was not only to enable a designer to specify elastomers with adequate strength and adhesive strength (at bond to metal lamina), but to determine the minimum thickness of metal lamina that could be used. Only by a mathematical analysis could the influence of compressive load, shear load (torsion), ratio of outside diameter to inside diameter, width-to-thickness ratio, and shear and bulk moduli of the elastomer be studied.

It was also anticipated that the deformations and strains within the elastomer were large and that the more general mathematical theories of elasticity would be required for an analysis, especially if the interaction of compressive load and bearing torsion observed in the experimental development were to be studied. Linear or classical theory is made linear by the assumptions that both deformation and strain are infinitesimal, thereby removing the ability to study the interaction of axial compression and torsion.

It was anticipated that in this study the state of stress and strain in the bearing could be investigated sufficiently with the use of large-deformation elasticity and viscoelasticity theories and that a computer program could be devised for the design of optimized bearing laminations for a given application and loading condition.

REVIEW OF EFFORT

When the present program was begun, the intention was to go directly to large-deformation elasticity because it appeared that sufficient theory was at hand. The intention was to use the linear or classical theory only for guidance and to establish certain "known points" in order to check the large-deformation work.

Soon after the large-deformation formulation was begun, it was discovered that a simplification could be made if it could be said that plane surfaces in the elastomer parallel to the bonded boundaries remained flat planes after deformation. Some preliminary work with the linear theory of elasticity indicated that these surfaces did not remain flat planes at the free edge where interest in stress and deformation is greater, and so the formulation of the large-deformation problem was made more general. By this time it was apparent that the equations for large deformations would be made up of many terms and that solutions would be time consuming.

At this time during the program, it was believed that an investigation should be made of the effects of small amounts of compressibility of the elastomer upon the displacement pattern and consequently upon stress magnitude and distribution. This was needed because the only constitutive relations (stress-strain relations) that were immediately available were the "Mooney Relations" (Reference 5) that were formulated for incompressible rubber-like materials. No relationships were at hand for compressible materials, because it seems that most rubber-like materials are used under such loading conditions that the hydrostatic component of total stress is relatively small. Hence, volume compression is small and may be neglected.

An investigation into the effect of small amounts of compressibility, using classical elasticity theory, was then begun. First, the literature was searched for solutions to similar elasticity problems (the problems of compression of a cylinder with ends fixed for no radial displacement). One similar problem (Reference 6) was discovered, and although the boundary condition was that of no radial displacement at the outside edge only, instead of along the whole bonded surface, it was believed that this solution from the literature would be useful. However, although it was an analytical solution, the effort necessary to extract information, even by a computer, for various conditions proved to be considerable. It was

much beyond the effort considered necessary to make an independent solution of the equilibrium equations of elasticity for the real boundary conditions (a bonded surface) by means of a digital computer. Therefore, this latter approach was started.

The first choice was to solve the equations of a compressible material. Appendix A shows the formulation of the equilibrium equations and the complete computer program (in FORTRAN) for the iteration procedure to yield deformations. In the theory of elasticity for homogeneous and isotropic materials, two constants are necessary to describe the material behavior. Of the sets of two constants that can be used, G, the shear modulus, and ν, Poisson's ratio, were chosen because the value of Poisson's ratio reflects the relative amount of compressibility: $\nu = 0.5$ for incompressible materials, and $\nu < 0.5$ for compressible materials. Since the equilibrium equations in terms of displacements contain the coefficient $1/1-2\nu$, which approaches infinity as ν approaches 0.5, it was decided for the first solution that a moderately compressible material, say, $\nu = 0.35$, should be used. The value of ν could then be increased to as close to 0.5 as possible in later solutions, after check-out and proof of the iteration procedure. Except for some initial trouble in programming finite-difference methods for the geometry of a very wide, but thin, annulus, the iteration procedure worked well. However, on subsequent trials, the iteration method would not converge to a reasonable solution of Poisson's ratios greater than $\nu = 0.4$. After this was worked with for a period of time, effort was stopped on this computer solution for the compressible elasticity theory, and all furtner investigation of compressible materials was accomplished with the use of an approximate solution described below.

The next effort was the formulation of the equations for incompressible elasticity theory and the computer solution thereof. This required a completely new formulation because the coefficient, $1/1-2\nu$, is infinite for $\nu = 0.5$. Consequently, the new formulation introduced a pressure term (Reference 1, page 79, problem 4) in the displacement relations instead of a volume compression term. Appendix B shows the derivation of these partial differential equations and the FORTRAN program for the solution by iteration.

The iteration scheme proved to be erratic. While effort was being made to improve the convergence of the method, an effort was made concurrently toward an approximate solution, also shown in Appendix B. It was not until some results of the approximate theory were used as the starting values for iteration that convergence seemed probable. By this time considerable effort had been expended upon this solution, because the approach was contined inasmuch as each innovation introduced seemed

to improve the results. The last computation made, of which partial computer output is included in Appendix B, appears to have been converging as planned, although the convergence was very slow.

In the meantime, approximate solutions, both analytical and computer, to the compressible cases of linear elasticity were investigated. It was recognized that the exact solution of the partial differential equations of equilibrium was not of major interest in this study but was only to guide the formulation and solution of the large-deformation case. The search for approximate solutions was fruitful. An analytical solution based upon the linear theory of elasticity was derived, not only for the incompressible case but for the compressible case as well; that is, flexible enough to accept Poisson's ratios in the neighborhood of almost negligible compressibility, $\nu = 0.495$ and $\nu = 0.499$. These values represent the order of magnitude of Poisson's ratios for elastomers of interest in the laminated elastomeric bearing. These approximations for incompressible and compressible theory are shown in Appendixes B and C, respectively.

Appendix C, for the approximate compressible theory, includes an attempt made to solve equations (C-17a) and (C-17b) by computer, using the Rung-Kutta-Gill method. The flow chart and computer program (FORTRAN) are shown at the end of Appendix C. The computer solution did not work, however, and the reason was discovered from the analytical solution: the solution consisted of e^{+x} and e^{-x}, with x assuming very large values. Inasmuch as this approximate solution for compressible materials, as shown by the analytical solution in Appendix C, was obtained by applying the variational theorems of linear elasticity, it satisfies equilibrium approximately, and it satisfies the stress boundary conditions on the average, that is, across the thickness.

Appendix D shows the derivation of equations, based upon large-deformation elastic theory, for the case of torsion only. This was done mainly for investigation of the large-deformation problem of combined compression and torsion. Note that it is infinitely less complex than the case of large-deformation compression, as shown in Appendix E.

The real stumbling block in this study is the set of large-deformation equations for axial loading (Appendix E). These involve constitutive relations (stress-strain relationships) that are, as yet, not known for the compressible case although they are generally known for the incompressible case. The real problem is the size and complexity of the highly nonlinear equations and the consequent lack of assurance that a solution is the right solution. This is the basis of the desire to establish solutions for light loads using classical elasticity, the establishment of a check point.

RESULTS OF LINEAR THEORY

While the computer solutions did finally yield satisfactory results for the incompressible case and for the compressible case where Poisson's ratio was 0.4 or less, these displacements or deformations were not translated into stress. By the time that it could have been done with reliable computer solutions, the analytical approximations were available. These yield a solution that is closer to the exact solution of the equations of linear elasticity than the application of linear elasticity theory is to the real bearing. Perhaps it should be mentioned here that the linear theory of elasticity is believed to yield a very close approximation to elastomer stress and deformation for light loads, but only for light loads, as will be shown presently.

SOLUTIONS FOR INCOMPRESSIBLE ELASTOMERS

Results of the computer solution (Appendix A) for the compressible case are not shown here because, for practical elastomers, the ratio of bulk modulus to shear modulus is so very large that the behavior is more like that of an incompressible material. While the computer solution in Appendix A worked, it only worked for materials with a Poisson's ratio up to 0.40. This is too low for elastomers of immediate interest.

Since meaninful results were obtained from the computer solution for an incompressible material and since a useful approximation was also derived, the results of the work shown in Appendix B are summarized as follows:

1. By examining successive iterations, it was found that the iterative procedure as programmed in Appendix B converges. The convergence is much improved if the iterative procedure is started with the results of the approximate solution, but it is still slow. This is evidenced by the fact that the computer output shown in Appendix B is from the 600th iteration, and certain inconsistencies are still present at the inner and outer boundaries. However, this disturbance is seen to exist at the free boundaries and propagates only about one-half a thickness (2h) into the bearing from either boundary; therefore, this was accepted as a good solution.

2. The influences of the boundaries $r = r_i$ and $r = r_o$ are manifested only near those boundaries. Away from the boundaries by, say, 5 thicknesses, the approximate solution and the computer coincide. Thus, the approximate solution can be used to predict all regions of the bearing except near the boundaries $r = r_i$ and $r = r_o$. These can be investigated by computer solutions using much funer gridworks.

3. Within the limits of the linear theory of elasticity, which has been grossly exceeded in the computer example, the analytical approximation, as shown in equations (B-7) through (B-23), is a very accurate method.

Figures 1 through 4 present data gained from the analytical approximations. Figure 1 shows for incompressible materials the relationship between average pressure, width-to-thickness ratio, and ratio of axial displacement to elastomer thickness. Thus, for

$$G = 80 \text{ psi (shear modulus)}$$

$$\frac{r_o - r_i}{2h} = 250 \text{ (width-to-thickness ratio)}$$

$$\frac{w_o}{h} = 0.002 \text{ (ratio of axial deformation to elastomer thickness)},$$

then

$$P_{ave} = 125G = 10{,}000 \text{ psi.}$$

However, from Figure 2, it is estimated that the associated maximum shear strain is $\epsilon_{rz} = \partial u/\partial z = 1.63$. This strain is far beyond the limits of the linear theory of elasticity because the linear theory is based on the assumption that $\partial u/\partial z$ is very small, as compared to 1.0.

From Figure 3, a measure of what is meant by "light loads" associated with linear elasticity theory can be obtained. Although $\partial u/\partial z = 0.2$ is still much beyond the values acceptable within the linear theory of elasticity, it is not sufficiently large to change the order of magnitude of the results. Assume, then, that $\partial u/\partial z = 0.2$ is accepted. From Figure 3, for a width-to-thickness ratio of 250, $P_{ave} = 15.5$ G. Thus, if G = 80 psi, as for the silicone rubber used in previous programs, then average pressure is 1240 psi. A bearing of this geometry and these materials was tested to an average pressure in excess of 40,000 psi. The rubber did not extrude or come unbonded, but the steel lamina broke. Thus, it is

known that elastomers with this geometry are capable of extreme pressures and extreme shear strains.

For strains that are not excessively large, say, $\epsilon_{rz} = 0.2$, a relationship between average pressure and the maximum shear stress as derived from the incompressible elastic approximation can be shown.

When

$$\frac{r_o - r_i}{2h} = 250 \text{ and } \frac{W_o}{h} = 0.00025,$$

then

$$\epsilon_{rz} = \frac{\partial u}{\partial z} = 0.2$$

and

$$\frac{P_{ave}}{G} = 15.5 \ .$$

Shear stress is

$$\sigma_{rz_{max}} = 2G\ \epsilon_{rz} = 0.4\ G\ ;$$

then

$$\frac{\sigma_{rz_{max}}}{P_{ave}} = \frac{0.4}{15.5}$$

and

$$\sigma_{rz_{max}} = 0.0258\ P_{ave}.$$

This provides some measure of the relationship between shear stress and average pressure, although the limits of linear elasticity have already been exceeded at this strain level.

APPROXIMATE SOLUTION FOR COMPRESSIBLE ELASTOMERS

The approximation shown in Appendix C has also yielded convenient expressions for the investigation of stress and displacement within the elastomer of a lamination. These equations are summarized as follows:

for $\sigma_{z_{average}}$, use equation (C-47);

for $\sigma_{z_{max}}$, use equation (C-48);

for ratio $\dfrac{\sigma_{z_{max}}}{\sigma_{z_{avg}}}$, use equation (C-49);

and

for ratio $\dfrac{\sigma_{rz_{max}}}{\sigma_{z_{avg}}}$, use equation (C-50).

Figure 4 shows a comparison of data for compressible and incompressible theory. The relative position of the curves indicates the influence of various amounts of compressibility, although the exact placement of the curves is not necessarily accurate because they have been calculated from linear theory where the strains exceed those allowable.

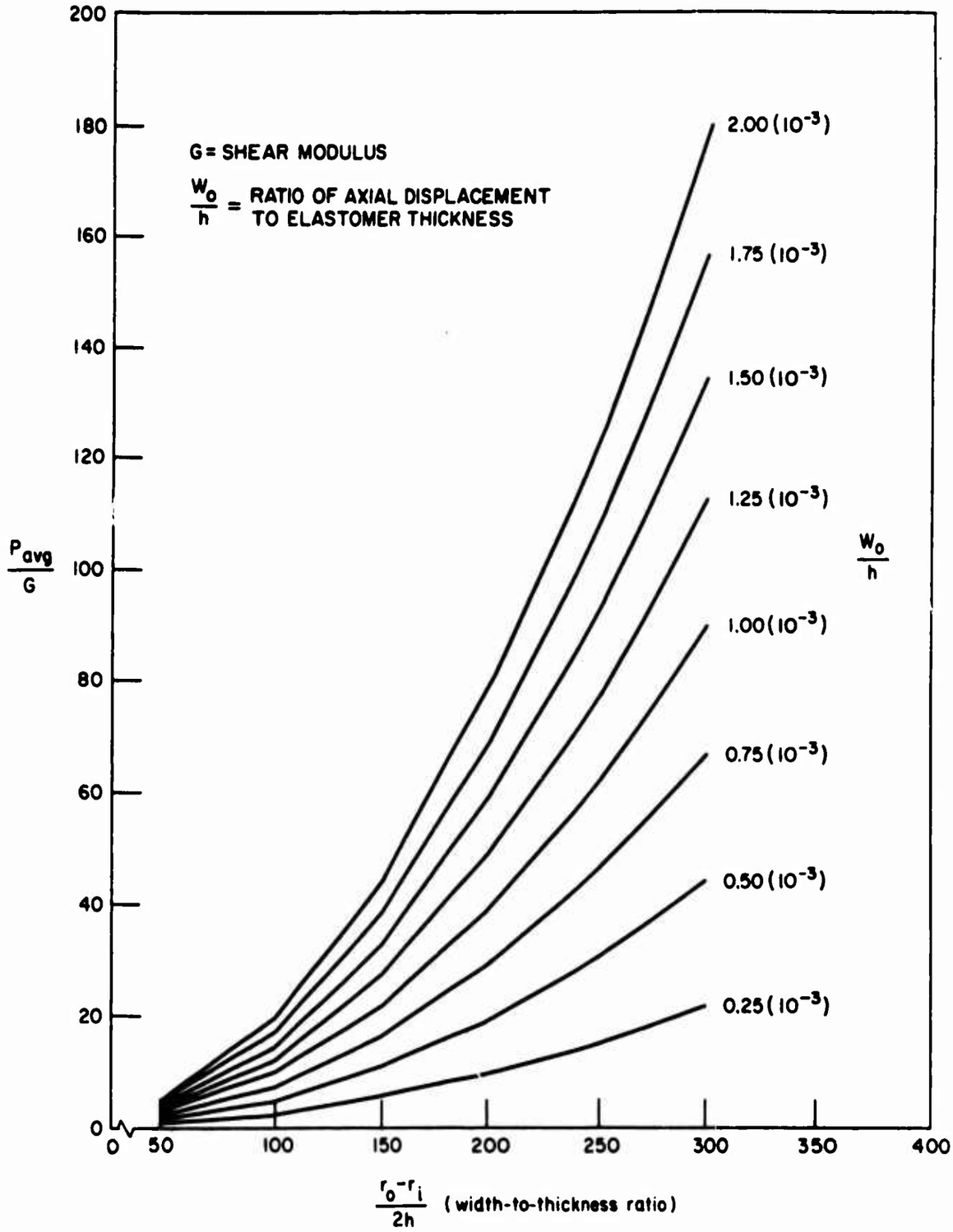


Figure 1. Average Pressure Versus Axial Compression (Incompressible Elasticity Theory).

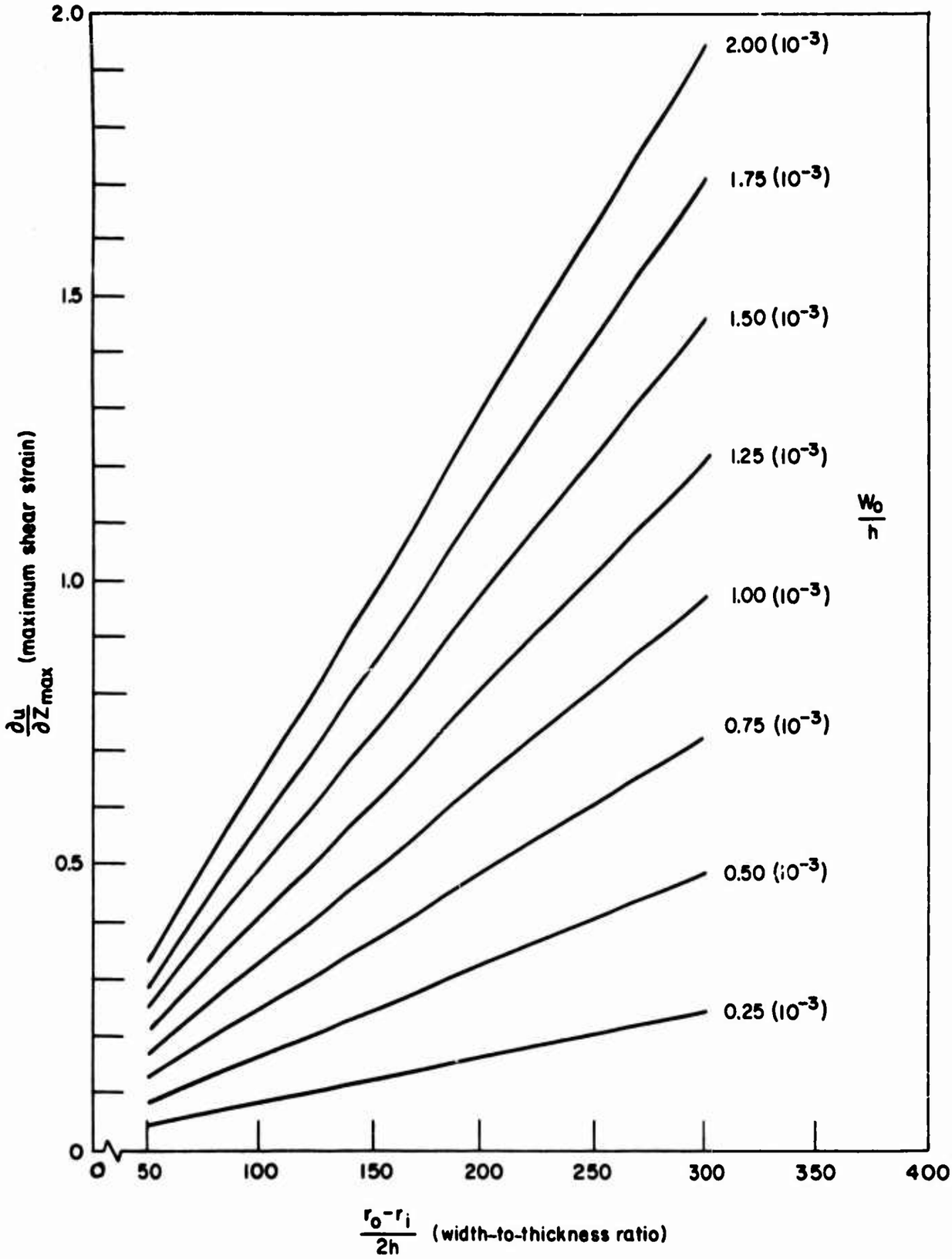


Figure 2. Shear Strain Versus Axial Compression (Incompressible Elasticity Theory).

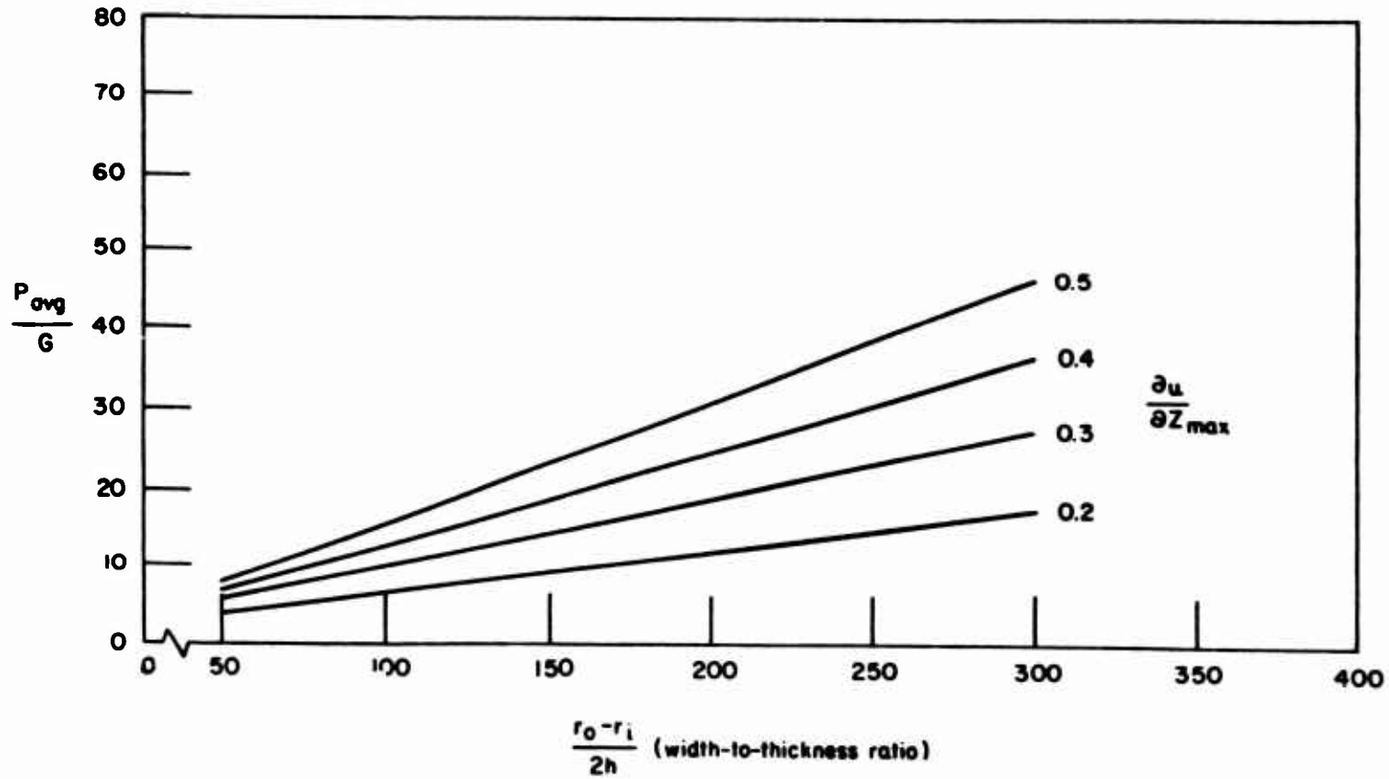
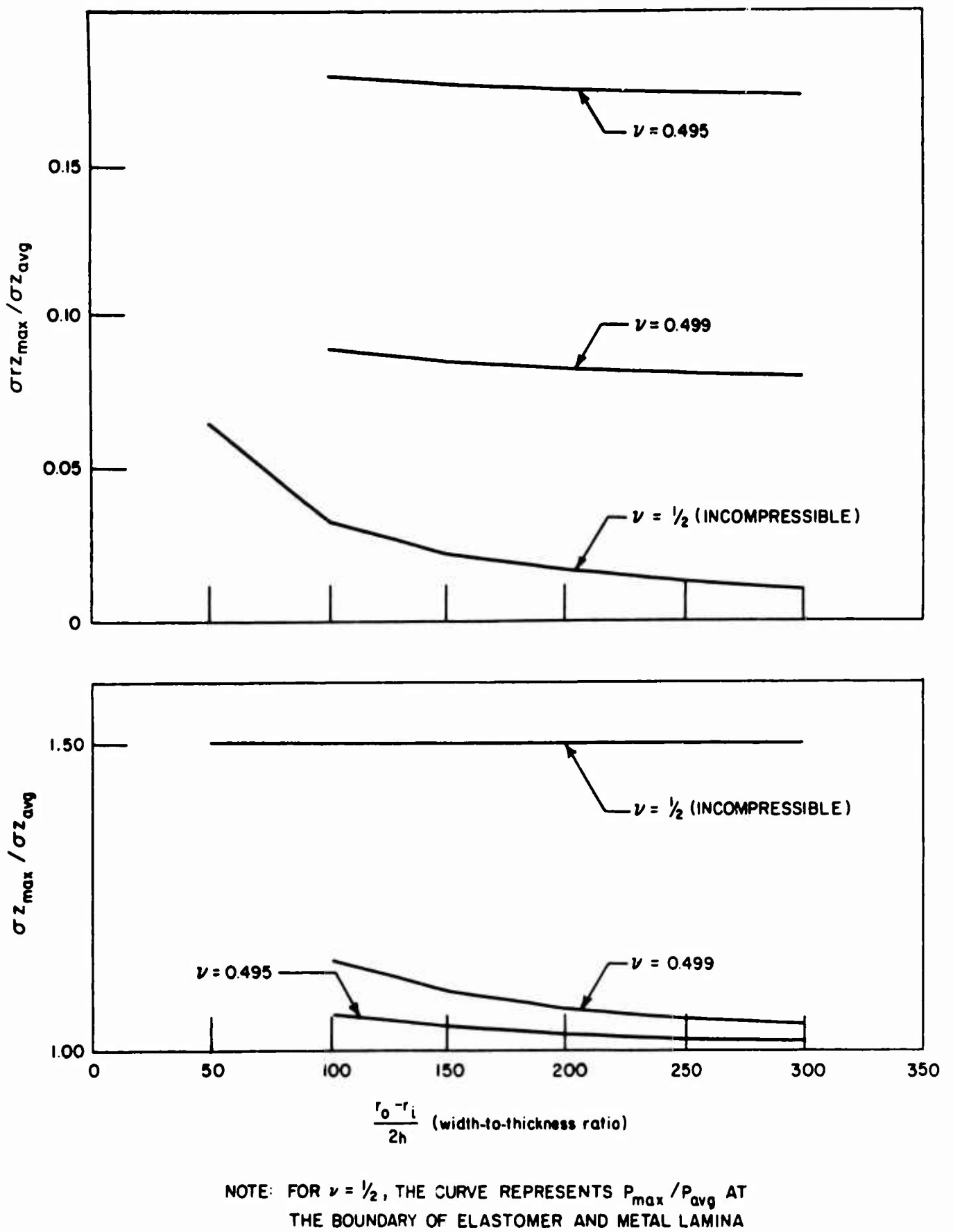


Figure 3. Average Pressure Versus Shear Strain (Incompressible Elasticity Theory).



NOTE: FOR $\nu = 1/2$, THE CURVE REPRESENTS P_{max} / P_{avg} AT THE BOUNDARY OF ELASTOMER AND METAL LAMINA

Figure 4. Comparative Data for Compressible and Incompressible Theory.

BIBLIOGRAPHY

1. Sokolnikoff, Ivan S., Mathematical Theory of Elasticity, 2nd Edition, McGraw-Hill Book Co., New York, N. Y., 1956.

2. Green, A. E., and Adkins, J. E., Large Elastic Deformations, Oxford University Press, London, England, 1960.

3. Green, A. E., and Zerna, W., Theoretical Elasticity, Oxford University Press, London, England, 1954.

4. Sokolnifoff, Ivan S., Tensor Analysis, John Wiley & Sons, Inc., New York, N. Y., 1951.

5. Mooney, M., A Theory of Large Elastic Deformation, Journal of Applied Physics, Vol. II, September, 1940.

6. Filon, L. N. G., The Elastic Equilibrium of Circular Cylinders Under Certain Practical Systems of Load, Philosophical Transactions of the Royal Society, Series A, Vol. 198, 1902, pages 147-233.

7. Fox, L., Numerical Solution of Ordinary and Partial Differential Equations, Addison-Wesley Publishing Co., Reading, Mass., 1962.

8. Forsythe, George E., and Wasow, Wolfgang R., Finite-Difference Methods for Partial Differential Equations, John Wiley & Sons, New York, N. Y., 1960.

9. Doshi, K. D., An Accelerated Iterative Procedure for Solving Partial Difference Equations of the Elliptic Type, Ph. D. Thesis, Civil Engineering Dept., Univ. of Illinois, Urbana, Ill., April 1962.

10. Hildebrand, F. B., Methods of Applied Mathematics, Prentice-Hall Inc., Englewood Cliffs, N. J., 1952.

11. Eirich, Frederick R., Rheology, Academic Press Inc., Volume 1, New York, N. Y., 1956.

12. Rivlin, R. S., and Saunders, D. W., Large Elastic Deformations of Isotropic Materials, VII. Experiments on the Deformation of Rubber, Phil. Trans. Royal Society (London), Vol. A243, 1951, page 251.

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APPENDIX A

COMPUTER SOLUTION FOR A COMPRESSIBLE ELASTOMER

This is the formulation of the pure axial compression problem of one lamination, following linear (or classical) elasticity theory, that was prepared for computer solution using finite-difference techniques and iteration schemes. The geometry is that of the flat thrust bearing.

Consider a thin disk of elastomer compressed between two rigid plates and assume that the disk is composed of a compressible material with Poisson's ratio, ν . The shape and the dimension of the disk are shown in Figure 5.

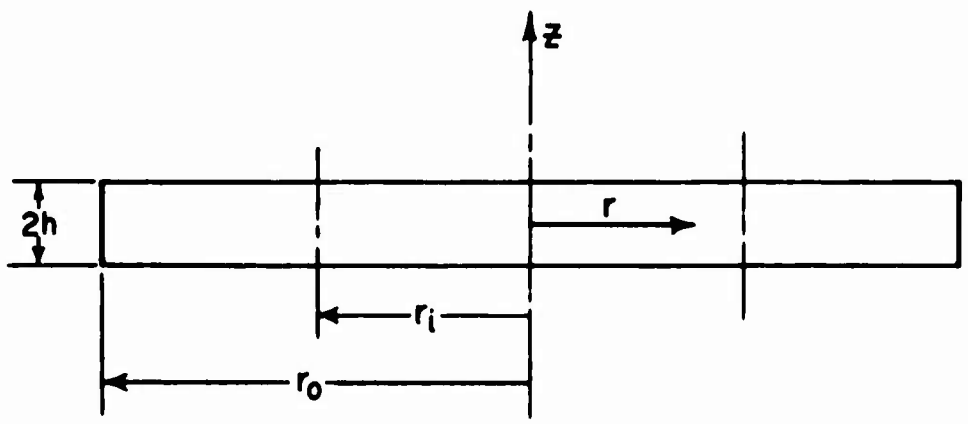


Figure 5. Sketch of Coordinates for One Lamination.

Since the disk is circular, it is convenient to use cylindrical coordinates (r, θ, z). The symmetry property of this problem indicates that stresses and displacements are functions of r, z only.

The equations of equilibrium are thus simplified to

$$\frac{\partial^2 w}{\partial r^2} + \frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 w}{\partial z^2} + \frac{1}{1-2\nu}\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{1-2\nu}\frac{1}{r}\frac{\partial u}{\partial z} = 0 \qquad \text{(A-1a)}$$

and

$$\frac{1}{1-2\nu}\frac{\partial^2 w}{\partial r \partial z} + \frac{\partial^2 u}{\partial z^2} + \frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 u}{\partial r^2} + \frac{2(1-\nu)}{1-2\nu}\frac{1}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = 0, \qquad \text{(A-1b)}$$

where u and w are the displacements in radial and axial directions, respectively.

The boundary conditions are

$$
\begin{aligned}
u &= 0, & w &= -\delta & &\text{at } z = +h \\
u &= 0, & w &= \delta & &\text{at } z = -h \\
\sigma_r &= 0, & \sigma_{rz} &= 0 & &\text{at } r = r_o \text{ and } r = r_i .
\end{aligned}
\tag{A-2}
$$

Since the mid-plane (z = 0) is a plane of symmetry, the boundary conditions can be replaced on z = -h by the symmetry conditions at z = 0.

$$\text{At } z = o, \quad w = 0 \quad \text{and} \quad \frac{\partial u}{\partial z} = 0 \; ; \tag{A-3}$$

thus, the problem can be solved only for the domain z = 0 to h and r = r_i to r_o.

Because of the complexity of this problem, a computer program is written to solve this set of simultaneous equations. The equations of equilibrium and the boundary conditions are written in finite-difference form as follows:

$$
\begin{aligned}
&[1 - \frac{b}{2R(I)}]\, w_{i-1,j} + [1 + \frac{b}{2R(I)}]\, w_{i+1,j} + \frac{2(1-\nu)}{1-2\nu}\, w_{i,j-1} \\
&\quad + \frac{2(1-\nu)}{1-2\nu}\, w_{i,j+1} + (\tfrac{1}{4})(\frac{1}{1-2\nu})(u_{i+1,j+1} - u_{i-1,j+1} + u_{i-1,j-1} - u_{i+1,j-1}) \\
&\quad + (\frac{b}{2R(I)})(\frac{1}{1-2\nu})(u_{i,j+1} - u_{i,j-1}) - 2(\frac{3-4\nu}{1-2\nu})\, w_{i,j} = 0
\end{aligned}
\tag{A-4a}
$$

and

$$
\begin{aligned}
&\tfrac{1}{4}\,(\frac{1}{1-2\nu})(w_{i+1,j+1} - w_{i-1,j+1} + w_{i-1,j-1} - w_{i-1,j+1}) + u_{i,j-1} \\
&\quad + u_{i,j+1} + \frac{2(1-\nu)}{1-2\nu}(1 + \frac{b}{2R(I)})\, u_{i+1,j} + \frac{2(1-\nu)}{1-2\nu}(1 - \frac{b}{2R(I)})\, u_{i-1,j} \\
&\quad - 2\,[\frac{3-4\nu}{1-2\nu} + \frac{b^2}{2R^2(I)}\,(\frac{2(1-\nu)}{1-2\nu})]\, u_{i,j} = 0.
\end{aligned}
\tag{A-4b}
$$

For a general point (i, j) in the body, and boundary conditions

$$\left.\begin{aligned} u_{i,mmm} &= 0 \\ w_{i,mmm} &= -\delta \end{aligned}\right\} z = h$$

$$\left.\begin{aligned} w_{i,2} &= 0 \\ u_{i,3} &= u_{i,1} \end{aligned}\right\} z = 0, \qquad \text{(A-5)}$$

$$\left.\begin{aligned} r = r_i & \begin{cases} \frac{2(1-\upsilon)}{1-2\upsilon}(u_{3,j} - u_{i,j}) + \frac{2\upsilon}{1-2\upsilon}\left(\frac{2b}{R_i}\right)u_{2,j} + \frac{2\upsilon}{1-2\upsilon}(w_{2,j+1} - w_{2,j-1}) = 0 \\ u_{2,j+1} - u_{2,j-1} = -(w_{3,j} - w_{1,j}) \end{cases} \\ r = r_o & \begin{cases} \frac{2(1-\upsilon)}{1-2\upsilon}(u_{nnn+1,j} - u_{nnn-1,j}) + \frac{2\upsilon}{1-2\upsilon}\left(\frac{2b}{R_o}\right) u_{nnn,j}) \\ \qquad + \frac{2\upsilon}{1-2\upsilon}(w_{nnn,j+1} - w_{nnn,j-1}) = 0 \\ u_{nnn,j+1} - u_{nnn,j-1} = -(w_{nnn+1,j} - w_{nnn-1,j}) \end{cases} \end{aligned}\right\} \text{(A-6)}$$

for points (i, j) on the boundary, where b is the grid size and

$$R(z) = r_i + i \times b.$$

It is noted that since the material is compressible, there is a region in the body where the material experiences only hydrostatic compression. This knowledge permits further reduction in the domain of the problem by combining the two end conditions into one. That is to say, the set of equations is solved from both ends ($r = r_i$ and $r = r_o$) simultaneously, and the condition of hydrostatic compression is used as a boundary condition.

Denote ends $r = r_i$ and $r = r_o$ by $k = 1$ and $k = 2$, respectively, and define coefficients as shown in expressions (A-7) through (A-9). Then simplify the set of equations to those that appear.

Coefficients for the computer program are

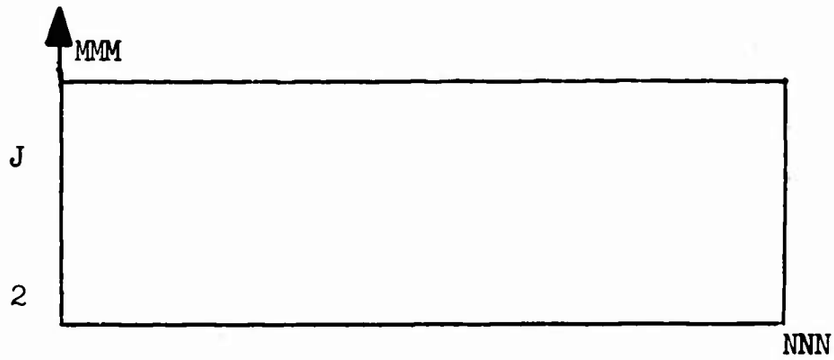
$$\left.\begin{aligned}
A(1) &= 1-\upsilon \\
A(2) &= \upsilon/(1-\upsilon) \\
A(3) &= 3-4\upsilon \\
A(4) &= 1-2\upsilon \\
A(5) &= H/FM \\
A(6) &= 1/(3-4\upsilon) \\
A(7) &= A(4) \times A(6) \\
A(8) &= A(1) \times A(6)
\end{aligned}\right\} \qquad \text{(A-7)}$$

k=1 RØ(I,K) = RZERØ + (FI-2.0) A(5)

$$\left.\begin{aligned}
C1(I,K) &= A(5)/RØ(I,K) \\
C2(I,K) &= C1(I,K) ** 2 \\
C3(I,K) &= 1.0 + 0.5 * C1(I,K) \\
C4(I,K) &= 1.0 - 0.5 * C1(I,K) \\
C5(I,K) &= A(3) + A(1) * C2(I,K) \\
C6(I,K) &= 1.0/C5(I,K) \\
D1(K) &= + 1.0
\end{aligned}\right\} \qquad \text{(A-8)}$$

k=2

$$\left.\begin{aligned}
RØ(I,K) &= RØUT - (RI - 2.0) * A(5) \\
C1(I,K) &= A(5)/RØ(I,K) \\
C2(I,K) &= C1(I,K) * C1(I,K) \\
C3(I,K) &= 1.0 - 0.5 * C1(I,K) \\
C4(I,K) &= 1.0 + 0.5 * C1(I,K) \\
C5(I,K) &= A(3) + A(1) * C2(I,K) \\
C6(I,K) &= 1.0/C5(I,K) \\
D1(K) &= - 1.0
\end{aligned}\right\} \qquad \text{(A-9)}$$



Note that in the following expressions the double sign ± or ∓ appears. The upper sign refers to the case where k = 1, while the lower sign refers to that where k = 2.

Boundary conditions are

$$\left.\begin{aligned} w_{i,j} &= w_{3,J} \pm (u_{2,J+1} - u_{2,J-1}) \\ u_{i,J} &= u_{3,J} \pm \frac{\upsilon}{1-\upsilon}\left(\frac{2b}{R(I,K)}\right) u_{2,J} + w_{2,J+1} - w_{2,J-1} \end{aligned}\right\}. \qquad \text{(A-10)}$$

Field equations are

$$\begin{aligned} w_{I,J} = \frac{1}{2(3+\frac{\lambda}{\mu})} \{ & [\mp \frac{b}{2*R\phi(I,K)}] w_{I-1,J} + [\pm \frac{b}{2*R\phi(I,K)}] w_{I+1,J} \\ & + (2 + \frac{\lambda}{\mu})(w_{I,J-1} + w_{I,J+1}) \pm \tfrac{1}{4} (1 + \frac{\lambda}{\mu})(u_{I+1,J+1} \\ & - u_{I-1,J+1} + u_{I-1,J-1} - u_{I+1,J-1}) + \frac{b}{2R\phi(I,K)} (1 + \frac{\lambda}{\mu}) \\ & (u_{I,J+1} - u_{I,J-1})\} \end{aligned} \qquad \text{(A-11a)}$$

$$\begin{aligned} u_{I,J} = \tfrac{1}{2} [& \frac{1}{(3+\frac{\lambda}{\mu}) + (2+\frac{\lambda}{\mu})(\frac{b^2}{2R\phi^2(I,K)})}] \{\pm \tfrac{1}{4}(1+\frac{\lambda}{\mu})(w_{I+1,J+1} \\ & - w_{I-1,J+1} + w_{I-1,J-1} - w_{I+1,J-1}) + u_{I,J-1} + u_{I,J+1} \\ & + (2 + \frac{\lambda}{\mu}) [(1 \pm \frac{b}{2R\phi(I,K)}] u_{I+1,J} + (1 \mp \frac{b}{2R\phi(I,K)}) u_{I-1,J}\}. \end{aligned} \qquad \text{(A-11b)}$$

Substituting $\lambda/\mu = 2\upsilon/1-2\upsilon$,

$$w_{I,J} = \frac{1-2\upsilon}{2(3-4\upsilon)} \left\{\left[1 \mp \frac{b}{2R\phi(I,K)}\right] w_{I-1,J} + \left[1 \pm \frac{b}{2R\phi(I,K)}\right] w_{I+1,J}\right\}$$

$$+ \frac{1-\upsilon}{3-4\upsilon} (w_{J,J-1} + w_{I,J+1}) \pm (\tfrac{1}{4}) \frac{1-\upsilon}{2(3-4\upsilon)} (u_{I+1,J+1} - u_{I-1,J+1}$$

$$+ u_{I-1,J-1} - u_{I+1,J-1}) + \frac{1-\upsilon}{2(3-4\upsilon)} \left(\frac{b}{2R\phi(I,K)}\right) (u_{I,J+1} - u_{I,J-1}) \qquad \text{(A-12a)}$$

$$u_{I,J} = \tfrac{1}{2} \left\{\frac{1}{\left[(3-4\upsilon) + (1-\upsilon) \dfrac{b^2}{R\phi^2(I,K)}\right.}\right\} \left\{\pm\tfrac{1}{4} (w_{I+1,J+1} - w_{I-1,J+1}\right.$$

$$+ w_{I-1,J-1} + w_{I+1,J-1}) + (1 - 2\upsilon)(u_{I,J-1} + u_{I,J+1})$$

$$+ 2(1-\upsilon) \left[\left(1 \pm \frac{b}{2R\phi(I,K)}\right) u_{I+1,J} + \left(1 \mp \frac{b}{2R\phi(I,K)}\right) u_{I-1,J}\right]\right\}. \qquad \text{(A-12b)}$$

From the definitions of the coefficients,

$$w_{I,J} = 0.5 * A(7) * \{C4(I,K)\ w_{I-1,J} + C3(I,K)\ w_{I+1,J}\}$$

$$+ A(8)* (w_{I,J-1} + w_{I,J+1}) + D1(K)* 0.125* A(8)* (u_{I+1,J+1}$$

$$- u_{I-1,J+1} + u_{I-1,J-1} - u_{I+1,J-1}) + 0.25* A(8)* C1(I,K)*$$

$$(u_{I,J+1} - u_{I,J-1}) \qquad \text{(A-13a)}$$

$$u_{I,J} = 0.5* C6(I,K)* D1(K)* 0.25* (w_{I+1,J+1} - w_{I-1,J+1} + w_{I-1,J-1}$$

$$- w_{I+1,J-1}) + A(4)* (u_{I,J-1} + u_{I,J+1}) + 2* A(1)$$

$$*(C3(I,K)\ u_{I+1,J} + C4(I,K)\ u_{I-1,J})). \qquad \text{(A-13b)}$$

Boundary conditions (end equation) are

$$w_{1,J} = w_{3,J} + D1(K)* (u_{2,J+1} - u_{2,J-1})$$

$$u_{1,J} = u_{3,J} + D1(K)* A(2)* (2* C1(2,K)\ u_{2,J} + w_{2,J+1} - w_{2,J-1}). \qquad \text{(A-14)}$$

Computation of stresses is as follows:

From the stress-strain relation

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2G\epsilon_{ij}$$

and the strain-displacement relation, stresses can be calculated at any point in the body when the displacement field is known. Since the displacements are given by numerical values, finite-difference methods must be used to calculate strains.

At the top and bottom boundaries, the strains are

$$\epsilon_r = \frac{\partial u}{\partial r} = 0$$

$$\epsilon_\theta = \frac{u}{r} = 0$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)$$

and the stresses are

$$\sigma_z = (\lambda + 2G)\ \epsilon_z = \frac{2G(1 - \nu)}{1 - 2\nu}\ \frac{\partial w}{\partial z}$$

$$\sigma_{rz} = G\ \frac{\partial u}{\partial z}\ .$$

In order to evaluate the derivatives there, backward or forward differences must be used. At the top surface, backward difference is used.

$$\frac{\partial f}{\partial z} = \frac{1}{2\Delta z}\ (3f(I,J) - 4f(I,J-1) + f(I,J-2))$$

Since in the computer result u and w are nondimensionalized with respect to h, $\Delta z = h/m$.

For m = 4,

$$\frac{\partial w}{\partial z} = 2(3w\ (I,mmm) - 4w(I,m) + w(I,m-1))$$

$$\frac{\partial u}{\partial z} = 2(3u\ (I,mmm) - 4u(I,m) + u(I,m-1));$$

hence,

$$\sigma_z = \frac{4(1-\nu)}{1-2\nu}\ G(3w\ (I,mmm) - 4w(I,m) + w(I,m-1)$$

$$\sigma_{rz} = 2G(3u\ (I,mmm) - 4u(I,m) + u(I,m-1).$$

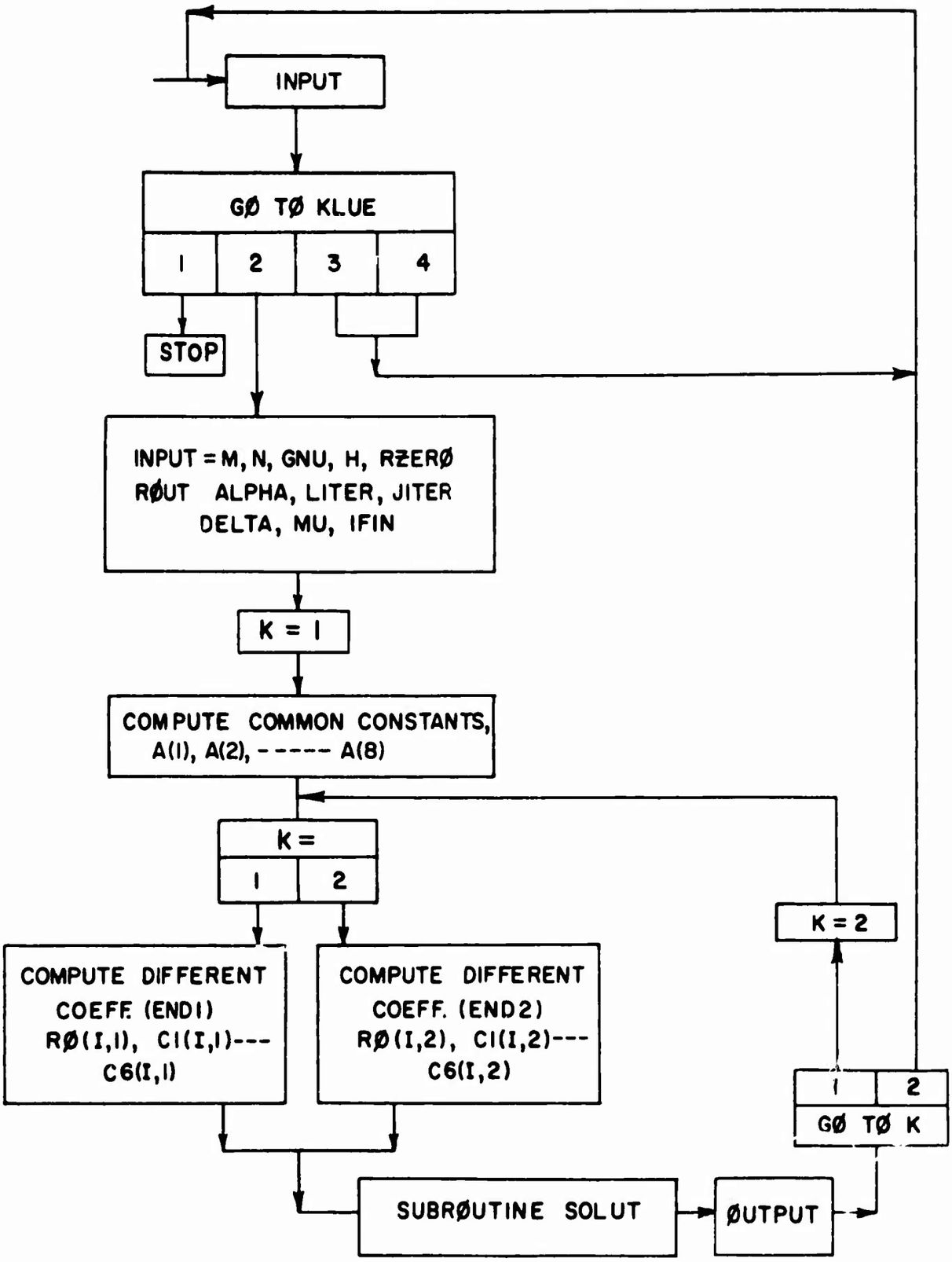


Figure 6. Main Flow Chart

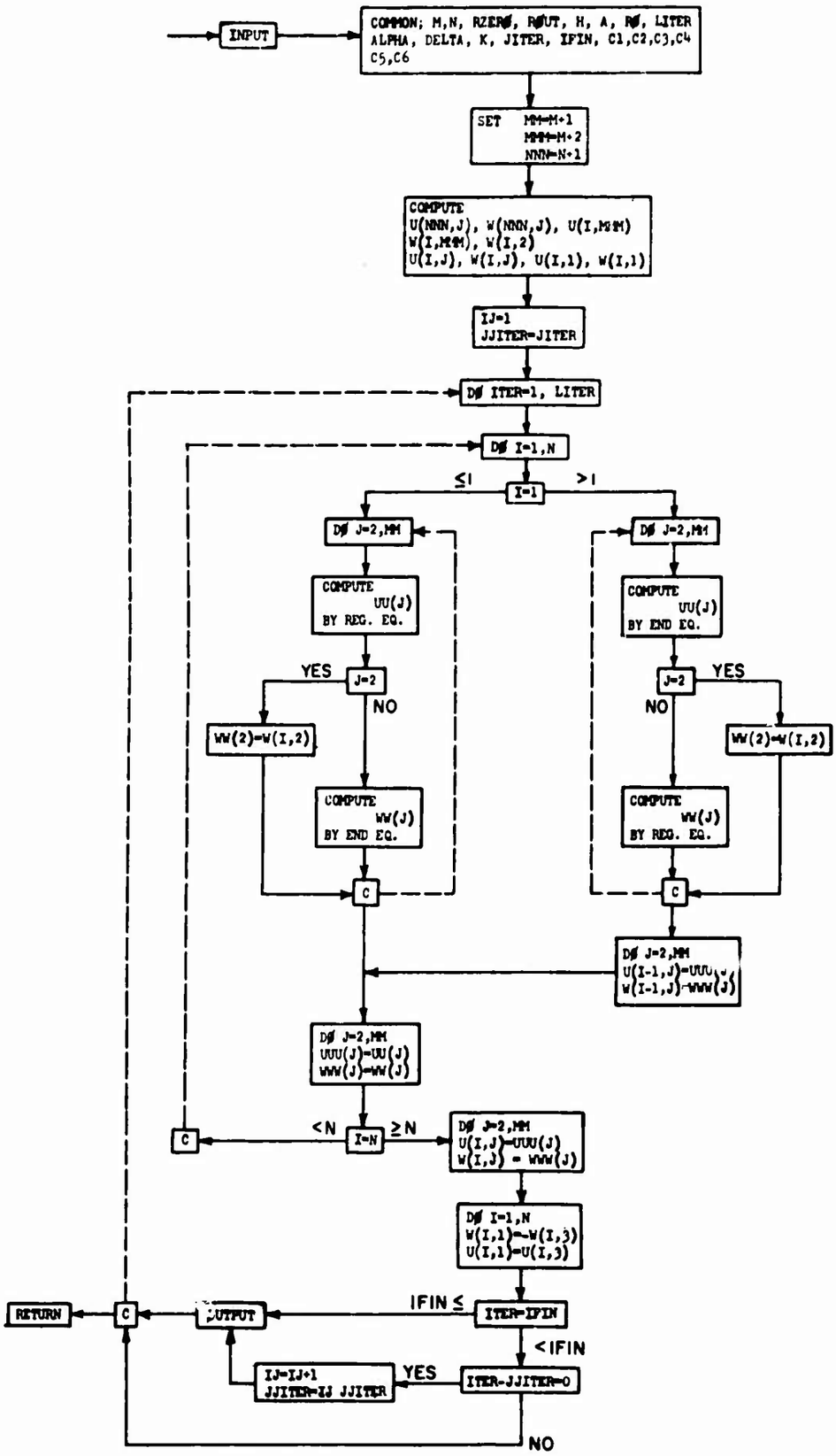


Figure 7. Subroutine Solut.

LAMINATED BEARING

```
C
C COMPRESSIBLE MATERIAL
C
      DIMENSION  A(8), RO(401,2), C1(401,2), C2(401,2),  C3(401,2),
     1C4(401,2), C5(401,2), C6(401,2), D1(2)
      COMMON  M, RZERO, ROUT, H, A, RO, N, LITER, ALPHA, DELTA,
     1CONST, K, JITER, IFIN, C1, C2, C3, C4, C5, C6, D1
  110 READ INPUT TAPE 2,1, KLUE , ICLUE
    1 FORMAT (2I5)
      GO TO (500,100,400,400),KLUE
  100 READ INPUT TAPE 2, 2,    N, M, RZERO, ROUT, GNU, H  ,JITER,IFIN
    2 FORMAT (2I5,4E15.8/2I6)
      READ INPUT TAPE 2, 3, LITER, ALPHA, DELTA,         CONST, PMU
    3 FORMAT (I5, 4E15.8 )
      A(1) = 1.0 - GNU
      A(2) = GNU/A(1)
      A(3) = 3.0 - 4.0*GNU
      A(4) = 1.0 - 2.0*GNU
      FM = M
      A(5) = H/FM
      A(6) = 1.0/A(3)
      A(7) = A(4)*A(6)
      A(8) = A(1)*A(6)
      CALL PDUMP (A,A(8),1)
      K = 1
   20 CONTINUE
      GO TO (30,40),K
   30 NNN = N+2
      DO 32 I = 2,NNN
      FI = I
      RO(I,K) = RZERO + (FI - 2.0) * A(5)
      C1(I,K) = A(5)/RO(I,K)
      C2(I,K) = C1(I,K) * C1(I,K)
      C3(I,K) = 1.0 + (0.5 * C1(I,K))
      C4(I,K) = 1.0 - (0.5 * C1(I,K))
      C5(I,K) = A(3) + A(1)*C2(I,K)
      C6(I,K) = 1.0/C5(I,K)
   32 CONTINUE
      D1(K) = 1.0
      GO TO 50
   40 NNN = N+2
      DO 42 I=2,NNN
      FI = I
      RO(I,K) = ROUT - (FI-2.0)*A(5)
      C1(I,K) = A(5)/RO(I,K)
      C2(I,K) = C1(I,K) * C1(I,K)
      C3(I,K) = 1.0 - ( 0.5 * C1(I,K))
      C4(I,K) = 1.0 + ( 0.5 * C1(I,K))
      C5(I,K) = A(3) + A(1)*C2(I,K)
      C6(I,K) = 1.0/C5(I,K)
   42 CONTINUE
      D1(K) = -1.0
      GO TO 60
   50 WRITE OUTPUT TAPE 10,10, RZERO,ROUT
   10 FORMAT (1H1 3X 8HRZERO = F10.5, 3X 7HROUT = F10.5//)
```

```
    WRITE OUTPUT TAPE 10,11, GNU, M, H, N
 11 FORMAT (3X 6HGNU = F10.5, 3X 4HM = I5, 3X 4HH = F10.5, 3X 4HN =
   1I5//)
    WRITE OUTPUT TAPE 10,12, LITER,ALPHA,DELTA,       CONST,PMU
 12 FORMAT (1X 8HLITER = I5, 3X  8HALPHA = F10.5, 3X 8HDELTA = F10.5
   1, 3X 8HCONST = F10.5, 3X 6HPMU = F10.5////)
 60 CALL  SOLUT
    GO TO (89,108),K
 89 GO TO (108,90) , ICLUE
 90 K = 2
    GO TO 20
108 CONTINUE
    GO TO 110
400 CONTINUE
    GO TO 110
500 CALL DUMP
    END(1,1,0,0,0,1,1,1,0,1,0,0,0,0,0)
```

```
      SUBROUTINE  SOLUT
      DIMENSION  A(8), RO(401,2), C1(401,2), C2(401,2),  C3(401,2),
     1C4(401,2), C5(401,2), C6(401,2), D1(2), U(401,10), W(401,10),
     1UUU(10), WWW(10),  UU(10), WW(10)
      COMMON  M, RZERO, ROUT, H, A, RO, N, LITER, ALPHA, DELTA,
     1CONST, K, JITER, IFIN, C1, C2, C3, C4, C5, C6, D1
      MM = M + 1
      MMM = M + 2
      NN = N+1
      NNN = N+2
      FM = M
      DO 10  J = 2,MMM
      U(NNN,J) = 0.0
      FJ = J
      W(NNN,J) = -(FJ-2.0)*DELTA/FM
   10 CONTINUE
      DO 15 I = 2,NN
      U(I,MMM) = 0.0
      W(I,MMM) = -DELTA
      W(I,2) = 0.0
   15 CONTINUE
      DO 20 I = 2,NN
      DO 20 J = 2,MM
      FJ = J
      U(I,J) = -D1(K)*CONST*DELTA*(1.0-((FJ-2.0)*A(5)/H)**2)
     1*EXPF(D1(K)*(RO(2,K)-RO(I,K)))
      W(I,J) = -(FJ-2.0)*DELTA/FM
   20 CONTINUE
      IJ = 1
      JJITER = JITER
      DO 110 ITER = 1,LITER
      DO 25 I = 2,NNN
      U(I,1) = U(I,3)
      W(I,1) = -W(I,3)
   25 CONTINUE
      DO 30 J = 2,MM
      U(1,J) = U(3,J)+D1(K)*A(2)*(2.0*C1(2,K)*U(2,J)+W(2,J+1)-W(2,J-1))
      W(1,J) = W(3,J)+D1(K)*(U(2,J+1)-U(2,J-1))
   30 CONTINUE
      U(1,1) = U(1,3)
      W(1,1) = -W(1,3)
      DO 80 I = 2,NN
      DO 50 J = 2,MM
      UEVAL = 0.5*C6(I,K)*(D1(K)*0.25*(W(I+1,J+1)-W(I-1,J+1)+W(I-1,J-1)
     1-W(I+1,J-1))+A(4)*(U(I,J-1)+U(I,J+1))+2.0*A(1)*(C3(I,K)*U(I+1,J)
     1+C4(I,K)*U(I-1,J)))
      UU(J) = ALPHA * UEVAL + (1.0-ALPHA) * U(I,J)
      IF (J-2) 45,40,45
   40 WW(2) = W(I,2)
      GO TO 50
   45 WEVAL = 0.5*A(7)*(C4(I,K)*W(I-1,J) + C3(I,K)*W(I+1,J))
     1+A(8)*(W(I,J-1)+W(I,J+1))+ D1(K)*0.125*A(8)*(U(I+1,J+1)
     1-U(I-1,J+1)+U(I-1,J-1)-U(I+1,J-1))+0.25*A(8)*C1(I,K)
     1*(U(I,J+1)-U(I,J-1))
      WW(J) = ALPHA * WEVAL + (1.0-ALPHA) * W(I,J)
   50 CONTINUE
```

```
      IF (I-2) 60,60,70
   60 DO 65 J = 2,MM
      UUU(J) = UU(J)
      WWW(J) = WW(J)
   65 CONTINUE
      GO TO 80
   70 DO 75 J=2,MM
      U(I-1,J) = UUU(J)
      W(I-1,J) = WWW(J)
   75 CONTINUE
      DO 79 J = 2,MM
      UUU(J) = UU(J)
      WWW(J) = WW(J)
   79 CONTINUE
   80 CONTINUE
      DO 85 J = 2,MM
      U(NN,J) = UUU(J)
      W(NN,J) = WWW(J)
   85 CONTINUE
      IF(ITER - IFIN) 101, 105, 105
  101 IF (ITER - JJITER) 110,104,104
  104 IJ = IJ+1
      JJITER = IJ*JITER
  105 INCR = 1
      JJ = 1
      KK = MMM - 6
  102 IF (KK)106,107,107
  106 JJJ= MMM
      GO TO 108
  107 JJJ = INCR * 6
  108 WRITE OUTPUT TAPE 10,1, ITER
    1 FORMAT (1H1 17HITERATION  NO. = I5///)
      WRITE OUTPUT TAPE 10,3
    3 FORMAT ( 7HU ARRAY//)
      WRITE OUTPUT TAPE 10,2,((U(I,J), J=JJ,JJJ),I=1,NNN)
      WRITE OUTPUT TAPE 10,4
    4 FORMAT (//7HW ARRAY//)
      WRITE OUTPUT TAPE 10,2,((W(I,J), J=JJ,JJJ),I=1,NNN)
    2 FORMAT (6(4X E15.8))
      IF (JJJ-MMM) 109,103,109
  109 KK = KK-6
      INCR = INCR + 1
      JJ = JJ+6
      GO TO 102
  103 CONTINUE
  110 CONTINUE
      RETURN
      END(1,1,0,0,0,1,1,1,0,1,0,0,0,0,0)
```

APPENDIX B

ANALYTICAL APPROXIMATION AND COMPUTER SOLUTION FOR AN INCOMPRESSIBLE ELASTOMER

Governing equations are again written in cylindrical coordinates, as shown in Appendix A. The equations of equilibrium remain the same, but the stress-strain relations become

$$\sigma_{ij} = P\,\delta_{ij} + 2G\epsilon_{ij} \tag{B-1}$$

where an additional unknown pressure, P, is introduced into the relation due to incompressibility. Furthermore, since there is no volume change,

$$\epsilon_{ii} = 0 \tag{B-2}$$

and strains are deviatoric. Taking into account the axial symmetry property, the nonvanishing strains are

$$\epsilon_r = \frac{\partial u}{\partial r}$$

$$\epsilon_\theta = \frac{u}{r}$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) . \tag{B-3}$$

Substituting the stress-strain relation and the strain-displacement relation into the equilibrium equation, we obtain the governing equation in terms of displacement.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{1}{G}\frac{\partial P}{\partial r} \; . \; . \; . \; . \tag{B-4a}$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} = -\frac{1}{G}\frac{\partial P}{\partial z} \; . \; . \; . \; . \tag{B-4b}$$

Since there are three unknowns, u, w, and P, three equations are needed; the third equation is furnished by equation (B-2),

$$\epsilon_{ii} = 0$$

or

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad \ldots \qquad \text{(B-4c)}$$

Equations (B-4a, b, and c) coupled with boundary conditions

$$\begin{aligned} u &= 0 & z &= \pm h \\ w &= -w_o & z &= +h \\ w &= w_o & z &= -h \\ \sigma_{rz} &= 0 & r &= r_i \text{ and } r_o \\ \sigma_{rr} &= 0 & r &= r_i \text{ and } r_o \end{aligned} \qquad \text{(B-5)}$$

completely define the problem. However, the deformations are extremely complex, owing to the imposed boundary conditions, and the mathematical analysis becomes intractable. Thus, the use of approximate methods and computer solutions becomes necessary.

APPROXIMATE SOLUTION

Because of the geometry of one lamination, $[\, h \ll (r_o - r_i)\,]$, the boundary conditions $\sigma_{rr} = 0$ and $\sigma_{rz} = 0$ may be replaced by an average condition.

Let

$$\left.\begin{aligned} \sigma_{rr} &= \frac{1}{2h} \int_{-h}^{h} \sigma_{rr}\, dz = 0 \\ \sigma_{rz} &= \frac{1}{2h} \int_{-h}^{h} \sigma_{rz}\, dz = 0 \end{aligned}\right\}; \qquad \text{(B-6)}$$

that is to say, the system of forces acting on the boundary is replaced by an equivalent system. By Saint Venant's principle, this replacement will produce negligible influence at some distance away from the boundary.

Consider a solution of the form

$$u = (ar - br^{-1})\ (1 - z^2/h^2)\ \ldots \tag{B-7}$$

$$w = -2az\left(1 - \frac{z^2}{3h^2}\right)\ \ldots \tag{B-8}$$

where a and b are constants. Substituting equations (B-7) and (B-8) into the governing equations, it is found that equation (B-4c) is identically satisfied and that equations (B-4a) and (B-4b) yield

$$P/G = a\,\frac{r^2}{h^2} - \frac{2b}{h^2}\ln r + 2a\left(1 - \frac{z^2}{h^2}\right) + C_2\ \ldots \tag{B-9}$$

where C_2 is a constant.

This solution satisfies boundary conditions on $z = \pm h$, identically, provided

$$a = \frac{3}{4}\,\frac{w_o}{h}\ . \tag{B-10}$$

The average boundary condition on $r = r_i$ and r_o gives

$$\frac{2}{2h}\int_{-h}^{h}\frac{\partial u}{\partial r}\,dz + \frac{1}{2h}\int_{-h}^{h} P/G\,dz = 0\ \ldots \tag{B-11}$$

$$\frac{1}{2h}\int_{-h}^{h}\frac{\partial u}{\partial z}\,dz + \frac{1}{2h}\int_{-h}^{h}\frac{\partial w}{\partial r}\,dz = 0\ \ldots. \tag{B-12}$$

Substituting equations (B-7), (B-8), and (B-9) into (B-11) and (B-12), the last equation is satisfied identically and equation (B-11) yields

$$\frac{4}{3}\,(a + br^{-2}) + \frac{2}{h^2}\left(\frac{ar^2}{2} - b\ln r\right) + \frac{4a}{3} + C_2 = 0$$

or

$$\frac{ar^2}{h^2} - \frac{2b}{h^2}\ln r + \frac{4}{3}\,br^{-2} + \frac{8}{3}\,a + C_2 = 0\,. \tag{B-13}$$

Hence,

$$\left\{\begin{array}{l} a\,\dfrac{r_i^2}{h^2} - \dfrac{2b}{h^2}\,\ln r_i + \dfrac{4}{3}\,b r_i^{-2} + \dfrac{8}{3}\,a + C_2 = 0 \\[2ex] a\,\dfrac{r_o^2}{h^2} - \dfrac{2b}{h^2}\,\ln r_o + \dfrac{4}{3}\,b r_o^{-2} + \dfrac{8}{3}\,a + C_2 = 0 \end{array}\right\} \qquad \text{(B-13a)}$$

$$b = \frac{a\,h^2\left(\dfrac{r_o^2}{h^2} - \dfrac{r_i^2}{h^2}\right)}{2\ln r_o/r_i + \dfrac{4}{3}\left[\dfrac{h^2}{r_i^2} - \dfrac{h^2}{r_o^2}\right]} \qquad \text{(B-14)}$$

$$C_2 = -\left[a\,\frac{r_i^2}{h^2} - \frac{2b}{h^2}\,\ln r_i + \frac{8}{3}\,a + \frac{4}{3}\,b\,r_i^{-2}\right]. \qquad \text{(B-15)}$$

The pressures at the top and bottom boundaries (bonded surface) can be calculated from equation (B-9).

$$P/G\Big|_{z\,=\,\pm h} = a\,\frac{r^2}{h^2} - \frac{2b}{h^2}\,\ln r + C_2 \qquad \text{(B-16)}$$

where a is substituted from (B-10) and b and C_2 from above.

The total axial force on the bearing is computed by integrating equation (B-16) over the surface of z = h (the bonded surface).

$$F/G = \int_{r_i}^{r_o} (P/G)\, r\, dr\, d\theta$$

$$= 2\pi \int_{r_i}^{r_o} \left[\frac{ar^3}{h^2} - \frac{2b}{h^2} r \ln r + C_2 r\right] dr$$

$$F/G = \pi \left[r^2 \left(\frac{ar^2}{2h^2} - \frac{2b}{h^2} \ln r + \frac{b}{h^2} + C_2 \right) \right]_{r_i}^{r_o} . \qquad \text{(B-17)}$$

Average pressure on the bonded surface is the total force divided by the area, or

$$\frac{P_{ave}}{G} = \frac{1}{r_o^2 - r_i^2} \left[r^2 \left(\frac{ar^2}{2h^2} - \frac{2b}{h^2} \ln r + \frac{b}{h^2} + C_2 \right) \right]_{r_i}^{r_o} . \qquad \text{(B-18)}$$

To find the maximum pressure on the bonded surface (z = h), radius is found at which pressure is maximum by differentiating equations (B-17) and then substituting that radius into equation (B-16).

$$\frac{1}{G}\frac{dP}{dr} = \frac{2ar}{h^2} - \frac{2b}{h^2}\frac{1}{r} = 0$$

and

$$r = \sqrt{\frac{b}{a}} \quad . \qquad \text{(B-19)}$$

Now

$$\left.\frac{P_{max}}{G}\right|_{z = h} = \frac{b}{h^2}\left[1 - 2 \ln\sqrt{b/a}\right] + C_2 . \qquad \text{(B-20)}$$

The stress distribution at the inner and outer radial boundaries is now approximated by

$$\frac{\sigma_r}{G} = 2\,\frac{\partial u}{\partial r} + \frac{P}{G}\,, \tag{B-21}$$

where from equation (B-7)

$$\frac{\partial u}{\partial r} = \left(a + \frac{b}{r^2}\right)\left(1 - \frac{z^2}{h^2}\right),$$

and by

$$\frac{\sigma_{rz}}{G} = \frac{\partial u}{\partial z}, \tag{B-22}$$

where from equation (B-7)

$$\frac{\partial u}{\partial z} = -\left(ar - \frac{b}{r}\right)\frac{2z}{h^2}\,.$$

Consider now a typical example:

r_i = 0.75 inch

r_o = 1.125 inches

h = 0.75 (10^{-3}) inch
(width-to-thickness ratio of 250).

If the ratio of axial compressive deformation to the elastomer thickness is chosen as

$$\frac{w_o}{h} = (10^{-3})\,,$$

then

$$a = 0.75\ (10^{-3})$$

$$b \approx 1.156\ (10^{-3})\ h$$

$$C_2 \approx -\ 1.415\ (10^{-3})\,.$$

The pressure on the bonded surfaces can be computed from equation (B-9):

$$P/G\Big|_{z=h} = a\,\frac{r^2}{h^2} - \frac{2b}{h^2}\,\ln r + C_2$$

$$= 0.75(10^{-3})\,\frac{r^2}{h^2} - 2.312(10^{-3})\,\ln r - 1.415(10^{-3})$$

and

r	r/h	ln r	P
0.75"	10^3	-0.2877	0
0.825"	1.1 (10^3)	-0.1924	$0.0629(10^3)G$
0.90"	1.2 (10^3)	-0.1054	$0.0915(10^3)G$
0.975"	1.3 (10^3)	-0.0253	$0.0892(10^3)G$
1.05"	1.4 (10^3)	+0.0488	$0.058(10^3)G$
1.125"	1.5 (10^3)	+0.1178	0

Average pressure is computed from equation (B-18):

$$\frac{P_{ave}}{G}\Big|_{z=h} = \frac{1}{r_o^2 - r_i^2}\left[r^2\left(\frac{ar^2}{2h} - \frac{2b}{h^2}\,\ln r + \frac{b}{h^2} + C_2\right)\right]_{r_i}^{r_o}$$

$$= -\,0.0629\,(10^{-3})$$

or, in magnitude,

$P_{ave} = 0.0629\,(10^{-3})G$ (psi when G is in units of psi).

The total axial force applied is the average pressure multiplied by the cross-sectional area of 2.2 inches^2.

$F = 0.1384\,(10^3)\,G$ (pounds when G is in units of psi)

Maximum pressure occurs from equation (B-19) at

$$r = \sqrt{\;/a} = 0.931 \text{ inch}$$

and is of magnitude, from equation (B-20),

$$P_{max} = -0.0941\ (10^3)\ G\ (psi).$$

The ratio of $\frac{P_{max}}{P_{ave}}$ is 1.5.

Radial stress on the inner radial boundary is

$$\frac{\sigma_r}{G} = 2\frac{\partial u}{\partial r} + \frac{P}{G}$$

z/h	σ_r
0	1.812 (10^{-3}) G (midway between bonded surface)
1/4	1.427 (10^{-3}) G
1/2	0.485 (10^{-3}) G
3/4	-1.176 (10^{-3}) G
1	-3.5 (10^{-3}) G .

Shear stress on the inner radial boundary is, from equation (B-22),

$$\frac{\sigma_{rz}}{G} = \frac{\partial u}{\partial z}$$

or

$$\sigma_{rz} = 0.812\ \frac{z}{h}\ G\ (psi).$$

This example is also a test of the approximation. From the above calculations for stress on the inner boundary, it can be seen that radial stress, σ_r, is of the order of $(10^{-3}\ G)$; and with G = 100, for example, the radial stress is still of the order 0.1 psi. This is small as compared to the pressure developed within the bearing. Shear stress on the boundary varies from zero to 0.812 G. This is not small as compared to other shear stresses; but since a shear stress on this boundary does not influence greatly the pressure within the bearing, this is not a source of serious error. The approximation is considered to be very good for a wide, thin annulus.

COMPUTER SOLUTION

Due to symmetry with respect to z = 0, only half of the bearing need be considered; if z is positive, the symmetry condition then states

$$\left.\begin{aligned} & w\,(r,0) = 0 \\ & \left.\frac{\partial u}{\partial z}\right|_{z=0} = 0 \\ & \left.\frac{\partial P}{\partial z}\right|_{z=0} = 0 \\ & w(r,0^{+}) = -\,w(r,0^{-}) \end{aligned}\right\} . \qquad \text{(B-23)}$$

To solve this set of equations (B-4a, b, and c) by the finite-difference method, it is convenient to find an equation involving P explicitly. This can be achieved by differentiating between equations (B-4a) and (B-4b) rather than by using equation (B-4c). Thus,

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{\partial^2 P}{\partial z^2} = 0 \;\ldots\ldots \qquad \text{(B-24)}$$

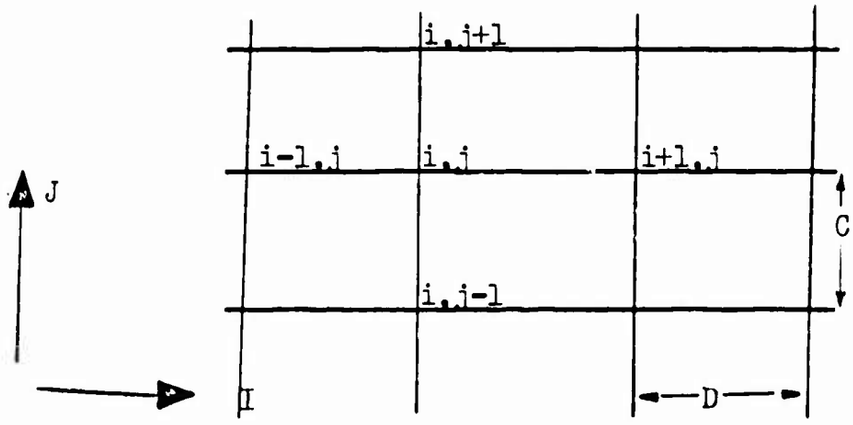
It is noted that equation (B-24) implies

$$\nabla^2 \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right) = 0 \qquad \text{(B-25)}$$

rather than

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \qquad \text{(B-26)}$$

It is thus necessary to specify $\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$ on the boundary.



Written in finite-difference form, equations (B-4a), (B-4b), and (B-24) become

$$\left[2\left(1+\frac{D^2}{C^2}\right)+\frac{D^2}{R(I)}\right]u_{i,j} = (u_{i+1,j}+u_{i-1,j})+\frac{D}{2R(I)}(_{i+1,j}-U_{i-1,j})$$

$$\frac{D^2}{C^2}(u_{i,j+1}+u_{i,j-1})+\frac{D}{2H}(P_{i+1,j}-P_{i-1,j}) \qquad \text{(B-27a)}$$

$$2\left(1+\frac{D^2}{C^2}\right)w_{i,j} = (w_{i+1,j}+w_{i-1,j})+\frac{D}{2R(I)}(w_{i+1,j}-w_{i-1,j})$$

$$+\frac{D^2}{C^2}(w_{i,j+1}+w_{i,j-1})+\frac{D^2}{2C,H}(P_{i,j+1}-P_{i,j-1}) \qquad \text{(B-27b)}$$

$$2\left(1+\frac{D^2}{C^2}\right)P_{i,j} = (P_{i+1,j}+P_{i-1,j})+\frac{D}{2R(I)}(P_{i+1,j}-P_{i-1,j})$$

$$+\frac{D^2}{C^2}(P_{i,j+1}+P_{i,j-1}) \qquad \text{(B-27c)}$$

where

$P = P/G$; $U_{i,j} = \frac{u}{H}$; $w_{i,j} = \frac{w}{H}$; and

$H = h$.

Treatment of equations at boundary:

Since the boundary conditions involve derivatives, fictitious points are introduced outside. The values of those points are found from boundary conditions.

At $r = r_i$

$$\left.\begin{aligned} w_{1,j} &= w_{3,j} - u_{2,j+1} + u_{2,j-1} \\ P_{2,j} &= u_{1,j} - u_{3,j} \\ u_{1,j} &= u_{3,j} + \frac{2}{R(Z)}\, u_{2,j} + \frac{D}{C}\,(w_{2,j+1} - w_{2,j-1}) \end{aligned}\right\}. \qquad \text{(B-28)}$$

At $r = r_o$

$$\left.\begin{aligned} w_{NNN,j} &= w_{NE,j} + u_{NN,j+1} - u_{NN,j-1} \\ P_{NN,j} &= u_{NE,j} - u_{NNN,j} \\ u_{NNN,j} &= u_{NE,j} - \frac{R}{R(NN)}\, u_{NN,j} - \frac{D}{C}\,(w_{NN,j+1} - w_{NN,j-1}) \end{aligned}\right\}. \qquad \text{(B-29)}$$

These equations, coupled with field equations (B-4a) and (B-4b), are enough to solve for the values at boundaries r_i, r_o and the fictitious points. However, it is not desirable to indroduce any fictitious points with P as unknown. Hence, $\frac{\partial P}{\partial r}$ is written in terms of forward or backward differences, and equation (B-4a) is modified accordingly.

$$\frac{\partial P}{\partial r} = \frac{1}{2D}\,(r\, P_{i+1,j} - 3\, P_{i,j} - P_{i+2,j}) \quad \text{(forward difference).} \qquad \text{(B-30)}$$

$$\frac{\partial P}{\partial r} = \frac{1}{2D}\,(3\, P_{i,j} - 4\, P_{i-1,j} + P_{i-2,j}) \quad \text{(backward difference).} \qquad \text{(B-31)}$$

At $z = h$, u and w are given but P is found through equation (B-24). In order to achieve this, it is necessary to introduce a fictitious point for P. The value of P at this point is found by using equations (B-4c) and (B-4b).

This set of finite-difference equations is solved by using the successive relaxation method. Values of u, w, and P at one point are calculated from values at neighboring points; they are u^{eva}, w^{eva}, and P^{eva}. New values of this point are obtained from u^{eva}, etc., through

$$u^{new} = \alpha\, u^{eva} + (1 - \alpha)\, u^{old}.$$

These new values can then be substituted for the old values immediately; then one can proceed to the next point.

Note on computer data:

Owing to the limitation of computer memory, the dimension of the bearing has been changed to

$$r_i = 0.75 \text{ inch}$$

$$(r_o - r_i)/2h = 50$$

$$h = 1 \times 10^{-3} \text{ inches}.$$

The approximate solution is used as the first guess.

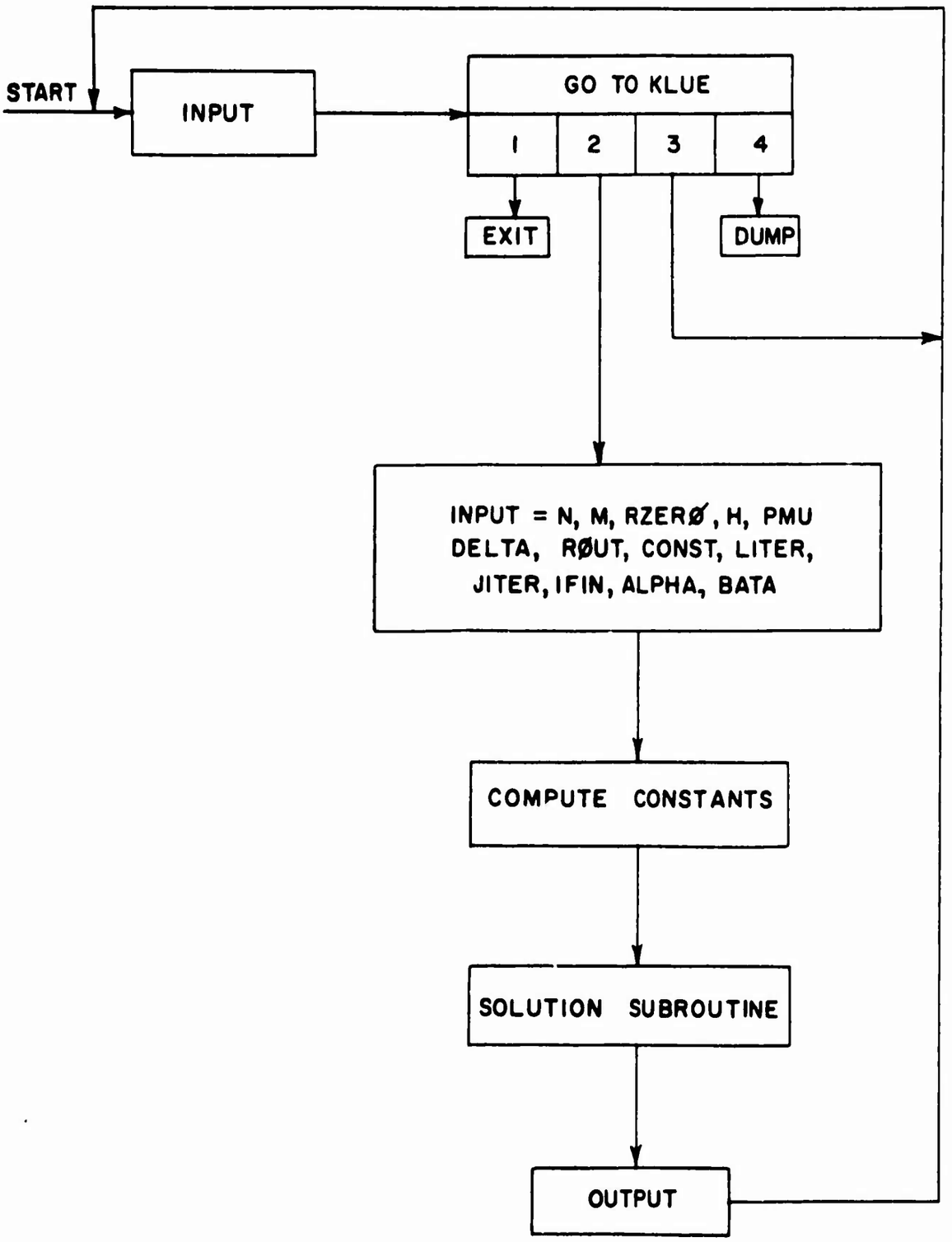


Figure 8. Main Flow Chart.

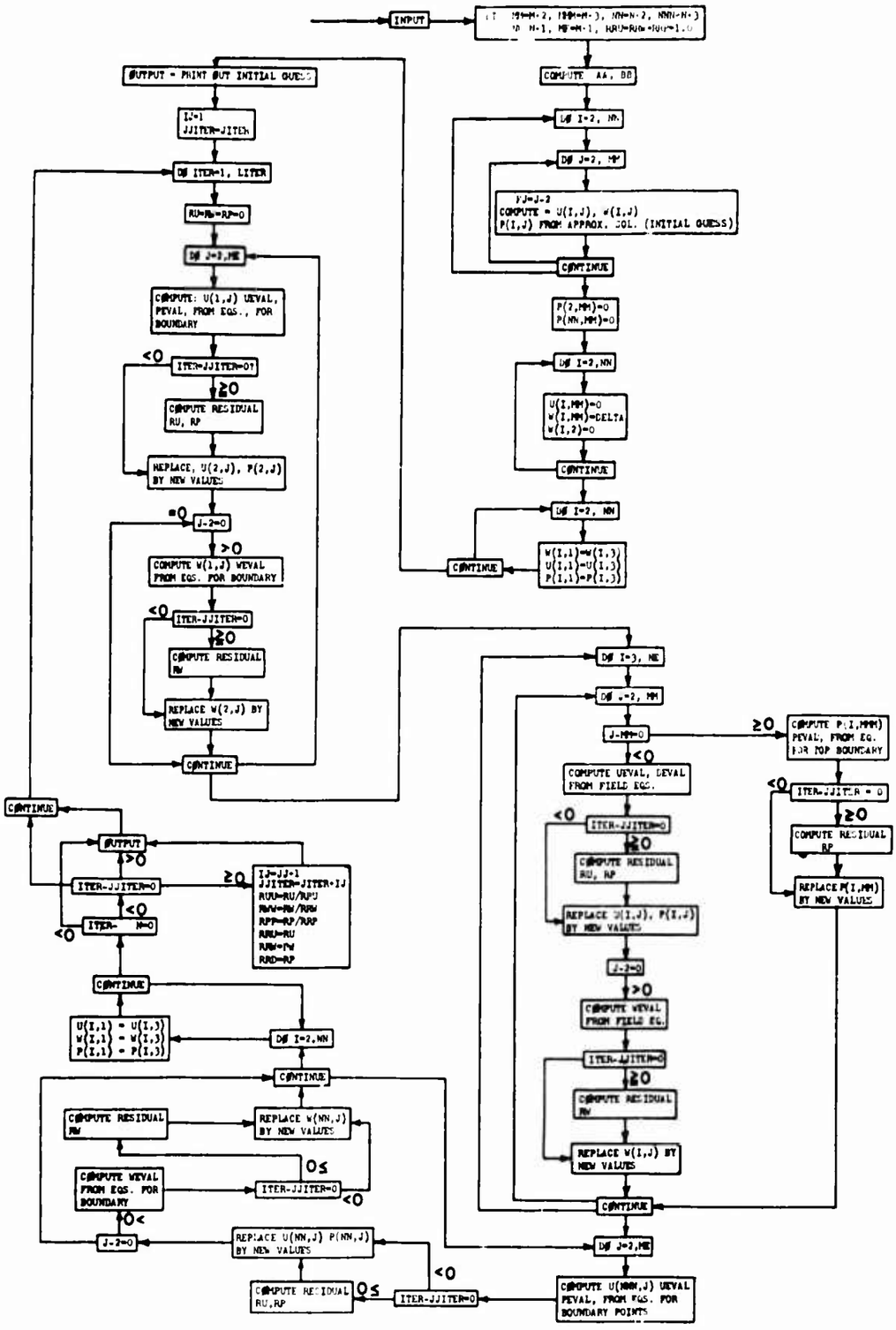


Figure 9. Solution Subroutine Flow Chart.

LAMINATED BEARING

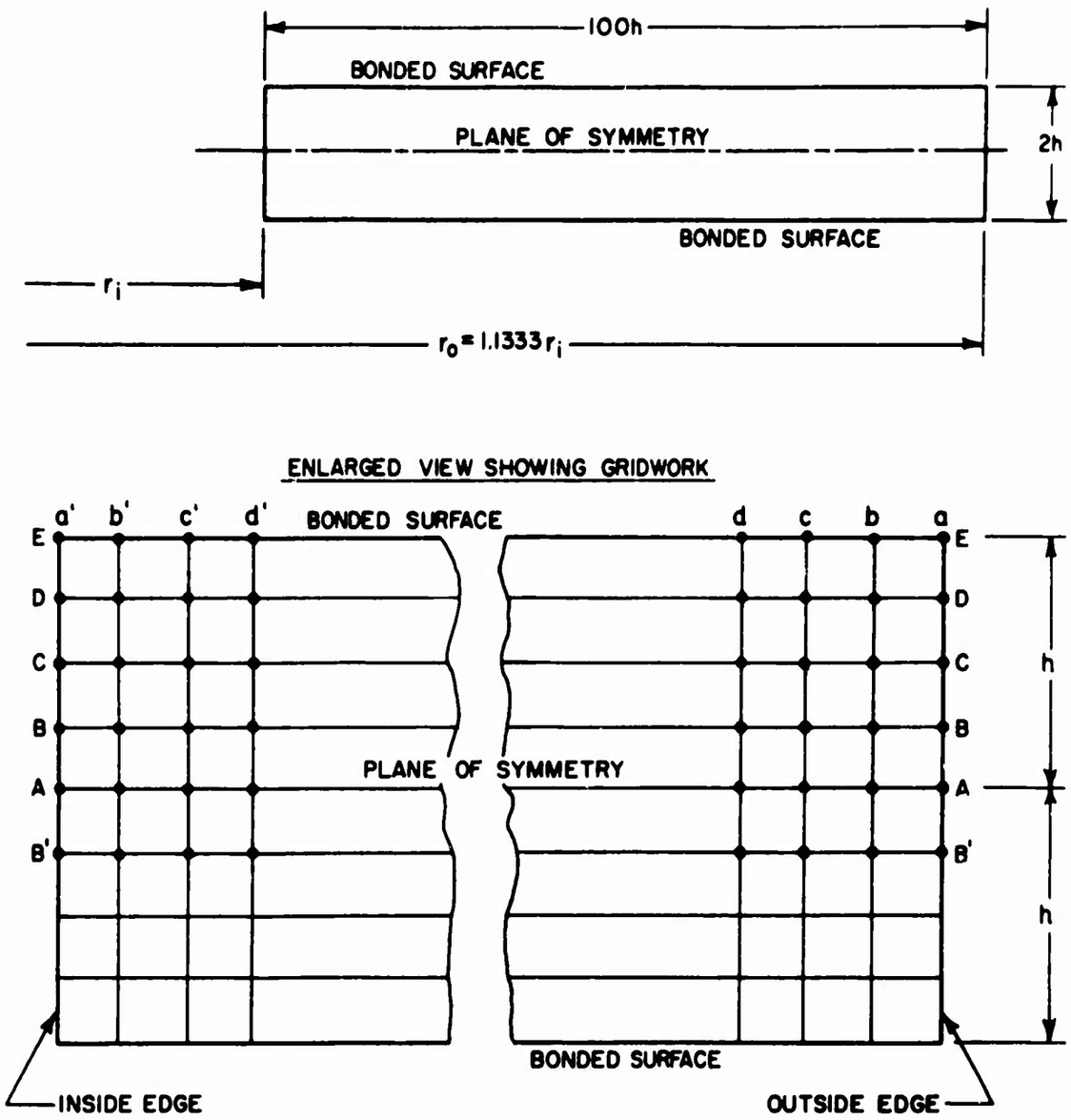
```
C
C  INCOMPRESSIBLE MATERIAL
C
      DIMENSION RO(500), B(8), AR1(500), AR2(500), AR3(500), AR4(500)
      COMMON  M, RZERO, H, RO, N, LITER, ALPHA, DELTA, BATA,ROUT,  B,C,D
     1,IFIN,JITER,AR1,AR2,AR3,AR4 ,CONST
  110 READ INPUT TAPE 2,1, KLUE
    1 FORMAT (I5)
      GO TO (500,100,300,400),KLUE
  100 READ INPUT TAPE 2,2,    N, M, RZERO, H, PMU, DELTA,ROUT,CONST
    2 FORMAT (2I5,4E15.8/2E15.8)
      READ INPUT TAPE 2,3,    LITER, JITER, IFIN, ALPHA, BATA
    3 FORMAT (3I5,2E15.8)
      WRITE OUTPUT TAPE 10,10, RZERO, H, PMU, DELTA,ROUT,CONST
   10 FORMAT (1H1 3X 8HRZERO = F10.5, 3X 4HH = F10.5,3X 6HPMU = F10.5,
     13X 8HDELTA = F10.5/3X 7HROUT = F10.2,3X 8HCONST = F10.5//)
      WRITE OUTPUT TAPE 10,11, N, M, LITER, JITER, IFIN, BATA, ALPHA
   11 FORMAT (1X 4HN = I5,3X 4HM = I5, 3X 8HLITER = I5, 3X 8HJITER = I5/
     13X 7HIFIN =  I5, 3X 7HBATA = F10.5, 3X 8HALPHA = F10.5////)
      FM = M
      NN = N+2
      C = H/FM
      D = C*BATA
      B(1) = D/C
      B(2) = B(1)**2
      B(3) = 1.0 + B(2)
      B(4) = 1.0/C
      B(5) = 1.0/D
      B(6) = C*B(1)
      B(7) = B(4)*B(1)
      B(8) = 1.0/B(3)
      DO 150 I = 2,NN
      FI = I - 2
      RO(I) = RZERO + FI*D
      AR1(I) = 1.0/RO(I)
      AR2(I) = D*AR1(I)
      AR3(I) = 2.0*B(3) + AR2(I)**2
      AR4(I) = 1.0/AR3(I)
  150 CONTINUE
      CALL PDUMP (RO,RO(10) ,1,B,B(8),1,AR1,AR1(10),1,AR2,AR2(10),
     11,AR3,AR3(10),1,AR4,AR4(10),1)
      CALL SOLUT
      GO TO 110
  500 CALL EXIT
  300 CONTINUE
  400 CALL DUMP
      END(1,1,0,0,0,1,1,1,0,1,0,0,0,0,0)
```

```
      SUBROUTINE  SOLUT
      DIMENSION  RO(500), U(500,13),W(500,13), P(500,13)
     1,B(8),AR1(500),AR2(500),AR3(500),AR4(500)
      COMMON  M, RZERO, H, RO, N, LITER, ALPHA, DELTA, BATA,ROUT,  B,C,D
     1,IFIN,JITER,AR1,AR2,AR3,AR4 ,CONST
      MM = M+2
      NN = N+2
      MMM = M+3
      NNN = N+3
      NE = N+1
      ME = M+1
      RRU = 1.0
      RRW = 1.0
      RRP = 1.0
      AA = 0.75*DELTA
      BB= AA*(ROUT**2 - RZERO**2) / (2.0*LOGF(ROUT/RZERO) -(4.0/3.0)*
     1((1.0 /ROUT)**2 - (1.0 /RZERO)**2))
      DO 10 I = 2,NN
      DO 10 J=2,MM
      FJ = J - 2
      U(I,J) = (AA * RO(I) - BB * AR1(I))* (1.0 -(FJ*C)**2)
      W(I,J) = - 2.0 * AA * (FJ*C) * (1.0 - (FJ*C)**2/3.0)
      P(I,J) = AA* (RO(I)**2 - RZERO**2)+2.0*BB* LOGF(RZERO*AR1(I))
     1+2.0 * AA* (1.0 - (FJ*C)**2) - 8.0*AA/3.0-4.0*BB/(3.0*RZERO**2)
   10 CONTINUE
      P(2,MM) = 0.0
      P(NN,MM) = 0.0
      DO 20 I = 2,NN
      U(I,MM) = 0.0
      W(I,MM) = -DELTA
      W(I,2)  = 0.0
   20 CONTINUE
      DO 30 I = 2,NN
      W(I,1) = -W(I,3)
      U(I,1) = U(I,3)
      P(I,1) = P(I,3)
   30 CONTINUE
      INCR = 1
      JJ=1
      KK=MM-6
 1128 IF (KK) 1130,1131,1131
 1130 JJJ=MM
      GO TO 1151
 1131 JJJ=INCR * 6
 1151 WRITE OUTPUT TAPE 10,3
      WRITE OUTPUT TAPE 10,2,((U(I,J), J=JJ,JJJ),I=1,NNN)
      WRITE OUTPUT TAPE 10,4
      WRITE OUTPUT TAPE 10,2,((W(I,J), J=JJ,JJJ),I=1,NNN)
      WRITE OUTPUT TAPE 10,5
      WRITE OUTPUT TAPE 10,2,((P(I,J),J=JJ,JJJ),I=1,NNN)
      IF (JJJ-MM) 1155,1160,1155
 1155 KK=KK-6
      INCR = INCR + 1
      JJ = JJ+6
      GO TO 1128
 1160 CONTINUE
```

```
      IJ = 1
      JJITER = JITER
      DO 200 ITER = 1,LITER
      RU = 0.0
      RW = 0.0
      RP = 0.0
      DO 35 J = 2,ME
      U(1,J) = U(3,J) + 2.0*AR2(2)*U(2,J) + B(1)*(W(2,J+1)-W(2,J-1))
      UEVAL  = AR4(2)*((U(3,J) + U(1,J)) + 0.5*AR2(2)*(U(3,J)-U(1,J))
     1+ B(2)*(U(2,J+1) + U(2,J-1)) + 0.5*D*(4.0*P(3,J) - 3.0*P(2,J)
     1-P(4,J)))
      PEVAL  = B(5)*(U(1,J) -U(3,J))
      IF ( ITER - JJITER) 34,29,29
   29 UU = ABSF(UEVAL - U(2,J))
      PP = ABSF(PEVAL - P(2,J))
      IF (RU - UU) 31,32,32
   31 RU = UU
   32 IF (RP - PP) 33,34,34
   33 RP = PP
   34 U(2,J)= ALPHA * UEVAL       + (1.0 - ALPHA) * U(2,J)
      P(2,J)= ALPHA * PEVAL       + (1.0 - ALPHA) * P(2,J)
      IF(J-2) 45,40,45
   40 GO TO 35
   45 W(1,J) =W(3,J)+B(1)*(U(2,J+1)-U(2,J-1))
      WEVAL = 0.5* B(8)           *((W(3,J)+W(1,J))+0.5*AR2(2)
     1*(W(3,J)-W(1,J))+B(2)      *(W(2,J+1)+W(2,J-1))+0.5*B(6)
     1*(P(2,J+1)-P(2,J-1)))
      IF ( ITER - JJITER) 38,36,36
   36 WW = ABSF(WEVAL - W(2,J))
      IF (RW - WW) 37,38,38
   37 RW = WW
   38 W(2,J)= ALPHA * WEVAL       + (1.0 - ALPHA) * W(2,J)
   35 CONTINUE
      DO 100 I = 3,NE
      DO 80  J = 2,MM
      IF(J - MM) 60,65,65
   60 UEVAL = AR4(I) * ((U(I+1,J)+U(I-1,J))+0.5*AR2(I)*(U(I+1,J)
     1-U(I-1,J))+B(2)*(U(I,J+1)+U(I,J-1))+0.5*D*(P(I+1,J)-P(I-1,J)))
      PEVAL = 0.5*B(8)*((P(I+1,J)+P(I-1,J))+0.5*AR2(I)*(P(I+1,J)
     1-P(I-1,J))+B(2)*(P(I,J+1)+P(I,J-1)))
      IF ( ITER - JJITER) 64,59,59
   59 UU = ABSF(UEVAL - U(I,J))
      PP = ABSF(PEVAL - P(I,J))
      IF (RU - UU) 61,62,62
   61 RU = UU
   62 IF (RP - PP) 63,64,64
   63 RP = PP
   64 U(I,J)= ALPHA * UEVAL       + (1.0 - ALPHA) * U(I,J)
      P(I,J)= ALPHA * PEVAL       + (1.0 - ALPHA) * P(I,J)
      IF (J-2) 75,70,75
   70 GO TO 80
   75 WEVAL = 0.5*B(8)*((W(I+1,J)+W(I-1,J))+0.5*AR2(I)*(W(I+1,J)
     1-W(I-1,J))+B(2)*(W(I,J+1)+W(I,J-1))+0.5*B(6)*(P(I,J+1)-P(I,J-1)))
      IF ( ITER - JJITER) 77,74,74
   74 WW = ABSF(WEVAL - W(I,J))
      IF (RW - WW) 76,77,77
```

```
 76 RW = WW
 77 W(I,J)= ALPHA * WEVAL      + (1.0 - ALPHA) * W(I,J)
    GO TO 80
 65 P(I,MMM) =P(I,ME)+4.0*B(4)*(W(I,MM)-W(I,ME))
    PEVAL = 0.5*B(8)*((P(I+1,MM)+P(I-1,MM))+0.5*AR2(I)*(P(I+1,MM)-
   1P(I-1,MM))+ B(2)*(P(I,MMM)+P(I,ME)))
    IF ( ITER - JJITER) 79,81,81
 81 PP = ABSF(PEVAL - P(I,J))
    IF (RP - PP) 78,79,79
 78 RP = PP
 79 P(I,MM) = ALPHA * PEVAL + (1.0 - ALPHA) * P(I,MM)
 80 CONTINUE
100 CONTINUE
    DO 95 J=2,ME
    U(NNN,J) = U(NE,J) - 2.0*AR2(NN)*U(NN,J) - B(1)*(W(NN,J+1)
   1-W(NN,J-1))
    UEVAL    = AR4(NN)*((U(NNN,J) + U(NE,J)) + 0.5*AR2(NN)*(U(NNN,J)
   1-U(NE,J)) + B(2)*(U(NN,J+1) + U(NN,J-1)) + 0.5*D*( 3.0*P(NN,J) -
   14.0*P(NE,J) + P(N,J)))
    PEVAL = B(5)*(U(NE,J) - U(NNN,J))
    IF ( ITER - JJITER) 85,86,86
 86 UU = ABSF(UEVAL-U(NN,J))
    IF (RU - UU) 82,83,83
 82 RU = UU
 83 PP = ABSF(PEVAL-P(NN,J))
    IF (RP - PP) 84,85,85
 84 RP = PP
 85 U(NN,J) = ALPHA * UEVAL + (1.0 - ALPHA) * U(NN,J)
    P(NN,J) = ALPHA * PEVAL + (1.0 - ALPHA) * P(NN,J)
    IF (J-2) 88,87,88
 87 GO TO 95
 88 W(NNN,J) =W(NE,J)-B(1)*(U(NN,J+1)-U(NN,J-1))
    WEVAL = 0.5*B(8)*((W(NNN,J)+W(NE,J))+0.5*AR2(NN)
   1*(W(NNN,J)-W(NE,J))+B(2)      *(W(NN,J+1)+W(NN,J-1))+0.5*B(6)
   1*(P(NN,J+1)-P(NN,J-1)))
    IF ( ITER - JJITER) 93,89,89
 89 WW = ABSF(WEVAL-W(NN,J))
    IF (RW - WW) 91,93,93
 91 RW = WW
 93 W(NN,J) = ALPHA * WEVAL + (1.0 - ALPHA) * W(NN,J)
 95 CONTINUE
    DO 110 I = 2,NN
    U(I,1) = U(I,3)
    W(I,1) = -W(I,3)
    P(I,1) = P(I,3)
110 CONTINUE
    IF (ITER-IFIN) 120,125,125
120 IF (ITER-JJITER) 200,135,135
135 IJ=IJ+1
    JJITER = IJ*JITER
    RUU = RU/RRU
    RWW = RW/RRW
    RPP = RP/RRP
    RRU = RU
    RRW = RW
    RRP = RP
```

```
125 INCR = 1
    JJ = 1
    KK = MM  - 6
128 IF (KK) 130,131,131
130 JJJ= MM
    GO TO 150
131 JJJ = INCR * 6
150 WRITE OUTPUT TAPE 10,1, ITER
  1 FORMAT (1H1 17HITERATION  NO. = I5///)
    WRITE OUTPUT TAPE 10,3
  3 FORMAT ( 7HU ARRAY//)
    WRITE OUTPUT TAPE 10,2,((U(I,J), J=JJ,JJJ),I=1,NNN)
    WRITE OUTPUT TAPE 10,4
  4 FORMAT (//7HW ARRAY//)
    WRITE OUTPUT TAPE 10,2,((W(I,J), J=JJ,JJJ),I=1,NNN)
    WRITE OUTPUT TAPE 10,5
  5 FORMAT (//7HP ARRAY//)
    WRITE OUTPUT TAPE 10,2,((P(I,J),J=JJ,JJJ),I=1,NNN)
  2 FORMAT (6(4X E15.8))
    WRITE OUTPUT TAPE 10,6, RU, RUU, RW, RWW, RP, RPP
  6 FORMAT (//2X 5HRU = E12.6, 2X 6HRUU = E12.6, 2X 5HRW = E12.6,
   12X 6HRWW = E12.6, 2X 5HRP = E12.6, 2X 6HRPP = E12.6)
    IF (JJJ - MM) 155,160,155
155 KK = KK-6
    INCR = INCR + 1
    JJ = JJ+6
    GO TO 128
160 CONTINUE
200 CONTINUE
    RETURN
    END(1,1,0,0,0,1,1,1,0,1,0,0,0,0,0)
```



Notes

1. Upper case letters on grid, Figure 8 refer to columns on computer output.
2. Lower case letters refer to row from either end as noted.
3. U array denotes radial displacement.
   example: 0.73755702 E - 01 means μ= 0.073755h
4. W array denotes axial displacement.
   example: 0.43044931 E - 02 means w = 0.004304h
5. P array denotes pressure.
   example: 0.13768537 E - 00 means P = 0.13768G

Figure 10. Geometry and Key to Computer Output.

ITERATION NO. = 600

U ARRAY

	Column B'	Column A	Column B	Column C	Column D	Column E
Inside Edge	0.	-0.41229375E-01	-0.41602026E-01	-0.40751156E-01	-0.64828150E-01	0.
Row a'	-0.71237640E-01	-0.76150560E-01	-0.71237640E-01	-0.56640506E-01	-0.32501135E-01	0.
Row b'	-0.71714811E-01	-0.76726006E-01	-0.71714811E-01	-0.56879040E-01	-0.32507309E-01	0.
Row c'	-0.69070283E-01	-0.73725202E-01	-0.69070283E-01	-0.55175163E-01	-0.32107724E-01	0.
Row d'	-0.67316884E-01	-0.71809946E-01	-0.67316884E-01	-0.53843167E-01	-0.31600556E-01	0.
	-0.65790301E-01	-0.70177127E-01	-0.65790301E-01	-0.52530898E-01	-0.30700341E-01	0.
	-0.64309352E-01	-0.68596926E-01	-0.64309352E-01	-0.51447152E-01	-0.30010595E-01	0.
	-0.62835054E-01	-0.67026239E-01	-0.62835054E-01	-0.50267886E-01	-0.29322827E-01	0.
	-0.61359814E-01	-0.65450599E-01	-0.61359814E-01	-0.49087747E-01	-0.28634455E-01	0.
	-0.59882776E-01	-0.63875055E-01	-0.59882776E-01	-0.47906150E-01	-0.27945212E-01	0.
	-0.58404551E-01	-0.62298249E-01	-0.58404551E-01	-0.46723595E-01	-0.27255405E-01	0.
	-0.56926022E-01	-0.60721120E-01	-0.56926022E-01	-0.45540794E-01	-0.26565450E-01	0.
	-0.55448083E-01	-0.59144630E-01	-0.55448083E-01	-0.44358461E-01	-0.25875766E-01	0.
	-0.53971490E-01	-0.57569576E-01	-0.53971490E-01	-0.43177202E-01	-0.25186707E-01	0.
	-0.52496862E-01	-0.55996624E-01	-0.52496862E-01	-0.41997511E-01	-0.24498561E-01	0.
	-0.51024680E-01	-0.54426286E-01	-0.51024680E-01	-0.40819773E-01	-0.23811550E-01	0.
	-0.49555320E-01	-0.52858963E-01	-0.49555320E-01	-0.39644288E-01	-0.23125853E-01	0.
	-0.48089026E-01	-0.51294913E-01	-0.48089026E-01	-0.38471256E-01	-0.22441586E-01	0.
	-0.46625824E-01	-0.49736165E-01	-0.46625824E-01	-0.37300695E-01	-0.21759760E-01	0.
	-0.45165636E-01	-0.48176631E-01	-0.45165636E-01	-0.36132544E-01	-0.21077339E-01	0.
	-0.43708359E-01	-0.46622209E-01	-0.43708359E-01	-0.34966719E-01	-0.20397272E-01	0.
	-0.42253885E-01	-0.45070775E-01	-0.42253885E-01	-0.33803136E-01	-0.19718513E-01	0.
	-0.40802070E-01	-0.43522176E-01	-0.40802070E-01	-0.32641680E-01	-0.19040994E-01	0.
	-0.39352730E-01	-0.41976219E-01	-0.39352730E-01	-0.31482203E-01	-0.18364630E-01	0.
	-0.37905699E-01	-0.40432724E-01	-0.37905699E-01	-0.30324575E-01	-0.17689344E-01	0.
	-0.36460824E-01	-0.38891528E-01	-0.36460824E-01	-0.29168673E-01	-0.17015067E-01	0.
	-0.35017957E-01	-0.37352473E-01	-0.35017957E-01	-0.28014378E-01	-0.16341726E-01	0.
	-0.33576968E-01	-0.35815421E-01	-0.33576968E-01	-0.26861588E-01	-0.15669261E-01	0.
	-0.32137783E-01	-0.34280293E-01	-0.32137783E-01	-0.25710233E-01	-0.14997539E-01	0.
	-0.30700307E-01	-0.32746980E-01	-0.30700307E-01	-0.24560250E-01	-0.14326815E-01	0.
	-0.29264434E-01	-0.31215392E-01	-0.29264434E-01	-0.23411549E-01	-0.13656738E-01	0.
	-0.27830150E-01	-0.29685492E-01	-0.27830150E-01	-0.22264120E-01	-0.12987404E-01	0.
	-0.26397418E-01	-0.28157245E-01	-0.26397418E-01	-0.21117936E-01	-0.12318797E-01	0.
	-0.24966154E-01	-0.26630561E-01	-0.24966154E-01	-0.19972925E-01	-0.11650874E-01	0.
	-0.23536307E-01	-0.25105394E-01	-0.23536307E-01	-0.18829048E-01	-0.10983612E-01	0.
	-0.22107886E-01	-0.23581743E-01	-0.22107886E-01	-0.17686310E-01	-0.10317014E-01	0.
	-0.20680898E-01	-0.22059624E-01	-0.20680898E-01	-0.16544719E-01	-0.96510866E-02	0.
	-0.19255328E-01	-0.20539017E-01	-0.19255328E-01	-0.15404263E-01	-0.89858206E-02	0.
	-0.17831162E-01	-0.19019906E-01	-0.17831162E-01	-0.14264930E-01	-0.83212099E-02	0.
	-0.16408396E-01	-0.17502289E-01	-0.16408396E-01	-0.13126717E-01	-0.76572518E-02	0.
	-0.14987014E-01	-0.15986148E-01	-0.14987014E-01	-0.11989611E-01	-0.69939406E-02	0.
	-0.13567010E-01	-0.14471477E-01	-0.13567010E-01	-0.10853608E-01	-0.63312714E-02	0.
	-0.12148383E-01	-0.12959276E-01	-0.12148383E-01	-0.97187060E-02	-0.56692447E-02	0.
	-0.10731136E-01	-0.11446545E-01	-0.10731136E-01	-0.85849085E-02	-0.50078627E-02	0.
	-0.93152600E-02	-0.99362777E-02	-0.93152600E-02	-0.74522085E-02	-0.43471216E-02	0.
	-0.79007465E-02	-0.84276638E-02	-0.79007465E-02	-0.63205979E-02	-0.36870158E-02	0.
	-0.64875924E-02	-0.69200983E-02	-0.64875924E-02	-0.51900745E-02	-0.30275438E-02	0.
	-0.50757927E-02	-0.54141794E-02	-0.50757927E-02	-0.40606344E-02	-0.23687032E-02	0.
	-0.36653633E-02	-0.39097209E-02	-0.36653633E-02	-0.29322902E-02	-0.17105029E-02	0.
	-0.22562941E-02	-0.24067135E-02	-0.22562941E-02	-0.18050358E-02	-0.10529377E-02	0.
	-0.84857836E-03	-0.90515018E-03	-0.84857836E-03	-0.67886223E-03	-0.39600253E-03	0.
	0.55779205E-03	0.59497707E-03	0.55779205E-03	0.44623374E-03	0.26030283E-03	0.
	0.19628300E-02	0.20936853E-02	0.19628300E-02	0.15702637E-02	0.91598744E-03	0.
	0.33665368E-02	0.35909724E-02	0.33665368E-02	0.26932302E-02	0.15710516E-02	0.
	0.47689098E-02	0.50868377E-02	0.47689098E-02	0.38151276E-02	0.22254904E-02	0.
	0.61699547E-02	0.65812863E-02	0.61699547E-02	0.49359626E-02	0.24793108E-02	0.
	0.75696887E-02	0.80743359E-02	0.75696887E-02	0.60557506E-02	0.35325208E-02	0.
	0.89681058E-02	0.95659795E-02	0.89681058E-02	0.71744444E-02	0.41851154E-02	0.
	0.10365207E-01	0.11056221E-01	0.10365207E-01	0.82921661E-02	0.48370969E-02	0.
	0.11761000E-01	0.12545068E-01	0.11761000E-01	0.94088005E-02	0.54884667E-02	0.
	0.13155483E-01	0.14032514E-01	0.13155483E-01	0.10524386E-01	0.61392260E-02	0.
	0.14548653E-01	0.15518563E-01	0.14548653E-01	0.11638924E-01	0.67893736E-02	0.
	0.15940522E-01	0.17003224E-01	0.15940522E-01	0.12752418E-01	0.74389111E-02	0.
	0.17331091E-01	0.18486497E-01	0.17331091E-01	0.13864873E-01	0.80878425E-02	0.
	0.18720360E-01	0.19968385E-01	0.18720360E-01	0.14976288E-01	0.87361674E-02	0.
	0.20108347E-01	0.21449904E-01	0.20108347E-01	0.16086677E-01	0.93838950E-02	0.
	0.21495059E-01	0.229[illegible]3E-01	0.21495059E-01	0.17196047E-01	0.10031028E-01	0.
	0.22880509E-01	0.244[illegible]7E-01	0.22880509E-01	0.18304408E-01	0.10677572E-01	0.
	0.24264712E-01	0.258[illegible]9E-01	0.24264712E-01	0.19411770E-01	0.11323533E-01	0.
	0.25647675E-01	0.27357519E-01	0.25647675E-01	0.20518141E-01	0.11968916E-01	0.
	0.27029392E-01	0.28831350E-01	0.27029392E-01	0.21623516E-01	0.12613719E-01	0.
	0.28409880E-01	0.30303869E-01	0.28409880E-01	0.22727906E-01	0.13257946E-01	0.
	0.29789177E-01	0.31775118E-01	0.29789177E-01	0.23831344E-01	0.13901618E-01	0.
	0.31167372E-01	0.33245191E-01	0.31167372E-01	0.24933902E-01	0.14544779E-01	0.
	0.32544500E-01	0.34714126E-01	0.32544500E-01	0.26035607E-01	0.15187441E-01	0.
	0.33920614E-01	0.36181980E-01	0.33920614E-01	0.27136499E-01	0.15829628E-01	0.
	0.35295805E-01	0.37648845E-01	0.35295805E-01	0.28236654E-01	0.16471387E-01	0.
	0.36670161E-01	0.39114821E-01	0.36670161E-01	0.29336143E-01	0.17112758E-01	0.
	0.38043844E-01	0.40580040E-01	0.38043844E-01	0.30435092E-01	0.17753413E-01	0.
	0.39416982E-01	0.42044755E-01	0.39416982E-01	0.31533605E-01	0.18394615E-01	0.
	0.40789706E-01	0.43508988E-01	0.40789706E-01	0.32631787E-01	0.19035223E-01	0.
	0.42162210E-01	0.44972989E-01	0.42162210E-01	0.33729795E-01	0.19675729E-01	0.
	0.43534682E-01	0.46436953E-01	0.43534682E-01	0.34827776E-01	0.20316221E-01	0.
	0.44907254E-01	0.47901025E-01	0.44907254E-01	0.35925838E-01	0.20956758E-01	0.
	0.46280066E-01	0.49365356E-01	0.46280066E-01	0.37024087E-01	0.21597404E-01	0.
	0.47653279E-01	0.50830116E-01	0.47653279E-01	0.38122658E-01	0.22238237E-01	0.
	0.49026918E-01	0.52295333E-01	0.49026918E-01	0.39221568E-01	0.22879268E-01	0.
	0.50400855E-01	0.53760873E-01	0.50400855E-01	0.40320713E-01	0.23520432E-01	0.
	0.51774877E-01	0.55226506E-01	0.51774877E-01	0.41419923E-01	0.24161635E-01	0.
	0.53148619E-01	0.56691866E-01	0.53148619E-01	0.42518906E-01	0.24802701E-01	0.
	0.54521587E-01	0.58156366E-01	0.54521587E-01	0.43617266E-01	0.25443403E-01	0.
	0.55893126E-01	0.59619366E-01	0.55893126E-01	0.44714478E-01	0.26083432E-01	0.
	0.57262438E-01	0.61079994E-01	0.57262438E-01	0.45809904E-01	0.26722618E-01	0.
	0.58628620E-01	0.62537288E-01	0.58628620E-01	0.46902825E-01	0.27359939E-01	0.
	0.59990775E-01	0.63990291E-01	0.59990775E-01	0.47992515E-01	0.27995569E-01	0.
	0.61348274E-01	0.65438339E-01	0.61348274E-01	0.49078459E-01	0.28628996E-01	0.
Row d	0.62701926E-01	0.66882335E-01	0.62701926E-01	0.50161217E-01	0.29260471E-01	0.
Row c	0.64059105E-01	0.68330501E-01	0.64059105E-01	0.51245975E-01	0.29892495E-01	0.
Row b	0.65457626E-01	0.69826590E-01	0.65457626E-01	0.52356053E-01	0.30533299E-01	0.
Row a	0.67080878E-01	0.71593422E-01	0.67080878E-01	0.53579802E-01	0.31179389E-01	0.
	0.69549051E-01	0.74408997E-01	0.69549051E-01	0.55161100E-01	0.31525523E-01	0.
	0.68998894E-01	0.73757702E-01	0.68998894E-01	0.54857050E-01	0.31477478E-01	0.
Outside Edge	0.	0.39838655E-01	0.42142928E-01	0.39400101E-01	0.62748731E-01	0.

W ARRAY

	Column B'	Column A	Column B	Column C	Column D	Column E
	0.	0.	0.77650482E-01	[illegible]	0.22573502E-00	0.
Row a'	-0.44561546E-02	0.	0.44561546E-02	0.70634960E-02	0.85164851E-02	[illegible]
Row b'	0.39795925E-03	0.	-0.39795925E-03	-0.69506700E-03	-0.82696050E-03	-0.99999999E-03
Row c'	0.39510076E-03	0.	-0.39510076E-03	-0.72181053E-03	-0.92927173E-03	-0.99999999E-03
	0.36709151E-03	0.	-0.36709151E-03	-0.68600374E-03	-0.91108605E-03	-0.99999999E-03
	0.36387092E-03	0.	-0.36387092E-03	-0.68208507E-03	-0.90920097E-03	-0.99999999E-03
	0.36364640E-03	0.	-0.36364640E-03	-0.68182776E-03	-0.90909364E-03	-0.99999999E-03
	0.36363532E-03	0.	-0.36363532E-03	-0.68181673E-03	-0.90908969E-03	-0.99999999E-03
	0.36363678E-03	0.	-0.36363678E-03	-0.68181822E-03	-0.90909073E-03	-0.99999999E-03
	0.36363779E-03	0.	-0.36363779E-03	-0.68181941E-03	-0.90909035E-03	-0.99999999E-03
	0.36363515E-03	0.	-0.36363515E-03	-0.68181561E-03	-0.90908911E-03	-0.99999999E-03
	0.36363824E-03	0.	-0.36363824E-03	-0.68182015E-03	-0.90909121E-03	-0.99999999E-03
	0.36363690E-03	0.	-0.36363690E-03	-0.68181858E-03	-0.90909052E-03	-0.99999999E-03
	0.36363646E-03	0.	-0.36363646E-03	-0.68181900E-03	-0.90909066E-03	-0.99999999E-03
	0.36363718E-03	0.	-0.36363718E-03	-0.68181794E-03	-0.90909110E-03	-0.99999999E-03
	0.36363465E-03	0.	-0.36363465E-03	-0.68181944E-03	-0.90909097E-03	-0.99999999E-03
	0.36363690E-03	0.	-0.36363690E-03	-0.68181988E-03	-0.90909022E-03	-0.99999999E-03
	0.36363500E-03	0.	-0.36363500E-03	-0.68181720E-03	-0.90908901E-03	-0.99999999E-03
	0.36363518E-03	0.	-0.36363518E-03	-0.68181788E-03	-0.90909190E-03	-0.99999999E-03
	0.36363602E-03	0.	-0.36363602E-03	-0.68181748E-03	-0.90909089E-03	-0.99999999E-03
	0.36364060E-03	0.	-0.36364060E-03	-0.68182177E-03	-0.90909290E-03	-0.99999999E-03
	0.36363818E-03	0.	-0.36363818E-03	-0.68181935E-03	-0.90909001E-03	-0.99999999E-03
	0.36363576E-03	0.	-0.36363576E-03	-0.68181691E-03	-0.90909047E-03	-0.99999999E-03
	0.36363231E-03	0.	-0.36363231E-03	-0.68181453E-03	-0.90908342E-03	-0.99999999E-03
	0.36363413E-03	0.	-0.36363413E-03	-0.68181606E-03	-0.90909006E-03	-0.99999999E-03
	0.36363038E-03	0.	-0.36363038E-03	-0.68181179E-03	-0.90908637E-03	-0.99999999E-03
	0.36363242E-03	0.	-0.36363242E-03	-0.68181368E-03	-0.90908546E-03	-0.99999999E-03
	0.36363734E-03	0.	-0.36363734E-03	-0.68181939E-03	-0.90909285E-03	-0.99999999E-03
	0.36363440E-03	0.	-0.36363440E-03	-0.68181644E-03	-0.90909044E-03	-0.99999999E-03
	0.36363376E-03	0.	-0.36363376E-03	-0.68181850E-03	-0.90909126E-03	-0.99999999E-03
	0.36363652E-03	0.	-0.36363652E-03	-0.68181518E-03	-0.90908975E-03	-0.99999999E-03
	0.36363440E-03	0.	-0.36363440E-03	-0.68181642E-03	-0.90909030E-03	-0.99999999E-03
	0.36363434E-03	0.	-0.36363434E-03	-0.68181649E-03	-0.90909033E-03	-0.99999999E-03
	0.36363871E-03	0.	-0.36363871E-03	-0.68181986E-03	-0.90909190E-03	-0.99999999E-03
	0.36363710E-03	0.	-0.36363710E-03	-0.68181752E-03	-0.90909083E-03	-0.99999999E-03
	0.36364010E-03	0.	-0.36364010E-03	-0.68181960E-03	-0.90909142E-03	-0.99999999E-03
	0.36363654E-03	0.	-0.36363654E-03	-0.68181684E-03	-0.90909128E-03	-0.99999999E-03
	0.36363354E-03	0.	-0.36363354E-03	-0.68181574E-03	-0.90908997E-03	-0.99999999E-03
	0.36363560E-03	0.	-0.36363560E-03	-0.68181586E-03	-0.90908998E-03	-0.99999999E-03
	0.36363079E-03	0.	-0.36363079E-03	-0.68181381E-03	-0.90908730E-03	-0.99999999E-03
	0.36363219E-03	0.	-0.36363219E-03	-0.68181511E-03	-0.90908790E-03	-0.99999999E-03
	0.36363214E-03	0.	-0.36363214E-03	-0.68181771E-03	-0.90908909E-03	-0.99999999E-03
	0.36363203E-03	0.	-0.36363203E-03	-0.68181337E-03	-0.90908705E-03	-0.99999999E-03
	0.36363453E-03	0.	-0.36363453E-03	-0.68181571E-03	-0.90908988E-03	-0.99999999E-03
	0.36363246E-03	0.	-0.36363246E-03	-0.68181188E-03	-0.90908812E-03	-0.99999999E-03
	0.36363441E-03	0.	-0.36363441E-03	-0.68181551E-03	-0.90908977E-03	-0.99999999E-03
	0.36363474E-03	0.	-0.36363474E-03	-0.68181596E-03	-0.90909012E-03	-0.99999999E-03
	0.36363561E-03	0.	-0.36363561E-03	-0.68181756E-03	-0.90909082E-03	-0.99999999E-03
	0.36363521E-03	0.	-0.36363521E-03	-0.68181646E-03	-0.90909032E-03	-0.99999999E-03
	0.36363336E-03	0.	-0.36363336E-03	-0.68181727E-03	-0.90908701E-03	-0.99999999E-03
	0.36363570E-03	0.	-0.36363570E-03	-0.68181784E-03	-0.90909097E-03	-0.99999999E-03
	0.36363541E-03	0.	-0.36363541E-03	-0.68181568E-03	-0.90908990E-03	-0.99999999E-03
	0.36363478E-03	0.	-0.36363478E-03	-0.68181514E-03	-0.90909141E-03	-0.99999999E-03
	0.36363307E-03	0.	-0.36363307E-03	-0.68181505E-03	-0.90908964E-03	-0.99999999E-03
	0.36362854E-03	0.	-0.36362854E-03	-0.68181256E-03	-0.90908671E-03	-0.99999999E-03
	0.36363185E-03	0.	-0.36363185E-03	-0.68181491E-03	-0.90908776E-03	-0.99999999E-03
	0.36363591E-03	0.	-0.36363591E-03	-0.68181714E-03	-0.90909063E-03	-0.99999999E-03
	0.36363613E-03	0.	-0.36363613E-03	-0.68181814E-03	-0.90908942E-03	-0.99999999E-03
	0.36363576E-03	0.	-0.36363576E-03	-0.68181799E-03	-0.90908928E-03	-0.99999999E-03
	0.36363827E-03	0.	-0.36363827E-03	-0.68181679E-03	-0.90909042E-03	-0.99999999E-03
	0.36363511E-03	0.	-0.36363511E-03	-0.68181808E-03	-0.90908936E-03	-0.99999999E-03
	0.36363679E-03	0.	-0.36363679E-03	-0.68181793E-03	-0.90909100E-03	-0.99999999E-03
	0.36363237E-03	0.	-0.36363237E-03	-0.68181189E-03	-0.90908518E-03	-0.99999999E-03
	0.36363324E-03	0.	-0.36363324E-03	-0.68181521E-03	-0.90908963E-03	-0.99999999E-03
	0.36363069E-03	0.	-0.36363069E-03	-0.68181378E-03	-0.90908723E-03	-0.99999999E-03
	0.36363566E-03	0.	-0.36363566E-03	-0.68181424E-03	-0.90908924E-03	-0.99999999E-03
	0.36363561E-03	0.	-0.36363561E-03	-0.68181917E-03	-0.90908967E-03	-0.99999999E-03
	0.36363430E-03	0.	-0.36363430E-03	-0.68181226E-03	-0.90908835E-03	-0.99999999E-03
	0.36363312E-03	0.	-0.36363312E-03	-0.68181412E-03	-0.90908916E-03	-0.99999999E-03
	0.36363371E-03	0.	-0.36363371E-03	-0.68181574E-03	-0.90908822E-03	-0.99999999E-03
	0.36363330E-03	0.	-0.36363330E-03	-0.68181627E-03	-0.90909019E-03	-0.99999999E-03
	0.36363442E-03	0.	-0.36363442E-03	-0.68181655E-03	-0.90909037E-03	-0.99999999E-03
	0.36363626E-03	0.	-0.36363626E-03	-0.68181735E-03	-0.90909255E-03	-0.99999999E-03
	0.36363217E-03	0.	-0.36363217E-03	-0.68181780E-03	-0.90908933E-03	-0.99999999E-03
	0.36363415E-03	0.	-0.36363415E-03	-0.68181890E-03	-0.90908969E-03	-0.99999999E-03
	0.36363224E-03	0.	-0.36363224E-03	-0.68181358E-03	-0.90908538E-03	-0.99999999E-03
	0.36363333E-03	0.	-0.36363333E-03	-0.68181655E-03	-0.90909209E-03	-0.99999999E-03
	0.36363646E-03	0.	-0.36363646E-03	-0.68182090E-03	-0.90909242E-03	-0.99999999E-03
	0.36363236E-03	0.	-0.36363236E-03	-0.68181475E-03	-0.90909136E-03	-0.99999999E-03
	0.36363663E-03	0.	-0.36363663E-03	-0.68181773E-03	-0.90908921E-03	-0.99999999E-03
	0.36363447E-03	0.	-0.36363447E-03	-0.68181670E-03	-0.90909041E-03	-0.99999999E-03
	0.36363589E-03	0.	-0.36363589E-03	-0.68181542E-03	-0.90908812E-03	-0.99999999E-03
	0.36363701E-03	0.	-0.36363701E-03	-0.68181807E-03	-0.90909102E-03	-0.99999999E-03
	0.36363422E-03	0.	-0.36363422E-03	-0.68181635E-03	-0.90909031E-03	-0.99999999E-03
	0.36363432E-03	0.	-0.36363432E-03	-0.68181641E-03	-0.90909025E-03	-0.99999999E-03
	0.36363663E-03	0.	-0.36363663E-03	-0.68181745E-03	-0.90909163E-03	-0.99999999E-03
	0.36363576E-03	0.	-0.36363576E-03	-0.68181645E-03	-0.90909120E-03	-0.99999999E-03
	0.36363474E-03	0.	-0.36363474E-03	-0.68181510E-03	-0.90908973E-03	-0.99999999E-03
	0.36363482E-03	0.	-0.36363482E-03	-0.68181597E-03	-0.90908922E-03	-0.99999999E-03
	0.36363475E-03	0.	-0.36363475E-03	-0.68181584E-03	-0.90908913E-03	-0.99999999E-03
	0.36363374E-03	0.	-0.36363374E-03	-0.68181585E-03	-0.90909001E-03	-0.99999999E-03
	0.36363536E-03	0.	-0.36363536E-03	-0.68181788E-03	-0.90909190E-03	-0.99999999E-03
	0.36363537E-03	0.	-0.36363537E-03	-0.68181740E-03	-0.90908998E-03	-0.99999999E-03
	0.36363456E-03	0.	-0.36363456E-03	-0.68181619E-03	-0.90909021E-03	-0.99999999E-03
	0.36363466E-03	0.	-0.36363466E-03	-0.68181725E-03	-0.90909030E-03	-0.99999999E-03
	0.36363684E-03	0.	-0.36363684E-03	-0.68181846E-03	-0.90909051E-03	-0.99999999E-03
	0.36364713E-03	0.	-0.36364713E-03	-0.68182995E-03	-0.90909409E-03	-0.99999999E-03
Row c	0.36386415E-03	0.	-0.36386415E-03	-0.68207657E-03	-0.90919720E-03	-0.99999999E-03
Row b	0.36699688E-03	0.	-0.36699688E-03	-0.68588906E-03	-0.91103103E-03	-0.99999999E-03
Row a	0.39423104E-03	0.	-0.39423104E-03	-0.72070380E-03	-0.92871379E-03	-0.99999999E-03
	0.38776365E-03	0.	-0.38776365E-03	-0.69541036E-03	-0.82993055E-03	-0.99999999E-03
	0.43044931E-02	0.	0.43044931E-02	0.68201099E-02	0.82197364E-02	-0.99999999E-03
Outside Edge	0.	0.	0.75201605E-01	0.14938126E-00	0.21860147E-00	0.

Inside Edge

Plane of Symmetry, Z = 0

P ARRAY

	Column B'	Column A	Column B	Column C	Column D	Column E
Inside Edge	0.	0.	0.	0.	0.	0.
Row a'	0.241133466E-01	0.359972316E-01	0.241133466E-01	0.160128324E-01	-0.323265060E-01	0.
Row b'	-0.143077916E-00	-0.142221546E-00	-0.143077916E-00	-0.145204550E-00	-0.141342605E-00	-0.149572460E-00
Row c'	-0.292485396E-00	-0.292335236E-00	-0.292485396E-00	-0.292897678E-00	-0.293468441E-00	-0.296049521E-00
	-0.437513006E-00	-0.437420566E-00	-0.437513006E-00	-0.437790286E-00	-0.438246438E-00	-0.438864160E-00
	-0.579382776E 00	-0.579293946E 00	-0.579382776E 00	-0.579653396E 00	-0.580105426E 00	-0.580739736E 00
	-0.718130466E 00	-0.718042146E 00	-0.718130466E 00	-0.718400586E 00	-0.718852516E 00	-0.719486246E 00
	-0.853746466E 00	-0.853656706E 00	-0.853746466E 00	-0.854014446E 00	-0.854466046E 00	-0.855099466E 00
	-0.986216306E 00	-0.986128066E 00	-0.986216306E 00	-0.986485796E 00	-0.986937116E 00	-0.987570256E 00
	-0.111553906E 01	-0.111945156E 01	-0.111553906E 01	-0.111580936E 01	-0.111625936E 01	-0.111649226E 01
	-0.124170906E 01	-0.124162176E 01	-0.124170906E 01	-0.124197816E 01	-0.124242916E 01	-0.124306196E 01
	-0.136472466E 01	-0.136463746E 01	-0.136472466E 01	-0.136499936E 01	-0.136544466E 01	-0.136607716E 01
	-0.148658616E 01	-0.148649896E 01	-0.148658616E 01	-0.148685506E 01	-0.148730596E 01	-0.148793856E 01
	-0.160129576E 01	-0.160120856E 01	-0.160129576E 01	-0.160156476E 01	-0.160201566E 01	-0.160264826E 01
	-0.171485716E 01	-0.171476986E 01	-0.171485716E 01	-0.171512616E 01	-0.171557706E 01	-0.171620966E 01
	-0.182527506E 01	-0.182518776E 01	-0.182527506E 01	-0.182554406E 01	-0.182599516E 01	-0.182662776E 01
	-0.193255516E 01	-0.193246776E 01	-0.193255516E 01	-0.193282636E 01	-0.193327546E 01	-0.193330926E 01
	-0.203670396E 01	-0.203661646E 01	-0.203670396E 01	-0.203697326E 01	-0.203742446E 01	-0.203805736E 01
	-0.213772806E 01	-0.213764056E 01	-0.213772806E 01	-0.213799746E 01	-0.213844866E 01	-0.213908176E 01
	-0.223563626E 01	-0.223554876E 01	-0.223563626E 01	-0.223590376E 01	-0.223635496E 01	-0.223698816E 01
	-0.233042846E 01	-0.233034096E 01	-0.233042846E 01	-0.233069806E 01	-0.233114946E 01	-0.233178266E 01
	-0.242211696E 01	-0.242202946E 01	-0.242211696E 01	-0.242238666E 01	-0.242283816E 01	-0.242346146E 01
	-0.251070566E 01	-0.251061816E 01	-0.251070566E 01	-0.251097536E 01	-0.251142686E 01	-0.251206026E 01
	-0.259619976E 01	-0.259611226E 01	-0.259619976E 01	-0.259646946E 01	-0.259692126E 01	-0.259755466E 01
	-0.267866046E 01	-0.267857296E 01	-0.267866046E 01	-0.267887456E 01	-0.267932626E 01	-0.267995966E 01
	-0.275792506E 01	-0.275783756E 01	-0.275792506E 01	-0.275819496E 01	-0.275864656E 01	-0.275928016E 01
	-0.283416526E 01	-0.283407776E 01	-0.283416526E 01	-0.283443516E 01	-0.283488686E 01	-0.283552046E 01
	-0.290732946E 01	-0.290724196E 01	-0.290732946E 01	-0.290759936E 01	-0.290805106E 01	-0.290868466E 01
	-0.297742136E 01	-0.297733386E 01	-0.297742136E 01	-0.297769126E 01	-0.297814306E 01	-0.297877666E 01
	-0.304446506E 01	-0.304435756E 01	-0.304446506E 01	-0.304471496E 01	-0.304516676E 01	-0.304580036E 01
	-0.310846036E 01	-0.310831156E 01	-0.310846036E 01	-0.310866736E 01	-0.310912566E 01	-0.310975926E 01
	-0.316931016E 01	-0.316921296E 01	-0.316931016E 01	-0.316957096E 01	-0.317002276E 01	-0.317065646E 01
	-0.322714026E 01	-0.322705206E 01	-0.322714026E 01	-0.322741016E 01	-0.322786206E 01	-0.322849566E 01
	-0.328192456E 01	-0.328193636E 01	-0.328192456E 01	-0.328219456E 01	-0.328264636E 01	-0.328327996E 01
	-0.333365706E 01	-0.333356896E 01	-0.333365706E 01	-0.333392706E 01	-0.333437886E 01	-0.333501256E 01
	-0.338234076E 01	-0.338225256E 01	-0.338234076E 01	-0.338261076E 01	-0.338306256E 01	-0.338369626E 01
	-0.342797866E 01	-0.342789046E 01	-0.342797866E 01	-0.342824856E 01	-0.342870036E 01	-0.342933406E 01
	-0.347057376E 01	-0.347048566E 01	-0.347057376E 01	-0.347084376E 01	-0.347129556E 01	-0.347192926E 01
	-0.351012926E 01	-0.351004106E 01	-0.351012926E 01	-0.351039926E 01	-0.351085106E 01	-0.351148466E 01
	-0.354664806E 01	-0.354655976E 01	-0.354664806E 01	-0.354691806E 01	-0.354736986E 01	-0.354800346E 01
	-0.358013296E 01	-0.358004476E 01	-0.358013296E 01	-0.358040306E 01	-0.358085486E 01	-0.358148846E 01
	-0.361058706E 01	-0.361049896E 01	-0.361058706E 01	-0.361085716E 01	-0.361130896E 01	-0.361194256E 01
	-0.363901346E 01	-0.363792526E 01	-0.363801346E 01	-0.363828346E 01	-0.363873526E 01	-0.363936896E 01
	-0.366261476E 01	-0.366232666E 01	-0.366261476E 01	-0.366268476E 01	-0.366313656E 01	-0.366177026E 01
	-0.368379416E 01	-0.368370606E 01	-0.368379416E 01	-0.369406416E 01	-0.368451596E 01	-0.368514966E 01
	-0.370215466E 01	-0.370206626E 01	-0.370215466E 01	-0.370242466E 01	-0.370287626E 01	-0.370350996E 01
	-0.371749866E 01	-0.371741036E 01	-0.371749866E 01	-0.371776856E 01	-0.371822006E 01	-0.371885366E 01
	-0.372982936E 01	-0.372974126E 01	-0.372982936E 01	-0.373009946E 01	-0.373055126E 01	-0.373118486E 01
	-0.373914986E 01	-0.373906166E 01	-0.373914986E 01	-0.373941976E 01	-0.373987166E 01	-0.374050526E 01
	-0.374546276E 01	-0.374537466E 01	-0.374546276E 01	-0.374573286E 01	-0.374618476E 01	-0.374681826E 01
	-0.374977126E 01	-0.374968306E 01	-0.374977126E 01	-0.374904126E 01	-0.374949306E 01	-0.375012666E 01
	-0.374907796E 01	-0.374898986E 01	-0.374907796E 01	-0.374934796E 01	-0.374979976E 01	-0.375043346E 01
	-0.374638586E 01	-0.374629776E 01	-0.374638586E 01	-0.374665586E 01	-0.374710776E 01	-0.374774146E 01
	-0.374069776E 01	-0.374060966E 01	-0.374069776E 01	-0.374096776E 01	-0.374141956E 01	-0.374205316E 01
	-0.373201656E 01	-0.373192836E 01	-0.373201656E 01	-0.373228656E 01	-0.373273836E 01	-0.373337196E 01
	-0.372034506E 01	-0.372025696E 01	-0.372034506E 01	-0.372061506E 01	-0.372106686E 01	-0.372170046E 01
	-0.370568606E 01	-0.370559776E 01	-0.370568606E 01	-0.370595596E 01	-0.370640776E 01	-0.370704146E 01
	-0.368804226E 01	-0.368795406E 01	-0.368804226E 01	-0.368831226E 01	-0.368876406E 01	-0.368939766E 01
	-0.366741666E 01	-0.366732846E 01	-0.366741666E 01	-0.366768666E 01	-0.366813846E 01	-0.366877206E 01
	-0.364381196E 01	-0.364372376E 01	-0.364381196E 01	-0.364408196E 01	-0.364453376E 01	-0.364516736E 01
	-0.361723086E 01	-0.361714276E 01	-0.361723086E 01	-0.361750096E 01	-0.361795276E 01	-0.361858636E 01
	-0.358767636E 01	-0.358758816E 01	-0.358767636E 01	-0.358794636E 01	-0.358839816E 01	-0.358903186E 01
	-0.355515116E 01	-0.355506296E 01	-0.355515116E 01	-0.355542116E 01	-0.355587296E 01	-0.355650666E 01
	-0.351965796E 01	-0.351956986E 01	-0.351965796E 01	-0.351992796E 01	-0.352037976E 01	-0.352101336E 01
	-0.348119976E 01	-0.348111156E 01	-0.348119976E 01	-0.348146976E 01	-0.348192156E 01	-0.348255516E 01
	-0.343977906E 01	-0.343969086E 01	-0.343977906E 01	-0.344004896E 01	-0.344050076E 01	-0.344113446E 01
	-0.339953986E 01	-0.339951106E 01	-0.339953986E 01	-0.339966986E 01	-0.339961206E 01	-0.339675406E 01
	-0.334806126E 01	-0.334797316E 01	-0.334806126E 01	-0.334833126E 01	-0.334878306E 01	-0.334941676E 01
	-0.329776976E 01	-0.329766156E 01	-0.329776976E 01	-0.329803976E 01	-0.329849156E 01	-0.329912516E 01
	-0.324452636E 01	-0.324443826E 01	-0.324452636E 01	-0.324479636E 01	-0.324524816E 01	-0.324588186E 01
	-0.318833416E 01	0.318824606E 01	-0.318833416E 01	-0.318860406E 01	-0.318905586E 01	-0.318968956E 01
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	-0.293412686E 01	-0.293403866E 01	-0.293412686E 01	-0.293439676E 01	-0.293484856E 01	-0.293548216E 01
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	-0.255030016E 01	-0.255021216E 01	-0.255030016E 01	-0.255056996E 01	-0.255102166E 01	-0.255165506E 01
	-0.246467326E 01	-0.246465536E 01	-0.246467326E 01	-0.246501296E 01	-0.246546456E 01	-0.246609786E 01
	-0.237625816E 01	-0.237617026E 01	-0.237625816E 01	-0.237552786E 01	-0.237697926E 01	-0.237761256E 01
	-0.228484506E 01	-0.228475726E 01	-0.228484506E 01	-0.228511466E 01	-0.228556606E 01	-0.228619936E 01
	-0.219050396E 01	-0.219041626E 01	-0.219050396E 01	-0.219077336E 01	-0.219122476E 01	-0.219185786E 01
	-0.209323436E 01	-0.209314676E 01	-0.209323436E 01	-0.209350376E 01	-0.209395496E 01	-0.209458806E 01
	-0.199303576E 01	-0.199294836E 01	-0.199303576E 01	-0.199330516E 01	-0.199375626E 01	-0.199638926E 01
	-0.188990706E 01	-0.188981966E 01	-0.188990706E 01	-0.189017626E 01	-0.189062726E 01	-0.189126016E 01
	-0.178384746E 01	-0.178376006E 01	-0.178384746E 01	-0.178411656E 01	-0.178456746E 01	-0.178520026E 01
	-0.167485626E 01	-0.167476906E 01	-0.167485626E 01	-0.167512536E 01	-0.167557626E 01	-0.167620896E 01
	-0.156293396E 01	-0.156284686E 01	-0.156293396E 01	-0.156320306E 01	-0.156365386E 01	-0.156428646E 01
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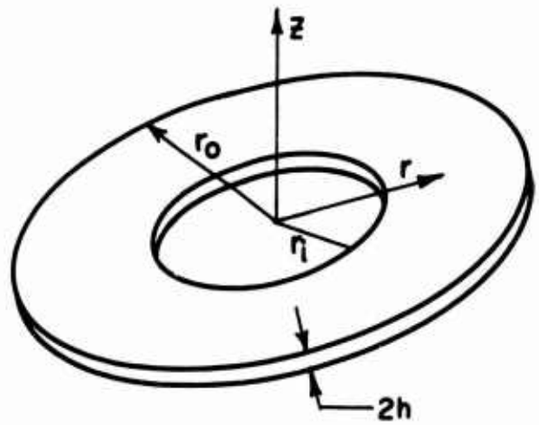
	-0.120959686E 01	-0.120950946E 01	-0.120959686E 01	-0.120986596E 01	-0.121031696E 01	-0.121094966E 01
	-0.108597436E 01	-0.108588686E 01	-0.108597436E 01	-0.108624356E 01	-0.108669476E 01	-0.108732746E 01
	-0.959441506E 00	-0.959353826E 00	-0.959441506E 00	-0.959711016E 00	-0.960162336E 00	-0.960795496E 00
	-0.830008446E 00	-0.829920466E 00	-0.830008446E 00	-0.830278246E 00	-0.830729866E 00	-0.831363316E 00
	-0.697687616E 00	-0.697599926E 00	-0.697687616E 00	-0.697957756E 00	-0.698409716E 00	-0.699043466E 00
	-0.562494266E 00	-0.562405416E 00	-0.562494266E 00	-0.562764906E 00	-0.563216946E 00	-0.563850326E 00
Row c	-0.424445816E-00	-0.424353446E-00	-0.424445816E-00	-0.424722936E-00	-0.425177016E-00	-0.425801116E-00
Row b	-0.283513906E-00	-0.283365446E-00	-0.283513906E-00	-0.283922136E-00	-0.284649436E-00	-0.285071956E-00
Row a	-0.138519306E-00	-0.137685376E-00	-0.138519306E-00	-0.140669186E-00	-0.143650066E-00	-0.144861236E-00
	0.274061226E-01	0.345703416E-01	0.274061226E-01	0.157609986E-01	-0.312232076E-01	0.
Outside Edge	0.	0.	0.	0.	0.	0.

RU = 0.599772E-06 RUU = 0.912181E 00 RW = 0.719447E-07 RWW = 0.907489E 00 RP = 0.746548E-05 RPP = 0.965318E 00

APPENDIX C

APPROXIMATION FOR A COMPRESSIBLE ELASTOMER

Consider a thin disk of rubber bonded between two rigid plates, the rubber being assumed as compressible.



The dimensions of the bearing are denoted by

r_i - inner radius

r_o - outer radius

2h - thickness.

Owing to the difficulties encountered in solving the equations of equilibrium, an approximate solution through variational methods will be derived. In this case, the theorem of minimum potential energy is used. A displacement field, u, w, satisfying the displacement boundary condition, is assumed and put into the potential energy:

$$U = \frac{1}{2} \int_V \sigma_{ij} \epsilon_{ij} \, dV \, . \tag{C-1}$$

The theorem of minimum potential energy states that

$$\delta U = 0 . \tag{C-2}$$

Thus, by taking the first variation of the expression U, two differential equations and the associated natural boundary conditions are obtained.

Assuming that

$$u = f(r)(1 - \frac{z^2}{h^2}), \tag{C-3}$$

$$w = \frac{Wo}{h} z + g(r) z (1 - \frac{z^2}{h^2}) \tag{C-4}$$

$$\frac{\partial u}{\partial r} = f'(r)(1 - \frac{z^2}{h^2})$$

$$\frac{\partial u}{\partial z} = - \frac{2z}{h^2} f(r)$$

$$\frac{\partial w}{\partial r} = g'(r) z (1 - \frac{z^2}{h^2})$$

$$\frac{\partial w}{\partial z} = \frac{Wo}{h} + g(r)(1 - 3 \frac{z^2}{h^2})$$

$$\frac{u}{\sigma} = \frac{f(r)}{r} (1 - \frac{z^2}{h^2}).$$

Let

$$\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} . \tag{C-5}$$

Then

$$\sigma_{ij} \epsilon_{ij} = W = \lambda\Delta^2 + 2G [\epsilon_r\epsilon_r + \epsilon_\theta\epsilon_\theta + \epsilon_z\epsilon_z + 2\epsilon_{rz}\epsilon_{rz}] \tag{C-6}$$

where

$$\Delta = (f'(r) + \frac{f}{r})(1 - \frac{z^2}{h^2}) + \frac{Wo}{h} + g(r)(1 - 3 \frac{z^2}{h^2}),$$

and

$$\Delta^2 = [f'(r) + \frac{f}{r}]^2 (1 - \frac{z^2}{h^2})^2 + \frac{Wo^2}{h^2} + g^2(r)(1 - 3 \frac{z^2}{h^2})^2$$

$$+ 2[f'(r)+ \frac{f}{r}] (1 - \frac{z^2}{h^2})(\frac{Wo}{h}) + 2 \frac{Wo}{h} g(r)(1 - 3 \frac{z^2}{h^2})$$

$$+ 2[f'(r) + \frac{f}{r}] g(r) (1 - \frac{z^2}{h^2}) (1 - 3 \frac{z^2}{h^2}).$$

The following are expanded for substitution into equation (C-1):

$$\epsilon_r\epsilon_r = [f'(r)]^2 \left(1 - \frac{z^2}{h^2}\right)^2$$

$$\epsilon_\theta\epsilon_\theta = \frac{f^2}{r^2} \left(1 - \frac{z^2}{h^2}\right)^2$$

$$\epsilon_z\epsilon_z = \left(\frac{Wo}{h}\right)^2 + g^2(r)\left(1 - 3\frac{z^2}{h^2}\right)^2 + 2\left(\frac{Wo}{h}\right) g(r)\left(1 - 3\frac{z^2}{h^2}\right)$$

$$\epsilon_{rz}\epsilon_{rz} = \tfrac{1}{4}\left[-\frac{2z}{h^2} f(r) + g'(r)\, z\left(1 - \frac{z^2}{h^2}\right)\right]^2$$

$$= \tfrac{1}{4}\left[4\frac{z^2}{h^4} f^2(r) + g'^2(r)\, z^2\left(1 - \frac{z^2}{h^2}\right)^2 - 4\frac{z^2}{h^2} f(r)\, g'(r)\left(1 - \frac{z^2}{h^2}\right)^2\right]$$

$$\int_{-h}^{h}\left(1 - \frac{z^2}{h^2}\right)^2 dz = \int_{-h}^{h}\left[1 - 2\frac{z^2}{h^2} + \frac{z^4}{h^4}\right]dz$$

$$= \left[z - \frac{2}{3}\frac{z^3}{h^2} + \frac{1}{5}\frac{z^5}{h^4}\right]_{-h}^{h}$$

$$= 2\left[h - \frac{2}{3}h + \frac{1}{5}h\right] = \frac{16}{15}h$$

$$\int_{-h}^{h}\left(1 - 3\frac{z^2}{h^2}\right)dz = \left(z - \frac{z^3}{h^2}\right)\Big]_{-h}^{h} = 0$$

$$\int_{-h}^{h}\left(1 - \frac{z^2}{h^2}\right)dz = \left[z - \frac{1}{3}\frac{z^3}{h^2}\right]_{-h}^{h} = \frac{4}{3}h$$

$$\int_{-h}^{h}(1 - 3\frac{z^2}{h^2})^2\,dz = \int_{-h}^{h}(1 - 6\frac{z^2}{h^2} + 9\frac{z^4}{h^4})dz$$

$$= z - 2\frac{z^3}{h^2} + \frac{9}{5}\frac{z^5}{h^4}]_{-h}^{h}$$

$$= 2(h - 2h + \frac{9}{5}) = \frac{8}{5}h$$

$$\int_{-h}^{h}(1 - \frac{z^2}{h^2})(1 - 3\frac{z^2}{h^2})dz = \int_{-h}^{h}(1 - 4\frac{z^2}{h^2} + 3\frac{z^4}{h^4})dz$$

$$= z - \frac{4}{3}\frac{z^3}{h^2} + \frac{3}{5}\frac{z^5}{h^4}]_{-h}^{h}$$

$$= 2(h - \frac{4}{3}h + \frac{3}{5}h) = \frac{8}{15}h$$

$$\int_{-h}^{h} z^2 dz = \frac{z^3}{3}]_{-h}^{h} = \frac{2}{3}h^3$$

$$\int_{-h}^{h} z^2(1 - \frac{z^2}{h^2})^2 = \int_{-h}^{h}(z^2 - z\frac{z^4}{h^4} + \frac{z^6}{h^4})\,dz$$

$$= (\frac{1}{3}z^3 - \frac{2}{5}\frac{z^5}{h^2} + \frac{1}{7}\frac{z^7}{h^4})_{-h}^{h}$$

$$= 2(\frac{h^3}{3} - \frac{2}{5}h^3 + \frac{1}{7}h^3) = \frac{16}{105}h^3$$

$$\int_{-h}^{h} z^2(1 - \frac{z^2}{h^2})d = [\frac{1}{3}z^3 - \frac{z^5}{5}h^2\,]_{-h}^{h}$$

$$= 2(\frac{1}{3}h^3 - \frac{1}{5}h^3) = \frac{4}{15}h^3$$

$$\int \Delta^2 \, dz = \frac{16}{15} h[f'(r) + \frac{f}{r}]^2 + 2 \frac{w^2}{h} + \frac{8}{5} h \, g^2(r) + \frac{8}{3} h(\frac{Wo}{h})[f'(r) + \frac{f}{r}]$$

$$+ \frac{16}{15} h \, [f'(r) + \frac{f}{r}] \, g(r)$$

$$\int \epsilon_r \epsilon_r \, dz = \frac{16}{15} h \, [f'(r)]^2$$

$$\int \epsilon_\theta \epsilon_\theta \, dz = \frac{16}{15} h \, f^2/r^2$$

$$\int \epsilon_z \epsilon_z \, dz = 2h(\frac{Wo}{h})^2 + \frac{8}{5} h \, g^2(r)$$

$$\int \epsilon_{rz} \epsilon_{rz} \, dz = \frac{1}{4}[\frac{8}{3h} f^2(r) + \frac{16}{105} h^3 \, g'^2(r) - \frac{16}{15} h \, f(r) \, g'(r)] \, .$$

Equation (C-1) is equivalent to

$$\int W dV = \{\lambda \Delta^2 + 2G \, (\epsilon_r \epsilon_r + \epsilon_\theta \epsilon_\theta + \epsilon_z \epsilon_z + 2\epsilon_{rz} \epsilon_{rz})\} dV$$

$$= \int \{\lambda \Delta^2 + 2G(\epsilon_r \epsilon_r + \epsilon_\theta \epsilon_\theta + \epsilon_z \epsilon_z + 2\epsilon_{rz} \epsilon_{rz})\} r dr d\theta dz$$

$$= 2\pi \int \lambda \, \{\frac{16}{15} h \, [f'(r) + \frac{f}{r}]^2 + 2 \frac{Wo^2}{h} + \frac{8}{5} h \, g^2(r) + \frac{8}{3} h(\frac{Wo}{h})[f'(r) + \frac{f}{r}]$$

$$+ \frac{16}{15} h \, [f'(r) + \frac{f}{r}] \, g(r)\} \, r dr$$

$$+ 2\pi \int 2G \, \{\frac{16}{15} h \, [f'^2(r) + \frac{f^2}{r^2}] + 2h \, (\frac{Wo^2}{h^2}) + \frac{8}{5} h \, g^2(r)$$

$$+ \frac{4}{3h} f^2(r) + \frac{8}{105} h^3 g'^2(r) - \frac{8}{15} h \, f(r) \, g'(r)\} r dr \, ; \qquad \text{(C-7)}$$

but

$$\delta \int W dV = 0 \qquad \text{(C-8)}$$

and

$$\int \lambda \{\frac{32}{15} h\,[f'(r) + \frac{f}{r}]\,[\delta\, f'(r) + \frac{\delta f}{r}] + \frac{16}{5} h\, g(r)\, \delta\, g(r)$$

$$+ \frac{8}{3} h(\frac{Wo}{h})\,[\delta f'(r) + \frac{\delta f}{r}] + \frac{16}{15} h\,[\delta f'(r) + \frac{\delta f}{r}]g(r)$$

$$+ \frac{16}{15} h\,[f'(r) + \frac{f}{r}]\,\delta g(r)\}rdr$$

$$+ \int 2G\,\{\frac{32}{15} h[f'(r)\,\delta f'(r) + \frac{f}{r^2}\,\delta f] + \frac{16}{5} h\, g(r)\,\delta g(r)$$

$$+ \frac{8}{3h} f(r)\,\delta f(r) + \frac{16}{105} h^3\, g'(r)\,\delta g'(r) - \frac{8}{15} h\, f(r)\delta g'(r)$$

$$- \frac{8}{15} h\, g'(r)\,\delta f(r)\}\, rdr = 0. \qquad \text{(C-8a)}$$

Expanding,

$$\int \lambda\,\{\frac{32}{15} h\,[f'(r) + \frac{f}{r}] + \frac{8}{3} h\,(\frac{Wo}{h}) + \frac{16}{15} h\, g(r)\}\,\delta f'(r)rdr$$

$$+ \int 2G\,\{\frac{32}{15} h\, f'(r)\}\,\delta f'(r)rdr$$

$$+ \int \lambda\,\{\frac{32}{15} h[f'(r) + \frac{f}{r}](\frac{1}{r}) + \frac{8}{3} h(\frac{Wo}{h})\frac{1}{r} + \frac{16}{15} h\,\frac{g(r)}{r}\}\delta f(r)rdr$$

$$+ \int 2G\{\frac{32}{15} h\,\frac{f}{r^2} + \frac{8}{3h} f(r) - \frac{8}{15} h\, g'(r)\}\,\delta f(r)\, rdr$$

$$+ \int 2G\{\frac{16}{105} h^3\, g'(r) - \frac{8}{15} h\, f(r)\}\,\delta g'(r)rdr$$

$$+ \int \lambda\,\{\frac{16}{5} h\, g(r) + \frac{16}{15} h\,[f'(r) + \frac{f}{r}]\}\,\delta g(r)rdr$$

$$+ \int 2G\{\frac{16}{5} h\, g(r)\}\,\delta g(r)dr = 0. \qquad \text{(C-8b)}$$

From the first line of equation (C-8b),

$$\int \lambda \left\{\frac{32}{15} h \left[f'(r) + \frac{f(r)}{r}\right] + \frac{8}{3} h \left(\frac{Wo}{h}\right) + \frac{16}{15} h\, g(r)\right\} \delta f'(r) r dr$$

$$= \int \lambda r \left\{\frac{32}{15} h\left[f'(r) + \frac{f(r)}{r}\right] + \frac{8}{3} h\left(\frac{Wo}{h}\right) + \frac{16}{15} h\, g(r)\right\} \delta f(r) \Big]_{r_i}^{r_o}$$

$$- \lambda \int \frac{1}{r}\left\{\frac{32}{15} h\left[\frac{\partial}{\partial r}[rf'(r)] + f'(r)\right] + \frac{8}{3} h\left(\frac{Wo}{h}\right) + \frac{16}{15} h \frac{\partial}{\partial r}(rg(r))\right]\right\} \delta f(r) r dr$$

$$= \lambda r \left\{\frac{32}{15} h\left[f'(r) + \frac{f(r)}{r}\right] + \frac{8}{3h}\left(\frac{Wo}{h}\right) + \frac{16}{15} h\, g(r)\right\} \delta f(r) \Big]_{r_i}^{r_o}$$

$$- \lambda \int \left\{\frac{32}{15} h \left[f''(r) + \frac{2}{r} f'(r)\right] + \frac{8}{3} h\left(\frac{Wo}{h}\right) \frac{1}{r} + \frac{16}{15} h \left[g'(r) + \frac{g(r)}{r}\right]\right\} \delta f(r) r dr.$$

From the second line of equation (C-8b),

$$\int 2G \left\{\frac{32}{15} h\, f'(r)\right\} \delta f'(r) r dr$$

$$= 2Gr\left\{\frac{32}{15} h\, f'(r)\right\} \delta f(r) \Big]_{r_i}^{r_o} - 2G \int \frac{32}{15} h \frac{1}{r} \frac{\partial}{\partial r} [r\, f'(r)] \delta f(r) dr$$

$$= 2Gr\left\{\frac{32}{15} h\, f'(r)\right\} \delta f(r) \Big]_{r_i}^{r_o} - 2G \left(\frac{32}{15} h\right) \int \left[f''(r) + \frac{f'(r)}{r}\right] \delta f(r) r dr.$$

From the fifth line of equation (C-8b),

$$\int 2G \left\{\frac{16}{105} h^3\, g'(r) - \frac{8}{15} h\, f(r)\right\} \delta\, g'(r)\, r dr$$

$$= 2Gr\left\{\frac{16}{105} h^3\, g'(r) - \frac{8}{15} h\, f(r)\right\} \delta g(r) \Big]_{r_i}^{r_o}$$

$$- 2G \int \frac{1}{r} \frac{\partial}{\partial r} \left\{\frac{16}{105} h^3\, rg'(r) - \frac{8}{15} h\, f(r)\right\} \delta g(r) r dr.$$

Grouping terms in equation (C-8b), we have the differential equations for f and g:

$$\lambda\{\frac{32}{15} h [f'' + \frac{1}{r}f' - \frac{f}{r^2}] + \frac{16}{15} hg'\} + 2G\{\frac{32}{15} [f'' + \frac{f'}{r} - \frac{f}{r^2}]$$

$$- \frac{8}{3h} f + \frac{8}{15} h g'\} = 0 \qquad \text{(C-9a)}$$

$$\lambda \{\frac{16}{15} h g + \frac{16}{15} h [f' + \frac{1}{r}f]\} + 2G \{\frac{16}{5} hg - \frac{16}{105} h^3 [g'' + \frac{1}{r} g']$$

$$+ \frac{8}{15} h [f' + \frac{1}{r}f]\} = 0. \qquad \text{(C-9b)}$$

After rearrangement, the equations become

$$f'' + \frac{1}{r} f' - \frac{1}{r^2} f - \frac{5}{2h^2} \frac{G}{\lambda+2G} f + \frac{1}{2} \frac{\lambda+G}{\lambda+2G} g' = 0 \qquad \text{(C-10a)}$$

$$g'' + \frac{1}{r} g' - \frac{21}{2h^2} \frac{\lambda+2G}{G} g - \frac{7}{2h^2} \frac{\lambda+G}{G} [f' + \frac{1}{r} f] = 0. \qquad \text{(C-10b)}$$

The appropriate boundary conditions are

$$f' + \frac{\lambda}{\lambda+2G} \{\frac{f}{r} + \frac{1}{2}g + \frac{5}{4} (\frac{Wo}{h})\} = 0 \qquad \text{(C-11a)}$$

$$\frac{2}{7} h^2 g' - f = 0 \qquad \text{(C-11b)}$$

at $r = r_i$ and r_o.

For the sake of convenience, these equations are nondimensionalized with respect to h.

Defining

$$\bar{f} = \frac{f}{h}$$

$$\bar{r} = \frac{r}{h}$$

$$\bar{g} = g(\frac{r}{h}), \qquad \text{(C-12)}$$

the set of equations becomes

$$\bar{f}'' + \frac{1}{\bar{r}}\bar{f}' - \frac{1}{\bar{r}^2}\bar{f} - \frac{5}{4}\frac{1-2\nu}{1-\nu}\bar{f} + \frac{1}{4}\frac{1}{1-\nu}\bar{g}' = 0 \tag{C-13a}$$

$$\bar{g}'' + \frac{1}{\bar{r}}\bar{g}' - \frac{21(1-\nu)}{1-2\nu}\bar{g} - \frac{7}{2}\frac{1}{1-2\nu}\left(\bar{f}' + \frac{1}{\bar{r}}f\right) = 0, \tag{C-13b}$$

and at r_o and r_i,

$$\bar{f}' + \frac{\nu}{1-\nu}\left\{\frac{\bar{f}}{\bar{r}} + \frac{1}{2}\bar{g} + \frac{5}{4}\left(\frac{W_o}{h}\right)\right\} = 0 \tag{C-14a}$$

$$\frac{2}{7}\bar{g}' - \bar{f} = 0, \tag{C-14b}$$

where prime denotes $d/d\bar{r}$, ν - Poisson's ratio and the following relation is used:

$$\lambda = \frac{2G\nu}{1-2\nu}\,. \tag{C-15}$$

In the following, the bar will be dropped from quantities b, q, etc., with the understanding that they are nondimensional.

Define

$$a = \frac{5}{4}\frac{1-2\nu}{1-\nu}$$

$$b = \frac{1}{4}\frac{1}{1-\nu}$$

$$c = \frac{21(1-\nu)}{1-2\nu}$$

$$d = \frac{7}{2}\frac{1}{1-2\nu}\,. \tag{C-16}$$

The differential equations can be written as

$$f'' + \frac{1}{r}f' - \frac{1}{r^2}f - af + bg' = 0 \tag{C-17a}$$

$$g'' + \frac{1}{r}g' - cg - d\left(f' + \frac{1}{r}f\right) = 0\,. \tag{C-17b}$$

Consider

$$f = A\ I_1(kr)$$

$$g = B\ I_o(kr). \tag{C-18}$$

Where I's are modified Bessel functions of the first kind,

$$I_1' = k\ I_o(kr) - \frac{1}{r}\ I_i(kr)$$

$$I_1'' = k^2\ I_1(kr) + \frac{2}{r^2}\ I_1(kr) - \frac{k}{r}\ I_o(kr)$$

$$I_o' = k\ I_1(kr)$$

$$I_o'' = k^2\ I_o(kr) - \frac{k}{r}\ I_1(kr). \tag{C-19}$$

Substituting into equations (C-17a) and (C-17b),

$$[(k^2 - a)A + kbB]\ I_1 = 0 \tag{C-20a}$$

$$[(k^2 - c)B - kdA]\ I_o = 0 \tag{C-20b}$$

or

$$k^4 - (a + c - bd)k^2 + ac = 0. \tag{C-21}$$

It is noted that this equation is the same as equation (C-24); hence, the same k_1, k_2 will satisfy this equation and

$$F = -\frac{dk_i}{k_i^2 - c}\ E\ . \tag{C-22}$$

Now the general solution of equation (C-17) can be written as

$$f = A\ I_1(k_1 r) + B\ I_1(k_2 r) + E\ K_1(k_1 r) + F\ K_1(k, r) \tag{C-23}$$

$$g = \frac{dk_1}{k_1^2 - c}\ A\ I_o(k_1 r) + \frac{dk_2}{k_2^2 - c}\ B\ I_o(k_2 r) - E\ \frac{dk_1}{k_1^2 - c}\ K_o(k_1 r)$$

$$- \frac{dk_2}{k_2^2 - c}\ F\ K_o(k_2 r). \tag{C-24}$$

The derivatives are

$$f' = A[k_1 I_o(k_1 r) - \frac{1}{r} I_1(k_1 r)] + B[k_2 I_o(k_2 r) - \frac{1}{r} I_1(k_2 r)] - E[k_1 K_o(k_1 r) + \frac{1}{r} K_1(k_1 r)] - F[k_2 K_o(k_2 r) + \frac{1}{r} K_1(k_2 r)] \tag{C-25}$$

$$g' = \frac{dk_1}{k_1^2 - c} A\, I_1(k_1 r) + \frac{dk_2^2}{k_2^2 - c} B\, I_1(k_2 r) + \frac{dk_1^2}{k_1^2 - c} E\, K_1(k_1 r) + \frac{dk_2^2}{k_2^2 - c} F\, K_1(k_2 r). \tag{C-26}$$

Thus,

$$(k^2 - a)A + kbB = 0$$

$$-kdA + (k^2 - c)B = 0\,. \tag{C-27}$$

In order to have a solution, it is required that

$$\begin{pmatrix} k^2 - a & bk \\ -dk & k^2 - c \end{pmatrix} = 0 \tag{C-28}$$

or

$$k^4 - (a + c - bd)k^2 + ac = 0 \tag{C-29}$$

$$k^2 = \tfrac{1}{2}[(a + c - bd) \pm \sqrt{(a + c - bd)^2 - 4ac}]$$

$$k = \pm \{\tfrac{1}{2}(a + c - bd) \pm \tfrac{1}{2}\sqrt{(a + c - bd)^2 - 4ac}\}^{\frac{1}{2}}\,. \tag{C-30}$$

Since $I_p(kr) = -I_p(-kr)$, it is noted that there are only two independent solutions corresponding to

$$k_1 = \{\tfrac{1}{2}(a + c - bd) + \tfrac{1}{2}\sqrt{(a + c - bd)^2 - 4ac}\}^{\frac{1}{2}} \tag{C-31a}$$

and

$$k_2 = \{\tfrac{1}{2}(a + c - bd) - \tfrac{1}{2}\sqrt{(a + c - bd)^2 - 4ac}\}^{\frac{1}{2}} . \tag{C-31b}$$

In both cases,

$$B = \frac{k_i d}{k_i^2 - c} A. \tag{C-32}$$

Consider next

$$\begin{aligned} f &= E\, K_1(kr) \\ g &= F\, K_o(kr). \end{aligned} \tag{C-33}$$

Where K's are modified Bessel functions of the second kind,

$$\begin{aligned} K_1' &= -k\, K_o(kr) - \frac{1}{r} K_1(kr) \\ K_1'' &= k^2 K_1(kr) + \frac{2}{r^2} K_1(kr) + \frac{k}{r} K_o(kr) \\ K_o' &= -k\, K_1(kr) \\ K_o'' &= k^2 K_o(kr) + \frac{k}{r} K_1(kr). \end{aligned} \tag{C-34}$$

Substituting into equations (C-17),

$$\begin{aligned} [(k^2 - a)E - bkF]\, K_1 &= 0 \\ [(k^2 - c)F + dkE]\, K_o &= 0 . \end{aligned} \tag{C-35}$$

Thus,

$$(k^2 - a)E - bkF = 0$$

$$dkE + (k^2 - c)F = 0. \qquad \text{(C-36)}$$

In order for this set to have a solution,

$$\begin{pmatrix} (k^2-a) & -bk \\ dk & (k^2-c) \end{pmatrix} = 0. \qquad \text{(C-37)}$$

Substituting into equations (C-11),

$$\left(\frac{2}{7}\frac{dk_1^2}{k_1^2-c} - 1\right)[A\, I_1(k_1r_i) + E\, K_1(k_1r_i)] + \left(\frac{2}{7}\frac{dk_2^2}{k_2^2-c} - 1\right)[B\, I_1(k_2r_i) + F\, K_1(k_2r_i)] = 0$$

$$\left(\frac{2}{7}\frac{dk_1^2}{k_1^2-c} - 1\right)[A\, I_1(k_1r_o) + E\, K_1(k_1r_o)] + \left(\frac{2}{7}\frac{dk_2^2}{k_2^2-c} - 1\right)[B\, I_1(k_2r_o) + FK_1(k_2r_o)] = 0$$

and

$$A\left[k_1I_o(k_1r_i) - \frac{1}{r_i}I_1(k_1r_i) + \frac{\nu}{1-\nu}\left(\frac{1}{r_i}I_1(k_1r_i) + \frac{1}{2}\frac{dk_1}{k_1^2-c}I_o(k_1r_i)\right)\right]$$

$$+ B\left[k_2I_o(k_2r_i) - \frac{1}{r_i}I_1(k_2r_i) + \frac{\nu}{1-\nu}\left(\frac{1}{r_i}I_1(k_2r_i) + \frac{1}{2}\frac{dk_2}{k_2^2-c}I_o(k_2r_i)\right)\right]$$

$$- E\left[k_1K_o(k_1r_i) + \frac{1}{r_i}K_1(k_1r_i) - \frac{\nu}{1-\nu}\left(\frac{1}{r_i}K_1(k_1r_i) - \frac{1}{2}\frac{dk_1}{k_1^2-c}K_o(k_1r_i)\right)\right]$$

$$- F\left[k_2K_o(k_2r_i) + \frac{1}{r_i}K_1(k_2r_i) - \frac{\nu}{1-\nu}\left(\frac{1}{r_i}K_1(k_2r_i) - \frac{1}{2}\frac{dk_2}{k_2^2-c}K_o(k_2r_i)\right)\right]$$

$$= -\frac{5}{4}\frac{\nu}{1-\nu}\frac{w_o}{h} \qquad \text{(C-38)}$$

$$A[k_1 I_o(k_1 r_o) - \frac{1}{r_o} I_1(k_1 r_o) + \frac{\nu}{1-\nu} (\frac{1}{r_o} I_1(k_1 r_o) + \frac{1}{2} \frac{dk_1}{k_1^2-c} I_o(k_1 r_o)]$$

$$+ B[k_2 I_o(k_2 r_o) - \frac{1}{r_o} I_1(k_2 r_o) + \frac{\nu}{1-\nu} (\frac{1}{r_o} I_1(k_2 r_o) + \frac{1}{2} \frac{dk_2}{k_2^2-c} I_o(k_2 r_o)]$$

$$- E[k_1 K_o(k_1 r_o) + \frac{1}{r_o} K_1(k_1 r_o) - \frac{\nu}{1-\nu} (\frac{1}{r_o} K_1(k_1 r_o) - \frac{1}{2} \frac{dk_1}{k_1^2-c} K_o(k_1 r_o)]$$

$$- F[k_2 K_o(k_2 r_o) + \frac{1}{r_o} K_1(k_2 r_o) - \frac{\nu}{1-\nu} (\frac{1}{r_o} K_1(k_2 r_o) - \frac{1}{2} \frac{dk_2}{k_2^2-c} K_o(k_2 r_o)]$$

$$= - \frac{5}{4} (\frac{\nu}{1-\nu}) \frac{w_o}{h}. \qquad \text{(C-39)}$$

For very large values of kr, the modified Bessel functions have the asymptotic behavior

$$I_\rho(kr) \sim \frac{e^{kr}}{\sqrt{2\pi kr}}$$

$$(kr \to \infty) \qquad \text{(C-40)}$$

$$I_\rho(kr) \sim \frac{e^{-kr}}{\sqrt{\frac{2}{\pi} kr}}.$$

Using this and a set of new constants defined by

$$A' = e^{k_1 r_o} \frac{1}{\sqrt{2\pi k_1 r_o}} A \qquad B' = e^{k_2 r_o} \frac{1}{\sqrt{2\pi k_2 r_o}} B$$

$$\text{(C-41)}$$

$$E' = e^{-k_1 r_i} \frac{1}{\sqrt{\frac{2}{\pi} k_1 r_u}} E \qquad F' = e^{-k_2 r_i} \frac{1}{\sqrt{\frac{2}{\pi} k_2 r_i}} F,$$

the four boundary conditions are written as

$$A'\left(\frac{2}{7}\frac{dk_1^2}{k_1^2-c}-1\right)\sqrt{\frac{r_o}{r_i}}\,e^{k_1(r_i-r_o)}+B'\left(\frac{2}{7}\frac{dk_2^2}{k_2^2-c}-1\right)\sqrt{\frac{r_o}{r_i}}\,e^{k_2(r_i-r_o)}$$

$$+E'\left(\frac{2}{7}\frac{dk_1^2}{k_1^2-c}-1\right)+F'\left(\frac{2}{7}\frac{dk_2^2}{k_2^2-c}-1\right)=0 \qquad \text{(C-42a)}$$

$$A'\left(\frac{2}{7}\frac{dk_1^2}{k_1^2-c}-1\right)+B'\left(\frac{2}{7}\frac{dk_2^2}{k_2^2-c}-1\right)+E'\left(\frac{2}{7}\frac{dk_1^2}{k_1^2-c}-1\right)\sqrt{\frac{r_i}{r_o}}\,e^{-k_1(r_o-r_i)}$$

$$+F'\left(\frac{2}{7}\frac{dk_2^2}{k_2^2-c}-1\right)\sqrt{\frac{r_i}{r_o}}\,e^{-k_2(r_o-r_i)}=0 \qquad \text{(C-42b)}$$

$$A'\left[k_1-\frac{1}{r_i}+\frac{\nu}{1-\nu}\left(\frac{1}{r_i}+\frac{1}{2}\frac{dk_1}{k_1^2-c}\right)\right]\sqrt{\frac{r_o}{r_i}}\,e^{k_1(r_i-r_o)}+B'\left[k_2-\frac{1}{r_i}+\frac{\nu}{1-\nu}\right.$$

$$\left.+\frac{1}{2}\frac{kd_2}{k_2^2-c}\right)\right]\sqrt{\frac{r_o}{r_i}}\,e^{k_2(r_i-r_o)}-E'\left[k_1+\frac{1}{r_i}-\frac{\nu}{1-\nu}\left(\frac{1}{r_i}-\frac{dk_1}{k_1^2-c}\right)\right]$$

$$-F'\left[k_2+\frac{1}{r_i}-\frac{\nu}{1-\nu}\left(\frac{1}{r_i}-\frac{1}{2}\frac{dk_2}{k_2^2-c}\right)\right]=-\frac{5}{4}\left(\frac{\nu}{1-\nu}\right)\frac{w_o}{h} \qquad \text{(C-42c)}$$

$$A'[k_1 - \frac{1}{r_o} + \frac{\nu}{1-\nu}(\frac{1}{r_o} + \frac{1}{2}\frac{dk_1}{k_1^2-c})] + B'[k_2 - \frac{1}{r_o} + \frac{\nu}{1-\nu}(\frac{1}{r_o} + \frac{1}{2}\frac{dk_2}{k_2^2-c})]$$

$$- E'[k_1 + \frac{1}{r_o} - \frac{\nu}{1-\nu}(\frac{1}{r_o} - \frac{1}{2}\frac{dk_1}{k_1^2-c})]\sqrt{\frac{r_i}{r_o}}\; e^{-k_1(r_o-r_i)} - F'[k_2$$

$$+ \frac{1}{r_o} - \frac{\nu}{1-\nu}(\frac{1}{r_o} - \frac{1}{2}\frac{dk_2}{k_2^2-c})]\sqrt{\frac{r_i}{r_o}}\; e^{-k_2(r_o-r_i)} = -\frac{5}{4}(\frac{\nu}{1-\nu})\frac{w_o}{h}. \qquad \text{(C-42d)}$$

For large values of $r_o - r_i$, A', B' in equations (C-42a and c) and E', F' in equations (C-42b and d) can be disregarded.

Hence,

$$E' = -\frac{\frac{2}{7}\frac{dk_2^2}{k_2^2-c} - 1}{\frac{2}{7}\frac{dk_1^2}{k_1^2-c} - 1}\; F',$$

$$E'[k_1 + \frac{1}{r_i} - \frac{\nu}{1-\nu}(\frac{1}{r_i} - \frac{dk_1}{k_1^2-c})] + F'[k_2 + \frac{1}{r_i} - \frac{\nu}{1-\nu}(\frac{1}{r_i} - \frac{1}{2}\frac{dk_2}{dk_2^2-c})]$$

$$= \frac{5}{4}(\frac{\nu}{1-\nu})\frac{w_o}{h}, \qquad \text{(C-43)}$$

and

$$A' = -\frac{\frac{2}{7}\frac{dk_2^2}{k_2^2-c} - 1}{\frac{2}{7}\frac{dk_1^2}{k_1^2-c} - 1}\; B',$$

$$A'[k_1 - \frac{1}{r_o} + \frac{\nu}{1-\nu}(\frac{1}{r_o} + \frac{1}{2}\frac{dk_1}{k_1^2-c})] + B'[k_2 - \frac{1}{r_o} + \frac{\nu}{1-\nu}(\frac{1}{r_o} + \frac{1}{2}\frac{dk_2}{k_2^2-c})]$$

$$= -\frac{5}{4}(\frac{\nu}{1-\nu})\frac{w_o}{h}. \qquad \text{(C-44)}$$

In terms of A's, the general solution can be written as

$$f = A' \sqrt{\frac{r_o}{r}}\, e^{k_1(r-r_o)} + B' \sqrt{\frac{r_o}{r}}\, e^{k_2(r-r_o)} + E' \sqrt{\frac{r_o}{r}}\, e^{-k_1(r-r_o)}$$

$$+ F' \sqrt{\frac{r_o}{r}}\, e^{-k_2(r-r_o)}$$

$$g = \frac{dk_1}{k_1^2-c} A' \sqrt{\frac{r_o}{r}}\, e^{k_1(r-r_o)} + \frac{dk_2}{k_2^2-c} B' \sqrt{\frac{r_o}{r}}\, e^{k_2(r-r_o)}$$

$$- E' \frac{dk_1}{k_1^2-c} \sqrt{\frac{r_o}{r}}\, e^{-k_1(r-r_i)} - F' \frac{dk_2}{k_2^2-c} \sqrt{\frac{r_i}{r}}\, e^{-k_2(r-r_i)}. \qquad \text{(C-45)}$$

When $r_o - r_i$ is large, the solution can be further simplified:

(1) Region near r_o,

$$f = A' \sqrt{\frac{r_o}{r}}\, e^{k_1(r-r_o)} + B' \sqrt{\frac{r_o}{r}}\, e^{k_2(r-r_o)}$$

$$g = A' \frac{dk_1}{k_1^2-c} \sqrt{\frac{r_o}{r}}\, e^{k_1(r-r_o)} + B' \frac{dk_2}{k_2^2-c} \sqrt{\frac{r_o}{r}}\, e^{k_2(r-r_o)}. \qquad \text{(C-46)}$$

(2) Region near r_i,

$$f = E' \sqrt{\frac{r_i}{r}}\, e^{-k_1(r-r_i)} + F' \sqrt{\frac{r_i}{r}}\, e^{-k_2(r-r_i)}$$

$$g = -E' \frac{dk_1}{k_1^2-c} \sqrt{\frac{r_i}{r}}\, e^{-k_1(r-r_i)} - F' \frac{dk_2}{k_2^2-c} \sqrt{\frac{r_i}{r}}\, e^{-k_2(r-r_i)}. \qquad \text{(C-47)}$$

(3) Away from ends,

$$f = 0$$

$$g = 0\,. \qquad \text{(C-48)}$$

Calculation of stresses is

$$\sigma_z = (\lambda + 2G)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)$$

$$\sigma_{rz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \qquad \text{(C-50)}$$

at points on bonded surfaces ($z = \pm h$)

$$\frac{\partial u}{\partial r} = \frac{u}{r} = 0$$

$$\frac{\partial w}{\partial r} = 0 \ ;$$

thus, at $z = h$,

$$\sigma_z = (\lambda + 2G)\frac{\partial w}{\partial z}$$

$$= \frac{2G(1-\nu)}{1-2\nu}\left(\frac{w_o}{h} - 2g\right)$$

$$\sigma_{rz} = -\ 2G\ f. \qquad \text{(C-51)}$$

Average σ_z at bonded surface is

$$\sigma_{z_{avg}}\Big|_{z=+h} = \frac{2}{r_o^2 - r^2}\int_{r_i}^{r_o} \sigma_z\Big|_{z=+h} r\,dr$$

$$= 2\left(\frac{2G(1-\nu)}{1-2\nu}\right)\left(\frac{1}{r_o^2 - r_i^2}\right)\int_{r_i}^{r_o} \frac{\partial w}{\partial z}\, r\,dr$$

$$= \left(\frac{2G(1-\nu)}{1-2\nu}\right)\left\{\left(\frac{w_o}{h}\right) - \frac{4}{r_o^2 - r_i^2}\int_{r_i}^{r_o} g\,r\,dr\right\}, \qquad \text{(C-52)}$$

since

$$g = \frac{dk_1}{k_1^2-c}\,[A\ I_o(k_1r) - E\ K_o(k_1r)]$$

$$+ \frac{dk_2}{k_2^2-c}\,[B\ I_o(k_2r) - F\ K_o(k_2r)]$$

$$\int_{r_i}^{r_o} grdr = \{\frac{d}{k_1^{\ 2}-c}\ [Ar\ I_1(k_1r) + Er\ K_1(k_1r)]$$

$$+ \frac{dr}{k_2^2-c}\ [B\ I_1(k_2r) + F\ K_1(k_2r)]\}\int_{r_o}^{r_o}$$

$$= r_o\ \{\frac{d}{k_1^{\ 2}-c}\ [A\ I_1(k_1r_o) + E\ K_1(k_1r_o)] + \frac{d}{k_2^{\ 2}-c}\ [BI_1(k_2r_o)$$

$$+ F\ K_1(k_2r_c)]\}$$

$$- r_i\ \{\frac{d}{k_1^{\ 2}-c}\ [A\ I_1(k_1r_i) + E\ K_1(k_1r_i)] + \frac{d}{k_2^{\ 2}-c}\ [B\ I_1(k_2r_i)$$

$$+ F\ K_1(k_1r_i)]\}.$$

For very large values of kr,

$$\int_{r_i}^{r_o} grdr = r_o\ \{\frac{d}{k_1^{\ 2}-c}\ [A\ \frac{e^{+k_1r_o}}{\sqrt{2\pi k_1r_o}} + E\ \frac{e^{-k_1r_o}}{\sqrt{\frac{2}{\pi}\,k_1r_o}}] + \frac{d}{k_2^{\ 2}-c}\ [B\ \frac{e^{k_1r_o}}{\sqrt{2\pi k_2r_o}}$$

$$+ F\ \frac{e^{-k_2r_o}}{\sqrt{\frac{2}{\pi}k_2r_o}}]\}$$

$$- r_i \left\{\frac{d}{k_1^2-c} \left[A \frac{e^{k_1 r_i}}{\sqrt{2\pi k_1 r_i}} + E \frac{e^{-k_1 r_i}}{\sqrt{\frac{2}{\pi} k_1 r_i}}\right] + \frac{d}{k_2^2-c} \left[B \frac{e^{k_2 r_i}}{\sqrt{2\pi k_2 r_i}}\right.\right.$$

$$\left. + F \frac{e^{-k_2 r_i}}{\sqrt{\frac{2}{\pi} k_2 r_i}}\right.$$

$$= r_o \left\{\frac{d}{k_1^2-c} \left(A' + E' \sqrt{\frac{r_i}{r_o}}\, e^{-k_1(r_o-r_i)}\right) + \frac{d}{k_2^2-c} \left[B'\right.\right.$$

$$\left.\left. + F' \sqrt{\frac{r_i}{r_o}} e^{-k(r_o-r_i)}\right]\right\}$$

$$- r_i \left\{\frac{d}{k_1^2-c} \left(A' \sqrt{\frac{r_o}{r_i}}\, e^{-k_1(r_o-r_i)} + E'\right)\right.$$

$$\left. + \frac{d}{k_2^2-c} \left[B' \sqrt{\frac{r_o}{r_i}}\, e^{-k_1(r_o-r_i)} + F'\right]\right\}$$

or

$$\int_{r_i}^{r_o} grdr = r_o \left(\frac{d}{k_1^2-c} A' + \frac{d}{k_1^2-c} B'\right) - r_i \left(\frac{d}{k_1^2-c} E' + \frac{d}{k_1^2-c} F'\right)$$

and

$$\sigma_{z_{max}} \Big|_{z=h} = \frac{2G(1-\nu)}{1-2\nu} \left(\frac{w_o}{h}\right) ; \qquad \text{(C-53)}$$

hence,

$$\frac{\sigma_{z\ max}}{\sigma_{z\ avg}} = \frac{\left(\frac{w_o}{h}\right)}{\frac{w_o}{h} \left(\frac{4}{r_o^2-r_i^2}\right) \int_{r_i}^{r_o} grdr} . \qquad \text{(C-54)}$$

It is also noted that the maximum shear stress occurs at the outer edge; i.e., $r = r_o$.

Hence,

$$\frac{\sigma_{rz\ max}}{\sigma_{z\ avg}} = \frac{+\ 2G\ f(r_o)}{\frac{2G(1-\nu)}{1-2\nu}\left(\frac{w_o}{h} - \frac{4}{r_o^2 - r_i^2}\int_{r_i}^{r_o} grdr\right)}$$

$$= \frac{f(r_o)}{\frac{1-\nu}{1-2\nu}\left(\frac{w_o}{h} - \frac{4}{r_o^2 - r_i^2}\int_{r_i}^{r_o} grdr\right)} \quad . \qquad \text{(C-55)}$$

This is programmed for the computer as follows:

Let

$$\bar{f} = X1 \qquad \bar{f}' = Z1$$

$$\bar{g} = X2 \qquad \bar{g}' = Z2.$$

The differential equations can be written as

$$(Z1)' + \frac{1}{r} Z1 - \frac{1}{r^2} X1 - \frac{5}{4}\frac{1-2\nu}{1-\nu} X1 + \frac{1}{4}\frac{1}{1-\nu} Z2 = 0$$

$$(X1)' = Z1$$

$$(Z2)' + \frac{1}{r} Z2 - \frac{21(1-\nu)}{1-2\nu} X2 - \frac{7}{2(1-2\nu)}\left(Z1 + \frac{1}{r} X1\right) = 0$$

$$(X2)' = Z2 .$$

Define

$$AA = \frac{1-2\nu}{1-\nu}$$

$$BB = \frac{1}{1-\nu}$$

$$CC = \frac{1}{1-2\nu}$$

DRAD = (RADN - RADØ)/FNR

RAD = RADØ + FI*RAD FI = I - 1

NRR = (NR/NP) + 1.0

ND - number of intervals skipped during printing

$$(Z1)' = \frac{1}{RAD} Z1 + \frac{1}{RAD^2} X1 + (1.25*AA)*(X1) - (0.25*BB)*z\ 2$$

$$(X1)' = Z1$$

$$(Z1)' = 1\ \frac{1}{RAD} Z2 + \frac{21}{AA} *Z2 + 3.5*CC*(Z1 + \frac{X1}{RAD})$$

$$(X2)' = Z2\ .$$

Define

$$DERZ1F\ (A,B,C,D) = DRAD*\left(\frac{A}{D} + \frac{B}{D*D} + (1.25*AA)*B - 0.25*BB*C)\right)$$

$$DERX1F\ (A) = DRAD*A$$

$$DERZ2F\ (A,B,C,D,E) = DRAD*\left(-\frac{C}{E} + \frac{21}{AA} *D+3.5*CC*(A + \frac{B}{E})\right)$$

$$DERX2F\ (A) = DRAD*A\ .$$

The four independent solutions of X1, Z1, X2, Z2 are

X1	S2(1,IJ),	S2(2,IJ),	S2(3,IJ),	S2(4,IJ)
Z1	S1(1,IJ),	S1(2,IJ),	S1(3,IJ),	S1(4,IJ)
X2	S4(1,IJ),	S4(2,IJ),	S4(3,IJ),	S4(4,IJ)
Z2	S3(1,IJ),	S3(2,IJ),	S3(3,IJ),	S3(4,IJ).

Where the first index denotes number of independent solutions,

F(IJ) = A(1) S2(1,IJ) + A(2) S2(2,IJ) + A(3) S2(3,IJ) + A(4) S2(4,I) = X1

FF(IJ) = A(1) S1(1,IJ) + A(2) S1(2,IJ) + A(3) S1(3,IJ) + A(4) S1(4,IJ) = Z1

G(IJ) = A(1) S4(1,IJ) + A(2) S4(2,IJ) + A(3) S4(3,IJ) + A(4) S4(4,IJ) = X2

GG(IJ) = A(1) S3(1,IJ) + A(2) S3(2,IJ) + A(3) S3(3,IJ) + A(4) S3(4,IJ) = Z2.

Since the boundary condition states

$$\bar{f}' + \left(\frac{\nu}{1-\nu}\right)\left(\frac{\bar{f}}{\bar{r}_i} + \tfrac{1}{2}\,\bar{g}\right) = -\frac{5}{4}\left(\frac{\nu}{1-\nu}\right)\frac{w_o}{h}$$

$$\bar{f}' + \left(\frac{\nu}{1-\nu}\right)\left(\frac{\bar{f}}{\bar{r}_o} + \tfrac{1}{2}\,\bar{g}\right) = -\frac{5}{4}\left(\frac{\nu}{1+\nu}\right)\frac{w_o}{h}$$

$$2/7\ \bar{g}' - \bar{f} = 0$$

$$2/7\ \bar{g}' - \bar{f} = 0,$$

using

$$DD = GNU/(1.0 - GNU)$$

$$w\emptyset = w_o/h$$

and

$$F,\ FF,\ G,\ GG,$$

the following are obtained:

$$\left(S1(1,1) + DD*\left(\frac{S2(1,1)}{RAD\emptyset} + 0.5*S4(1,1)\right)\right) A_1$$

$$+\left(S1(2,1) + DD*\left(\frac{S2(2,1)}{RAD\emptyset} + 0.5*S4(2,1)\right)\right) A_2$$

$$+\left(S1(3,1) + DD*\left(\frac{S2(3,1)}{RAD\emptyset} + 0.5*S4(3,1)\right)\right) A_3$$

$$+\left(S1(4,1) + DD*\left(\frac{S2(4,1)}{RAD\emptyset} + 0.5*S4(4,1)\right)\right) A_4 = -\ DD*w\emptyset*1.25$$

$$\left(S1(1,NRR) + DD*\left(\frac{S2(1,NRR)}{RADN} + 0.5*\ S4(1,NRR)\right)\right) A_1$$

$$+\left(S1(2,NRR) + DD*\left(\frac{S2(2,NRR)}{RADN} + 0.5*\ S4(2,NRR)\right)\right) A_2$$

$$+\left(S1(3,NRR) + DD*\left(\frac{S2(3,NRR)}{RADN} + 0.5*\ S4(3,NRR)\right)\right) A_3$$

$$+\left(S1(4,NRR) + DD*\left(\frac{S2(4,NRR)}{RADN} + 0.5*\ S4(4,NRR)\right)\right) A_4 = -\ DD*\ 1.25*\ W\emptyset$$

$$\left(\frac{2}{7}\,S3(1,1) - S2(1,1)\right)A_1 + \left(\frac{2}{7}\,S3(2,1) - S2(Z,1)\right)A_2$$

$$+\left(\frac{2}{7}\,S3(3,1) - S2(3,1)\right)A_3 + \left(\frac{2}{7}\,S3(4,1) - S2(4,1)\right)A_4 = 0$$

$$\left(2/7\ S3(1,NRR) - S2(1,NRR)\right)A_1 + \left(2/7\ S3(2,NRR) - S2(2,NRR)\right)A_2$$

$$+\left(2/7\ S3(3,NRR) - S2(3,NRR)\right)A_3 + \left(2/7\ S3(4,NRR) - S2(4,NRR)\right)A_4 = 0.$$

Let these equations be represented as

$$\{TRX\}\ \{A\} = \{B\}\ .$$

Solving for A,

$$\{TRX\}^{-1}\ \{B\} = \{A\}.$$

Since B can be written as

$$-\ DD*\ 1.25*\ w\emptyset \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix},$$

$$\{A\} = -\ DD*\ 1.25*\ w\emptyset\ \{TRX\}^{-1}\begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}.$$

Calculation of stresses is

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2G\ \epsilon_{ij}$$

$$\sigma_z = \lambda\Delta + 2G\ \epsilon_z$$

$$= (\lambda+2G)\epsilon_z + \lambda(\epsilon_\theta + \epsilon_r)$$

$$= (\lambda+2G)\ \frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right).$$

$$\sigma_{rz} = 2G\epsilon_{rz}$$

$$= G\ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right).$$

Consider points on flat surface ($z = \pm h$),

$$\frac{\partial u}{\partial r} = \frac{u}{r} = 0$$

$$\frac{\partial w}{\partial r} = 0 \cdot$$

Thus,

$$\sigma_z = (\lambda+2G)\,\frac{\partial w}{\partial z} = (\lambda+2G)\,\left[\frac{w_o}{h} - 2g(r)\right]$$

$$= (\lambda+2G)\,(w\phi - 2G) = \frac{2G(1-\nu)}{1-2\nu}\,(w\dot{\phi} - 2G)$$

$$\sigma_{rz} = G\,\frac{\partial u}{\partial z} = -\,2G\,\frac{f(r)}{h} = -\,2G\,\bar{f}\ .$$

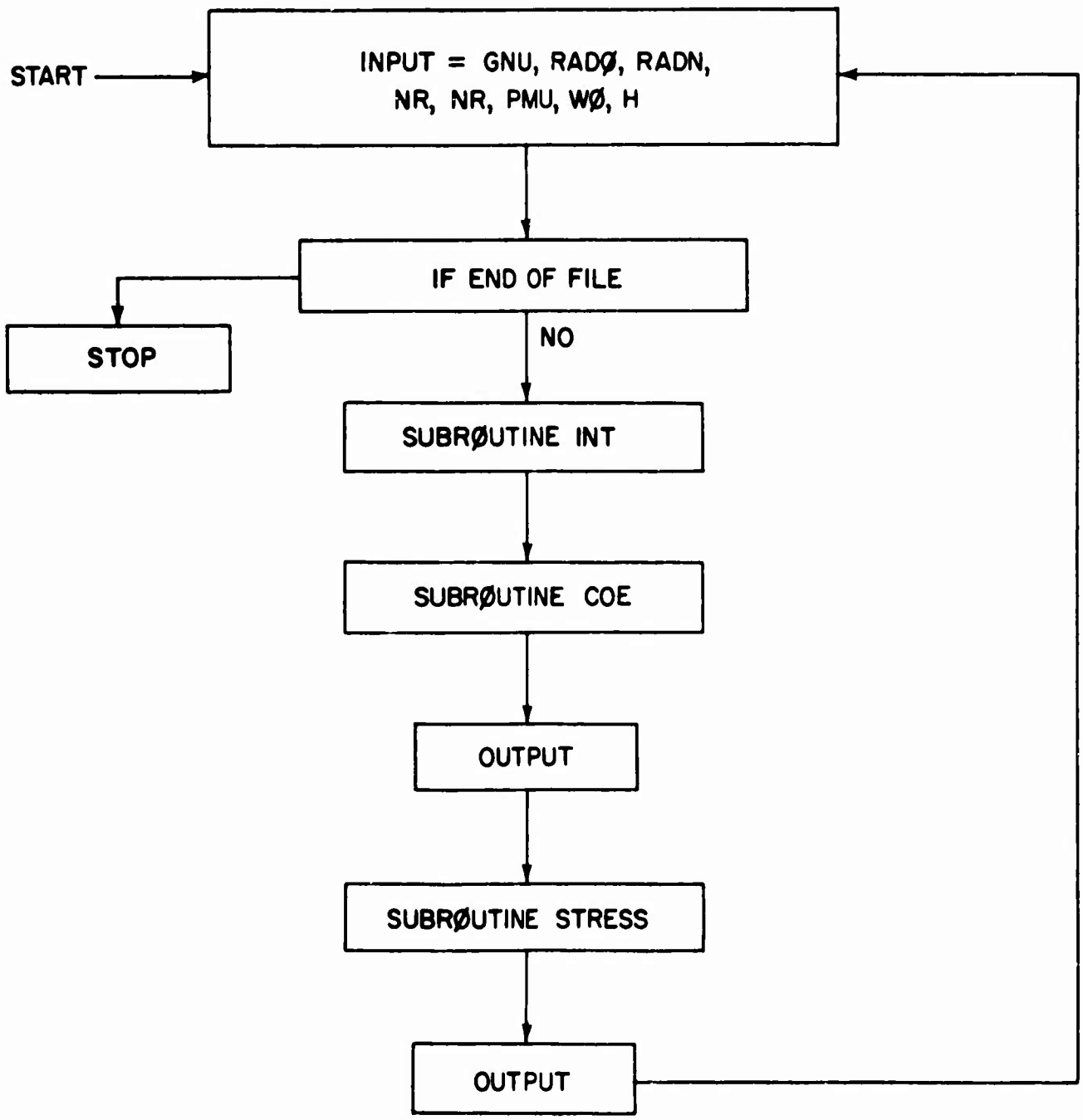


Figure 11. Main Flow Chart.

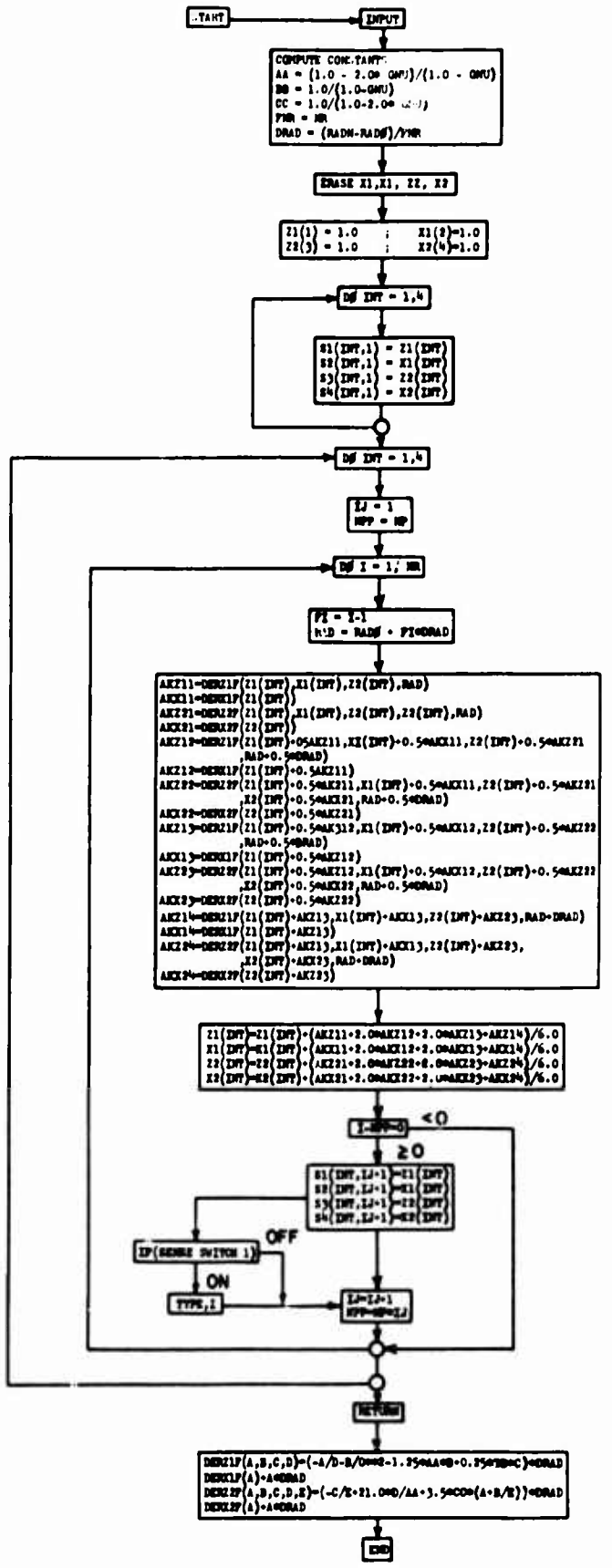


Figure 12. Subroutine Int.

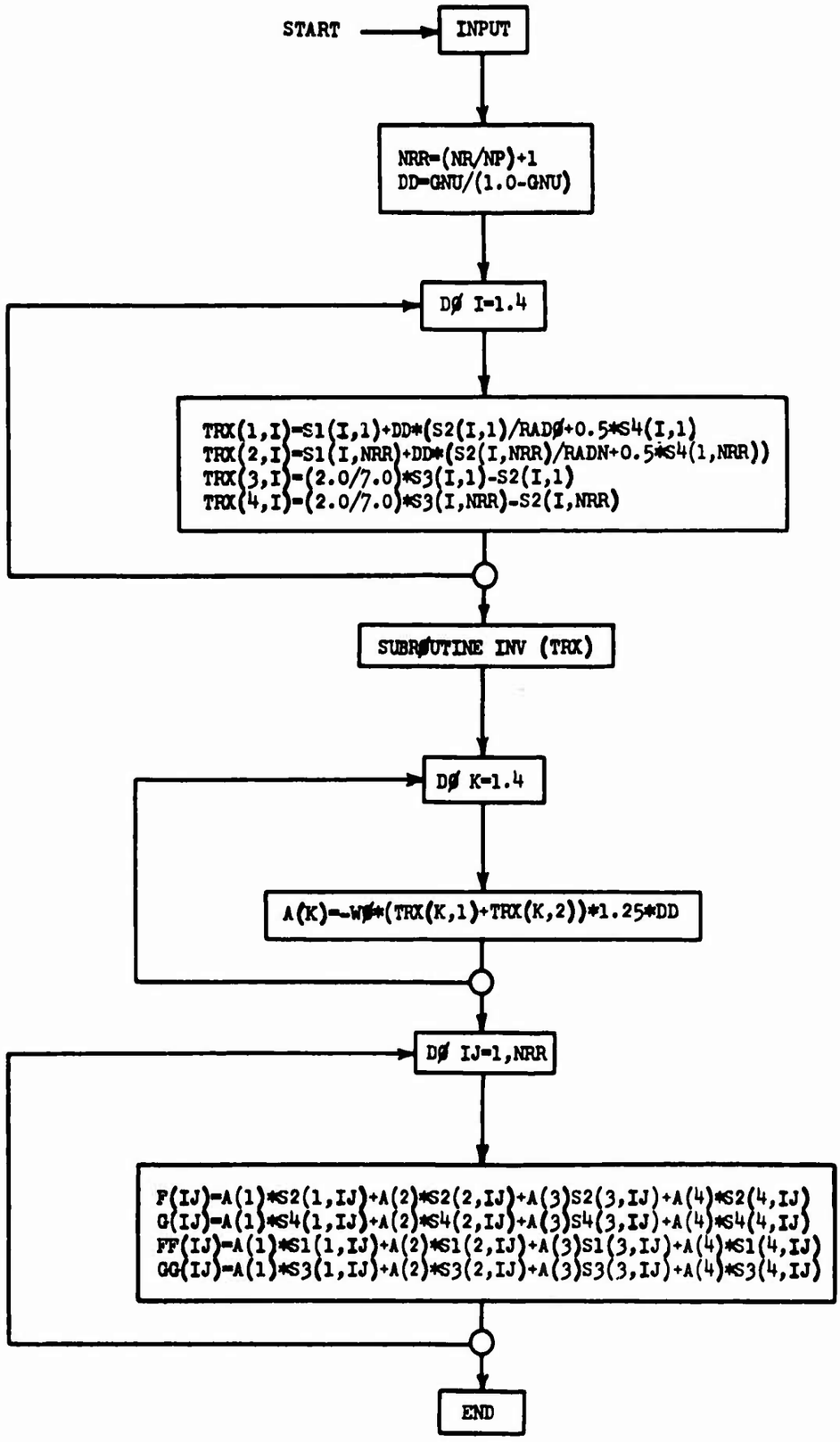


Figure 13. Subroutine Coe.

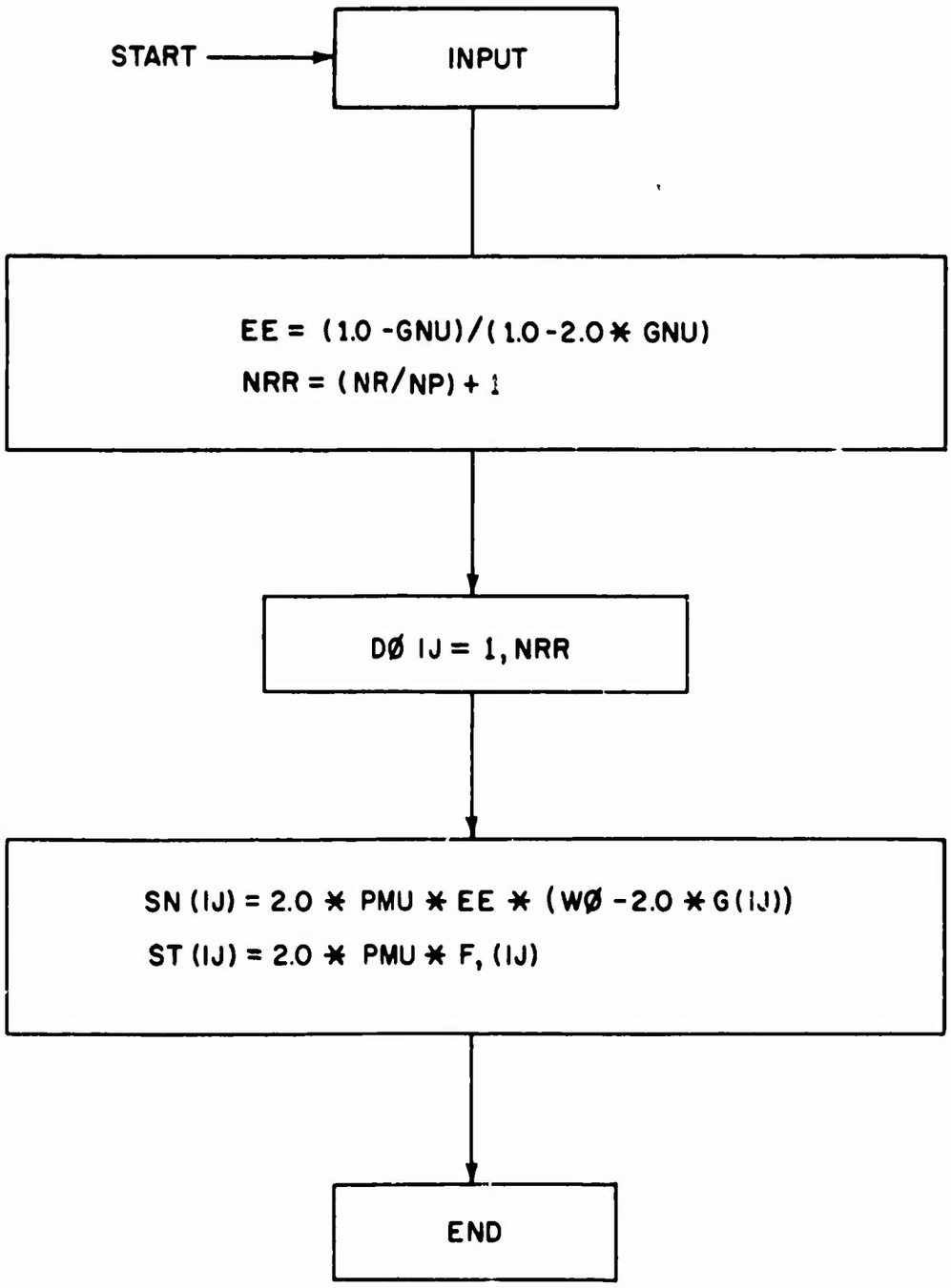


Figure 14. Subroutine Stress.

* * * S O U R C E P R O G R A M L I S T I N G * * *

```
TITLEBEARING
C     MAIN PROGRAM  LAMINATED BEARING
      COMMON  GNU, RADO, RADN, NR, NP, PMU, WO, H,X1,Z1,X2,Z2,
     1S1, S2, S3, S4, F, FF, G, GG, SN, ST
      DIMENSION  X1(4),Z1(4),Z2(4),X2(4), S1(4,51 ),S2(4,51 ),S3(4,51),
     1S4(4,51 ), F(51),  FF(51), G(51), GG(51), SN(51), ST(51)
   10 READ 4,GNU,RADO,RADN,NR,NP,PMU,WO,H
    4 FORMAT (3E15.8, 2I5/3E15.8)
      IF END OF FILE 200,201
  200 STOP
  201 PRINT 6, GNU,RADO,RADN,NR
    6 FORMAT (1H1/3X,6HGNU = F10.5, 3X,7HRADO = F10.3, 3X,7HRADN = F10.3
     1, 3X,5HNR = I5//)
      PRINT 7,NP, PMU, WO, H
    7 FORMAT (3X,5HNP = I5, 3X,6HPMU = F10.5, 3X,5HWO = F10.5,
     13X,4HH = F10.8////)
   20 CALL INT
   40 CALL COE
      PRINT 1
    1 FORMAT  H1/45X,29HDISPLACEMENTS AND DERIVATIVES//)
      PRINT 2
    2 FORMAT (// 18X,2HIJ 6X,11HF  FUNCTION 7X,11HG  FUNCTION 18X,
     113H F DERIVATIVE  7X,13H G DERIVATIVE//)
      NRR =(NR/NP)+1
      PRINT 3,(IJ, F(IJ), G(IJ), FF(IJ), GG(IJ),IJ=1,NRR)
    3 FORMAT (18X,I2,2X,E15.8,3X,E15.8, 16X,E15.8,5X,E15.8)
   60 CALL STRESS
      PRINT 101
  101 FORMAT (1H1/47X 23H STRESS AT FLAT SURFACE)
      PRINT 102
  102 FORMAT (// 39X 2HIJ 7X 14H NORMAL STRESS 10X 14H SHEAR  STRESS)
      PRINT 103, (IJ, SN(IJ), ST(IJ), IJ=1,NRR)
  103 FORMAT (39X I2, 6X E15.8, 9X E15.8)
      GO TO 10
      END
```


PAGE001 #01L301X AUTOMATH SYSTEM

```
      SUBROUTINEINT

      COMMON  GNU, RADO, RADN, NR, NP, PMU, WO, H,X1,Z1,X2,Z2,
     1S1, S2, S3, S4, F, FF, G, GG, SN, ST

      DIMENSION  X1(4),Z1(4),Z2(4),X2(4), S1(4,51 ),S2(4,51 ),S3(4,51),
     1S4(4,51 ), F(51),  FF(51), G(51), GG(51), SN(51), ST(51)

      AA = (1.0 - 2.0*GNU)/(1.0 - GNU)

      BB = 1.0 /(1.0 - GNU)

      CC = 1.0 /(1.0 - 2.0*GNU)

      FNR = NR

      DRAD = (RADN - RADO)/FNR

      ERASE (Z1,X1,Z2,X2)

      Z1(1)    =1.0

      X1(2)    = 1.0

      Z2(3)    = 1.0

      X2(4)    = 1.0

      DO 20 INT = 1,4

      S1(INT,1) = Z1(INT)

      S2(INT,1) = X1(INT)

      S3(INT,1) = Z2(INT)

      S4(INT,1) = X2(INT)

   20 CONTINUE

      DO 100  INT=1,4

      IJ=1

      NPP=NP

      DO 60 I= 1,NR

      FI = I -1

      RAD = RADO + FI*DRAD

      AKZ11 = DERZ1F(Z1(INT),X1(INT),Z2(INT),RAD)

      AKX11 = DERX1F(Z1(INT))

      AKZ21 = DERZ2F(Z1(INT)  , X1(INT)  ,Z2(INT)  ,X2(INT)  , RAD)

      AKX21 = DERX2F(Z2(INT))

      AKZ12 = DERZ1F(Z1(INT) + 0.5*AKZ11,X1(INT) + 0.5*AKX11, Z2(INT)
     1+0.5*AKZ21, RAD+0.5*DRAD)

      AKX12 = DERX1F(Z1(INT) + 0.5*AKZ11)

      AKZ22 = DERZ2F(Z1(INT) + 0.5*AKZ11, X1(INT) + 0.5*AKX11,Z2(INT)
```

A

```
      AKZ21 = DERZ2F(Z1(INT), X1(INT) + 0.5*AKX11, Z2(INT)
     1+0.5*AKZ11, X2(INT) + 0.5*AKX11, RAD+0.5*DRAD)

      AKX21 = DERX2F(Z2(INT))

      AKZ12 = DERZ1F(Z1(INT) + 0.5*AKZ11,X1(INT) + 0.5*AKX11, Z2(INT)
     1+0.5*AKZ21, RAD+0.5*DRAD)

      AKX12 = DERX1F(Z1(INT) + 0.5*AKZ11)

      AKZ22 = DERZ2F(Z1(INT) + 0.5*AKZ11, X1(INT) + 0.5*AKX11,Z2(INT)
     1+0.5*AKZ21, X2(INT) + 0.5*AKX21, RAD+0.5*DRAD)

      AKX22 = DERX2F(Z2(INT) + 0.5*AKZ21)

      AKZ13 = DERZ1F(Z1(INT) + 0.5*AKZ12, X1(INT) + 0.5*AKX12, Z2(INT)
     1+0.5*AKZ22, RAD+0.5*DRAD)

      AKX13 = DERX1F(Z1(INT) + 0.5*AKZ12)

      AKZ23 = DERZ2F(Z1(INT) + 0.5*AKZ12, X1(INT) + 0.5*AKX12, Z2(INT)
     1+0.5*AKZ22, X2(INT) + 0.5*AKX22,RAD+0.5*DRAD)

      AKX23 = DERX2F(Z2(INT) + 0.5*AKZ22)

      AKZ14 = DERZ1F(Z1(INT) + AKZ13, X1(INT) + AKX13, Z2(INT) + AKZ23,
     1RAD+DRAD)

      AKX14 = DERX1F(Z1(INT) + AKZ13)

      AKZ24 = DERZ2F(Z1(INT) + AKZ13, X1(INT) + AKX13, Z2(INT) + AKZ23,
     1X2(INT) + AKX23, RAD+DRAD)

      AKX24 = DERX2F(Z2(INT) + AKZ23)

      Z1(INT)      = Z1(INT)    + (AKZ11+2.0*AKZ12+2.0*AKZ13+AKZ14)/6.0

      X1(INT)      = X1(INT)    + (AKX11+2.0*AKX12+2.0*AKX13+AKX14)/6.0

      Z2(INT)      = Z2(INT)    + (AKZ21+2.0*AKZ22+2.0*AKZ23+AKZ24)/6.0

      X2(INT)      = X2(INT)    + (AKX21+2.0*AKX22+2.0*AKX23+AKX24)/6.0

      IF(I-NPP)60,30,30

   30 S1(INT,IJ+1) = Z1(INT)

      S2(INT,IJ+1) = X1(INT)

      S3(INT,IJ+1) = Z2(INT)

      S4(INT,IJ+1) = X2(INT)

      IF (SENSE SWITCH 1) 200,300

  200 TYPE,I

  300 CONTINUE

      IJ=IJ+1

      NPP= NP*IJ

   60 CONTINUE

  100 CONTINUE

      RETURN

      DERZ1F(A,B,C,D) = (-A/D+ B/D**2 + 1.25*AA*B - 0.25*BB*C)*DRAD

      DERX1F(A) = A * DRAD

      DERZ2F(A,B,C,D,E) = (-C/E + 21.0*D/AA + 3.5*CC*(A + B/E))*DRAD

      DERX2F(A) = A * DRAD

      END
```

* * * S O U R C E P R O G R A M L I S T I N G * * *

```
      SUBROUTINE  INV(A)
C ROUTINE TO INER  A 4 BY 4 MATRIX, A(4, 4)
      DIMENSION A(4, 4)
      DO 5 I = 1, 4
      B = A(I, I)
      DO 1 K = 1, 4
    1 A(I, K) = A(I, K)/B
      DO 4 K = 1, 4
      IF(K-I) 2, 4, 2
    2 C = A(K, I)
      DO 3 L = 1, 4
    3 A(K, L) = A(K, L) - C * A(I, L)
      A(K, I) = - C / B
    4 CONTINUE
    5 A(I, I) = 1.0 / B
      RETURN
      END
```

59 60

DERX2F(A) = A * DRAD

END

AUTOMATH SYSTEM 06/22/64

* * * S O U R C E P R O G R A M L I S T I N G * * *

```
      SUBROUTINECOE

      COMMON  GNU, RADO, RADN, NR, NP, PMU, WO, H,X1,Z1,X2,Z2,
     1S1, S2, S3, S4, F, FF, G, GG, SN, ST

      DIMENSION  X1(4),Z1(4),Z2(4),X2(4), S1(4,51 ),S2(4,51 ),S3(4,51),
     1S4(4,51 ), F(51),  FF(51), G(51), GG(51), SN(51), ST(51)
     1,TRX(4,4), A(4)

      NRR =(NR/NP)+1

      DD = GNU/(1.0 - GNU)

      DO 10 I = 1,4

      TRX(1,I) = S1(I,1) +DD * (S2(I,1)/RADO + 0.5*S4(I,1))

      TRX(2,I) = S1(I,NRR) + DD * (S2(I,NRR)/RADN + 0.5* S4(I,NRR))

      TRX(3,I) = (2.0/7.0)*S3(I,1) - S2(I,1)

      TRX(4,I) = (2.0/7.0)*S3(I,NRR) - S2(I,NRR)

   10 CONTINUE

      CALL INV(TRX)

      DO 20 K =1,4

      A(K)=-WO*(TRX(K,1) + TRX(K,2))*1.25*DD

   20 CONTINUE

      DO 30 IJ = 1, NRR

      F(IJ) = A(1)*S2(1,IJ)+A(2)*S2(2,IJ)+A(3)*S2(3,IJ)+A(4)*S2(4,IJ)

      G(IJ) = A(1)*S4(1,IJ)+A(2)*S4(2,IJ)+A(3)*S4(3,IJ)+A(4)*S4(4,IJ)

      FF(IJ) =A(1)*S1(1,IJ)+A(2)*S1(2,IJ)+A(3)*S1(3,IJ)+A(4)*S1(4,IJ)

      GG(IJ) =A(1)*S3(1,IJ)+A(2)*S3(2,IJ)+A(3)*S3(3,IJ)+A(4)*S3(4,IJ)

   30 CONTINUE

      RETURN

      END
```

* * * S O U R C E P R O G R A M L I S T I N G * * *

```
      SUBROUTINESTRESS

      COMMON  GNU, RADO, RADN, NR, NP, PMU, WO, H,X1,Z1,X2,Z2,
     1S1, S2, S3, S4, F, FF, G, GG, SN, ST

      DIMENSION  X1(4),Z1(4),Z2(4),X2(4), S1(4,51 ),S2(4,51 ),S3(4,51),
     1S4(4,51 ), F(51),  FF(51), G(51), GG(51), SN(51), ST(51)

      EE =(1.0 - GNU)/(1.0 - 2.0*GNU)                                   1

      NRR =(NR/NP)+1                                                    2

      DO 10  IJ = 1,NRR                                                 3

      SN(IJ) = 2.0 * PMU * EE * (WO - 2.0 * G(IJ))                      4

      ST(IJ) = -2.0 * PMU * F(IJ)                                       5

   10 CONTINUE                                                          6

      RETURN                                                            7

      END                                                               8
```

APPENDIX D

PURE TORSION IN LARGE-DEFORMATION ELASTICITY OF AN INCOMPRESSIBLE ELASTOMER

Take as a reference frame cylindrical coordinates r, θ, z in the deformed body. Then a point r, θ, z was initially at the point $\rho, \theta-\zeta z, z$, where ρ is assumed to be a function of r only, and the point of the undeformed body is given by

$$x_1 = \rho \cos(\theta-\zeta z),$$
$$x_2 = \rho \sin(\theta-\zeta z),$$
$$x_3 = z, \tag{D-1}$$

where ζ is the angle of rotation in the z direction.

The components G_{ij}, G^{ij} of the metric tensor of the deformed body are

$$G_{ij} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{Bmatrix}, \qquad G^{ij} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{Bmatrix}. \tag{D-2 \& D-3}$$

The components g_{ij}, g^{ij} of the metric tensor of the undeformed body are

$$g_{ij} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & r^2 & -\zeta r^2 \\ 0 & -\zeta r^2 & 1+\zeta^2 r^2 \end{Bmatrix}, \qquad g^{ij} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 + 1/r^2 & \zeta \\ 0 & \zeta & 1 \end{Bmatrix}. \tag{D-4 \& D-5}$$

Since $G = g = r^2$, the incompressibility equation $I_3 = 1$ is satisfied.

Consider the Mooney stress-strain relation for which the strain energy functions are given in terms of two constants, C_1 and C_2.

$$W = C_1(I_1 - 3) + C_2(I_2 - 3). \tag{D-6}$$

Thus, from equations (E-7) and (E-7a),

$$\Phi = 2C_1, \qquad \bar{\Psi} = 2C_2$$

$$Q_{ik} = \begin{Bmatrix} 2+\zeta^2 r^2 & 0 & 0 \\ 0 & \zeta^2 + \frac{2}{r^2} & \zeta \\ 0 & \zeta & 2 \end{Bmatrix}. \tag{D-7}$$

The contravariant stress tensor is

$$\left.\begin{aligned} \tau^{11} &= 2(C_1 + 2C_2) + 2C_2\,\zeta^2\,r^2 + P \\ r^2\tau^{22} &= 2(C_1 + 2C_2) + 2(C_1 + C_2)\,\zeta^2 r^2 + P \\ \tau^{33} &= 2(C_1 + 2C_2) + P \\ \tau^{33} &= 2(C_1 + C_2)\zeta \\ \tau^{31} &= \tau^{12} = 0 \end{aligned}\right\}. \tag{D-8}$$

The equilibrium equations are

$$\tau^{ik}_{\ \ i} + \Gamma^{\ i}_{ir}\,\tau^{rk} + \Gamma^{\ k}_{ir}\,\tau^{ir} = 0. \tag{D-9}$$

From the metric tensor for a deformed body,

$$\begin{aligned} \Gamma^1_{22} &= -r \\ \Gamma^2_{12} &= \Gamma^2_{21} = 1/r. \end{aligned} \tag{D-10}$$

The rest are zero.

The equilibrium equations are

$$\left.\begin{aligned} &\frac{\partial \tau^{11}}{\partial r} + \frac{1}{r}\tau^{11} - r\tau^{22} = 0 \\ &\frac{\partial \tau^{22}}{\partial \theta} = 0 \\ &\frac{\partial \tau^{33}}{\partial z} = 0 \end{aligned}\right\} \qquad \text{(D-11)}$$

or

$$\left.\begin{aligned} &\frac{\partial P}{\partial r} + 4C_2\zeta^2 r + \frac{1}{r}\{2C_2\zeta^2 r^2 - 2(C_1 + C_2)\zeta^2 r^2\} = 0 \\ &\frac{\partial P}{\partial \theta} = 0 \\ &\frac{\partial P}{\partial z} = z \end{aligned}\right\}. \qquad \text{(D-12)}$$

From equation (D-7),

$$\frac{\partial P}{\partial r} + (4C_2 - 2C_1)\zeta^2 r = 0$$

and

$$P = (C_1 - 2C_2)\zeta^2 r^2 + \alpha. \qquad \text{(D-13)}$$

Since

$$\tau^{11} = 0 \quad \text{at } r = r_o \quad \text{(outer radius)}$$

$$P = -2(C_1 + 2C_2) - 2C_2\zeta^2 r_o^{\,2}$$

$$\alpha = -2(C_1 + 2C_2) - C_1\zeta^2 r_o^{\,2} \qquad \text{(D-14)}$$

$$\therefore \; P = -2(C_1 + 2C_2) - C_1\zeta^2 r_o^{\,2} + (C_1 - 2C_2)\zeta^2 r^2 \qquad \text{(D-15)}$$

$$\left.\begin{aligned}
\tau^{11} &= -C_1\zeta^2 r_o^{\,2} + C_1\zeta^2 r^2 = C_1\zeta^2(r^2 - r_o^{\,2}) \\
r^2\tau^{22} &= 3C_1\zeta^2 r^2 - C_1\zeta^2 r_o^{\,2} = C_1\zeta^2(3r^2 - r_o^{\,2}) \\
\tau^{33} &= -C_1\zeta^2 r_o^{\,2} + (C_1 - 2C_2)\zeta^2 r^2 \\
\tau^{23} &= 2(C_1 + C_2)\zeta
\end{aligned}\right\}. \tag{D-16}$$

The physical components of stress are

$$\sigma_{ij} = \sqrt{G_{ij}/G^{ii}}\ \tau^{ij} \quad \text{(no sum)} \tag{D-17}$$

$$\left.\begin{aligned}
\sigma_{11} &= \tau^{11} = C_1\zeta^2(r^2 - r_o^{\,2}) \\
\sigma_{22} &= r^2\tau^{22} = C_1\zeta^2(3r^2 - r_o^{\,2}) \\
\sigma_{33} &= \tau^{33} = C_1\zeta^2(r^2 - r_o^{\,2}) - 2C_2\zeta^2 r^2 \\
\sigma_{23} &= r\tau^{23} = 2(C_1 + C_2)r\zeta
\end{aligned}\right\}. \tag{D-18}$$

$$\sigma_{31} = \sigma_{12} = 0$$

The surface tractions at plane end (bonded surface) are

$$\left.\begin{aligned}
T_r &= 0 \\
T_\theta &= r\tau^{22} \\
T_z &= \tau^{33}
\end{aligned}\right\}. \tag{D-19}$$

The moment required for the twist is

$$M = \int (rT\theta) r dr d\theta = \int r^3 \tau^{22} dr d\theta$$

$$= 2\pi \int_b^a C_1 \zeta^2 (3r^3 - r_o^2) dr$$

$$= 2\pi C_1 \zeta^2 \left[\frac{3r^4}{4} - \frac{r_o^2 r^2}{2}\right]_{r_i}^{r_o} = 2\pi C_1 \zeta^2 \left[+ \frac{r_o^4}{4} - \frac{r_i^4}{4} + \frac{r_o^2 r_i^2}{2}\right]$$

$$M = \frac{\pi}{2} C_1 \zeta^2 [r_o^4 - r_i^4 + 2r_o^2 r_i^2] . \qquad \text{(D-20)}$$

Force is also required to prevent elongation:

$$N = 2\pi \int_a^b \tau^{33} r dr = 2\pi \int_a^b [C_1 \zeta^2 (r^2 - r_o^2) - 2C_2 \zeta^2 r^2] r dr$$

$$= 2\pi \{-C_1 \zeta^2 r_o^2/2 \, (r_o^2 - r_i^2) + (C_1 - 2C_2)\zeta^2 \tfrac{1}{4}(r_o^4 - r_i^4)\}$$

$$= 2\pi \zeta^2 \{- \frac{C_1}{4} r_o^4 + \frac{C_1}{2} r_o^2 r_i^2 - \frac{C_2}{2} (r_o^4 - r_i^4)\} . \qquad \text{(D-21)}$$

It is found that in order to maintain this state of deformation, pressure must be applied at the inner curve surface. The magnitude of the pressure is

$$-C_1 \zeta^2 (\Gamma_o^2 - \Gamma_i^2)$$

where

Γ_i = inner radius

Γ_o = outer radius.

APPENDIX E

LARGE-DEFORMATION ELASTICITY THEORY FOR AXIAL COMPRESSION

In order to characterize the deformation of an elastic body, the position of each point in the body before and after deformation must be described. That is, if a point in the undeformed body is at y^1, y^2, y^3, then in the deformed body it will occupy the point x^1, x^2, x^3. For the current problem, it is convenient to employ cylindrical coordinates for this description; therefore, an undeformed system (r, θ', z') is defined by

$$y^1 = r \cos\theta', \qquad y^2 = r \sin\theta', \qquad y^3 = z \tag{E-1}$$

and a deformed system (ρ, θ, z) by

$$x^1 = \rho \cos\theta, \qquad x^2 = \rho \sin\theta, \qquad x^2 = z. \tag{E-2}$$

In particular, since the loading in the present problem is axially symmetric, the deformation may be considered in the form

$$\rho = \rho(r, z'), \qquad z = z(r, z') \text{ and } \theta = \theta' + t(r, z'). \tag{E-3}$$

In order to determine these functions, it is useful to consider another description of the deformation. For this purpose, the distortion which the initial polar coordinate system undergoes as the body deforms can be taken into account if these coordinates are considered to be embedded in the material. To characterize this distortion, the metric tensors G_{ij} and g_{ij}, which are associated with the polar coordinates before and after deformation, are defined as

$$G_{ij} = \frac{\partial y^r}{\partial \varphi^i} \frac{\partial y^r}{\partial \varphi^j} \quad \text{and} \quad g_{ij} = \frac{\partial x^r}{\partial \varphi^i} \frac{\partial x^r}{\partial \varphi^j} \tag{E-4}$$

where $\varphi^1 = r$, $\varphi^2 = \theta'$, $\varphi^3 = z'$. Using equation (E-1), we obtain

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{E-5}$$

and from equations (E-2) and (E-3),

$$g_{ij} = \begin{pmatrix} [(\frac{\partial\rho}{\partial r})^2 + \rho^2 (\frac{\partial t}{\partial r})^2 + (\frac{\partial z}{\partial r})^2] & \rho^2 \frac{\partial t}{\partial r} & [\frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial z'} + \rho^2 \frac{\partial t}{\partial r}\frac{\partial t}{\partial z'} + \frac{\partial}{\partial r}\frac{\partial z}{\partial z'}] \\ \rho^2 \frac{\partial t}{\partial r} & \rho^2 & \rho^2 \frac{\partial t}{\partial z'} \\ [\frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial z'} + \rho^2 \frac{\partial t}{\partial r}\frac{\partial t}{\partial z} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}] & \rho^2 \frac{\partial t}{\partial z'} & [(\frac{\partial z}{\partial z'})^2 + \rho^2 (\frac{\partial t}{\partial z'}) + (\frac{\partial\rho}{\partial z'})^2] \end{pmatrix}. \tag{E-6}$$

In terms of the above metrics, G_{ij} and g_{ij}, the stresses τ^{ij} in the body may now be determined. As a first approach to the solution of the problem under consideration, we will consider the material to be incompressible and use constitutive relations given by

$$\tau^{ij} = \Phi G^{ij} + \Psi Q^{ij} + g_{ij} \tag{E-7}*$$

where

$$Q^{ij} = G^{rs} g_{rs} G^{ij} - G^{ir} G^{is} g_{rs} , \tag{E-7a}*$$

Ψ and Φ are constants, and ρ is a scalar function of position. Furthermore, from equations (E-5) and (E-6), the contravariant components of the undeformed and deformed coordinates which appear in equation (E-7) are found to be

$$G^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E-8}$$

and

$$g^{ij} = \begin{pmatrix} \text{Matrix where each component} \\ \text{is the cofactor of the cor-} \\ \text{responding component in array} \\ \text{(E-6). This is shown later as} \\ \text{(E-21).} \end{pmatrix}. \tag{E-9}$$

*See reference 2, equations 1.15.13 and 1.15.9, page 28.

Since the stresses are defined, it is now possible to obtain a set of equations whose solution will yield the desired functions (E-3). These equations are obtained by using the equilibrium equations

$$\tau_{,i}^{ij} + \Gamma_{ri}^{\ i}\, \tau^{rj} + \Gamma_{im}^{\ j}\, \tau^{im} = 0 \qquad \text{(E-10)}$$

where the $\Gamma_{st}^{\ r}$ are the Christoffel symbols of the second kind defined by

$$\Gamma_{st}^{\ r} = \tfrac{1}{2}\, g^{kr}\, [g_{sk,t} + g_{tk,s} - g_{st,k}] \qquad \text{(E-11)}$$

and the stresses are given by equations (E-7).

It may be observed that, due to the complexity of the tensors g^{ij} and g_{ij}, the equilibrium equations as defined above will be extremely cumbersome (see Appendix D equations). It may also be observed that if we had, in relation (E-4), identified φ^i's with the deformed rather than the undeformed coordinates, the resulting form of the equilibrium equations would have been greatly simplified. The following considerations, however, indicate that this approach is not practical, since it introduces difficulties in the boundary conditions. If the undeformed coordinates appear as the independent variables in the equilibrium equations, as is currently the case, the boundary is located by specifying its undeformed position. If, however, the equilibrium equations are simplified, as described above, the deformed coordinates will appear as the independent variable and the boundary must be located by specifying its deformed position, which is unknown until the problem has been solved. It is observed that the three equations (E-10) contain four unknowns: ρ, t, z, and P. However, using the fact that the material is incompressible, a fourth equation is obtained,

$$\frac{|G_{ij}|}{|g_{ij}|} = 1\,, \qquad \text{(E-12)}$$

which states that the volume of any portion of the body remains constant.

Up to this point, the loading history of the body has not been discussed. However, it is apparent that unlike in a linear elastic theory, the order of loading must be specified. In order to build this information into the equations, the following considerations are used: Let ρ, t, z, and P be made up of contributions due to compression and torsion (i.e., let $\rho = \rho_c + \rho_t$, $z = z_c + z_t$, etc., where ρ_c and ρ_t represent components

after compression and tension, respectively). Since the compression is applied first, equations (E-10) are solved for ρ_c, z_c, and ρ_c (i.e., $t_c = 0$) with $\rho_t = t_c = z_t = P_t = 0$. Having solved for the compression contribution, the torsion contribution is now determined. It is observed that the torsion considerations are influenced by the initial compression by way of the compression-torsion coupling terms which appear in the equilibrium equations.

Having set up the field equations, we must now consider the boundary conditions.

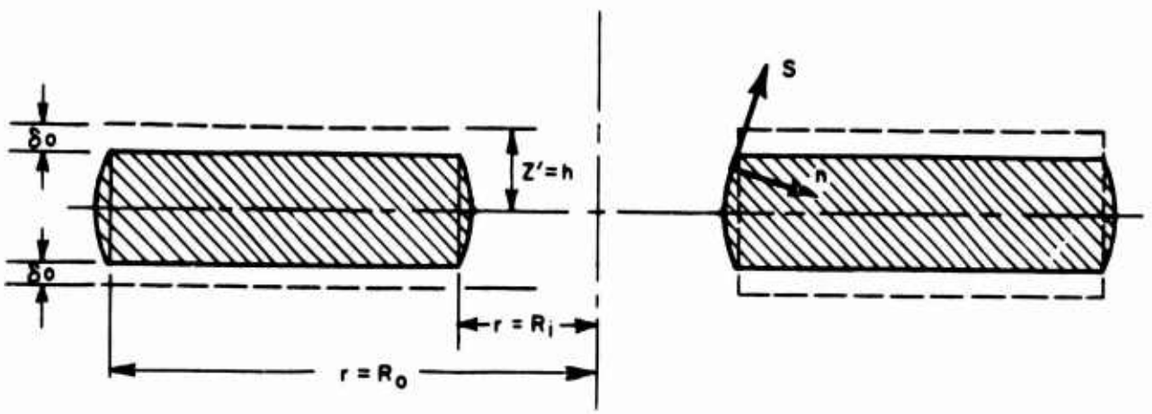


Figure 15. Sketch of Coordinates for One Lamination.

For the compression problem, the boundary conditions which must be satisfied are as follows:

at

$$z' = \pm h, \qquad z_c = \pm (h - \delta_o), \qquad \rho_c = r \tag{E-13}$$

at $r = R_o, R_i$, traction is zero, or in terms of a coordinate system on the boundary, $\tau^{nn} = \tau^{ns} = 0$,

or

$$\tau_c^{nn} = \frac{\left(\frac{d\rho}{dz}\right)^2 \tau_c^{rr} + 2\left(\frac{d\rho}{dz}\right) \tau_c^{rz'} + \tau_c^{z'z'}}{1 + \left(\frac{d\rho}{dz}\right)^2} = 0$$

$$= \tau_i^{ns} = \frac{-\left(\frac{d\rho}{dz}\right) \tau_c^{rr} + \left[\left(\frac{d\rho}{dz}\right)^2 - 1\right] \tau_c^{rz'} + \left(\frac{d\rho}{dz}\right) \tau_c^{z'z'}}{1 + \left(\frac{d\rho}{dz}\right)^2} = 0. \tag{E-14}$$

And for the torsion problem,

at $\quad z' = \pm h, \quad z_t = z_c \pm (h - \delta_o), \quad \rho_t = \rho_c = r$

$\tau_t = \tau_o \quad$ (specified twist angle)

at $\quad r = R_o, R_i, \quad \tau_t^{nn} = \tau_c^{nn} = 0, \ \tau_t^{ns} = \tau_c^{ns} = 0$ and $\tau_t^{\theta'\theta'} = 0.$

Having formulated the problem, a method for obtaining the solution will now be considered. For this purpose, consider rewriting the equilibrium equations and boundary conditions using the displacements

$$\rho_c - r = u_c, \qquad \rho_t - r = u_t$$

$$z_c - z' = w_c, \qquad z_t - z' = w_t, \text{ etc.,}$$

rather than the deformed positions, as dependent variables.

If it is considered now that the linear elastic equations are given in the form

$$f(u, w, P) = 0, \ g(u,w,p) = 0, \quad h(u,w) = 0,$$

then, with the above change of variables, the nonlinear equations will be

$$f(u,w,P) = F(u,w,P), \qquad g(u,w,P) = G(u,w,P)$$

and

$$h(u,w) = H(u,w)$$

where F, G, and H represent nonlinear contributions. The solution to these equations can now be obtained by using a numerical technique similar to that used in the solution of the linear equations. To do this, the nonlinear terms, F, G, H, which appear in the equations are treated as if they represent a nonhomogeneous portion of the linear elastic equations. That is, in each iteration it is considered that the nonlinear terms are known, the value being given by the value obtained from the previous iteration.

Combined Compression and Torsion

Assume that a rubber annulus is placed under axisymmetric compression followed by torsion. Consider deformation described by

deformed coordinates	(ρ, θ, z)
undeformed coordinates	(r, θ', z')
where	$\rho = \rho(r, z')$
	$z = z(r, z')$
	$\theta = \theta' + t(r, z')$.

The metric of φ in the undeformed coordinates is

$$G_{ij} = \frac{\partial y^r}{\partial \varphi^i} \frac{\partial y^r}{\partial \varphi^j} \tag{E-15}$$

where

$$y^1 = r \cos\theta'$$
$$y^2 = r \sin\theta'$$
$$y^3 = z'$$

and

$$\varphi^1 = r$$
$$\varphi^2 = \theta'$$
$$\varphi^3 = z';$$

then

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E-16}$$

$$|G_{ij}| = r^2 \tag{E-16a}$$

and

$$G^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E-17}$$

$$|G^{ij}| = 1/r^2. \tag{E-17a}$$

The metric of φ in the deformed coordinates is

$$g_{ij} = \frac{\partial x^r}{\partial \varphi^i} \frac{\partial x^r}{\partial \varphi^j} \tag{E-18}$$

where

$$x^1 = \rho \cos\theta$$
$$x^2 = \rho \sin\theta$$
$$x^3 = z .$$

Expanding the separate derivatives in equation (E-18) yields

$$\frac{\partial x^1}{\partial \varphi^1} = \frac{\partial x^1}{\partial r} = \frac{\partial x^1}{\partial \rho} \frac{\partial \rho}{\partial r} + \frac{\partial x^1}{\partial \theta} \frac{\partial \theta}{\partial r} = \cos\theta \frac{\partial \rho}{\partial r} - \rho \sin\theta \frac{\partial t}{\partial r}$$

$$\frac{\partial x^2}{\partial \varphi^1} = \frac{\partial x^2}{\partial r} = \frac{\partial x^2}{\partial \rho} \frac{\partial \rho}{\partial r} + \frac{\partial x^1}{\partial \theta} \frac{\partial \theta}{\partial r} = \sin\theta \frac{\partial \rho}{\partial r} + \rho \cos\theta \frac{\partial t}{\partial r}$$

$$\frac{\partial x^3}{\partial \varphi^1} = \frac{\partial x^3}{\partial r} = \frac{\partial x^3}{\partial z} \frac{\partial z}{\partial r} = \frac{\partial z}{\partial r}$$

$$\frac{\partial x^1}{\partial \varphi^2} = \frac{\partial x^1}{\partial \theta} = \frac{\partial x^1}{\partial \theta} \frac{\partial \theta}{\partial \theta'} = - \rho \sin\theta$$

$$\frac{\partial x^2}{\partial \varphi^2} = \frac{\partial x^2}{\partial \theta'} = \frac{\partial x^2}{\partial \theta} \frac{\partial \theta}{\partial \theta'} = \rho \cos\theta$$

$$\frac{\partial x^3}{\partial \varphi^2} = \frac{\partial x^3}{\partial \theta'} = 0$$

$$\frac{\partial x^1}{\partial \varphi^3} = \frac{\partial x^1}{\partial z'} = \frac{\partial x^1}{\partial \rho}\frac{\partial \rho}{\partial z'} + \frac{\partial x^1}{\partial \theta}\frac{\partial \theta}{\partial z'} = \cos\theta \frac{\partial \rho}{\partial z'} - \rho \sin\theta \frac{\partial t}{\partial z'}$$

$$\frac{\partial x^2}{\partial \varphi^3} = \frac{\partial x^2}{\partial z'} = \frac{\partial x^2}{\partial \rho}\frac{\partial \rho}{\partial z'} + \frac{\partial x^2}{\partial \theta}\frac{\partial \theta}{\partial z'} = \sin\theta \frac{\partial \rho}{\partial z'} + \rho \cos\theta \frac{\partial t}{\partial z'}$$

$$\frac{\partial x^3}{\partial \varphi^3} = \frac{\partial x^3}{\partial z'} = \frac{\partial x^3}{\partial z}\frac{\partial z}{\partial z'} = \frac{\partial z}{\partial z'} ;$$

then

$$g_{ij} = \begin{pmatrix} [(\frac{\partial \rho}{\partial r})^2 + \rho^2 (\frac{\partial t}{\partial r})^2 + (\frac{\partial z}{\partial r})^2] & \rho^2 \frac{\partial t}{\partial r} & [\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \rho^2 \frac{\partial t}{\partial r}\frac{\partial t}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}] \\ \rho^2 \frac{\partial t}{\partial r} & \rho^2 & \rho^2 \frac{\partial t}{\partial z'} \\ [\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \rho^2 \frac{\partial t}{\partial r}\frac{\partial t}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}] & \rho^2 \frac{\partial t}{\partial z'} & [(\frac{\partial z}{\partial z'})^2 + \rho^2 (\frac{\partial t}{\partial z'})^2 + (\frac{\partial \rho}{\partial z'})^2] \end{pmatrix}, \tag{E-19}$$

which implies

$$|g_{ij}| = -\rho^2 t_r \{\rho^2 t_r [Z_{z'}^2 + \rho^2 t_{z'}^2 + \rho_{z'}^2] - \rho^2 t_{z'} [\rho_r \rho_{z'} + \rho^2 t_r t_{z'} + Z_r Z_{z'}]\}$$

$$+ \rho^2 \{[\rho_r^2 + \rho^2 t_r^2 + Z_r^2][Z_r^2 + \rho^2 t_{z'}^2 + \rho_{z'}^2] - [\rho_r \rho_{z'} + \rho^2 t_r t_{z'} + Z_r Z_{z'}][\rho_r \rho_{z'}$$

$$+ \rho^2 t_r t_{z'} + Z_r Z_{z'}]\} - \rho^2 t_{z'} \{\rho^2 t_{z'} [\rho_r^2 + \rho^2 t_r^2 + Z_r^2]$$

$$- \rho^2 t_r [\rho_r \rho_{z'} + \rho^2 t_r t_{z'} + Z_r Z_{z'}]\}$$

$$= \rho^2 [\rho_r^2 Z_{z'}^2 + Z_r^2 Z_{z'}^2 + \rho_r^2 \rho_{z'}^2 + Z_r \rho_{z'}^2 - \rho_r^2 \rho_{z'}^2 - Z_r^2 Z_{z'}^2 - 2\rho_r \rho_{z'} Z_r Z_{z'}]$$

and

$$|g_{ij}| = \rho^2 [\frac{\partial \rho}{\partial r}\frac{\partial z}{\partial z'} - \frac{\partial z}{\partial r}\frac{\partial \rho}{\partial z'}]^2 \equiv [\rho h(r,z')]^2. \tag{E-20}$$

Thus,

$$g_{ij} = \frac{1}{[\rho h]^2} \begin{pmatrix} \rho^2 \left[\left(\frac{\partial z}{\partial z'}\right)^2 + \left(\frac{\partial \rho}{\partial z'}\right)^2\right] & \rho^2 \left\{\frac{\partial t}{\partial z'}\left[\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}\right] + \frac{\partial t}{\partial r}\left[\left(\frac{\partial z}{\partial z'}\right)^2 + \left(\frac{\partial \rho}{\partial z'}\right)^2\right]\right\} & -\rho^2 \left[\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}\right] \\ \rho^2 \left\{\frac{\partial t}{\partial z'}\left[\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}\right] + \frac{\partial t}{\partial r}\left[\left(\frac{\partial z}{\partial z'}\right)^2 + \left(\frac{\partial \rho}{\partial z'}\right)^2\right]\right\} & \rho^2 \left[\left(\frac{\partial \rho}{\partial r}\frac{\partial t}{\partial z'} - \frac{\partial t}{\partial r}\frac{\partial \rho}{\partial z'}\right)^2 + \left(\frac{\partial t}{\partial r}\frac{\partial z}{\partial z'} - \frac{\partial t}{\partial z'}\frac{\partial z}{\partial r}\right)^2\right] + \left(\frac{\partial \rho}{\partial r}\frac{\partial z}{\partial z'} - \frac{\partial t}{\partial r}\frac{\partial \rho}{\partial z'}\right) & \rho^2 \left\{\frac{\partial t}{\partial z'}\left[\left(\frac{\partial \rho}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right] + \frac{\partial t}{\partial r}\left[\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}\right]\right\} \\ -\rho^2 \left[\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}\right] & \rho^2 \left\{\frac{\partial t}{\partial z'}\left[\left(\frac{\partial \rho}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right] + \frac{\partial t}{\partial r}\left[\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}\right]\right\} & \rho^2 \left[\left(\frac{\partial \rho}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right] \end{pmatrix}. \qquad \text{(E-21)}$$

Now the strain invariants are

$$I_1 = G^{ij} g_{ij} = g_{11} + \frac{1}{r^2} g_{22} + g_{33} \qquad \text{(E-22)}$$

and

$$I_3 = \frac{|G_{ij}|}{|g_{ij}|} = \frac{r^2}{\rho^2 h^2}, \qquad \text{(E-23)}$$

but for incompressibility, $I_3 = 1$; hence,

$$\rho = \frac{r}{h(r,z')}. \qquad \text{(E-24)}$$

Also, from equation (E-7a),

$$Q^{ij} = I_1 \, G^{ij} - G^{ir} \, G^{js} \, g_{rs}$$

where G^{ij} is the diagonal $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Thus, the components of G^{ij} are

$$\left.\begin{aligned}
Q^{11} &= I_1 \, G^{11} - G^{11} \, G^{11} \, g_{11} = 1/r^2 \, g_{22} + g_{33} \\
Q^{22} &= I_1 \, G^{22} - G^{22} \, G^{22} \, g_{22} = 1/r^2 \, (g_{11} + g_{33}) \\
Q^{33} &= I_1 \, G^{33} - G^{33} \, G^{33} \, g_{33} = g_{11} + 1/r^2 \, g_{22} \\
Q^{12} &= Q^{21} = G^{11} \, G^{22} \, g_{12} = - 1/r^2 \, g_{12} \\
Q^{13} &= Q^{31} = - G^{11} \, G^{33} \, g_{13} = - g_{13} \\
Q^{23} &= Q^{32} = - G^{22} \, G^{33} \, g_{23} = - 1/r^2 \, g_{23}
\end{aligned}\right\} \quad \text{(E-25)}$$

Thus,

$$\underbrace{\tau^{ij}}_{\text{deformed coordinates}} = \Phi \, \underbrace{G^{ij}}_{\text{undeformed coordinates}} + \Psi \, Q^{ij} + P \, \underbrace{g^{ij}}_{\text{deformed coordinates}}$$

which is then substituted into the equilibrium equations, (E-10).

Compression Only

For the problem of axial compression only, the formulation is carried further as follows:

For the case of notation, let

$$\left.\begin{aligned} u_c &= u \\ z &= z' + \\ \rho &= r + \end{aligned}\right\}. \qquad \text{(E-26)}$$

From equation (E-17),

$$G^{ij}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From equation (E-20),

$$h = \left(\frac{\partial\rho}{\partial r}\frac{\partial z}{\partial z'} - \frac{\partial z}{\partial r}\frac{\partial\rho}{\partial z'}\right) = \left[\left(1 + \frac{\partial u}{\partial r}\right)\left(1 + \frac{\partial w}{\partial z'}\right) - \frac{\partial w}{\partial r}\frac{\partial u}{\partial z'}\right]; \qquad \text{(E-27)}$$

and from equation (E-23), h is also given by

$$h = r/\rho\,. \qquad \text{(E-27a)}$$

From equation (E-21), the components of g^{ij} are

$$\left.\begin{aligned}
g^{11} &= \frac{1}{h^2}\left[\left(\frac{\partial z}{\partial z'}\right)^2 + \left(\frac{\partial\rho}{\partial z'}\right)^2\right] = \frac{1}{h^2}\left[\left(1 + \frac{\partial w}{\partial z'}\right)^2 + \left(\frac{\partial w}{\partial z'}\right)^2\right] \\
g^{22} &= \frac{1}{\rho^2 h^2}\left[\frac{\partial\rho}{\partial r}\frac{\partial z}{\partial z'} - \frac{\partial z}{\partial r}\frac{\partial\rho}{\partial z'}\right]^2 = \frac{1}{\rho^2} = \frac{1}{r^2\left(1+\frac{u}{r}\right)^2} = \frac{h^2}{r^2} \\
g^{33} &= \frac{1}{h^2}\left[\left(\frac{\partial\rho}{\partial r}\right)^2 + \left(\frac{\partial t}{\partial r}\right)^2\right] = \frac{1}{h^2}\left[\left(1 + \frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right] \\
g^{13} &= g^{31} = \frac{1}{h^2}\left[\frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}\right] = -\frac{1}{h^2}\left[\left(1 + \frac{\partial u}{\partial r}\right)\frac{\partial u}{\partial z'} + \frac{\partial w}{\partial r}\left(1 + \frac{\partial w}{\partial z'}\right)\right] \\
g^{12} &= g^{21} = g^{23} = g^{32} = 0 \\
&\qquad \text{(Since torsion is zero, } \quad 0)
\end{aligned}\right\}. \qquad \text{(E-28)}$$

From equation (E-19), the components of g_{ij} are

$$\left.\begin{aligned}
g_{11} &= \left(\frac{\partial\rho}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \left(1 + \frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2 = h^2\, g^{33} \\
g_{22} &= \rho^2 = (u + r)^2 = r^2(1 + u/r)^2 = r^2/h^2 \\
g_{33} &= \left(\frac{\partial z}{\partial z'}\right)^2 + \left(\frac{\partial\rho}{\partial z'}\right)^2 = \left(1 + \frac{\partial w}{\partial z'}\right)^2 + \left(\frac{\partial u}{\partial z'}\right)^2 = h^2\, g^{11} \\
g_{31} &= g_{13} = \frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'} = \left(1 + \frac{\partial u}{\partial r}\right)\frac{\partial u}{\partial z'} + \frac{\partial w}{\partial z'}\left(1 + \frac{\partial w}{\partial z'}\right) = -h^2 g^{31} = h^2 g^{13} \\
g_{23} &= g_{32} = g_{21} = g_{12} \equiv 0
\end{aligned}\right\} . \tag{E-29}$$

Now the components of τ^{ij} from equation (E-7), by substituting from equations (E-17), (E-25), (E-28), and (E-29), are

$$\left.\begin{aligned}
\tau^{11} &= \Phi + \frac{\Psi}{h^2} + \left[\left(1 + \frac{\partial W}{\partial z}\right)^2 + \left(\frac{\partial U}{\partial z'}\right)^2\right]\left(\Psi + \frac{P}{h^2}\right) \\
\tau^{22} &= \frac{\Phi}{r^2} + \frac{\Psi}{r^2}\left(g_{11} + g_{33}\right) + P\,\frac{h^2}{r^2} \\
&= \frac{1}{r^2}\left\{\Phi + \Psi\left[\left(1 + \frac{\partial U}{\partial r}\right)^2 + \left(1 + \frac{\partial W}{\partial z'}\right)^2 + \left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{\partial U}{\partial z'}\right)^2\right] + Ph^2\right\} \\
\tau^{33} &= \Phi + \Psi\left(g_{11} + 1/r^2\, g_{22}\right) + P\,\frac{g_{11}}{h^2} = \Phi + \frac{\Psi}{h^2} \\
&\quad + \left[\left(1 + \frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial W}{\partial r}\right)^2\right]\left(\Psi + P/h^2\right) \\
\tau^{13} &= \tau^{31} = \left(\Psi + \frac{P}{h^2}\right)\left[\left(1 + \frac{\partial U}{\partial r}\right)\left(\frac{\partial U}{\partial z'}\right) + \frac{\partial W}{\partial r}\left(1 + \frac{\partial W}{\partial z'}\right)\right] \\
\tau^{23} &= \tau^{32} = \tau^{12} = \tau^{21} \equiv 0
\end{aligned}\right\} . \tag{E-30}$$

The equilibrium equations are now

$$\tau^{ij}/_{i} = 0 \tag{E-31}$$

or

$$\tau^{ij}_{,i} + \Gamma^{i}_{ri}\,\tau^{rj} + \Gamma^{j}_{im}\,\tau^{im} = 0\,,$$

as shown in Equation (E-10),

where

$$\Gamma^{r}_{st} = \tfrac{1}{2}\, g^{kr}\,[g_{sk,t} + g_{tk,s} - g_{st,k}]\,. \tag{E-32}$$

Note: $\Gamma^{r}_{st} = \Gamma^{r}_{ts}$.

In the current problem of compression only, the only nonzero Christoffel symbols are

$$\left.\begin{aligned}
\Gamma^{1}_{11} &= \tfrac{1}{2}\,[g^{11}\,g_{11,1} + g^{13}\,(2g_{13,1} - g_{11,3})] \\
\Gamma^{1}_{22} &= -\tfrac{1}{2}\,(g^{11}\,g_{22,1} + g^{13}\,g_{22,3}) \\
\Gamma^{1}_{13} &= \Gamma^{1}_{31} = \tfrac{1}{2}\,(g^{11}\,g_{11,3} + g^{13}\,g_{33,1}) \\
\Gamma^{1}_{33} &= \tfrac{1}{2}\,[g^{13}\,g_{33,3} + g^{11}\,(2g_{13,3} - g_{33,1}) \\
\Gamma^{2}_{12} &= \Gamma^{2}_{21} = \tfrac{1}{2}g^{22}\,g_{22,1} \\
\Gamma^{2}_{23} &= \Gamma^{2}_{32} = \tfrac{1}{2}g^{22}\,g_{22,3} \\
\Gamma^{3}_{11} &= \tfrac{1}{2}\,[g^{13}\,g_{11,1} + g^{33}\,(2g_{13,1} - g_{11,3})] \\
\Gamma^{3}_{22} &= -\tfrac{1}{2}\,[g^{31}\,g_{22,1} + g^{33}\,g_{22,3}] \\
\Gamma^{3}_{33} &= \tfrac{1}{2}\,[\,g^{33}\,g_{33,3} + g^{31}\,(2g_{13,3} - g_{33,1})] \\
\Gamma^{3}_{31} &= \Gamma^{3}_{13} = \tfrac{1}{2}\,[g^{13}\,g_{11,3} + g^{33}\,g_{33,1}]
\end{aligned}\right\} \tag{E-33}$$

Now the equilibrium equations become

$$\underline{\tau^{11}_{,1}} + \underline{\tau^{31}_{,2}} + 2\,\Gamma^1_{11}\tau^{11} + 3\,\Gamma^1_{31}\tau^{31} + \underline{\Gamma^2_{12}\tau^{11}} + \Gamma^2_{32}\tau^{31}$$

$$+ \Gamma^3_{33}\tau^{31} + \Gamma^3_{13}\tau^{11} + \Gamma^1_{33} + \underline{\Gamma^1_{22}\tau^{22}} = 0 \qquad \text{(E-34a)}$$

and

$$\underline{\tau^{33}_{,3}} + \underline{\tau^{13}_{,1}} + 2\,\Gamma^3_{33}\tau^{33} + 3\,\Gamma^3_{13}\tau^{13} + \Gamma^2_{32}\tau^{33} + \underline{\Gamma^2_{12}\tau^{13}}$$

$$+ \Gamma^1_{11}\tau^{13} + \Gamma^1_{31}\tau^{33} + \Gamma^3_{11}\tau^{11} + \Gamma^3_{22}\tau^{22} = 0\,. \qquad \text{(E-34b)}$$

(Note: The quantities underlined are those that contribute linear terms.)

The third equilibrium is satisfied identically, as expected, since there is no dependence upon angle θ (axisymmetric). Now consider rewriting the stress component in the form of linear terms plus the nonlinear contribution. To do this, consider first the continuity equation from equations (E-27) and (E-27a).

For convenience, the following notation will be used:

$$\frac{\partial u}{\partial r} = u_r, \qquad \frac{\partial u}{\partial z'} = u_{z'}, \text{ etc.} \qquad \text{(E-35)}$$

From equations (E-27) and (E-27a)

$$(1 + u_r)\,(1 + w_{z'}) - w_r\,u_{z'} = \frac{1}{1 + \frac{u}{r}}$$

or

$$(1 + u_r)\,(1 + w_z)\,(1 + \tfrac{u}{r}) - w_r\;{}_{z'}\,(1 + \tfrac{u}{r}) = 1. \qquad \text{(E-36)}$$

Expanding, this yields

$$u_r + w_{z'} + \frac{u}{r} = -\,(u_r\,w_{z'} + \frac{u}{r}w_{z'} + \frac{u}{r}u_r + u_r\,w_{z'}\,\frac{u}{r} + w_r\,u_{z'}(1 + \frac{u}{r})$$

or

$$u_r + w_{z'} + \frac{u}{r} = A \qquad \text{(E-37)}$$

where $A = -\,u_r\,w_{z'} + \frac{u}{r}w_{z'} + \frac{u}{r}u_r + u_r\,w_{z'}\,\frac{u}{r} + w_r\,u_{z'}\,(1 + \frac{u}{r})$.

Let (r, z) be initial coordinates and

(r', z') be final coordinates.

Equilibrium equations are written with r and z as independent variables. A location is then described by its initial coordinates; one then solves for r' and z'. The stresses are expressed in the final coordinates; and on the traction-free boundary, the free inner and outer edge, the normal and shear stresses, σ_{nn} and σ_{ns}, are required to be zero. To express this in terms of $\sigma_{r'r'}$, $\sigma_{z'z'}$, and $\sigma_{z'r'}$, which are in turn functions of the coordinates, one needs the boundary shape. That is, the quantity $\frac{d\rho}{dz}$ is needed where the boundary location is given as r = R. If a finite-difference method is to be employed, the derivative, $\frac{d\rho}{dz}$, is equivalent to $\frac{\rho_{i+1} - \rho'_{i-1}}{z_{j+1} - z'_{j-1}}$.

With this approach, the grid shape is constant and the equilibrium equations are written in the deformed coordinates, which is the more convenient form.

Now, if we let $P = P' - (\Phi + 24)$, (E-38)

then expressions (E-30) yield

$$\tau^{11} = \Phi + \Psi\left(1 + \frac{u}{r}\right)^2 + \left(1 + 2w_{z'} + w_{z'}^2 + u_{z'}^2\right)\left[\Psi + (P' - \Phi - 2\Psi)\left(1+\frac{u}{r}\right)^2\right]$$

$$\tau^{11} = \left(\Phi + \Psi + 2\Psi\frac{u}{r} + \left(\frac{u}{r}\right)^2\Psi\right) + \left[\Psi + P'-\Phi-2\Psi+2\frac{u}{r}P'-2\frac{u}{r}(\Phi+2\Psi)\right.$$

$$\left. + (P'-\Phi-2\Psi)\left(\frac{u}{r}\right)^2\right] + 2w_{z'}\Psi + 2w_{z'}(u'-\Phi-2\Psi) + 2w_{z'}(P'-\Phi-2\Psi)\left[2\frac{u}{r} + \left(\frac{u}{r}\right)^2\right]$$

$$+ \left(w_{z'}^2 + u_{z'}^2\right)\left[\Psi + (P'-\Phi-2\Psi)\left(1 + \frac{u}{r}\right)^2\right].$$

$$\tau^{11} = 2\Psi\frac{u}{r} - 2\frac{u}{r}(\Phi + 2\Psi) + 2w_{z'}(\Phi+2\Psi) + P' + B \qquad \text{(E-39)}$$

where

$$B = \left(\frac{u}{r}\right)^2\Psi + 2\frac{u}{r}P' + (P'-\Phi-2\Psi)\left(\frac{u}{r}\right)^2 + 2w_{z'}(P'-\Phi-2\Psi)\left[2\frac{u}{r} + \left(\frac{u}{r}\right)^2\right]$$

$$+ \left(w_{z'}^2 + u_{z'}^2\right)\left[\Psi + (P'-\Phi-2\Psi)\left(1 + \frac{u}{r}\right)^2\right].$$

Then

$$\tau^{11} = -2(\Phi + \Psi)\left(w_{z'} + \frac{u}{r}\right) + P' + B. \qquad \text{(E-40)}$$

and from continuity, equation (E-37),

$$\Phi - (w_{z'} + \frac{u}{r}) = u_r - A$$

or

$$\tau^{11} = \underbrace{2(\Phi + \Psi)\, u_r + P'}_{\substack{\text{linear terms can} \\ \text{identify Lame's} \\ \text{constant} \\ \mu = \Phi + \Psi}} - \underbrace{2(\Phi + \Psi)A + B}_{\substack{\text{nonlinear} \\ \text{contribution}}}; \qquad \text{(E-41)}$$

also,

$$\tau^{13} = - [\Psi + (P' - \Phi - 2\Psi)(1 + \frac{u}{r})^2 [u_{z'} + w_r + u_r\, u_{z'} + w_r\, w_{z'}]$$

$$= - \{-(\Phi + \Psi)(u_{z'} + w_r) + [P'(1 + \frac{u}{r})^2 - (\Phi + 2\Psi)(2\frac{u}{r} + (\frac{u}{r})^2)]$$

$$[u_{z'} + w_r + u_r\, u_{z'} + w_r\, w_{z'}] + \Psi\, [u_r\, u_{z'} + w_r\, w_{z'}]\}$$

$$= \underbrace{(\Phi + \Psi)\, (u_{z'} + w_r)}_{\substack{\text{linear terms with} \\ \mu = \Phi + \Psi}} + \underbrace{C}_{\substack{\text{nonlinear} \\ \text{contribution}}} \qquad \text{(E-42)}$$

where

$$C = -\{[P'(1 + \frac{u}{r})^2 - (\Phi + 2\Psi)(2\frac{u}{r} + (\frac{u}{r})^2)]\, [u_{z'} + w_r + u_r\, u_{z'} + w_r\, z']$$

$$+ \Psi[u_r\, u_{z'} + w_r\, w_{z'}]\};$$

also,

$$\tau^{22} = \frac{1}{r^2 (1 + \frac{u}{r})^2} \{\Phi\, (1 + \frac{u}{r})^2 + \Psi(1 + \frac{u}{r})^2\, [z + z\, u_r + 2w_{z'} + u_r^2 + w_{z'}^2$$

$$+ w_r^2 + u_{z'}^2] + P' - \Phi - 2\Psi\}$$

$$= \frac{1}{r^2 (1 + \frac{u}{r})^2} \{2\frac{u}{r}\, (\Phi + 2\Psi) + 2\Psi(u_r + w_{z'}) + P' + 2\Psi\, (\frac{u}{r})^2$$

$$+ \Psi[2\frac{u}{r} + (\frac{u}{r})^2][2u_r + 2w_{z'} + u_r^2 + w_{z'}^2 + w_r^2 + u_{z'}^2]\}$$

from continuity $u_r + w_{z'} = A - \frac{u}{r}$, or

$$\tau^{22} = \frac{1}{r^2\left(1 + \frac{u}{r}\right)^2}\Big[\underbrace{2(\Phi + \Psi)\frac{u}{r} + P'}_{\text{linear}} + \underbrace{2\Psi A + D}_{\text{nonlinear}}\Big] \tag{E-43}$$

where

$$D = 2\Psi\left(\frac{u}{r}\right)^2 + \Psi\left[2\frac{u}{r} + \left(\frac{u}{r}\right)^2\right]\left[2u_r + 2w_{z'} + u_r^2 + w_{z'}^2 + w_r^2 + u_{z'}^2\right].$$

It is seen here that the factor $\dfrac{1}{r^2\left(1 + \frac{u}{r}\right)^2}$ causes the linear terms in τ^{22} to be different from the expression for τ^{22} encountered in a linear analysis. This is due to the fact that the physical and tensoral (τ^{22}) components of the stress tensor are different. Their contribution in the equilibrium equations, however, will be the same.

Also, from the similarity in the form of τ^{33} and τ^{11}, the following is immediately obtained:

$$\tau^{33} = 2(\Phi + \Psi)\, w_{z'} + P' - 2(\Phi + \Psi)\, A + E \tag{E-44}$$

where

$$E = \left(\frac{u}{r}\right)^2 + 2\frac{u}{r}P' + (P'-\Phi-2\Psi)\left(\frac{u}{r}\right)^2 + 2u_r(P'-\Phi-2\Psi)\left[2\frac{u}{r} + \left(\frac{u}{r}\right)^2\right]$$
$$+ \left(u_r^2 + w_r^2\right)\left[\Psi + (P'-\Phi-2\Psi)\left(1 + \frac{u}{r}\right)^2\right]$$

and the first of the equilibrium equations, (E-34a and b), yields

$$2(\Phi+\Psi)\frac{\partial^2 u}{\partial r^2} + \frac{\partial P'}{\partial r} - 2(\Phi+\Psi)\frac{\partial A}{\partial r} + \frac{\partial B}{\partial r} + (\Phi+\Psi)\left(\frac{\partial^2 u}{\partial z'^2} + \frac{\partial^2 w}{\partial z'\partial r}\right) + \frac{\partial C}{\partial z'}$$
$$+ \frac{2}{r}(\Phi+\Psi)\frac{\partial u}{\partial r} + \frac{P'}{r} + \left[\frac{\partial u}{\partial r} - \frac{u}{r}\right]\left[\frac{1}{r\left(1 + \frac{u}{r}\right)}\right]\left[2(\Phi+\Psi)\frac{\partial u}{\partial r} + P'\right]$$
$$+ \left[1 + \frac{\partial u}{\partial r}\right]\left[\frac{1}{r\left(1 + \frac{u}{r}\right)}\right]\left[-2(\Phi+\Psi)A + B\right] - 1/r\left[2(\Phi+\Psi)\frac{u}{r} + P'\right]$$

$$- \frac{1}{r^2} \{[1 + \frac{\partial w}{\partial z'})^2 + (\frac{\partial u}{\partial z'})^2][\frac{\partial u}{\partial r} (1 + \frac{u}{r}) + \frac{u}{r}] + [2 \frac{\partial w}{\partial z'} + (\frac{\partial w}{\partial z'})^2$$

$$+ (\frac{\partial u}{\partial f'})^2][(1 + \frac{u}{r})(1 + \frac{\partial u}{\partial r})]\} \{2(\Phi+\Psi) \frac{u}{r} + P' + 2\Psi A + D\}$$

$$- \frac{1}{r} [2\Psi A+D] + \frac{1}{r^2 (1 + \frac{u}{r})^2} g^{13} g_{22,3} [2(\Phi+\Psi) \frac{u}{r} + P' + 2\Psi A + D]$$

+ [other nonlinear terms from expression E-34a].

This equilibrium equation may be written

$$2(\Phi+\Psi) \frac{\partial^2 u}{\partial r^2} + \frac{\partial P'}{\partial r} + (\Phi+\Psi) (\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 w}{\partial z' \partial r}) + \frac{2}{r} (\Phi+\Psi)(\frac{\partial u}{\partial r} - \frac{u}{r})$$

$$-2(\Phi+\Psi) \frac{\partial A}{\partial r} + \frac{\partial B}{\partial r} + \frac{\partial C}{\partial z'} + [\frac{\partial u}{\partial r} - \frac{u}{r}][\frac{1}{r(1 + \frac{u}{r})}][2(\Phi+\Psi) \frac{\partial u}{\partial r} + P']$$

$$-[1 + \frac{\partial u}{\partial r}][\frac{1}{r(1 + \frac{u}{r})}][2(\Phi+\Psi) A - B] - \frac{1}{r^2} \{[(1 + \frac{\partial w}{\partial z'}) + (\frac{\partial u}{\partial z'})^2]$$

$$[\frac{\partial u}{\partial r}(1+ \frac{u}{r}) + \frac{u}{r}] + [2 \frac{\partial w}{\partial z'} + (\frac{\partial w}{\partial z'})^2 + (\frac{\partial u}{\partial z'})^2][(1 + \frac{u}{r})(1 + \frac{\partial u}{\partial r})]\}$$

$$\{2(\Phi+\Psi) \frac{u}{r} + P' + 2\Psi A + D\} - \frac{1}{r} [2\Psi A + D] + \frac{1}{r^2(1 + \frac{u}{r})^2} g^{13} g_{22,3}$$

$$[2(\Phi+\Psi) \frac{u}{r} + P' + 2\Psi A + D] + 2\Gamma^1_{11}\tau^{11} + 3\Gamma^1_{31}\tau^{31} + \Gamma^2_{32}\tau^{31} + \Gamma^3_{33}\tau^{31}$$

$$+ \Gamma^3_{13} \tau^{11} + \Gamma^1_{33} \tau^{33} = 0 ,$$

(E-45)

and the second of the equilibrium equations, (E-37b), yields

$$2(\Phi + \Psi)\frac{\partial^2 w}{\partial z'} + \frac{\partial P'}{\partial z'} - 2(\Phi + \Psi)\frac{\partial A}{\partial z'} + \frac{\partial E}{\partial z'} + (\Phi + \Psi)\left(\frac{\partial^2 u}{\partial z'^2} + \frac{\partial^2 w}{\partial z' \partial r}\right)$$

$$+ \frac{\partial C}{\partial z'} + \frac{1}{r}(\Phi+\Psi)\left(\frac{\partial u}{\partial z'} + \frac{\partial w}{\partial r}\right) + \frac{C}{r} + \left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\left[\frac{1}{r(1 + \frac{u}{r})}\right]$$

$$\left[(\Phi+\Psi)\left(\frac{\partial u}{\partial z'} + \frac{\partial w}{\partial r}\right) + C\right] + 2\Gamma^3_{33}\,\tau^{33} + 3\Gamma^3_{13}\,\tau^{13} + \Gamma^2_{32}\,\tau^{33} + \Gamma^1_{11}\,\tau^{13}$$

$$+ \Gamma^1_{13}\,\tau^{33} + \Gamma^3_{11}\,\tau^{11} + \Gamma^3_{22}\,\tau^{22} = 0. \qquad \text{(E-46)}$$

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The purpose of this study was the establishment of analytical design procedures for laminated elastomeric bearings. This was approached with the application of the linear mathematical theory of elasticity and later with nonlinear large-deformation elasticity theory.

The linear theory yielded analytical approximations that are close to exact solutions and which are easily applied and evaluated. This analysis of one typical lamination yields the distribution of stress and deformation in the elastomer between "rigid" metal lamina. However, the limits of the linear elasticity theory are exceeded for greater than small bearing loads, indicating the need for the application of the more comprehensive large-deformation elasticity theory.

The large-deformation theory was stated and the equilibrium equations were derived, but the solution of these equations was not carried out.

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