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California Institute of Technology
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THE EFFECT OF WALL ELASTICITY AND SURFACE TENSION ON THE FORCED OSCILLATIONS OF A LIQUID IN A CYLINDRICAL CONTAINER

By P. Tong and Y. C. Fung
California Institute of Technology

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ABSTRACT

The stability of a fluid contained in a circular cylindrical tank with a flat, flexible bottom under a periodic axial excitation is studied. A variational approach is formulated. An approximate solution results in a pair of coupled ordinary differential equations with periodic coefficients. A method of handling the stability of the solutions of such a system of equations is presented. Numerical results are discussed.

NOMENCLATURE

English Symbols

A_n, B_n	Constants
B_M	Membrane number
B_o	Bond number

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c_n , d_n	Amplitude of the n^{th} sloshing mode
F_1	Potential of the solid-liquid-gas interface
F_3	Potential of the edge load acting on the rim of the tank bottom
$g(t)$	Gravitational acceleration, time dependent
g_0	Mean local gravitational acceleration
g_1	Amplitude of the imposed axial acceleration
$G(\tau)$	Nondimensional gravitational acceleration
h	Membrane thickness
H	Nondimensional free surface shape
I, I_1, I_2, I_3, I_4	Functionals
J_0 , J_1	Bessel functions of first kind
k_n	n^{th} root of the equation $J_1(k_n) = 0$
ℓ	Depth of liquid
L	Nondimensional depth of liquid
L_1	Pressure energy in nondimensional form
L_2, L_3	Lagrangians in nondimensional form
$\underline{M}_1$, $\underline{M}_2$, $\underline{M}_3$	Matrix
N_r	Midplane stress resultant
${}_n\omega$	n^{th} sloshing frequency for rigid tank
p	Pressure
P	Nondimensional pressure
r_0	Radius of the tank
(r , θ , z)	Cylindrical coordinates
(R , Θ , Z)	Nondimensional cylindrical coordinates

S_1 , S_2 , S_3	Surfaces
t	Time
$\underline{U}$	Modal matrix
w	Transverse deflection of membrane
W	Nondimensional transverse deflection of membrane
z	Vertical coordinate
Z	Nondimensional vertical coordinate

Greek Symbols

α	Nondimensional axial acceleration
γ_n	Constants
Γ_1 , Γ_3	Boundary curves of S_1 , S_3 respectively
δ_1 , δ_2 , δ_3	Constants
η	Free surface shape
θ	Azimuthal coordinate
λ	Mass ratio
Λ_n	Constants
μ_n , ν_n	Constants
ρ	Density of membrane
ρ_0	Density of liquid
σ	Surface tension
τ	Nondimensional time
ϕ	Velocity potential
Φ	Nondimensional velocity potential
ω	Forcing frequency
Ω	Nondimensional frequency

Ω_1^2, Ω_2^2 (n = m , . . . , N) — Eigenvalues of Ω^2 where M , . . . , N indicate the fluid modes which are chosen for the approximate solution

Ω_M^2 — Frequency parameter for the membrane

Ω_σ^2 — Frequency parameter for surface tension

∇^2 , $\bar{\nabla}^2$ — Two-dimensional Laplace operator $\left(\frac{1}{r_0^2}\nabla^2 = \bar{\nabla}^2 = \frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right.$ in cylindrical coordinates$\Big)$

∇ , $\bar{\nabla}$ — Three-dimensional gradient operator $\left(\frac{1}{r_0}\nabla = \bar{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \vec{e}_z\frac{\partial}{\partial z}\right.$ in cylindrical coordinates; for the free surface and the membrane, ignore the $\vec{e}_z\frac{\partial}{\partial z}$ term$\Big)$

THE EFFECT OF WALL ELASTICITY AND SURFACE TENSION ON THE FORCED OSCILLATIONS OF A LIQUID IN A CYLINDRICAL CONTAINER

1 INTRODUCTION

Dynamics of large liquid-fuel rockets naturally involve the motion of a liquid in a flexible container. The symmetric modes of the fluid motion, which influences the pressure at the tank bottom, and therefore influences the pressure in the pump and in the combustion chamber, as well as thrust and rocket acceleration, have an important effect on the structural dynamics of a rocket. In some instances the longitudinal oscillations were so serious as to affect the safety of the vehicle. For this reason, the analysis of the forced oscillations of the liquid container is important.

At ground level, perhaps the effects on fuel sloshing of the flexibility of the tank wall and the surface tension of the free surface are negligible. At reduced gravity conditions, these effects become more evident. It is the purpose of this article to evaluate the effects of tank flexibility and surface tension on the stability of liquid motion in the symmetric modes.

Sloshing of liquids has been studied by many authors. Although most of them considered rigid containers (Ref. 1), Miles (Ref. 2) considered bending modes of a flexible container, and Bleich (Ref. 3) investigated the longitudinal modes. Recently, Bhuta and Koval (Refs. 4 and 5) studied the coupled oscillations of a liquid in a tank with a flexible bottom. They defined the normal modes of the system, and treated the orthogonality and expansion theorems. Bhuta and Yeh (Refs. 6 and 7) considered the problem of arbitrarily assigned velocity distribution on the tank bottom.

On the other hand, there is substantial literature about the influence of surface tension on sloshing, e.g., Yeh's bibliography (Ref. 8) and papers by Bond and Newton (Ref. 9) and Reynolds (Ref. 10). Most of these studies, however, are concerned with free oscillations. Very little has been done about the influence of surface tension on forced oscillations, and no work seems to have been done on coupling with the flexibility of the tank.

In the present paper, a circular tank with a flexible bottom under vertical periodic excitation is studied. The problem is first formulated in the form of differential equations and then in the form of a variational principle. An approximate solution is presented, which results in a pair of coupled ordinary differential equations with periodic coefficients. The stability of the solutions of these equations is discussed.

2 STATEMENT OF THE PROBLEM

A circular cylindrical container with rigid side walls and a flat, flexible, bottom contains a liquid with a free surface. The tank walls are subjected to an oscillatory axial acceleration, in addition to a constant mean-local-gravitational acceleration directed along the axis of the cylinder. Above the liquid surface is a gas with constant pressure. No external force acts underneath the tank bottom. The situation is pictured in Fig. 1. The problem is to determine the motion of the liquid and, in particular, its stability.

The fluid properties, including the surface tension, are assumed to be uniform, constant, incompressible, and inviscid.

The mean free surface of the liquid is assumed to be a plane perpendicular to the cylinder axis. In low-gravity and finite surface tension, one may have to consider a curved mean free surface. The governing criterion is the Bond number defined below. In this paper, we assume that the Bond number is sufficiently large so that the free surface is approximately a plane. The case of low Bond number is discussed later.

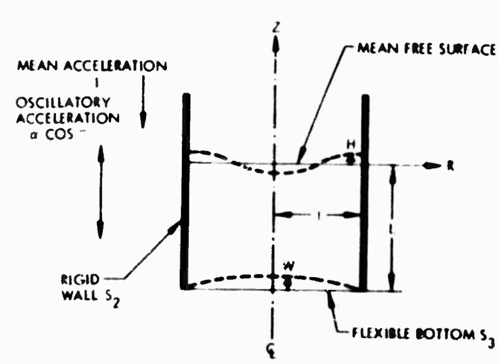

Fig. 1 Geometry of the Problem in Nondimensional Variables

As a further simplification, we assume that the deviation from the static equilibrium condition is small, so that the deflections of the free surface and of the tank bottom, the fluid velocity, and hence the velocity potential, may be considered infinitesimal quantities of the first order. Under this assumption, all the equations can be linearized, and the mathematical problem is relatively simple. A number of interesting nonlinear problems are ruled out by this assumption. But, as an investigation of the initial tendency toward instability, the linearized theory should be adequate.

3 MATHEMATICAL FORMULATION

Consider a quantity of inviscid liquid situated in a cylindrical container of radius r_0 as is shown in Fig. 1. The cylindrical polar coordinate system is chosen so that the $+z$ direction is directed upward away from the liquid, the zero on this axis being fixed on the mean free surface. If the fluid is assumed inviscid and incompressible, and the motion irrotational, the equation of continuity may be expressed in terms of the velocity potential ϕ,

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\, r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0 \tag{1}$$

and the velocity components $\bar{u}$, $\bar{v}$, $\bar{w}$ are

$$\bar{u} = \phi_r \quad , \quad \bar{v} = \frac{1}{r}\phi_\theta \quad , \quad \bar{w} = \phi_z \tag{2}$$

The usual subscript notation is used to denote partial differentiation.

The kinematic conditions at the tank walls and the free surface are

$$\bar{u} = \frac{\partial \phi}{\partial r} = 0 \quad \text{on} \quad r = r_0 \tag{3}$$

$$\bar{w} = \frac{\partial \phi}{\partial z} = \frac{\partial w}{\partial t} \quad \text{on} \quad z = -\ell \tag{4}$$

$$\bar{w} = \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{on} \quad z = 0 \tag{5}$$

where w denotes the deflection of the tank bottom, and η denotes the deflection of the free surface, both positive in the $+z$ direction, and both assumed to be infinitesimal.

Since the motion is irrotational, Bernoulli's equation is satisfied throughout the liquid domain. In particular, at the free surface, we have

$$\frac{p}{\rho_0} = -\frac{1}{2}(\overline{\nabla}\phi)^2 - g(t)\eta - \phi_t + c(t) \tag{6a}$$

$$g(t) = g_0 + g_1 \cos \omega t \tag{6b}$$

where

$c(t)$	=	arbitrary function of time	
g_0	=	mean local gravitational acceleration	Taken as positive if they are directed toward the tank bottom (along the $-z$ direction)
$g_1 \cos \omega t$	=	imposed axial acceleration*	
p	=	pressure just inside the interface	

The pressure p is related to the pressure just outside the liquid, p_G, by the relation

$$p_G - p = \sigma K \tag{7}$$

where

σ = surface tension

K = total curvature of the free surfaces

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*Here we just write out a special form of imposed axial acceleration. The method developed later can be applied to a general periodic imposed axial acceleration.

In linearized form, under the assumptions that $\eta/r_0 << 1$ and $|\text{grad}\ \eta| < 1$, we have

$$K = \bar{\nabla}^2 \eta \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \eta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2} \tag{8}$$

If the pressure of the gas p_G is a constant, then without loss of generality we may set $p_G = 0$. The function $c(t)$ can be absorbed in ϕ_t. We can also neglect $|\bar{\nabla}\phi|^2$ in Eq. (6) and evaluate ϕ_t on the surface $z = 0$ under the scheme of linearization. Thus we obtain the linearized free surface condition,

$$\frac{\sigma}{\rho_0}\bar{\nabla}^2 \eta = \left(\frac{\partial \phi}{\partial t}\right)_{z=0} + g(t)\,\eta \tag{9}$$

Similarly, Bernoulli's equation gives the pressure on top of the bottom wall

$$(p)_{z=-\ell+w} = \rho_0\left[-\left(\phi_t\right)_{z=-\ell} - g(t)\,(-\ell + w)\right] \tag{10}$$

No other forces are assumed to be acting on the tank bottom. If the tank bottom is very thin and is prestressed so that it behaves like a membrane, then the equation of motion of the bottom is

$$N_r \bar{\nabla}^2 w = \rho h\left[\frac{\partial^2 w}{\partial t^2} + g(t)\right] + (p)_{z=-\ell+w} \tag{11}$$

where

N_r = tensile stress resultant in the tank bottom (assumed to be a constant)
ρ = density of the tank bottom material
h = tank bottom wall thickness
ρh = mass per unit area of the tank bottom

A combination of Eqs. (10) and (11) gives the linearized equation of motion of the elastic bottom as a membrane:

$$N_r \bar{\nabla}^2 w = \rho h \frac{\partial^2 w}{\partial t^2} - \rho_0 g(t) w - \rho_0 \left(\frac{\partial \phi}{\partial t}\right)_{z=-\ell} + (\rho h + \rho_0 \ell) g(t) \tag{12}$$

In reality, a tank with flat bottom develops both bending and stretching stresses under fluid pressure. Equation (12) is a good approximation only if a membrane tension is built in at the edges by stretching the bottom onto a rigid cylinder before the two are welded together.

It is necessary to specify the boundary conditions for η and w at the edge $r = r_0$. We choose

$$w = 0 \quad \text{when} \quad r = r_0 \tag{13}$$

$$\frac{\partial \eta}{\partial r} = 0 \quad \text{when} \quad r = r_0 \tag{14}$$

The last condition is a special case of zero capillary-hysteresis. It is consistent with the simplifying assumption that the undisturbed free surface is a plane $z = 0$. In a very-low-gravity condition, the mean free surface is curved, and Eq. (14) should be replaced by the condition $\partial\eta/\partial r = \gamma\eta$ at the wall where γ is a physical constant.

These equations define the linear, inviscid problem of sloshing under appropriate initial or periodicity conditions.

4 DIMENSIONLESS EQUATIONS

Taking the radius of the cylinder r_0 as the characteristic length, the gravitational acceleration g_0 as the characteristic acceleration, and ω as the characteristic frequency, we define the dimensionless variables as follows:

$$\left.\begin{aligned}
R &= \frac{r}{r_0} & Z &= \frac{z}{r_0} \\
L &= \frac{\ell}{r_0} & \tau &= \omega t \\
\Phi &= \frac{\phi}{\omega\, r_0^2} & H &= \frac{\eta}{r_0} \\
W &= \frac{w}{r_0} & \alpha &= \frac{g_1}{g_0} \\
G(\tau) &= \frac{g_0 + g_1 \cos \omega t}{g_0} & P &= \frac{p}{\rho_0\, \omega^2\, r_0^2}
\end{aligned}\right\} \tag{15}$$

We define the dimensionless parameters as follows:

$$\left.\begin{aligned}
&\text{Bond number} = B_\sigma = \rho_0\, g_0\, r_0^2/\sigma \\
&\text{Membrane number} = B_M = \rho_0\, g_0\, r_0^2/N_r \\
&\text{Frequency parameter for surface tension} = \Omega_\sigma^2 = \rho_0\, r_0^3\, \omega^2/\sigma \\
&\text{Frequency parameter for the membrane} = \Omega_M^2 = \rho_0\, r_0^3\, \omega^2/N_r \\
&\text{Mass ratio} = \lambda = \frac{\rho h}{\rho_0\, r_0}
\end{aligned}\right\} \tag{16}$$

and the operator

$$\nabla^2 = 1/R\, \partial/\partial R\, (R\, \partial/\partial R) + 1/R^2\, \partial^2/\partial\theta^2 = r_0^2\, \bar{\nabla}^2 \tag{17}$$

Then the equations become

$$\left(\frac{\partial^2}{\partial Z^2} + \nabla^2\right)\Phi = 0 \tag{18}$$

$$\nabla^2 H - \Omega_\sigma^2 \left(\frac{\partial \Phi}{\partial \tau}\right)_{Z=0} - B_\sigma G H = 0 \tag{19}$$

$$\nabla^2 W - \lambda \Omega_M^2 \frac{\partial^2 W}{\partial \tau^2} + \Omega_M^2 \left(\frac{\partial \Phi}{\partial \tau}\right)_{Z=-L} + B_M G(\tau) W - (\lambda + L) B_M G(\tau) = 0 \tag{20}$$

with the boundary conditions

$$\frac{\partial \Phi}{\partial R} = 0 \quad \text{on} \quad R = 1 \tag{21}$$

$$\frac{\partial \Phi}{\partial Z} = \frac{\partial W}{\partial \tau} \quad \text{on} \quad Z = -L \tag{22}$$

$$\frac{\partial \Phi}{\partial Z} = \frac{\partial H}{\partial \tau} \quad \text{on} \quad Z = 0 \tag{23}$$

$$W = 0 \quad \text{on} \quad R = 1 \tag{24}$$

and

$$\frac{\partial H}{\partial R} = 0 \quad \text{on} \quad R = 1 \ \text{(assuming } \gamma = 0) \tag{25}$$

Equations (18) through (25) show that the problem of sloshing depends on the parameters Ω_σ^2, B_σ, α, Ω_M^2, B_M, λ, and L.

These dimensionless parameters are not all independent; since

$$\Omega_M^2 = \frac{\sigma}{N_r} \Omega_\sigma^2 \qquad \text{and} \qquad B_M = \frac{\sigma}{N_r} B_\sigma \tag{26}$$

therefore

$$\frac{B_M}{B_\sigma} = \frac{\Omega_M^2}{\Omega_\sigma^2} \tag{27}$$

However, we retain the sets of symbols Ω_σ^2, B_σ and Ω_M^2, B_M because these two pairs of parameters are not both likely to be important. The conditions are:

- Ω_M^2, $B_M \to 0$ if the tank bottom is rigid
- Ω_σ^2, $B_\sigma \to \infty$ if the surface tension has no effect

5 DISCUSSION OF ANALYTICAL SOLUTIONS

Consider symmetric modes of motion in which Φ, H, W are independent of the angular coordinates θ. A solution of Eq. (18) may be posed as

$$\Phi = \dot{d}_0(\tau)Z + \dot{c}_0(\tau) + \sum_{n=1}^{\infty} J_0(k_n R)\left[\dot{c}_n(\tau)\frac{\cosh k_n Z}{\sinh k_n L} + \dot{d}_n(\tau)\frac{\sinh k_n Z}{\cosh k_n L}\right] \tag{28}$$

Then Eqs. (22) and (23) give

$$H = d_0(\tau) + \sum_{n=1}^{\infty} d_n(\tau)\frac{k_n J_0(k_n R)}{\cosh k_n L} \tag{29}$$

and

$$W = d_0(\tau) + f(R) + \sum_{n=1}^{\infty} k_n\left[d_n(\tau) - c_n(\tau)\right] J_0(k_n R) \tag{30}$$

Both Eqs. (21) and (25) are satisfied if the k_n's are the roots of the equation

$$J_1(k_n) = 0 \qquad n = 1, 2, 3, \ldots\ldots \tag{31}$$

Eq. (24) is satisfied by taking

$$d_0(\tau) + \sum_{n=1}^{\infty} k_n \left[d_n(\tau) - c_n(\tau)\right] J_0(k_n) = 0 \tag{32}$$

and

$$f(R) = (\lambda + L)\left[1 - \frac{J_0(\sqrt{B_M}\,R)}{J_0(\sqrt{B_M})}\right]$$

Here, $f(R)$ is the static deflection of the membrane. We assume that, if B_M is positive, $\sqrt{B_M}$ is less than the first root of $J_0(x) = 0$, namely, 2.4048. To satisfy Eqs. (19) and (20), we substitute Φ, H, W from Eqs. (28) through (30), collect terms, and represent the lefthand side as a Fourier-Bessel series in $J_0(k_n R)$. Since the series vanishes, every coefficient of $J_0(k_n R)$, $n = 1, 2, \ldots\ldots$, must vanish. Thus, from Eq. (19) we obtain the necessary conditions

$$\Omega_\sigma^2 \ddot{c}_0(\tau) + B_\sigma G(\tau) d_0(\tau) = 0 \tag{33a}$$

and

$$\Omega_\sigma^2 \ddot{c}_n(\tau) + k_n \tanh k_n L\left[k_n^2 + B_\sigma G(\tau)\right] d_n(\tau) = 0 \tag{33b}$$

Multiplying Eq. (20) by R and integrating with respect to R from 0 to R, we get

$$R \frac{\partial W}{\partial R} + B_M G \int_0^R RW(R,\tau)\, dR - \lambda \Omega_M^2 \frac{\partial^2}{\partial \tau^2} \int_0^R W(R,\tau)\, R\, dR$$

$$- \frac{(\lambda + L)}{2} B_M GR^2 + \Omega_M^2 \int_0^R \frac{\partial \Phi}{\partial \tau}\bigg|_{Z=-L} R\, dR = 0 \quad (34)$$

Any function $W(R,\tau)$ of class C^2 in the closed interval 0 to 1 for R satisfying the above equation will satisfy Eq. (20). A substitution of Eqs. (28), (29), and (30) into it gives

$$\sum_{n=1}^{\infty} k_n^2 (c_n - d_n) J_1(k_n R) + B_M G \left[\frac{d_0 R}{2} + \sum_{n=1}^{\infty} (d_n - c_n) J_1(k_n R) \right]$$

$$-\lambda \Omega_M^2 \left[\frac{\ddot{d}_0 R}{2} + \sum_{n=1}^{\infty} (\ddot{d}_n - \ddot{c}_n) J_1(k_n R) \right] + \Omega_M^2 \left[\frac{-L\ddot{d}_0 + \ddot{c}_0}{2} R + \sum_{n=1}^{\infty} \frac{J_1(k_n R)}{k_n} \right.$$

$$\left. (\ddot{c}_n \coth k_n L - \ddot{d}_n \tanh k_n L) \right] - \alpha \frac{(\lambda + L)\sqrt{B_M}}{J_0(\sqrt{B_M})} J_1(\sqrt{B_M} R) = 0$$

(35)

By expanding R and $J_1(\sqrt{B_M} R)$ in terms of $J_1(k_n R)$, we can collect the coefficients of $J_1(k_n R)$ and set them equal to zero, to obtain

$$\frac{\Omega_M^2}{k_n} \left[\ddot{c}_n (\coth k_n L + \lambda k_n) - \ddot{d}_n (\tanh k_n L + \lambda k_n) \right] + \left(k_n^2 - B_M G \right) (c_n - d_n)$$

$$= - \frac{\Omega_M^2}{k_n J_0(k_n)} \left[(\lambda + L) \ddot{d}_0 + \ddot{c}_0 + \frac{B_M G}{\Omega_M^2} d_0 \right] + \frac{2\alpha (\lambda + L) \sqrt{B_M}\, J_1(\sqrt{B_M})}{k_n J_0(k_n) J_0(\sqrt{B_M})} \frac{k_n}{B_M - k_n^2}$$

(36)

Now, if we want to truncate the infinite series in Eqs. (28) through (30) by taking $n = m, \ldots\ldots, N$, we see that Eqs. (32), (33), and (36) always involve $2(N - m + 2)$ unknowns and $2(N - m + 2)$ equations, which in general have solutions.

Of course, we can do the same thing for Eq. (19) to obtain

$$\Omega_\sigma^2 \ddot{c}_n + \left(k_n^2 + B_\sigma G\right) k_n (\tanh k_n L) d_n = \frac{\sinh k_n L}{J_0(k_n)} \left(\Omega_\sigma^2 \ddot{c}_0 + B_\sigma G d_0\right) \tag{37}$$

However, since we have assumed $\partial H/\partial R = 0$ at $R = 1$, the series of Eq. (29), after twice term-by-term differentiation, is still convergent. Thus Eq. (37) coincides with Eq. (33).

We shall now discuss the solution for Eqs. (32), (33), and (36). For the case of free vibration, i.e. $G = 1$ or $\alpha = 0$, solutions for c_n, d_n can be obtained rather easily. If we let

$$c_n = C_n e^{i\Omega\tau}$$

and (38)

$$d_n = D_n e^{i\Omega\tau}$$

then, from Eq. (33), we obtain

$$C_0 = \frac{B_\sigma}{\Omega_\sigma^2 \Omega^2} D_0$$

and (39)

$$C_n = \frac{k_n \left(k_n^2 + B_\sigma\right) \tanh k_n L}{\Omega_\sigma^2 \Omega^2} D_n = \frac{p_n^2}{\Omega^2} D_n$$

and from Eqs. (36) and (39),

$$\Omega_M^2 \, \Omega^2 \frac{(L + \lambda)}{J_0(k_n)} D_0 = \beta_n D_n \tag{40}$$

where

$$\beta_n = -\left[\Omega_M^2 p_n^2 (\coth k_n L + \lambda k_n) - \Omega_M^2 \Omega^2 (\tanh k_n L + \lambda k_n) + k_n \left(k_n^2 - B_M\right)\left(\frac{p_n^2}{\Omega^2} - 1\right)\right] \tag{41}$$

Substituting into Eq. (30), we have

$$W = e^{i\Omega\tau} D_0 \left[1 + \sum_{n=1}^{\infty} k_n \frac{\Omega_M^2 \Omega^2 (L + \lambda)}{\beta_n} \left(1 - \frac{p_n^2}{\Omega^2}\right) \frac{J_0(k_n R)}{J_0(k_n)}\right] + f(R) \tag{42}$$

Now we can easily see that the coefficient of $J_o(k_n R)$ in the above series is of order $1/(k_n^{3/2})$ for large n ; so the series is actually divergent after twice term-by-term differentiation with respect to R. At $R = 1$, $W = 0$, we have $D_o \not\equiv 0$ for $\Omega_M^2 \not\equiv 0$ ($\Omega_M^2 \equiv 0$ corresponds to rigid tank); therefore,

$$\zeta(\Omega^2) = 1 + \sum_{n=1}^{\infty} \frac{\Omega_M^2 (L + \lambda) k_n}{\beta_n} \left(\Omega^2 - p_n^2\right) = 0 \tag{43}$$

This equation will determine the eigenvalue Ω . It can be shown that, for $p_n^2 \geq 0$, $\chi_n \geq 0$, all the roots Ω are real, and no double roots exist. In the case $\Omega_M^2 \to 0$, some Ω's can be obtained asymptotically in a rather simple fashion. Let

$$\Omega^2 = \Omega_\ell^2 = p_\ell^2 \left(1 - \delta_1 \Omega_M^2 - \delta_2 \Omega_M^4 - \delta_3 \Omega_M^6 \cdots \right) \tag{44}$$

Then we have

$$\begin{aligned}\beta_\ell = & -\Omega_M^2 \Big[p_\ell^2 (\coth k_\ell L + \lambda k_\ell) - p_\ell^2 \left(1 - \delta_1 \Omega_M^2 - \delta_2 \Omega_M^4 \cdots \right)(\tanh k_\ell L + \lambda k_\ell) \\ & \qquad + \chi_\ell \, \delta_1 \Omega_M^2 + \left(\delta_2 + \delta_1^2\right) \Omega_M^4 + \cdots \Big] \\ = & \left[\delta_1 \chi_\ell - p_\ell^2 (\coth k_\ell L - \tanh k_\ell L)\right] \Omega_M^2 \\ & + \left[\left(\delta_1^2 + \delta_2\right) \chi_\ell - p_\ell^2 \delta_1 (\tanh k_\ell L + \lambda k_\ell)\right] \Omega_M^4 \\ & + \left[\left(\delta_1^2 + 2\delta_1 \delta_2 + \delta_3\right) \chi_\ell - p_\ell^2 \delta_2 (\tanh k_\ell L + \lambda k_\ell)\right] \Omega_M^6 + 0\left(\Omega_M^6\right)\end{aligned} \tag{45}$$

where

$$\chi_\ell = k_\ell \left(k_\ell^2 - B_M\right) \tag{46}$$

Equation (43) becomes

$$1 - \frac{p_\ell^2 (L + \lambda)\left(\delta_1 + \delta_2 \Omega_M^2 + \cdots \right) \Omega_M^2}{\delta_1 \chi_\ell - p_\ell^2 (\coth k_\ell L - \tanh k_\ell L) + \left[\left(\delta_1^2 + \delta_2\right)\chi_\ell - p_\ell^2 \delta_1 (\tanh k_\ell L + \lambda k_\ell)\right] \Omega_M^2 + \cdots}$$

$$- \Omega_M^2 p_\ell^2 (L + \lambda) \left[\sum_{n=1}^{\infty} \frac{1}{k_n^2 - B_M} - \frac{1}{k_\ell^2 - B_M}\right] + 0\left(\Omega_M^4\right) = 0 \tag{47}$$

In order that the above equation be valid as $\Omega_M^2 \to 0$, we must have

$$\delta_1 = \frac{\coth k_\ell L - \tanh k_\ell L}{\chi_\ell} p_\ell^2 \tag{48}$$

and, since

$$\sum_{n=1}^{\infty} \frac{1}{k_n^2 - B_M} = \frac{1}{B_M} - \frac{J_0\left(\sqrt{B_M}\right)}{2\sqrt{B_M}\, J_1\left(\sqrt{B_M}\right)} \tag{49}$$

Eq. (43) becomes

$$1 - \frac{p_\ell^2 (L + \lambda)\left(\delta_1 + \delta_2 \Omega_M^2 + \ldots\right)}{\left(\delta_1^2 + \delta_2\right)\chi_\ell - p_\ell^2 \delta_1 (\tanh k_\ell L + \lambda k_\ell) + \left[\chi_\ell\left(\delta_1^2 + 2\delta_1\delta_2 + \delta_3\right) - p_\ell^2 \delta_2 (\tanh k_\ell L + \lambda k_\ell)\right]\Omega_M^2 + \ldots}$$
$$- \Omega_M^2 p_\ell^2 (L + \lambda)\left[\frac{1}{B_M} - \frac{J_0\left(\sqrt{B_M}\right)}{2\sqrt{B_M}\, J_1\left(\sqrt{B_M}\right)} - \frac{1}{k_\ell^2 - B_M}\right] + 0\left(\Omega_M^4\right) = 0 \tag{50}$$

Expanding the lefthand side in power series of Ω_M^2, and putting the coefficient of Ω_M^{2n} equal to zero, we get

$$1 - \frac{p_\ell^2 (L + \lambda)\,\delta_1}{\left(\delta_1^2 + \delta_2\right)\chi_\ell - p_\ell^2 \delta_1 (\tanh k_\ell L + \lambda k_\ell)} = 0 \tag{51}$$

and

$$p_\ell^2 (L + \lambda)\,\delta_2 - (\delta_1^2 + 2\delta_1\delta_2 + \delta_3)\,\chi_\ell + p_\ell^2 \delta_2 (\tanh k_\ell L + \lambda k_\ell)$$
$$+ \left[p_\ell^2 (L + \lambda)\right]^2 \left[\frac{1}{B_M} - \frac{J_0\left(\sqrt{B_M}\right)}{2\sqrt{B_M}\, J_1\left(\sqrt{B_M}\right)} - \frac{1}{k_\ell^2 - B_M}\right]\delta_1 = 0$$

Then

$$\delta_2 = \frac{\delta_1 p_\ell^2 \nu_\ell - \delta_1^2 \chi_\ell}{\chi_\ell}$$

and (52)

$$\delta_3 = \frac{1}{\chi_\ell}\left\{\delta_2 p_\ell^2 \nu_\ell - \left(\delta_1^2 + 2\delta_1\delta_2\right)\chi_\ell + \left[p_\ell^2 (L + \lambda)\right]^2 \left[\frac{1}{\sqrt{B_M}} - \frac{J_0\left(\sqrt{B_M}\right)}{2\sqrt{B_M}\, J_1\left(\sqrt{B_M}\right)} - \frac{1}{k_\ell^2 - B_M}\right] 1\right\}$$

Therefore, after some rearrangement, we get

$$\Omega_\ell^2 = p_\ell^2\left[1 - \left(\frac{\mu_\ell}{\nu_\ell} - 1\right)\left(\frac{\nu_\ell p_\ell^2}{\Lambda_\ell}\right) - \left(2 - \frac{\mu_\ell}{\nu_\ell}\right)\left(\frac{\mu_\ell}{\nu_\ell} - 1\right)\left(\frac{\nu_\ell p_\ell^2}{\Lambda_\ell}\right)^2 + 0\left(\frac{1}{\Lambda_\ell^3}\right)\right] \tag{53}$$

where

$$\Lambda_\ell = \frac{\chi_\ell}{\Omega_M^2} \tag{54}$$

From Eq. (53) we make two interesting observations:

- The term p_ℓ is the ℓ^{th} nondimensional natural sloshing frequency of the liquid in a rigid tank. Since $(\mu_\ell/\nu_\ell) > 1$, we conclude that $\Omega_\ell^2 < p_\ell^2$.
- The elastic effect is at least of order

$$\frac{\Omega_M^2}{\sinh 2k_\ell L}\left[= 2\,\frac{\chi_\ell}{\Lambda_\ell}\left(\mu_\ell - \nu_\ell\right)\right]$$

Because of the presence of the elastic bottom, the sloshing mode shapes are no longer as simple as those in a rigid tank; but, in rigid tanks, the influence of cross coupling of different fluid modes on the natural frequency is of order Ω_M^6 .

The other limiting case is: ρ_0 , the density of the fluid, tends to zero while B_σ , Ω_σ^2 are bounded away from zero; then Eq. (43) will tend to the free-vibration frequency equation for a circular membrane – i.e.,

$$\begin{aligned} \xi(\Omega^2) &= 1 + \sum_{n=1}^{\infty} \frac{k^2\Omega^2}{k_n^2\Omega^2 - k_n^2} \\ &= \frac{k\Omega J_o(k\Omega)}{2J_1(k\Omega)} = 0 \end{aligned} \tag{55}$$

or

$$J_o(k\Omega) = 0$$

where

$$k^2 = \frac{\rho h r_o^2 \omega^2}{N_r}$$

6 STABILITY OF THE SOLUTION

To study the stability of the solution (Refs. 11 and 12), we shall consider the following more general system of equations:*

$$\begin{aligned} \ddot{y}_1 + \sum_{n=-\infty}^{\infty} A_{2n} e^{i2nt} y_1 + \sum_{n=-\infty}^{\infty} A_{2n-1} e^{i2nt} y_2 &= 0 \\ \ddot{y}_2 + \sum_{n=-\infty}^{\infty} B_{2n} e^{i2nt} y_2 + \sum_{n=-\infty}^{\infty} B_{2n+1} e^{i2nt} y_1 &= 0 \end{aligned} \tag{56}$$

*This is an extension of Hill's method (see Ref. 12, p. 413) to a system of two equations.

where $B_o \neq A_o$, $A_o > 0$, $B_o > 0$, $A_{-1} = B_1 = 0$, $\sum_{n=-\infty}^{\infty} \beta_n$ (where $\beta_n = |A_n|$ or $|B_n|$) is an absolutely convergent series. Equations (56) are invariant when t is changed to $t + \pi$; therefore if $y(t)$ is a solution of Eqs. (58), $y(t+\pi)$ is also a solution. By the Floquet theorem (Ref. 13), Eqs. (56) have solutions of the following form:

$$\underline{y}(t) = e^{\xi t} \underline{\psi}(t)$$

where

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad , \quad \underline{\psi} = -\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

The term $\underline{\psi}(t)$ is periodic function mod(π). If Re $\xi > 0$, $y \to \infty$ as $t \to \infty$ an unbounded solution exists, which is said to be unstable. For a periodic solution mod (π) to exist, Im ξ must be equal to an integer, whereas Re $\xi = 0$

Let us assume a solution of the following form:

$$y_1(t) = e^{\xi t} \sum_{n=-\infty}^{\infty} \gamma_{2n} e^{i2nt}$$

$$y_2(t) = e^{\xi t} \sum_{n=-\infty}^{\infty} \gamma_{2n+1} e^{i2nt}$$

where $\sum_{n=-\infty}^{\infty} n^2 \gamma_{2n}$, $\sum_{n=-\infty}^{\infty} n^2 \gamma_{2n+1}$ are absolutely convergent series.

Substituting into Eqs. (56), we get

$$\sum_{n=-\infty}^{\infty} \gamma_n (2ni + \xi)^2 e^{(2ni+\xi)t} + \sum_{m=-\infty}^{\infty} A_{2m} e^{2mit} \sum_{n=-\infty}^{\infty} \gamma_{2n} e^{(2ni+\xi)t}$$

$$+ \sum_{m=-\infty}^{\infty} A_{2m-1} e^{2mit} \sum_{n=-\infty}^{\infty} \gamma_{2n+1} e^{(2ni+\xi)t} = 0$$

(57)

$$\sum_{n=-\infty}^{\infty} \gamma_{2n+1} (2ni + \xi)^2 e^{(2ni+\xi)t} + \sum_{m=-\infty}^{\infty} B_{2m} e^{2mit} \sum_{n=-\infty}^{\infty} \gamma_{2n+i} e^{(2ni+\xi)t}$$

$$+ \sum_{m=-\infty}^{\infty} B_{2m+1} e^{2mit} \sum_{n=-\infty}^{\infty} \gamma_{2n} e^{(2ni+\xi)t} = 0$$

On rearranging the terms of the absolute convergent series, and equating the coefficients of $e^{(2ni+\xi)t}$ to zero, we obtain

$$-\gamma_{2n} \frac{(i\xi - 2n)^2}{A_0 - 4n^2} + \sum_{m=-\infty}^{\infty} \frac{A_m}{A_0 - 4n^2} \gamma_{2n-m} = 0$$

(58)

$$-\gamma_{2n+1} \frac{(i\xi - 2n)^2}{B_0 - 4n^2} + \sum_{m=-\infty}^{\infty} \frac{B_m}{B_0 - 4n^2} \gamma_{2n+1-m} = 0$$

provided that $A_0 - 4n^2 \neq 0$, $B_0 - 4n^2 \neq 0$. The divisors $A_0 - 4n^2$ and $B_0 - 4n^2$ are introduced in order to make an infinite determinant, which will be formed below, to be convergent.

Equations (58) are a set of homogeneous equations. For γ_n to have nontrivial solutions, the determinant formed by the coefficients of the equations must vanish. Call this determinant $\Delta(i\xi)$; then

$$\Delta(i\xi) = |\alpha_{ij}| = 0 \tag{59}$$

where

$$\alpha_{2m,2m} = \frac{A_0 - (i\xi - 2m)^2}{A_0 - 4m^2}$$

$$\alpha_{2m+1,2m+1} = \frac{B_0 - (i\xi - 2m)^2}{B_0 - 4m^2}$$

$$\alpha_{2m,n} = \frac{A_{2m-n}}{A_0 - 4m^2} \qquad \text{for } 2m - n \neq 0$$

$$\alpha_{2m+1,n} = \frac{B_{2m+1-n}}{B_0 - 4m^2} \qquad \text{for } 2m + 1 - n = \neq 0$$

$$m, n = 0, \pm 1, \pm 2, \ldots$$

We consider another infinite determinant $\Delta_1(i\xi) = |\beta_{ij}|$ where

$$\beta_{m,m} = 1 \tag{60a}$$

$$\beta_{2m,n} = \frac{\alpha_{2m,n}}{\alpha_{2m,2m}} = \frac{A_{2m-n}}{A_0 - (i\xi - 2m)^2} \quad \text{for } 2m - n \neq 0 \tag{60b}$$

$$\beta_{2m+1,n} = \frac{\alpha_{2m+1,n}}{\alpha_{2m+1,2m+1}} = \frac{B_{2m+1-n}}{B_0 - (i\xi - 2m)^2} \quad \text{for } 2m+1-n \neq 0 \tag{60c}$$

Since

$$\prod_{m=-\infty}^{\infty} \beta_{m,m} = 1 \quad , \quad \sum_{\substack{m,n=-\infty \\ m \neq n}}^{\infty} |\beta_{m,m}|$$

converges, provided ξ does not have such a value that one of the denominators of $\beta_{m,n}(i\xi)$ vanishes. Thus, the infinite determinant $\Delta_1(i\xi)$ is absolutely convergent. Then (Ref. 14),

$$\Delta(i\xi) = \Delta_1(i\xi) \lim_{m\to\infty} \prod_{n=-m}^{m} \frac{\left[A_0 - (i\xi - 2n)^2\right]\left[B_0 - (i\xi - 2n)^2\right]}{\left(A_0 - 4n^2\right)\left(B_0 - 4n^2\right)}$$

$$= \Delta_1(i\xi) \frac{\sin\frac{\pi}{2}(i\xi - \sqrt{A_0}) \sin\frac{\pi}{2}(i\xi + \sqrt{A_0}) \sin\frac{\pi}{2}(i\xi - \sqrt{B_0}) \sin\frac{\pi}{2}(i\xi + \sqrt{B_0})}{\sin^2\left(\frac{\pi}{2}\sqrt{A_0}\right) \sin^2\left(\frac{\pi}{2}\sqrt{B_0}\right)} \tag{61}$$

We note some interesting properties of $\Delta_1(i\xi)$: (1) $\Delta_1(i\xi)$ is a meromorphic function of ξ and tends to 1 as $\operatorname{Re}\xi \to \pm\infty$; (2) $\Delta_1(\xi)$ is a periodic function of ξ with period $2i$. If we form another function,

$$\begin{aligned} F(\xi) = \Delta_1(i\xi) &- K_1\left[\cot\frac{\pi}{2}(i\xi + \sqrt{A_0}) - \cot\frac{\pi}{2}(i\xi - \sqrt{A_0})\right] \\ &- K_2\left[\cot\frac{\pi}{2}(i\xi + \sqrt{A_0}) + \cot\frac{\pi}{2}(i\xi - \sqrt{A_0})\right] \\ &- K_3\left[\cot\frac{\pi}{2}(i\xi + \sqrt{B_0}) - \cot\frac{\pi}{2}(i\xi - \sqrt{B_0})\right] \\ &- K_4\left[\cot\frac{\pi}{2}(i\xi + \sqrt{B_0}) + \cot\frac{\pi}{2}(i\xi - \sqrt{B_0})\right] \end{aligned} \tag{62}$$

where K_j's are so chosen that $F(\xi)$ has no poles at $i\xi = \pm\sqrt{A_0}$, $\pm\sqrt{B_0}$, then, since $\Delta_1(i\xi)$ is a periodic function of ξ, it follows that $F(\xi)$ has no poles at

$$i\xi = 2n \pm \sqrt{A_0}, \quad 2n \pm \sqrt{B_0}, \quad n = \pm 1, \pm 2, \ldots.$$

Thus, $F(\xi)$ is a meromorphic function with no pole on the entire plane. $F(\xi)$ is certainly bounded, therefore, by Liouville's theorem, $F(\xi)$ must be a constant, say C. As $\mathrm{Re}\,\xi \to \pm\infty$, $\Delta_1(i\xi) = 1$. Therefore,

$$C = 1 + 2(K_2 + K_4)i \quad \text{as } \mathrm{Re}\,\xi \to \infty$$

$$= 1 - 2(K_2 + K_4)i \quad \text{as } \mathrm{Re}\,\xi \to -\infty$$

Hence, $K_2 + K_3 = 0$, and $F(\xi) = 1$ for all ξ. Using this result and Eqs. (61) and (62), we get

$$\Delta(i\xi) = \frac{\sin^4 \frac{\pi\xi i}{2} - 2\delta_2 \sin^2 \frac{\pi\xi i}{2} + \delta_1 + \delta_3 \sin \frac{\pi\xi i}{2}}{\sin^2\left(\frac{\pi}{2}\sqrt{A_0}\right) \sin^2\left(\frac{\pi}{2}\sqrt{B_0}\right)} \tag{63}$$

where δ_i are some constants relating to K_j's, A_0, and B_0. Put $i\xi = 0$, $1/2$ and 1 in Eq. (63) and we get

$$\delta_1 = \Delta(0) \sin^2\left(\frac{\pi}{2}\sqrt{A_0}\right) \sin^2\left(\frac{\pi}{2}\sqrt{B_0}\right)$$

$$2\delta_2 = 1 + \sin^2\left(\frac{\pi}{2}\sqrt{A_0}\right) \sin^2\left(\frac{\pi}{2}\sqrt{B_0}\right)\left[\Delta(0) - \Delta(1)\right] \tag{64}$$

and

$$2\delta_3 = \frac{1}{2} + \sin^2\left(\frac{\pi}{2}\sqrt{A_0}\right) \sin^2\left(\frac{\pi}{2}\sqrt{B_0}\right)\left[2\Delta\left(\frac{1}{2}\right) - \Delta(0) - \Delta(1)\right]$$

In a special case, if the coefficients of Eq. (56) are even functions of t, then Eq. (56) is unchanged when we change t to $-t$; we see that, if ξ is a solution, then $-\xi$ is also a solution. Therefore, if $\Delta(i\xi_0) = 0$ and $\sin(i\xi_0\pi) \neq 0$, then $\delta_3 = 0$. Therefore, when we want to find the roots for $\Delta(i\xi) = 0$, we always have $\delta_3 \sin(\pi_i \xi) = 0$ and the roots of Eq. (63) can be written out in a simple form,

$$\sin^2 \frac{\pi\xi i}{2} = \delta_2 \pm \sqrt{\delta_2^2 - \delta_1^2} \tag{65}$$

For a bounded solution, i.e. for $\mathrm{Re}\,\xi = 0$, we must have

$$1 \geq \delta_2 \pm \sqrt{\delta_2^2 - \delta_1^2} \geq 0 \tag{66}$$

In Eq. (63), by putting $\Delta(i\xi) = 0$, we can compute ξ, and determine whether this is an unbounded solution or not. Then, from Eq. (58), we can compute γ_n, and obtain the complete solution of Eqs. (56). For a periodic solution, we must have $\Delta(0) = 0$ or $\Delta(1) = 0$.

If a periodic solution of an inhomogeneous counterpart of Eqs. (56) is considered, e.g.

$$\ddot{\underline{y}} + \underline{A}(t)\underline{y} = \underline{B}(t)$$

where $B(t)$ is a column matrix with its elements as periodic function, we can use Eq. (57) (with non-zero righthand side) by putting $\xi = 0$, whereas Eqs. (58) become inhomogeneous. If $\Delta(0)$ is not equal to 0, we can solve for γ_n uniquely. If $\Delta(0) = 0$ we are on the boundary where Eqs. (56) have an unbounded solution. Therefore, for such an inhomogeneous equation as in our problem, the zone of instability is determined by the homogeneous solution.

7 APPROXIMATE SOLUTIONS

We choose

$$\Phi = d_o(\tau)Z + c_o(\tau) + \sum_{n=m}^{N} J_o(k_n R) \left[\dot{c}_n(\tau) \frac{\cosh k_n Z}{\sinh k_n L} + \dot{d}_n(\tau) \frac{\sinh k_n Z}{\cosh k_n L} \right] \tag{67}$$

$$H = d_o(\tau) + \sum_{n=m}^{N} d_n(\tau) \frac{k_n J_o(k_n R)}{\cosh k_n L} \tag{68}$$

and

$$W = d_o(\tau) + \sum_{n=m}^{N} k_n \left[d_n(\tau) - c_n(\tau) \right] J_o(k_n R) + f(R) \tag{69}$$

where

$$d_o(\tau) = \sum_{n=m}^{N} k_n \left[c_n(\tau) - d_n(\tau) \right] J_o(k_n) \tag{70}$$

These equations satisfy Eqs. (18), (19), and (21) through (25). Then the equations

$$\Omega_\sigma^2 \ddot{c}_o(\tau) + B_\sigma G(\tau) d_o(\tau) = 0 \tag{71a}$$

$$\Omega_\sigma^2 \ddot{c}_n(\tau) + k_n \tanh k_n L \left[k_n^2 + B_\sigma G(\tau) \right] d_n(\tau) = 0 \tag{71b}$$

hold for $n = m, m+1, \ldots N$. Now, $N - (m-1)$ of the c_n, d_n are left arbitrary. To determine these arbitrary functions, we use the Ritz method by substituting Φ, H, and W of Eqs. (67) through (70) into the variational equation

$$\delta \int_{\tau_1}^{\tau_2} I\left(\Phi, H, W; \tau\right) d\tau = 0 \tag{72}$$

The Euler equations are exactly Eq. (36) for N = m ... N, with c_o, d_o satisfying Eq. (71a).

For free vibration, by taking a single term m = N, we get the "one-term approximation":

$$1 + \frac{\Omega_M^2 (L + \lambda)\left(\Omega^2 - p_m^2\right) k_m}{\beta_m} = 0 \tag{73a}$$

or

$$\begin{matrix}\Omega_1 \\ \Omega_2\end{matrix} = \frac{1}{2} p_m^2 \left\{ \frac{\Lambda_m}{\nu_m p_m^2} + \frac{\mu_m}{\nu_m} \mp \sqrt{\left(\frac{\Lambda_m}{\nu_m p_m^2} + \frac{\mu_m}{\nu_m}\right)^2 - \frac{4\Lambda_m}{\nu_m p_m^2}} \right\} \tag{73b}$$

which is the same as neglecting all the terms in the summation of Eq. (43) except the m^{th} term. If we take two terms, i.e., N = m + 1 ("two-term approximation"), we get the following characteristic equation for Ω :

$$1 + \sum_{n=m}^{m+1} \frac{\Omega_M^2 (L + \lambda)\left(\Omega^2 - p_n^2\right) k_n}{\beta_n} = 0 \tag{74}$$

8 NUMERICAL RESULTS

Some numerical results are obtained using the method discussed above. In Figs. 2 through 5, the natural frequencies normalized by the sloshing frequency p_1, which equals

$$\left[\frac{k_1\left(k_1^2 + B_\sigma\right)\tanh k_1 L}{\Omega_\sigma^2}\right]^{1/2}$$

of the corresponding rigid tank, versus the membrane number B_M, which equals $\left(\rho_o g_o r_o^2\right)/N_r$, were plotted for parameters B_σ, L, λ. Since Ω is normalized by p_1, the parameters Ω_σ^2 and Ω_M^2, which equals $\Omega_\sigma^2\left(B_M/B_\sigma\right)$ can be eliminated. The computation was based on Eq. (73) (one-term approximation), with $m = 1$, and on Eq. (43). In most of the calculations, the one-term approximation and exact solution gave almost identical results for the lowest frequency, which cannot be distinguished in the figures shown.

From these figures, we see the general trend clearly that frequencies decrease as B_M increases. In Fig. 2, the Bond number is large ($B_\sigma = 10$), and B_M does not have much effect on the lowest frequency. As B_σ decreases (see Figs. 3 and 4, $B_\sigma = 1,\ 0.1$), the great effect of B_M can be seen. At certain ranges of B_M, the lowest frequency decreases sharply as B_M increases; and the range is approximately determined by the ratio of B_σ to B_M, and the depth of liquid L. This relationship can be seen clearly from Eq. (43), if the terms in the summation sig. are normalized in a slightly different form, i.e., for B_σ, B_M small,

$$1 + \sum_{n=1}^{\infty} \frac{k_n(L+\lambda)\left[1 - \left(\frac{\Omega}{p_1}\right)^2 \frac{\left(k_1^2 + B_\sigma\right) k_1 \tanh k_1 L}{\left(k_n^2 + B_\sigma\right) k_n \tanh k_n L}\right]}{\coth k_n L + \lambda k_n - \frac{\Omega_1^2 \left(k_1^2 + B_\sigma\right) k_1 \tanh k_1 L}{p_1^2 \left(k_n^2 + B_\sigma\right) k_n \tanh k_n L} + \frac{k_n^2 - B_M}{\left(k_n^2 + B_\sigma\right) \tanh k_n L} \frac{B_\sigma}{B_M}\left(\frac{\Omega^2}{p_n^2} - 1\right)}$$

$$\simeq 1 + \sum_{n=1}^{\infty} \frac{k_n(L+\lambda)\left[1 - \left(\frac{\Omega}{p_1}\right)^2 \frac{k_1^3 \tanh k_1 L}{k_n^3 \tanh k_n L}\right]}{\coth k_n L + \lambda k_n - \frac{\Omega^2 k_1^3 \tanh k_1 L}{p_1^2 k_n^3 \tanh k_n L} + \frac{1}{\tanh k_n L} \frac{B_\sigma}{B_M}\left[\left(\frac{\Omega}{p_1}\right)^2 \frac{k_1^3 \tanh k_1 L}{k_n^3 \tanh k_n L} - 1\right]} = 0 \qquad (75)$$

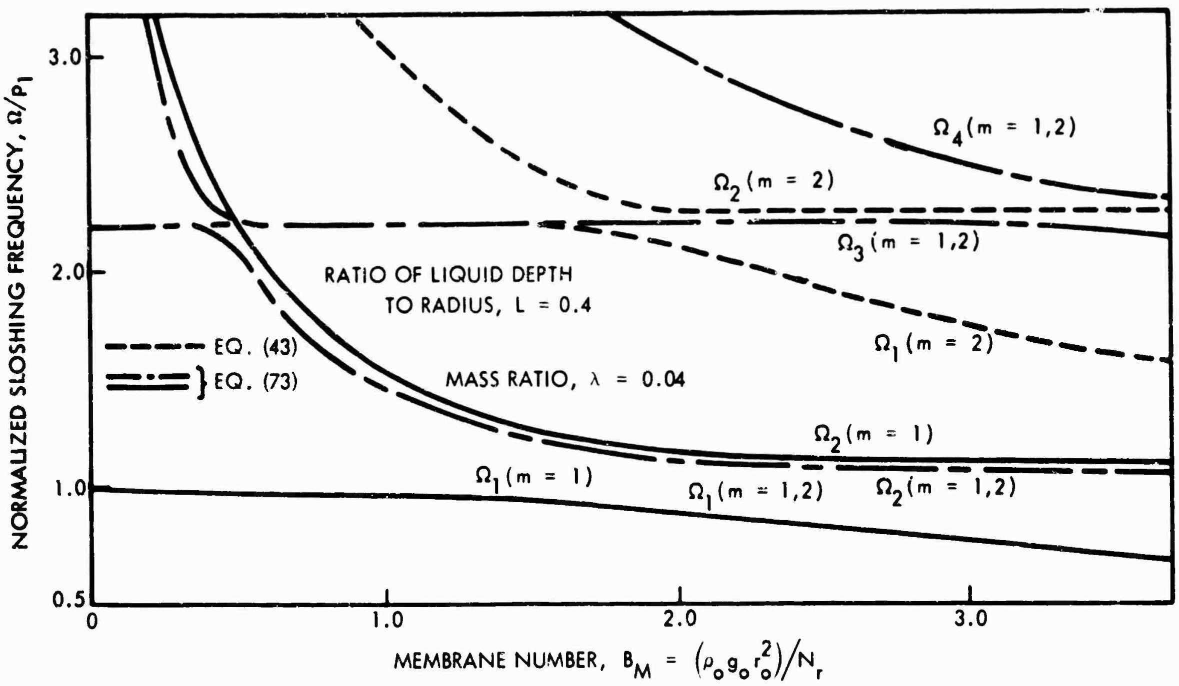

Fig. 2 Frequencies Computed From Eqs. (73) and (43), With Bond Number 10

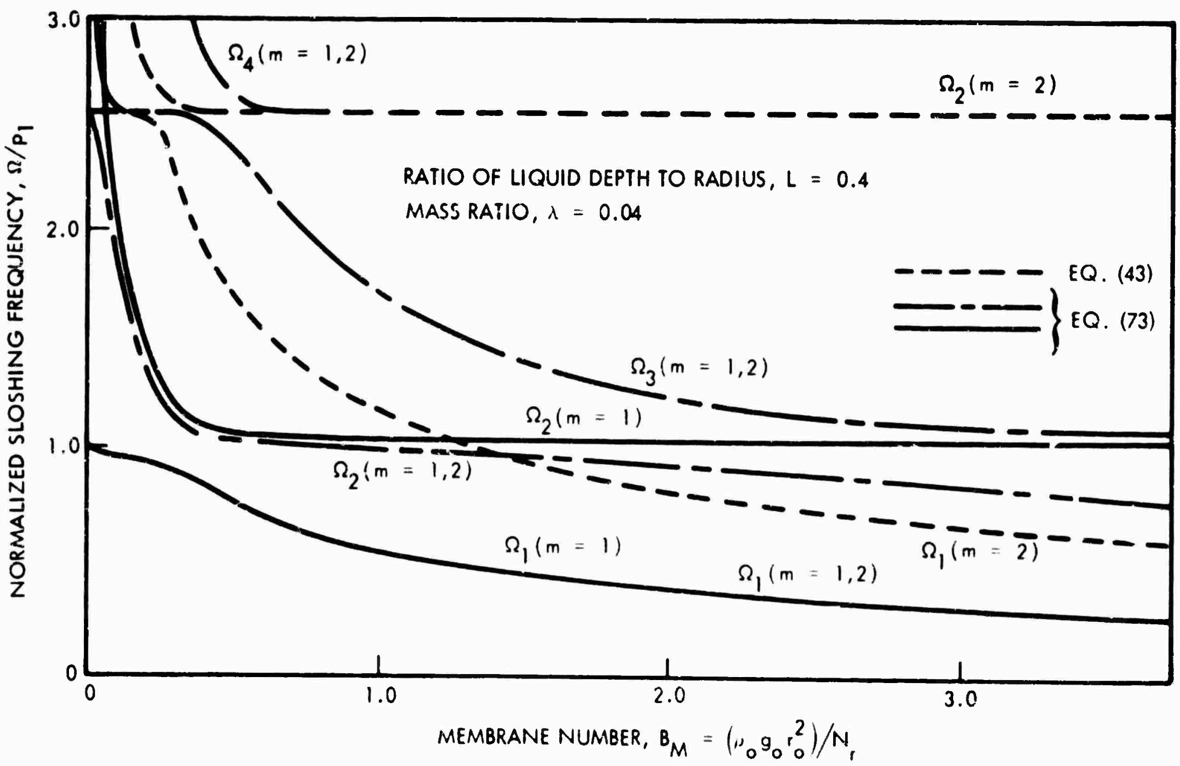

Fig. 3 Frequencies Computed From Eqs. (73) and (43), With Bond Number 1.0

Apparently the roots for Ω/p_1 depend on B_σ/B_M only for fixed L, λ. Physically it says that, at low Bond number, the surface tension plays an important role; its effect depends mainly on B_M/B_σ (the ratio of the surface tension to midplane stress resultant of the membrane). The argument holds for any N-term approximation.

In Figs. 4 and 5 we can see the effect of the depth of the liquid to radius ratio L. It causes the sharp decrease of frequency occurring at smaller B_M for larger L.

In Fig. 6 (a and b), the stability boundary with α $(= g_1/g_0)$ is plotted versus forcing frequency normalized by the rigid tank first sloshing frequency, according to Eq. (A-9) in the Appendix, with $i = j = 1$ for one-term approximation only.

9 CONCLUSIONS

From the exact solution for free vibration, we make four conclusions for the eigenvalue Ω: (1) Ω is always real, for p_n^2, $\chi_n > 0$; (2) the elastic effect lowers the natural vibration frequency; (3) the elastic effect on each sloshing frequency is of order $\Omega_M^2/\sinh 2k_n L$; (4) because of the presence of the elastic bottom, coupling of different sloshing modes occurs, and the effect of one sloshing mode on the natural frequency of another due to this elastic coupling is of order Ω_M^6.

Numerical results indicate a great effect of surface tension on the natural frequency and stability boundary at low Bond numbers. If B_σ, B_M are both small (<1), then the frequency depends on the combination of parameters B_M/B_σ.

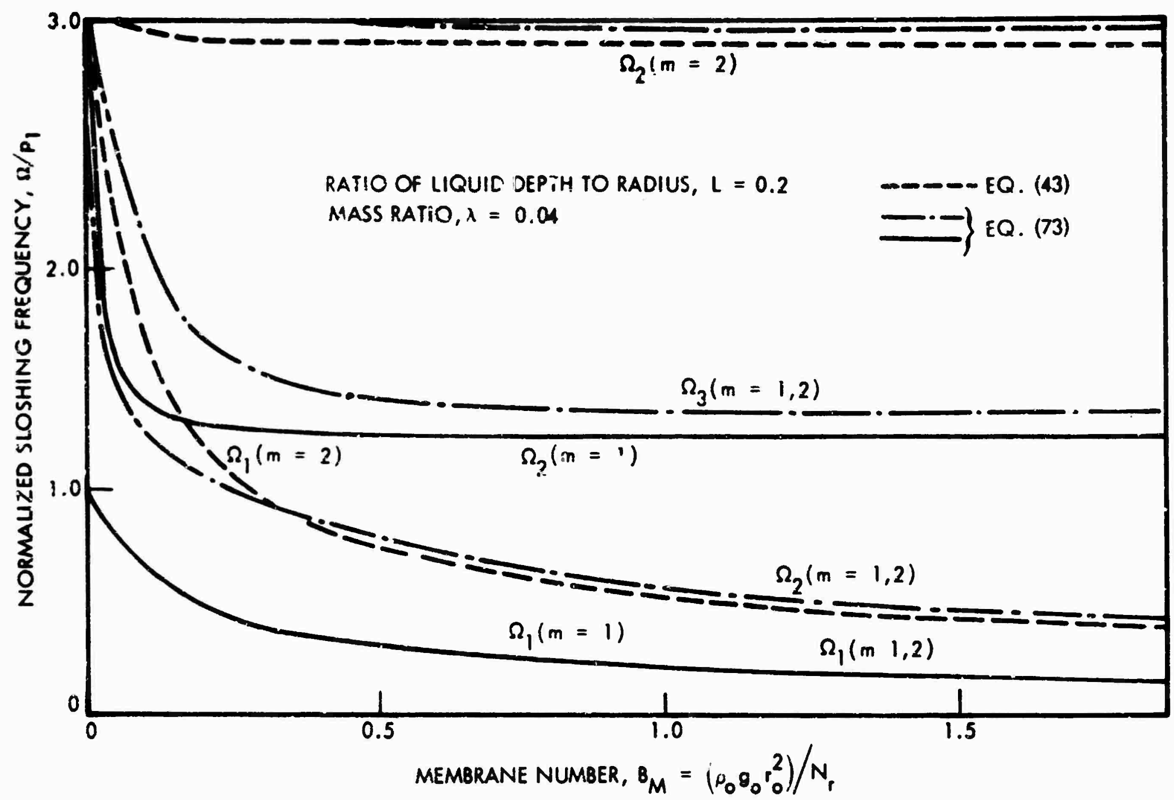

Fig. 4 Frequencies Computed From Eqs. (73) and (43), With Bond Number 0.1 and Lower Ratio of Liquid Depth to Radius

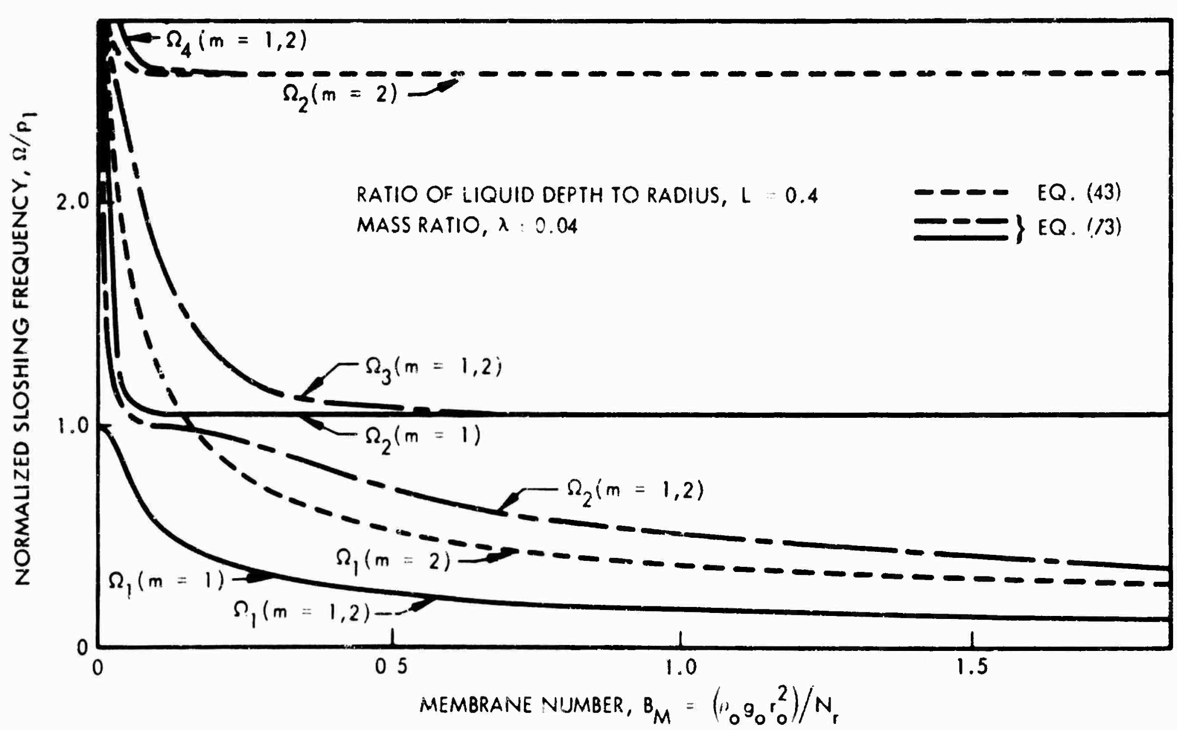

Fig. 5 Frequencies Computed From Eqs. (73) and (43), With Bond Number 0.1

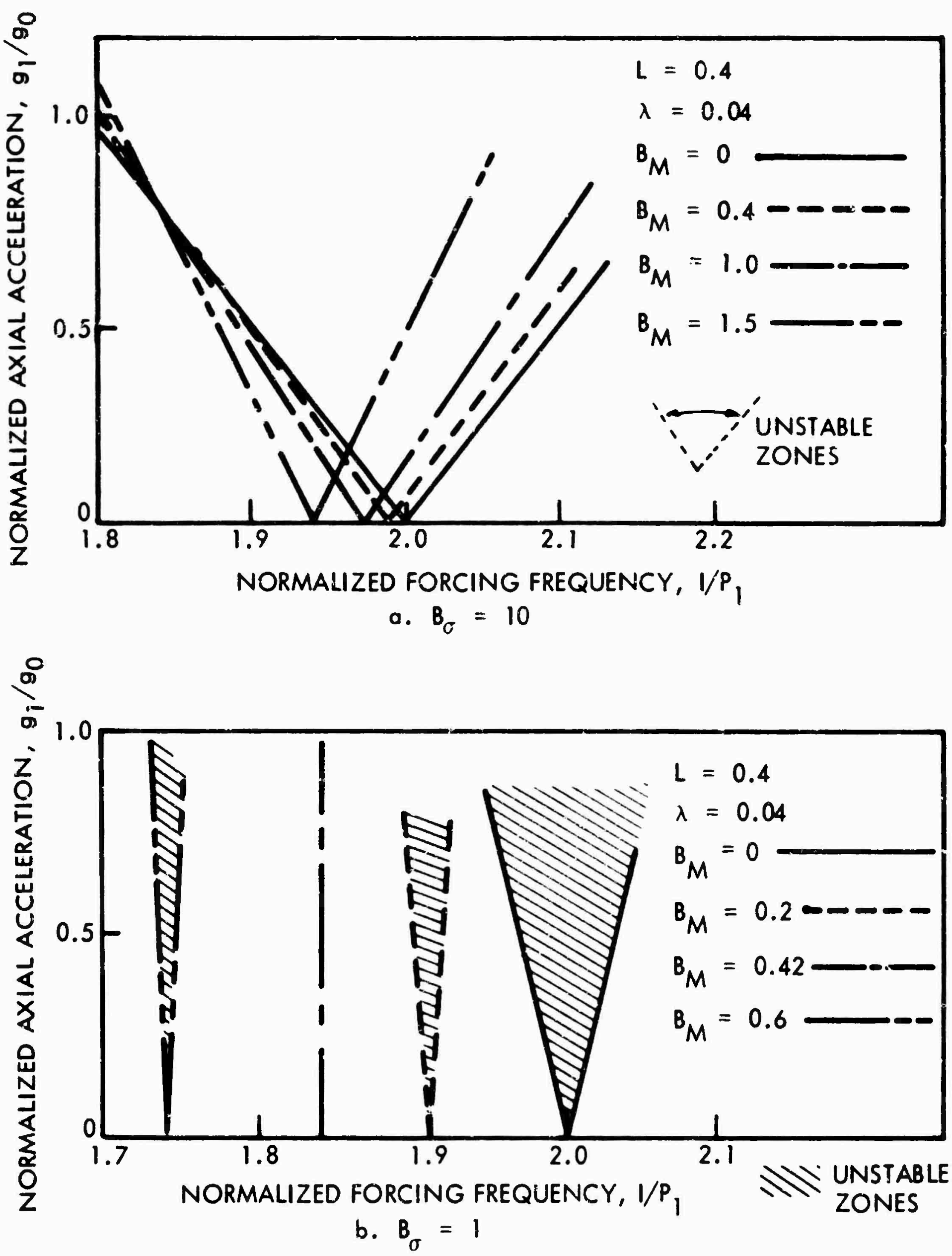

Fig. 6 Stability Boundary Computed From Eq. (75), for Various Values of B_σ and B_M

10 CITED REFERENCES

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Appendix A

APPROXIMATE BOUNDARY FOR A SYSTEM OF DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS TO HAVE UNBOUNDED SOLUTION

Following Struble's (Ref. 15) approximate stability boundary for Mathieu's equation, we extend it to a system of N equations:

$$\underline{\ddot{y}} + \underline{D}y = \alpha \cos t\, \underline{Ay} \tag{A-1}$$

where

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\underline{D} = \begin{bmatrix} \Omega_1^2 & & & 0 \\ & \Omega_2^2 & & \\ & & \ddots & \\ 0 & & & \Omega_n^2 \end{bmatrix} \tag{A-2}$$

$$\underline{A} = (a_{ij})$$

and Ω_i's are different from zero, α is a small parameter, a_{ij} are constants. We assume that y_i's have the expansions

$$y_i = E_i(t) \cos(\Omega_i t - \theta_i t) + \alpha y_i^{(1)} + \alpha^2 y_i^{(2)} + \ldots\ldots \tag{A-3}$$

where $E_i(t)$, $\theta_i(t)$ are slowly varying parameters, and that $\dot{E}_i^2$, $\dot{\theta}_i^2$, $\ddot{E}_i$, $\dot{\theta}_i \dot{E}_i$, $\ddot{\theta}_i$ can be neglected as compared with $\dot{E}_i$, $\dot{\theta}_i$. Substituting (A-3) into (A-1), we get

$$2\Omega_i \left[E_i \dot{\theta}_i + 0\left(\dot{\theta}_i^2, \dot{E}_i^2 \ldots\right)\right] \cos(\Omega_i t - \theta_i) - 2\Omega_i (\dot{E}_i + \ldots\ldots) \sin(\Omega_i t - \theta_i) + \alpha \left[\ddot{y}_i^{(1)} + \Omega_i^2 y_i^{(1)}\right]$$

$$= \frac{\alpha}{2} \sum_{j=1}^{n} a_{ij} E_j \left\{\cos\left[(1 + \Omega_j) t - \theta_j\right] + \cos\left[(1 - \Omega_j) t + \theta_j\right]\right\} + 0(\alpha^2) \qquad i = 1, 2, \ldots\ldots, n \tag{A-4}$$

For $\Omega_i \neq 1 \pm \Omega_j$, we may solve for $y_i^{(1)}$ immediately by taking E_i, θ_i to be constants. But there is a possibility that the expansion for $y_i^{(1)}$ breaks down because of zero or small divisors for the solution of $y_i^{(1)}$.* In order to avoid this difficulty, we can use E_i, θ_i to cancel those terms that cause trouble. We shall consider the case that $\Omega_i + \Omega_j = 1 + 0(\alpha)$ for only one pair of specific i, j.

In this case, we remove the troublesome terms in (A-4) by putting

$$\begin{aligned}
2\Omega_i \dot{\theta}_i E_i \cos(\Omega_i t - \theta_i) - 2\Omega_i \dot{E}_i \sin(\Omega_i t - \theta_i) \\
&= \frac{\alpha}{2} a_{ij} E_j \cos\left[(1 - \Omega_j) t + \theta_j\right] \\
&= \frac{\alpha}{2} a_{ij} E_j \Big\{ \cos\left[(1 - \Omega_i - \Omega_j) t + \theta_i + \theta_j\right] \cos(\Omega_i t - \theta_i) \\
&\qquad - \sin\left[(1 - \Omega_i - \Omega_j) t + \theta_i + \theta_j\right] \sin(\Omega_i t - \theta_i) \Big\}
\end{aligned} \tag{A-5}$$

and a similar equation with the subscripts i, j interchanged. Then by equating the coefficients of $\cos(\Omega_r t - \theta_r)$, $\sin(\Omega_r t - \theta_r)$, $r = i, j$ we get

$$\begin{aligned}
2\Omega_i \dot{\theta}_i E_i &= \frac{\alpha}{2} a_{ij} E_j \cos\left[(1 - \Omega_i - \Omega_j) t + \theta_i + \theta_j\right] \\
2\Omega_i \dot{E}_i &= \frac{\alpha}{2} a_{ij} E_j \sin\left[(1 - \Omega_i - \Omega_j) t + \theta_i + \theta_j\right] \\
2\Omega_j \dot{\theta}_j E_j &= \frac{\alpha}{2} a_{ji} E_i \cos\left[(-\Omega_i - \Omega_j) t + \theta_i + \theta_j\right] \\
2\Omega_j \dot{E}_j &= \frac{\alpha}{2} a_{ji} E_i \sin\left[(1 - \Omega_i - \Omega_j) t + \theta_i + \theta_j\right]
\end{aligned} \tag{A-6}$$

*Struble, R.A., Nonlinear Differential Equations, New York, McGraw-Hill Book Company, 1962, Chapter 8, pp. 221–227

It can be easily shown that

$$\begin{aligned} \Omega_i \, a_{ji} \, E_i^2 &= \Omega_j \, a_{ij} \, E_j^2 \\ \theta_i &= \theta_j \end{aligned} \tag{A-7}$$

Substituting into Eq. (A-6), it becomes

$$\begin{aligned} 2\Omega_i \, \dot{\theta}_i &= \frac{\alpha}{2}\left(\frac{a_{ij} \, q_{ji} \, \Omega_i}{\Omega_j}\right)^{1/2} \cos\left[(1 - \Omega_i - \Omega_j)t + 2\theta_i\right] \\ 2\Omega_i \, \dot{E}_i &= \frac{\alpha}{2}\left(\frac{a_{ij} \, a_{ji} \, \Omega_i}{\Omega_j}\right)^{1/2} E_i \cos\left[(1 - \Omega_i - \Omega_j)t + 2\theta_i\right] \end{aligned} \tag{A-8}$$

The values of $\Omega_i, \Omega_j, a_{ij}, a_{ji}, \alpha$ will determine whether E_i, E_j in the above equation, has unbounded solutions or not. We note that Eq. (A-8) has exactly the same form as that obtained by Struble for a single Mathieu's equation. If $a_{ij} \, a_{ji} \, \Omega_i/\Omega_j$ is less than 0 from Struble's result, all solutions of E_i are bounded; if $a_{ij} a_{ji} \Omega_i \Omega_j$ is greater than 0, its stability criterion is

$$|1 - \Omega_i - \Omega_j| \begin{array}{c} > \\ < \end{array} \left|\frac{\alpha}{2}\left(\frac{a_{ij} \, a_{ji}}{\Omega_i \, \Omega_j}\right)^{1/2}\right| \qquad \begin{array}{l} \text{Stable} \\ \text{Unstable} \end{array} \tag{A-9}$$