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THE METHOD OF LINES FOR NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

by

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John T. Harrison

March 1966

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RDT&E Project No. 1P014501A14B

A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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THE METHOD OF LINES FOR NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

ABSTRACT

In the method of lines for solving certain kinds of boundary value problems in rectangular or trapezoidal regions one of the variables, say y, is discretized while the other variable x is left continuous. When suitable finite difference approximations are substituted for the partial derivatives with respect to y the differential equation is changed into a simultaneous system of ordinary differential equations in the variable x. The method used very little in the USA is used extensively in the Soviet Union and nearly all the literature on this subject is in Russian. The method has been tried in BRL and it seems to be a very useful one. This report does not pretend to be a monograph on the subject. It intends to be a practical guide to computations.

TABLE OF CONTENTS

THE PRINCIPLE OF THE METHOD OF LINES

We shall explain the method of lines for the following differential equation of the second order in two variables which is to be integrated in the rectangular region.

$$R;\ \alpha \le x \le \beta;\ y_o \le y \le y_o + L$$

with boundary C.

Our boundary value problem is

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \quad \text{in } R \tag{1}$$

$$u(x, y_o) = q_o(x); \qquad u(x, y_o + L) = q_1(x) \tag{2}$$

on C

$$u(\alpha, y) = p_o(y); \qquad u(\beta, y) = p_1(y) \tag{3}$$

where a, b, c, d, e, f, g are functions of x and y, and q_i and p_i are prescribed functions of x and y and all of these functions are continuous.

To solve the above boundary value problem by the method of lines we shall use the following procedure:

Subdivide the interval $L = y_{n+1} - y_o$ into n + 1 equal subintervals of width $h = L/(n + 1)$, then draw n lines parallel to the x axis

$$y = y_k = y_o + kh; \qquad k = 1, 2, ..., n.$$

which form the grid shown in Figure 1.

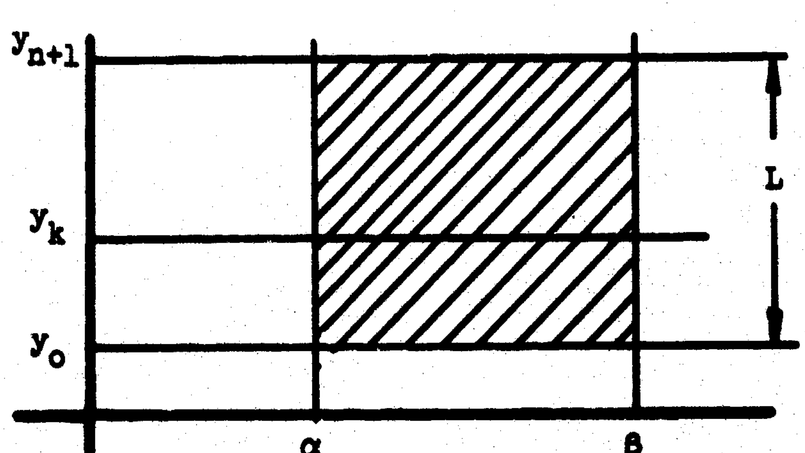


Figure 1

We assume that both the first and the second order partial derivatives are continuous in x and y. Then we substitute in Equation (1)

$$y = y_k; \quad (k = 1, 2, ., ., n)$$

and replace the partial derivatives with respect to y by the central differences

$$u_y(x, y_k) \sim (2h)^{-1}\left[U_{k+1}(x) - U_{k-1}(x)\right]$$

$$u_{yy}(x, y_k) \sim (h)^{-2}\left[U_{k+1}(x) - 2U_k(x) + U_{k-1}(x)\right]$$

$$u_{xy'}(x, y_k) \sim (2h)^{-1}\left[U'_{k+1}(x) - U'_{k-1}(x)\right]$$

where

$u_k(x) = U(x, y_k)$; $U'_k(x) = (d/dx)(U(x,yk))$, and $U_k(x)$ is an approximation of $u(x, yk)$ on the line $y = y_k$.

When we perform these substitutions we obtain a system of n simultaneous differential equations of the second order which approximates the system Equations (1) and (2):

$$a_k U''_k + (2h)^{-1} b_k (U'_{k+1} - U'_{k-1}) + h^{-2} c_k (U_{k+1} - 2U_k + U_{k-1}) + d_k U'_k \qquad (4)$$

$$+ (2h)^{-1} e_k (U_{k+1} - U_{k-1}) + f_k u_k = g_k; \; k = 1, 2, ., ., n.$$

The boundary conditions become

$$U_o(x) = q_o(x); \qquad U_{n+1}(x) = q_1(x) \qquad (5)$$

$$U_k(\alpha) = p_o(y_k); \qquad U_k(\beta) = p_1(y_k) \; . \qquad (6)$$

Now Equations (5) are no longer considered to be boundary conditions. They determine certain terms in the equation for k = 1 and for k = n belonging to system (4). The Equations (6) are the 2n boundary conditions for the n second order differential Equations (4).

The simultaneous system of ordinary differential Equations (4) and (5) together with the 2n boundary conditions Equation (6) approximate the boundary value problem Equations (1), (2), and (3). The general solution of Equations (4) and (5) depends linearly on the 2n arbitrary constants of integration which are determined from the 2n boundary conditions Equation (6).

The convergence of the approximating system Equations (4), (5), (6) to the original system Equations (1), (2) and (3) when h approaches zero under certain restrictions on the coefficients and on the boundary conditons has been proven by various Soviet mathematicians[1*].

THE LAPLACE EQUATION

Consider the boundary value problem Equations (1), (2) and (3) when the left member of Equation (1) is the Laplacian

$$\Delta u = u_{xx} + u_{yy} = g(x) \ . \qquad (1A)$$

In this case the approximating system of Equation (4) takes the form

$$U_k'' + h^{-2}(U_{k+1} - 2U_k + U_{k-1}) = g_k; \qquad k = 1, 2, ., n. \qquad (4A)$$

The Equations (5A) and (6A) would be the same as Equations (5) and (6).

A HIGHER ORDER OF APPROXIMATION FOR THE LAPLACE (OR POISSON) EQUATION

The order of approximation for the system Equations (4) or (4A) is $O(h^2)$. For the Laplace equation we can derive an approximating system of the order $O(h^6)$. To obtain it we expand u_{k+1} and u_{k-1} in Taylor Series about y_k, keeping the fourth order terms, and after eliminating the fourth partial derivative u_{yyyy} we get

$$\begin{aligned}(5/6)U_k'' + (1/12)(U_{k+1}'' + U_{k-1}'') + h^{-2}(U_{k+1} - 2U_k + U_{k-1}) \\ = (5/6)g_k + (1/12)(g_{k+1} + g_{k-1}) \ .\end{aligned} \qquad (4B)$$

[*] *Superscript numbers denote references which may be found on page 30.*

THE CLOSED SOLUTION

For the Laplace equation and when the prescribed values of u on the lines $y = y_o$ and $y = y_{n-1}$ are zero, we can obtain a simple closed solution for each line.

Consider the boundary value problem Equations (1), (2), and (3), when $q_1 = 0$, which is approximated by

$$U_k'' + h^{-2}(U_{k+1} - 2U_k + U_{k-1}) = 0; \qquad k = 1, 2, .,., n \tag{4A}$$

$$U_o(x) = U_{n+1}(x) = 0 , \tag{5A}$$

$$U(\alpha, y_k) = p_o(y_k); \qquad U(\beta, y_k) = p_1(y_k); \qquad k = 1, 2, .,., n. \tag{6A}$$

Applying the separation of variables we assume the following form

$$U_k(x) = q(k)v(x)$$

and substitute it into the above homogeneous equation. This yields the following equation:

$$q(k)v''(x) + h^{-2}v(x)\left[q(k + 1) - 2q(k) + q(k - 1)\right] = 0$$

$$q(0) = q(n + 1) = 0$$

or

$$v''(x)/v(x) = \left[q(k + 1) - 2q(k) + q(k - 1)\right]/ - h^2q(k) = \delta^2 = \text{constant.}$$

To find q we must solve the homogeneous difference equation

$$q(k - 1) - \left[2 - h^2\delta^2\right]q(k) + q(k - 1) = 0$$

with the boundary conditions

$$q(0) = q(n + 1) = 0 .$$

The general solution of this difference equation has the form

$$q(k) = c_1\lambda_1^k + c_2\lambda_2^k$$

where C_1 and C_2 are arbitrary constants and λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 - \left[2 - h^2\delta^2\right]\lambda + 1 = 0 .$$

From the boundary conditions we have

$$q(0) = C_1 + C_2 = 0, \text{ hence } C_2 = - C_1$$

$$q(k + 1) = C_1(\lambda_1{}^{n+1} - \lambda_2{}^{n+1}) = 0, \text{ hence } (\lambda_1/\lambda_2)^{n+1} = 1$$

and

$$(\lambda_1/\lambda_2) = \exp\ (2\pi\ is/(n + 1)) .$$

From the characteristic equation we have that

$$\lambda_1\lambda_2 = 1$$

consequently

$$\lambda_1 = \exp(\pi\ is/(n + 1)) \quad ;$$

$$\lambda_2 = \exp(- \pi\ is/(n + 1)) ,$$

$$s = (y_s - y_o)/h = 1, 2, .. n .$$

From the characteristic equation we have also that

$$\lambda_1 + \lambda_2 = 2 - h^2\delta^2$$

consequently

$$2 - h^2\delta_s^2 = \exp(\pi\ is/(n + 1)) + \exp(- \pi\ is/(n + 1)) = 2 \cos(\pi s/(n + 1))$$

$$h^2\delta_s^2 = 2 - 2 \cos(\pi s/(n + 1)) = 4 \sin^2(\pi(y_s - y_o)/2L$$

$$q_s(k) = C\left[\exp(\pi\ is^k/(n + 1)) - \exp(- \pi\ is^k/(n + 1))\right] = C \sin(\pi s(y_k - y_o)/L).$$

Then taking

$$v''(x) - \delta_s^2 v(x) = 0$$

we obtain

$$v_s(x) = C_s \exp(\delta_s z) + D_s \exp(-\delta_s x) .$$

Thus, we have a set of linearly independent solutions

$$U_{k,s}(x) = \left[C_s \exp(\delta_s x) + D_s \exp(-\delta_s x)\right] \sin(\pi s(y_k - y_o)/L); \quad s = 1, 2, \ldots n$$

and the general solution is

$$U_k(x) = \sum_{s=1}^{n} \left[C_s \exp(\delta_s x) + D_s \exp(-\delta_s x)\right] \sin(\pi s(y_k - y_o)/L)$$

where C_s and D_s are arbitrary constants.

In a similar way it can be shown that the solution of the homogeneous system corresponding to the higher order approximation for the Laplace Equation (4B) is

$$U_k(x) = \sum_{s=1}^{n} \left[C'_s \exp(\delta'_s x) + D'_s \exp(-\delta'_s x)\right] \sin(\pi s(y_k - y_o)/L)$$

where

$$\delta_s'^2 = 24 \sin^2(\pi/2L)(y_s - y_o)/h^2(5 + \cos(\pi/L)(y_s - y_o))$$

and C'_s and D'_s are arbitrary constants.

Having the general solution of homogeneous system we may be able in many concrete cases to find the particular integral corresponding to the given right member, $g(x, y)$.

For example if g is a constant or a function of y only, then

$$\Delta u = g(y) \tag{1}$$

and

$$U''_k + h^{-2}(U_{k+1} - 2U_k + U_{k-1}) = g_k; \qquad (g_k = g(y_k)) . \tag{4A}$$

Assume that the particular integral on the k-th line is

$$U_k = A_k \text{ (a constant).}$$

Substituting it in Equation (4A) we obtain the linear system of equations

$$A_{k+1} - 2A_k + A_{k-1} = g_k h^{-2}$$

from which the values of A_k are easily determined.

When g is a n-th degree polynomial in x, we can assume the solution in the form of another n-th degree polynomial whose coefficients are determined by substituting it in Equation (4A). Let for example

$$g = m_o x^2 + m_1 x + m_2 .$$

Assume the solution to be

$$U_k = A_k x^2 + B_k x + C_k .$$

Substituting it in Equation (4A) we obtain

$$2A_k + (A_{k+1} - 2A_k + A_{k-1})x^2 + (B_{k+1} - 2B_k + B_{k-1})x$$

$$+ C_{k+1} - 2C_k + C_{k-1} = m_o x^2 + m_1 x + m_2 .$$

Comparing coefficients of the same powers of x we have

$$A_{k+1} - 2A_k + A_{k-1} = m_o$$

$$B_{k+1} - 2B_k + B_{k-1} = m_1$$

$$C_{k+1} - 2C_k + C_{k-1} + 2A_k = m_2$$

from which the values of A_k, B_k, C_k can be determined.

NUMERICAL EXAMPLE 1

Solve the following boundary value problem:

$\Delta u = -1$ in the region R which is the rectangle $|y| \le \frac{1}{2}$; $|x| \le \frac{1}{2}$ (1)

$$u(\tfrac{1}{2}, y) = u(-\tfrac{1}{2}, y) = 0 \quad \text{on the boundary C;} \qquad (2)$$

$$u(x, \tfrac{1}{2}) = u(x, -\tfrac{1}{2}) = 0 . \qquad (3)$$

Applying the method of lines we shall use the three lines

$$y_1 = -\tfrac{1}{4}; \quad y_2 = 0; \quad y_3 = \tfrac{1}{4}; \quad (h = \tfrac{1}{4}; \quad n = 3; \ L = 1) .$$

We shall compute $U_k(x)$ on these lines using the approximating system of Equations (4A), which in our case is

$$U_1''(x) + 16[U_2(x) - 2U_1(x)] = -1; \qquad U_0(x) = U_4(x) = 0.$$

$$U_2''(x) + 16[U_3(x) - 2U_2(x) + U_1(x)] = -1 \qquad (1a)$$

$$U_3''(x) + 16[U_2(x) - 2U_3(x)] = -1 ,$$

with the boundary conditions

$$U_i(-\tfrac{1}{2}) = U_i(\tfrac{1}{2}) = 0; \quad i = 1, 2, 3 .$$

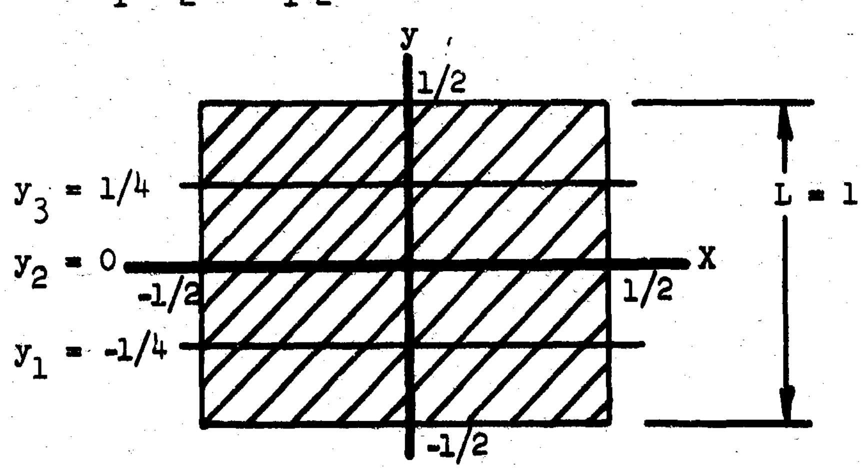


Figure 2

THE PARTICULAR INTEGRALS

We shall assume that the particular integrals of our system are the constants

$$U_i = A_i \ .$$

Substituting the above in the system Equation (1a) we obtain

$$A_1 = A_3 = 3/32; \quad A_2 = 1/8$$

THE GENERAL SOLUTIONS

Since the prescribed values of u on the lines $y = -\frac{1}{2}$ and $y = \frac{1}{2}$ are zero we can use the closed solution. Adding the complementary functions to the particular integrals we can write the general solutions on each line as

$$U_1(x) = \sin\frac{\pi}{4}[C_1\exp(d_1x) + D_1\exp(-d_1x)] + \sin\frac{\pi}{2}[C_2\exp(d_2x) + D_2\exp(-d_2x)]$$

$$+ \sin(3\pi/4)[C_3\exp(d_3x) + D_3\exp(-d_3x)] + 3/32 =$$

$$= (\sqrt{2}/2)(C_1\exp(d_1x) + D_1\exp(-d_1x) + C_3\exp(d_3x) + D_3\exp(-d_3x)]$$

$$+ C_2\exp(d_2x) + D_2\exp(-d_2x) + 3/32 \ ;$$

$$U_2(x) = \sin\frac{1}{2}\pi[C_1\exp(d_1x) + D_1\exp(-d_1x)] - \sin\pi[C_2\exp(d_2x) + D_2\exp(-d_2x)]$$

$$+ \sin(3\pi/2)[C_3\exp(d_3x) + D_3\exp(-d_3x)] + 1/8$$

$$= C_1\exp(d_1x) + D_1\exp(-d_1x) - C_3\exp(d_3x) - D_3\exp(-d_3x) + 1/8$$

$$U_3(x) = \sin(3\pi/4)[C_1\exp(d_1x) + D_1\exp(-d_1x)] + \sin(3\pi/2)[C_2\exp(d_2x)$$

$$+ D_2\exp(-d_2x)] + \sin(9\pi/4)[C_3\exp(d_3x) + D_3\exp(-d_3x)] + 3/32$$

$$= (\sqrt{2}/2)[C_1\exp(d_1x) + D_1\exp(-d_1x)] - [C_2\exp(d_2x) + D_2\exp(-d_2x)]$$

$$+ (\sqrt{2}/2)[C_3\exp(d_3x) + D_3\exp(-d_3x)] + 3/32$$

where

$$d_1^2 = 64\sin^2(\pi/8); \qquad d_2^2 = 64\sin^2(\pi/4); \qquad d_3^2 = 64\sin^2(3\pi/8) .$$

Since the region R and the boundary conditions are symmetrical with respect to the y axis that is $U_k(x) = U_k(-x)$, we have

$$C_i = D_i$$

and using the boundary conditions we obtain the system of algebraic equations which determine C_i

$$(\sqrt{2}\cosh(d_1/2))C_1 + (2\cosh(d_2/2))C_2 + (\sqrt{2}\cosh(d_3/2))C_3 = -3/32$$

$$(2\cosh(d_1/2))C_1 - (2\cosh(d_3/2))C_3 = -1/8$$

$$(\sqrt{2}\cosh(d_1/2))C_1 - (2\cosh(d_2/2))C_2 + (\sqrt{2}\cosh(d_3/2))C_3 = -3/32$$

Hence

$$C_1 = -(3\sqrt{2}+4)\operatorname{sech}(d_1/2)/128; \qquad C_3 = -(3\sqrt{2}-4)\operatorname{sech}(d_3/2)/128; \qquad C_2 = 0.$$

Thus finally

$$U_1(x) = \sqrt{2}C_1\cosh(d_1x) + \sqrt{2}C_3\cosh(d_3x) + 3/32$$

$$U_2(x) = 2C_1\cosh(d_1x) - 2C_3\cosh(d_3x) + 1/8$$

$$U_3(x) = \sqrt{2}C_1\cosh(d_1x) + \sqrt{2}C_3\cosh(d_3x) + 3/32$$

NUMERICAL EXAMPLE 2

Solve the boundary value problem

$$\Delta u = 0 \text{ in R} \qquad (1)$$

$$\left.\begin{aligned} u(x,0) = u(x,8) &= 100\,x\,(12-x) \qquad (2)\\ u(0,y) = u(12,y) &= 100\,y\,(8-y) \qquad (3)\end{aligned}\right\}\text{ on C}$$

$$R; \qquad 0 \le x \le 12; \qquad 0 \le y \le 8$$

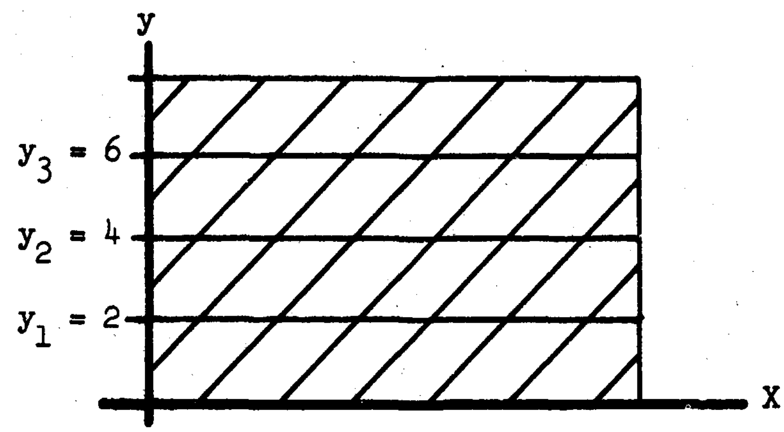


Figure 3

Applying the method of lines we shall use the three lines

$$y_1 = 2,\ y_2 = 4,\ y_3 = 6; \qquad (h = 2,\ n = 3,\ L = 8).$$

We shall compute $U_k(x)$ on these lines using the approximating system Equation (4B) which in our case is

$$(5/6)U_k'' + (1/12)(U_{k+1}'' + U_{k-1}'') + h^{-2}(U_{k+1} - 2U_k + U_{k-1}) = 0 \qquad (4B)$$

$$U_o(x) = U_4(x) = 100\ x\ (12 - x) \qquad (5B)$$

$$U_k(0) = U_k(12) = 100\ y_k(8 - y_k); \qquad k = 1,\ 2,\ 3. \qquad (6B)$$

We shall rewrite the system combining Equations (4B) and (5B)

$$(5/6)U_1'' + (1/12)(U_2'' - 200) + (1/4)(U_2 - 2U_1 + 100\ x\ (12 - x)) = 0$$

$$(5/6)U_2'' + (1/12)(U_3'' + U_2'') + (1/4)(U_3 - 2U_2 + U_1) = 0 \qquad (4B)$$

$$(5/6)U_3'' + (1/12)(-200 + U_2'') + (1/4)(100\ x\ (12 - x) - 2U_3 + U_2) = 0$$

$$y_1 = 2; \qquad U_1(0) = 1200; \qquad U_1(12) = 1200$$

$$y_2 = 4; \qquad U_2(0) = 1600; \qquad U_2(12) = 1600 \qquad (6B)$$

$$y_3 = 6; \qquad U_3(0) = 1200; \qquad U_3(12) = 1200.$$

For digital computations the second order system must be reduced to an initial value first order system. To achieve this let us set

$$U_1' = V_1 \quad ; \qquad \text{(I)}$$

$$U_2' = V_2 \quad ; \qquad \text{(II)}$$

$$U_3' = V_3 \quad ; \qquad \text{(III)}$$

Substituting Equation (I), (II), (III) in Equation (4B) we obtain

$$(5/6)V_1' + (1/12)(V_2' - 200) + (1/4)(U_2 - 2U_1 + 100\,x\,(12 - x)) = 0 \qquad \text{(IV)}$$

$$(5/6)V_2' + (1/12)(V_3' + V_1') + (1/4)(U_3 - 2U_2 + U_1) = 0 \qquad \text{(V)}$$

$$(5/6)V_3' + (1/12)(-200 + V_2') + (1/4)(100\,x\,(12 - x) - 2U_3 + U_2) = 0 \qquad \text{(VI)}$$

or

$$V_1' + (1/10)V_2' + (3/10)U_2 - (6/10)U_1 = 20 - 30\,x\,(12 - x) \qquad \text{(IV)}$$

$$V_2' + (1/10)(V_3' + V_1') + (3/10)(U_3 - 2U_2 + U_1) = 0 \qquad \text{(V)}$$

$$V_3' + (1/10)V_2' - (6/10)U_3 + (3/10)U_2 = 20 - 30\,x\,(12 - x) \qquad \text{(VI)}$$

The Equations (I) through (VI) form the system of six first order equations, two of them nonhomogeneous. The conditions Equation (6B), however, are not all initial and we have to arrange for that. The solution on any line k will be the linear combination of independent solutions of the homogeneous sytem U_k^i plus the particular integral U_k^p

$$U_k(x) = \sum_{i=1}^{6} C^i U_k^i(x) + U_k^p(x) \qquad k = 1, 2, 3, \qquad \text{(VII)}$$

where C^i are constants to be determined from the boundary conditions.

The independent solutions of the homogeneous system Equations (I) through (VI), where the right members in the Equations (IV) and (VI) are replaced by zeros, are obtained from the following set of initial conditions at $x = 0$:

TABLE I

INITIAL CONDITIONS, x = 0

Symbols of the Ind. Solutions		$U_1(0)$	$U_2(0)$	$U_3(0)$	$V_1(0)$	$V_2(0)$	$V_3(0)$	
V_k^1	U_k^1	1	0	0	0	0	0	1st set the initial conditions
V_k^2	U_k^2	0	1	0	0	0	0	2nd set the initial conditions
V_k^3	U_k^3	0	0	1	0	0	0	3rd set the initial conditions
V_k^4	U_k^4	0	0	0	1	0	0	4th set the initial conditions
V_k^5	U_k^5	0	0	0	0	1	0	5th set the initial conditions
V_k^6	U_k^6	0	0	0	0	0	1	6th set the initial conditions
V_k^p	U_k^p	0	0	0	0	0	0	

The particular* integrals U_k^p are obtained from the non-homogeneous system Equations (I) through (VI) with initial conditions all zero (as shown in the above table). The constants C^i are determined from the linear system arising from substituting in Equation (VII) the boundary conditions at x = 0 and at x = 12

$$\sum_{I=1}^{6} C^i U_k^i(0) = U_k(0) - U_k^p(0)$$

$$\sum_{I=1}^{6} C^i U_k(12) = U_k(12) - U_k^p(12); \; k = 1, 2, 3.$$

---

* *We shall explain Table I on the example. The first line indicates that the initial conditions are $U_1(0) = 1$; $U_2(0) = U_3(0) = V_1(0) = V_2(0) = V_3(0) = 0$. The solutions resulting from these initial conditions are V_K^1 and U_K^1.*

CURVILINEAR BOUNDARIES

If the region of integration R is of a shape of a curvilinear trapezoid shown in Figure 2 which is bounded by the lines

$$y = y_o; \quad \text{and} \quad y = y_{n+1}$$

and by the curves

$$x = \alpha(y) \quad \text{and} \quad x = \beta(y); \quad y_o \le y \le y_{n-1} \quad ,$$

then the procedure of the method of lines remains essentially the same as for a rectangular region. The proof of convergence, however, requires that the third partial derivative with respect to y be continuous. The curvilinear boundaries will be explained in the following example.

EXAMPLE 3

Solve the boundary value problem

$$\Delta u = 0 \text{ in R} \tag{1}$$

$$u(x, 0) = u(x, 4) = 0 \tag{2}$$

$$\text{on the curve } \alpha \ u = \varphi_1(x, y) = x + y \tag{3}$$

$$\text{on the curve } \beta \ u = \varphi_2(x, y) = x - y$$

where $\quad$ R is $\alpha(y) \le x \le \beta(y); \quad 0 \le y \le 4$

$$x = \alpha(y) = \frac{1}{8} y^2 + 1; \quad x = \beta(y) = \frac{1}{10} e^y + 3$$

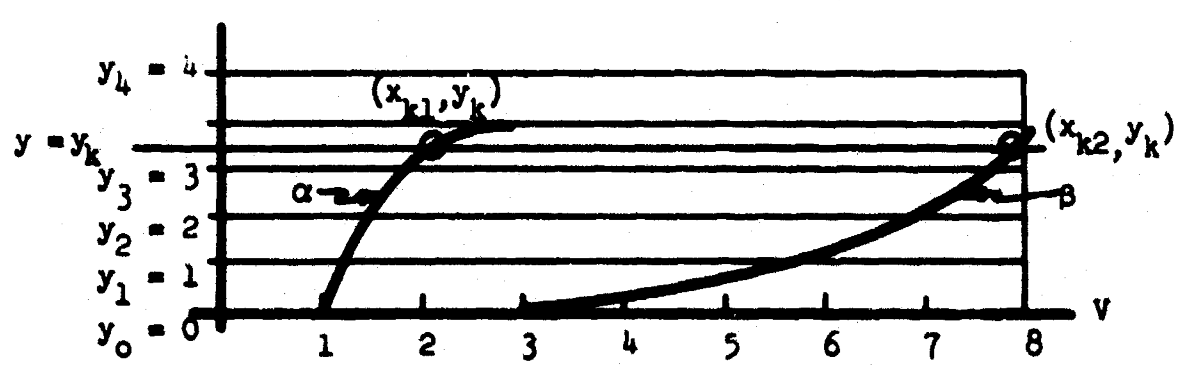


Figure 4

Applying the method of lines we shall use the three lines

$$y_1 = 1; \qquad y_2 = 2; \qquad y_3 = 3 (h = 1,\ n = 3,\ L = 4) .$$

We shall compute $U_k(x)$ on these lines using the approximating system of Equations (4A) which in our case is

$$U_1'' + U_2 - 2U_1 + U_o = 0; \qquad U_o = U_4 = 0;$$

$$U_2'' + U_3 - 2U_2 + U_1 = 0$$

$$U_3'' + U_4 - 2U_3 + U_2 = 0$$

On the curve α; $\quad U_k(\alpha_k) = x_{k1} + y_k = \frac{1}{8} y_k^2 + y_k + 1$

On the curve β; $\quad U_k(\beta_k) = x_{k2} - y_k = \frac{1}{10} e^{y_k} - y_k + 3$

where x_{k1} is the abscissa of the point of intersection of the line $y = y_k$ and the curve $x = \alpha(y)$, and x_{k2} is the abscissa of the point of intersection of the line y_k and the curve $x = \beta(y)$.

Like in the previous examples we obtain six independent solutions from the assumed initial values shown in Table I and form the general solutions

$$U_k(x) = \sum_{L=1}^{6} c^i U_k^i(x)$$

We determine the constants C^i from the linear system arising from substituting the boundary conditions

$$U_k(x_{k1}) = \sum c^i U_k^i(X_{k1})$$

$$U_k(x_{k2}) = \sum c^i U_k^i(X_{k2}); \quad k = 1,\ 2,\ 3.$$

MACHINE COMPUTATION

In order to compare the method of lines with the conventional grid method, Numerical Example 2 has been programmed using the two methods. The programming, with notation included, for the method of lines is given. Figure 5 is a flow chart showing computer operations.

CONCLUSIONS

The method of lines and the conventional grid methods have been compared on two high-speed digital computers at Ballistic Research Laboratories, Computing Laboratory, Aberdeen Proving Ground, Maryland, with respect to run time, computer limitations, and one known solution, U(4, 4) = 2428. Let "H" be the step size and "N" be the number of points. The comparisons follow:

First Method - Conventional Grid Method

A. H = 2 N = 15 15 x 15 Matrix

ORDVAC - Run Time 5 min.
No Limitations

BRLESC - Run Time 1 min.
No Limitations

U(4, 4) = 2419.53919

B. H = 1 N = 77 77 x 77 Matrix

ORDVAC - Memory too small

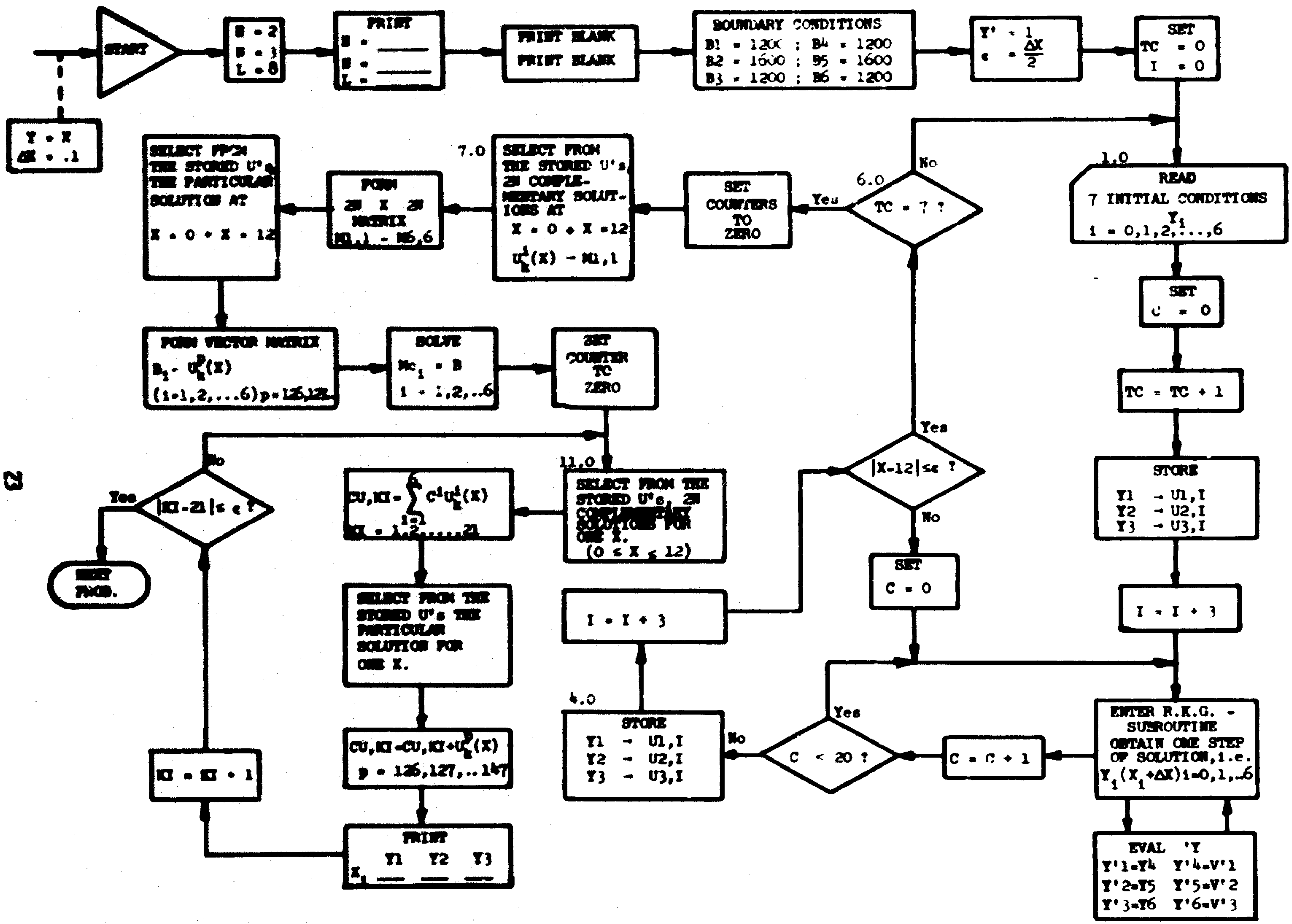
BRLESC - Run Time 1 min
No Limitations

U(4, 4) = 2420.69488

C. H = .5 N = 308 308 x 308 Matrix

Memory too small on both computers

FIGURE 5 FLOW CHART



Second Method - Method of Lines

A. H = 2 N = 3 $\Delta X = .1$ 6 x 6 Matrix

ORDVAC - Run Time 5 min.
Limitations, smaller ΔX's consume too much run time

BRLESC - Run Time 1 min.
No Limitations

U(4, 4) 2420.4529

B. H = 1 N = 7 $\Delta X = .1$ 14 x 14 Matrix

ORDVAC - Run Time 10 min.
Limitations, same as A.

BRLESC - Run Time 1 min.
No Limitations

U(4, 4) 2420.7435

The method of lines needs approximately ten times less storage than the conventional finite difference methods. In some cases it may be faster and more accurate. Another advantage of this method is its applicability to analog computers.

TADEUSZ LESER JOHN T. HARRISON

FORAST PROGRAM

METHOD OF LINES

PROGRAMMER- J. T. HARRISON

```
COMM    GIVEN-

COMM    DEL U = 0     IN R
COMM    U(X,0) = U(X,8) = 100X(12-X)
COMM    U(0,Y) = U(12,Y) = 100Y(8-Y)

COMM    FIND  U(X,Y)  IN REGION R

COMM    NOTATION-

COMM    H = STEP SIZE
COMM    N =  NUMBER OF LINES TO FIND  (0<Y<8).
COMM    L = LENGTH OF Y   (0<=Y<=8).

COMM    B1 = U1(0) = 1200 ,     B4 = U1(12) = 1200
COMM    B2 = U2(0) = 1600 ,     B5 = U2(12) = 1600
COMM    B3 = U3(0) = 1200 ,     B6 = U3(12) = 1200

COMM    Y = X
COMM    YI = UK(X) ,          (I=1,2,3) , (K=1,2,3).
COMM    YI = VK(X) = U'K(X),(I=4,5,6) , (K=1,2,3).
COMM    Y'I = D(UK)/DX ,     (I=1,2,3) ,  (K=1,2,3).
COMM    Y'I = D(VK)/DX ,     (I=4,5,6) ,  (K=1,2,3).

COMM    M1,1 - 2N X 2N MATRIX TO FINE THE C'S.
COMM    CI   - (I=1,2,...6) CONSTANTS TO BE DETERMINED.
COMM    CUI  - (I=1,2,...21) FINAL SOLUTIONS FOR UK(X).
COMM    QI   - (I=0,1,...6) ERROR TERMS FOR SUBROUTINE.
COMM    UI   - (I=1,2,...230) TEMP. STORAGE.
```

```
      BLOC(Y-Y6)(Y'-Y'6)(Q-Q6)(B1-B6)%
      BLOC(U1-U230)(CU1-CU45)(M1,1-M6,7)%
      BLOC(C1-C6)%
      SYN (X=Y)%
DELX  DEC (.1)%
START     INT(H=2)%  INT(N=3)%   INT(L=8)%
          PRINT-FORMAT(F3)-<    H = >(H)<    N = >(N)
      CONT<    L = >(L)%
          ENTER(PRINTB)%  ENTER(PRINTB)%

      COMM    BOUNDARY CONDITIONS

          B1=1200% B2=1600% B3=1200%
          B4=1200% B5=1600% B6=1200%

      COMM    INTERGRATE  (0<=X<=12), BY MEANS OF
      COMM    A SUBROUTINE, RUNGE-KUTTA GILL TO APPROXIMATE
      COMM    ORDINARY DIFF. EQS. FROM 7 INITIAL CONDITIONS

          Y'=1%
          EPS=DELX*.5%
          SET(TC=0)(I=0)%
1.0       READ-FORMAT(F1)-(7)NOS.AT(Y0)% INC(TC=TC+1)%
          SET(C=0)%
          U1,I=Y1% U2,I=Y2% U3,I=Y3%
          INC(I=I+3)%
          ENTER(R.K.G.)(DELX)(7)(EVAL'Y)(Y)(Y')(Q)%
          COUNT(20 )IN(C)GOTO(R.K.G1)%     GOTO(4.0)%

      COMM    EVALUATE THE Y'S.

EVAL'Y    Y'1=Y4%    Y'2=Y5%    Y'3=Y6%
          IF-INT(TC=7)GOTO(2.0)%
          HOM = 0%  GOTO( 3.0)%
2.0       HOM =  20-30*X(12-X)%
3.0       AA=.6*Y1-.3*Y2+HOM%      BB=-.3*(Y3-2*Y2+Y1)%
          CC=.6*Y3-.3*Y2+HOM%
          V'1=(99*AA-10*BB+CC)/98%
          V'2=(100*BB-10*CC-10*AA)/98%
          V'3=(AA-10*BB+99*CC)/98%
          Y'4=V'1%    Y'5=V'2%    Y'6=V'3%
          GOTO(R.K.GD)%
```

```
        COMM    STORE DATA.

4.0          U1,I=Y1% U2,I=Y2% U3,I=Y3%
             INC(I=I+3)%
             IF(X=12)WITHIN(EPS)GOTO(6.0)%
             SET(C=0)%  GOTO(R.K.G1)%
6.0          IF-INT(TC=7)GOTO(FINDC)%     GOTO(1.0)%

        COMM    DETERMINE THE C'S FROM  UK(0)  AND  UK(12).
        COMM    FORM A 2N X 2N MATRIX.

FINDC        SET(II=0)(JJ=0)(T=0)(J=0)(KK=0)%
7.0          M1,1,II=U1,JJ% INC(JJ=JJ+21)(II=II+1)%
             COUNT(6)IN(T)GOTO(7.0)%
             M1,1,II=B1,KK-U1,JJ% INC(KK=KK+1)(II=II+1)%
             SET(T=0)%
             COUNT(3)IN(J)GOTO(8.0)% GOTO(9.0)%
8.0          IF-INT(KK<=2)GOTO(8.1)% GOTO(8.2)%
8.1          SET(JJ=0)% GOTO(8.3)%
8.2          SET(JJ=18)%
8.3          INT(JJ=JJ+J)% GOTO(7.0)%
9.0          IF-INT(KK=6)GOTO(10.0)%  SET(J=0)%  GOTO(8.0)%
10.0         ENTER(S.N.E.)(M1,1)(6)(C1)%

        COMM    FIND  UK(X)  -(K=1,2,3)  IN THE REGION R.

             SET(K=0)(II=0)(JJ=0)(I=0)(P=126)%
             CLEAR(45)NOS.AT(CU1)% SET(KK=1)(CT=0)%
             PRINT<>11><Y=2>9><Y=4>9><Y=6>%
             ENTER(PRINTB)%
11.0         INT(KI=K+I)%  SET(II=0)%
12.0         CU1,KI=C1,II*U1,JJ+CU1,KI%
             INC(JJ=JJ+21)%
             COUNT(6)IN(II)GOTO(12.0)%
             CU1,KI=CU1,KI+U1,P% INC(P=P+1)%
             INT(JJ=I+KK)%
             COUNT(3)IN(I)GOTO(11.0)%
             PRINT-FORMAT(F2)-<X=>(CT)(3)NOS.AT(CU1,K)%
             SET(I=0)%  INC(K=K+3)(CT=CT+2)(KK=KK+3)%
             IF-INT(K=21)GOTO(N.PROB)%  ENTER(PRINTB)% GOTO(11.0)%
             GOTO(11.0)%
F1      FORM(10-10)(1-7)%
F2      FORM(4-3)(3-2)(1-1)(12-4-10)(3-2)(1-3)%
F3      FORM(4-3)(1-3)%
        END GOTO(START)%
```

METHOD OF LINES

INPUT

Y=X	Y1	Y2	Y3	Y4	Y5	Y6
0.	1.	0.	0.	0.	0.	0.
0.	0.	1.	0.	0.	0.	0.
0.	0.	0.	1.	0.	0.	0.
0.	0.	0.	0.	1.	0.	0.
0.	0.	0.	0.	0.	1.	0.
0.	0.	0.	0.	0.	0.	1.
0.	0.	0.	0.	0.	0.	0.

METHOD OF LINES

OUTPUT

H = 2 N = 3 L = 8

	Y=2	Y=4	Y=6
X=	1200.0000	1600.0000	1200.0000
X= 2	1907.5268	1944.4311	1907.5268
X= 4	2581.9081	2420.4529	2581.9081
X= 6	2839.0200	2620.3148	2839.0200
X= 8	2581.9081	2420.4529	2581.9081
X= 10	1907.5268	1944.4311	1907.5268
X= 12	1200.0000	1600.0000	1200.0000

REFERENCES

1. Mikhlin, S. G. and Smolitskii, K. L. Approximate Methods for Solving Differential and Integral Equations. Moscow 1965, pp 329-335.

2. Lebedev, V. I. The Equations and Convergence of a Differential-Difference Method (The Method of Lines), Amer. Math. Sec. Transl. (2)29(1963), pp 255-270.

3. Mikhlin, S. G. Variational Methods in Mathematical Physics. Moscow 1957, pp 402-422.

4. Vlasova, Z. A. A Numerical Realization of the Method of Reduction to Ordinary Differential Equations. Sibirsk. Math. Za. 4(1963), 475-479.

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13. ABSTRACT

In the method of lines for solving certain kinds of boundary value problems in rectangular or trapezoidal regions one of the variables, say y, is discretized while the other variable x is left continuous. When suitable finite difference approximations are substituted for the partial derivatives with respect to y the differential equation is changed into a simulataneous system of ordinary differential equations in the variable x. The method used very little in the USA is used extensively in the Soviet Union and nearly all the literature on this subject is in Russian. The method has been tried in BRL and it seems to be a very useful one. This report does not pretend to be a monograph on the subject. It intends to be a practical guide to computations.

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